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Qualitative properties of generalized principal eigenvalues for superquadratic viscous Hamilton–Jacobi equations

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Abstract. This paper is concerned with the ergodic problem for superquadratic viscous Hamilton–Jacobi equations with exponent $m > 2$. We prove that the generalized principal eigenvalue of the equation converges to a constant as $m \to \infty$, and that the limit coincides with the generalized principal eigenvalue of an ergodic problem with gradient constraint. We also investigate some qualitative properties of the generalized principal eigenvalue with respect to a perturbation of the potential function. It turns out that different situations take place according to $m = 2$, $2 < m < \infty$, and the limiting case $m = \infty$.

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1. Introduction

In this paper we study the ergodic problem for the following superquadratic viscous Hamilton–Jacobi equation with exponent $m > 2$:

$$\lambda - \Delta u + \frac{1}{m}|Du|^m - f = 0 \quad \text{in } \mathbf{R}^N, \tag{1.1}$$

where Du and Δu denote the gradient and the Laplacian of $u : \mathbf{R}^N \to \mathbf{R}$, respectively, and $f : \mathbf{R}^N \to \mathbf{R}$ is assumed to be continuous on $\mathbf{R}^N$ and to vanish as $|x| \to \infty$. The unknown of (1.1) is the pair of a real constant λ and a function u. We denote by λ_m the generalized principal eigenvalue of (1.1) which is defined by

$$\lambda_m := \sup\{\lambda \in \mathbf{R} \,|\, (1.1) \text{ has a continuous viscosity subsolution} u\}. \tag{1.2}$$

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Here and in what follows, unless otherwise specified, every solution (subsolution, supersolution) u is understood in the viscosity sense. We refer, for instance, to [5,13] for the definition and fundamental properties of viscosity solutions.

The objective of this paper consists of two parts, which we present as A and B below.

A. Convergence as $m \to \infty$. We study the convergence of λ_m as $m \to \infty$. More precisely, let us consider the following ergodic problem with gradient constraint:

$$\max\left\{\lambda - \Delta u - f, |Du| - 1\right\} = 0 \quad \text{in } \mathbf{R}^N. \tag{1.3}$$

Let λ_∞ denote the generalized principal eigenvalue of (1.3) defined, similarly as (1.2), by the supremum of $\lambda \in \mathbf{R}$ such that (1.3) has a continuous viscosity subsolution u. Then we prove that λ_m converges to λ_∞ as $m \to \infty$. In this sense, ergodic problem (1.3) can be regarded as the extreme case of (1.1) where $m = \infty$. Note that (1.3) has been studied by [7,8] for functions f that are smooth, convex, and of superlinear growth as $|x| \to \infty$. In these papers, λ_∞ is derived from the limit of $\delta v_\delta(0)$ as $\delta \to 0$, where v_δ is the solution to the following equation:

$$\max\left\{\delta v_\delta - \Delta v_\delta - f, |Dv_\delta| - 1\right\} = 0 \quad \text{in } \mathbf{R}^N.$$

The present paper provides another characterization of λ_∞ in terms of λ_m under a different type of assumptions on f. We mention that gradient constraint problems also arise from other types of limiting procedures, e.g., the limit of p-Laplace equations as $p \to \infty$. See, for instance, [12] and references therein for this topic.

B. Qualitative properties. We introduce a real parameter β and consider (1.1) and (1.3) with βf in place of f. We are interested in qualitative properties of the generalized principal eigenvalue $\lambda_m = \lambda_{m,\beta}$ with respect to β. In order to illustrate our main results briefly, we assume, for a moment, that f is nonnegative in $\mathbf{R}^N$ with compact support (this can be relaxed, see Sect. 4). Then it turns out that there exists a critical value $\beta_c \leq 0$ such that $\lambda_{m,\beta} = 0$ for all $\beta \geq \beta_c$, while $\lambda_{m,\beta} < 0$ for all $\beta < \beta_c$. Notice here that the value of β_c, especially, its negativity depends sensitively on m and N. More specifically, the following three situations occur according to the choice of m:

(a) if $m = 2$, then $\beta_c = 0$ for $N = 1, 2$ and $\beta_c < 0$ for all $N \geq 3$;
(b) if $2 < m < \infty$, then $\beta_c = 0$ for $N = 1$ and $\beta_c < 0$ for all $N \geq 2$;
(c) if $m = \infty$, then $\beta_c < 0$ for all $N \geq 1$.

The quadratic case (a) has been proved in [10, Theorem 2.5], and the second claim in (b) (i.e., the case where $2 < m < \infty$ and $N \geq 2$) is also suggested by [11, Theorem 2.4] in a slightly different context. The essential novelty of this paper, compared with [10,11], lies in the simultaneous derivation of (b) and (c) in combination with the convergence result obtained in part A. In particular, claim (c) for $N \geq 2$ can be derived by passing to the limit in (b) as $m \to \infty$. To the best of our knowledge, such a qualitative analysis of $\lambda_{m,\beta}$, especially for $m = \infty$, seems to be new. We remark that we consider not only nonnegative functions f but also sign-changing ones, which lead to a more complex picture where two critical parameters $\beta_- \le \beta_+$ will play the role of the above β_c. For instance, if $N \ge 2$ and $2 < m \le \infty$, then there exist $\beta_- < 0 < \beta_+$ such that $\lambda_{m,\beta} = 0$ for any $\beta \in [\beta_-, \beta_+]$, while $\lambda_{m,\beta} < 0$ outside this interval. See Sect. 4 for details.

Our study of critical value β_c is strongly motivated by the stochastic control interpretation of $\lambda_{m,\beta}$. Loosely speaking, if $2 \le m < \infty$, then the principal eigenvalue $\lambda_{m,\beta}$ coincides with the optimal value of the following ergodic stochastic control problem:

$$\begin{aligned} &\text{Minimize} \quad \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left\{\frac{1}{m^*}|\xi_t|^{m^*} + \beta f(X_t^\xi)\right\} dt\right], \\ &\text{subject to} \quad X_t^\xi = \sqrt{2} W_t + \int_0^t \xi_s ds, \quad t \ge 0, \end{aligned} \tag{1.4}$$

where $m^* := m/(m-1)$, and $W = (W_t)$ and $\xi = (\xi_t)$ denote, respectively, an N-dimensional standard Brownian motion and an $(\mathcal{F}_t)$-adapted control process defined on some filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$. If $f \ge 0$ in $\mathbf{R}^N$ and $\beta \ge 0$, then this is nothing but a minimization problem of the total cost $(1/m^*)|\xi_t|^{m^*} + \beta f(X_t^\xi)$. The situation becomes delicate as far as $\beta < 0$. Intuitively, the controller of the optimization problem (1.4) falls into a trade-off situation between minimizing the cost $(1/m^*)|\xi_t|^{m^*}$ and maximizing the reward $|\beta| f(X_t^\xi)$. The dominant term depends on the magnitude of $|\beta|$, and the critical value β_c is determined as the threshold at which the controller changes his/her optimal choice: either "minimize cost" or "maximize reward". In particular, the negativity of β_c implies the existence of such "phase transition", which we intend to characterize in the present paper.

As to the limiting case where $m = \infty$, the value $\lambda_{\infty,\beta}$ is related to the following singular ergodic stochastic control problem:

$$\begin{aligned} &\text{Minimize} \quad \limsup_{T\to\infty} \frac{1}{T} E\left[|\eta|_T + \int_0^T \beta f(X_t^\eta)\, dt\right], \\ &\text{subject to} \quad X_t^\eta = \sqrt{2} W_t + \eta_t, \quad t \ge 0, \end{aligned}$$

where $\eta = (\eta_t)$ stands for an $(\mathcal{F}_t)$-adapted control process of bounded variations, and $|\eta|_T$ denotes its bounded variation norm. We refer, for instance, to [17] and references therein for more information on singular ergodic stochastic control and associated PDEs with gradient constraint. See also [9–11] for the stochastic control interpretation of $\lambda_{m,\beta}$ for $2 \le m < \infty$. In this paper, we focus only on the PDE aspect and do not discuss its probabilistic counterpart.

Before closing this introductory section, we mention that (1.2) can be regarded as a nonlinear extension of the generalized principal eigenvalue in the sense of [3,19], where such notion is defined for linear elliptic operators (see also [2,18]). More specifically, Let Ω be a domain in $\mathbf{R}^N$ and let $L := \sum_{i,j} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x)$ be an elliptic operator in Ω. Then, under suitable assumptions on the coefficients, the generalized principal eigenvalue is defined by

$$\lambda^* := \sup\{\lambda \in \mathbf{R} \,|\, \exists\, \phi > 0 \ \text{ such that } \ L\phi + \lambda\phi \le 0 \ \text{ in } \Omega\},$$

where the meaning of the solution depends on the context (classical solution, strong solution, viscosity solution, etc). Note that λ_2 (i.e. λ_m for $m = 2$) coincides with λ^* when $\Omega = \mathbf{R}^N$ and $L = 2\Delta - f$. Indeed, if $m = 2$ in (1.1), then u is a subsolution of (1.1) if and only if the positive function $\phi := e^{-u/2}$ is a supersolution of $(2\Delta - f)\phi + \lambda\phi \le 0$ in $\mathbf{R}^N$. In this sense, λ_m is a generalization of the above λ^*, and it plays, in our context, the role of the generalized principal eigenvalue for nonlinear additive eigenvalue problem (1.1).

The organization of the paper is as follows. In the next section, we discuss the solvability of (1.1). Specifically, we prove that, for any $\lambda \le \lambda_m$, there exists a viscosity solution u of (1.1). In Sect. 3, we prove the convergence of λ_m as $m \to \infty$. Section 4 is devoted to qualitative properties of $\lambda_{m,\beta}$ with respect to β.

2. Solvability of (1.1)

We collect some notation used throughout the paper. For any $R > 0$, B_R stands for the open ball of radius R, centered at the origin. For an integer $k \ge 0$ and $p \in [1, \infty]$, we denote by $W^{k,p}(\mathbf{R}^N)$ the standard Sobolev space. For a given open set $\Omega \subset \mathbf{R}^N$ and any integer $k \ge 0$ and $\gamma \in (0, 1)$, we use the notation $C^{k,\gamma}(\overline{\Omega})$ to denote the Hölder space (or Lipschitz space if $k = 0$ and $\gamma = 1$) which consists of all $f \in C^k(\overline{\Omega})$ such that

$$|f|_{k,\gamma;\Omega} := \sum_{|\alpha| \le k} \max_{x \in \Omega} |D^\alpha f(x)| + \sum_{|\alpha| = k} \sup_{x,y \in \Omega,\ x \ne y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\gamma} < \infty,$$

where α is the multi-index of $D = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. Furthermore, we denote by $C^{k,\gamma}(\mathbf{R}^N)$ the set of functions $f \in C^k(\mathbf{R}^N)$ such that $|f|_{k,\gamma;\Omega} < \infty$ for any compact set $\overline{\Omega}$. Notice here that functions in $C^{k,\gamma}(\mathbf{R}^N)$ may not be bounded on $\mathbf{R}^N$, in general. We also denote by $C_c^\infty(\mathbf{R}^N)$ the set of smooth functions with compact support. Finally, let $C_0(\mathbf{R}^N)$ stand for the totality of continuous functions $f \in C(\mathbf{R}^N)$ vanishing at infinity, namely, $\sup_{|x| \ge r} |f(x)| \to 0$ as $r \to \infty$ (we express this property simply as $f(x) \to 0$ as $|x| \to \infty$).

Let $m > 2$ and consider the ergodic problem

$$\lambda - \Delta u + \frac{1}{m}|Du|^m = f \quad \text{in } \ \mathbf{R}^N, \quad u(0) = 0, \tag{2.1}$$

where the constraint $u(0) = 0$ is imposed to avoid the ambiguity of additive constants with respect to u. Throughout this paper, we assume without mentioning that f satisfies the following:

(A1) $f \in C_0(\mathbf{R}^N)$.

To begin with, we recall some regularity estimates that will be needed repeatedly.

Theorem 2.1. *Let* $\alpha := (m-2)/(m-1)$.

(i) *For any $R > 0$, there exists a constant $M_R > 0$ such that*

$$|u(x) - u(y)| \leq M_R|x - y|^\alpha, \quad x, y \in B_R,$$

for any locally bounded upper semicontinuous viscosity subsolution u of (2.1), *where M_R depends on $\max_{B_R}|f - \lambda|$, but is independent of any large $m > 2$.*

(ii) *Suppose that $f \in C^{0,1}(\mathbf{R}^N)$. Then, for any $R > 0$, there exists a constant $K_R > 0$ such that*

$$|u(x) - u(y)| \leq K_R|x - y|, \quad x, y \in B_R,$$

for any continuous viscosity solution u of (2.1), *where K_R may depend on the sup-norm and the Lipschitz norm of $f - \lambda$ over a larger ball, say B_{R+1}, but is independent of any large $m > 2$.*

Proof. This theorem is a direct consequence of [4, Theorems 1.1 and 3.1]. Notice here that the gradient term of the equation in [4] is not $(1/m)|Du|^m$ but $|Du|^p$ with $p > 2$. However, by a careful reading of their proofs, one can see that M_R and K_R can be taken uniformly with respect to any large $m > 2$. □

It is obvious from Theorem 2.1 that any locally bounded upper semicontinuous viscosity subsolution of (2.1) belongs to $C^{0,\alpha}(\mathbf{R}^N)$ with $\alpha = (m - 2)/(m - 1)$. Taking this fact into account, one can redefine the generalized principal eigenvalue of (2.1) by

$$\lambda_m := \sup\{\lambda \in \mathbf{R} \,|\, (2.1) \text{ has a viscosity subsolution } u \in C^{0,\alpha}(\mathbf{R}^N)\}. \quad (2.2)$$

Note here that $\lambda_m \neq -\infty$. Indeed, $(\lambda, u) = (\inf_{\mathbf{R}^N} f, 0)$ is a viscosity subsolution of (2.1), so that $\lambda_m \geq \inf_{\mathbf{R}^N} f > -\infty$. It is also easy to see that (2.1) has a viscosity subsolution in $C^{0,\alpha}(\mathbf{R}^N)$ for any $\lambda \in (-\infty, \lambda_m)$.

We first observe a few properties of λ_m that can be verified by its very definition. In what follows, we often use the notation $\lambda_m(f)$ to emphasize the dependence of λ_m on the function f.

Proposition 2.2. *Let $f, g \in C_0(\mathbf{R}^N)$. We denote by $\lambda_m(f), \lambda_m(g)$ the associated generalized principal eigenvalues of* (2.1), *respectively. Then the following* (i)–(iii) *hold.*

(i) *$f \leq g$ in $\mathbf{R}^N$ implies $\lambda_m(f) \leq \lambda_m(g)$.*
(ii) *$(1 - \delta)\lambda_m(f) + \delta\lambda_m(g) \leq \lambda_m((1 - \delta)f + \delta g)$ for any $\delta \in (0, 1)$.*
(iii) *$\lambda_m(f + c) = \lambda_m(f) + c$ for any $c \in \mathbf{R}$.*

Proof. We first show (i). Let $u \in C^{0,\alpha}(\mathbf{R}^N)$ be a viscosity subsolution of (2.1) with f. Then it is also a viscosity subsolution of (2.1) with g in place of f. Hence, $\lambda_m(f) \leq \lambda_m(g)$ by definition. We next prove (ii). Fix any $\varepsilon > 0$. Let $u_0 \in C^{0,\alpha}(\mathbf{R}^N)$ be a viscosity subsolution of (2.1) with $\lambda = \lambda_m(f) - \varepsilon$, and let $u_1 \in C^{0,\alpha}(\mathbf{R}^N)$ be a viscosity subsolution of (2.1) with g in place of f and $\lambda = \lambda_m(g) - \varepsilon$. Note that such solutions exist by the very definition of λ_m.

Then, in view of the convexity of $|p|^m$ with respect to p, one can easily see that, for any $\delta \in (0,1)$, the function $u_\delta := (1-\delta)u_0 + \delta u_1$ satisfies

$$(1-\delta)(\lambda_m(f)-\varepsilon) + \delta(\lambda_m(g)-\varepsilon) - \Delta u_\delta + \frac{1}{m}|Du_\delta|^m \le (1-\delta)f + \delta g$$

in $\mathbf{R}^N$ in the viscosity sense. This implies that $(1-\delta)\lambda_m(f) + \delta\lambda_m(g) - \varepsilon \le \lambda_m((1-\delta)f + \delta g)$. Since ε is arbitrary, we obtain (ii). The validity of (iii) is obvious from the definition of λ_m. Hence, we have completed the proof. □

The following result implies that, if $f \in C^{0,1}(\mathbf{R}^N)$, then "viscosity subsolution" in the definition of λ_m can be replaced by "classical subsolution".

Proposition 2.3. *Suppose that $f \in C^{0,1}(\mathbf{R}^N)$. Then, for any $\lambda < \lambda_m$, there exists a classical subsolution $u \in C^\infty(\mathbf{R}^N)$ of (2.1).*

Proof. Fix any $\lambda < \lambda_m$ and construct a smooth subsolution u of (2.1). To this end, we follow the ingenious idea due to [1,14]. Set $f_\varepsilon(x) := \min_{|e|<\varepsilon} f(x+e)$ for $\varepsilon > 0$. Then, $f_\varepsilon \in C^{0,1}(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$, $f_\varepsilon \le f$ in $\mathbf{R}^N$, and $\{f_\varepsilon\}$ converges to f uniformly in $\mathbf{R}^N$ as $\varepsilon \to 0$. Let $\lambda_m^{(\varepsilon)}$ be the generalized principal eigenvalue of (2.1) with f_ε in place of f. Then, in view of Proposition 2.2 and by choosing $\varepsilon > 0$ sufficiently small, we may assume that $\lambda < \lambda_m^{(\varepsilon)} \le \lambda_m$. In particular, for the above λ, there exists a viscosity subsolution $u^{(\varepsilon)} \in C^{0,\alpha}(\mathbf{R}^N)$ of (2.1) with f_ε in place of f. Since $f_\varepsilon(\,\cdot - e) \le f$ in $\mathbf{R}^N$ for any $|e| < \varepsilon$, one can also see that $u^{(\varepsilon)}(\,\cdot - e)$ is a viscosity subsolution of (2.1) for any $|e| < \varepsilon$.

Now, let $\{\rho_\delta\}_{\delta>0} \subset C_c^\infty(\mathbf{R}^N)$ be a family of mollifier functions, i.e., $\rho_\delta \ge 0$ in $\mathbf{R}^N$, $\int_{\mathbf{R}^N} \rho_\delta(x)\,dx = 1$, and $\operatorname{supp}\rho_\delta \subset B_\delta$ for all $\delta > 0$. Set $u_\delta^{(\varepsilon)}(x) := (u^{(\varepsilon)} * \rho_\delta)(x)$ for $\delta < \varepsilon$, where $*$ stands for the usual convolution. Then, by noting the convexity of $p \mapsto (1/m)|p|^m$, one can see, similarly as in the proof of [1, Lemma 2.7], that $u := u_\delta^{(\varepsilon)}$ is a smooth viscosity subsolution of (2.1). Since a smooth viscosity subsolution is a classical subsolution, we have completed the proof. □

We next verify that λ_m is nonpositive.

Proposition 2.4. *One has $\lambda_m \le 0$. In particular, λ_m is finite.*

Proof. It suffices to consider the case where $f \in C^{0,1}(\mathbf{R}^N)$. Indeed, for any $f \in C_0(\mathbf{R}^N)$, one can always find a function $g \in C_0(\mathbf{R}^N) \cap C^{0,1}(\mathbf{R}^N)$ such that $f \le g$ in $\mathbf{R}^N$. In particular, in view of Proposition 2.2 (i), we have $\lambda_m(f) \le 0$ provided $\lambda_m(g) \le 0$. So, hereafter, we assume that $f \in C_0(\mathbf{R}^N) \cap C^{0,1}(\mathbf{R}^N)$.

Fix any $\lambda < \lambda_m$, and let $u \in C^\infty(\mathbf{R}^N)$ be a classical subsolution of (2.1). Existence of such u is guaranteed by virtue of Proposition 2.3. Then, for any nonnegative test function $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} \eta(x)^{m^*}\,dx = 1$, where $m^* := m/(m-1)$, we have

$$\lambda \int_{\mathbf{R}^N} \eta^{m^*}\,dx + \int_{\mathbf{R}^N} Du \cdot D(\eta^{m^*})\,dx + \frac{1}{m}\int_{\mathbf{R}^N} |Du|^m \eta^{m^*}\,dx \le \int_{\mathbf{R}^N} f\eta^{m^*}\,dx.$$

Noting $D(\eta^{m^*}) = m^*\eta^{m^*/m} D\eta$ and

$$Du \cdot D(\eta^{m^*}) = (\eta^{m^*/m} Du)\cdot(m^* D\eta) \le \frac{1}{m}|Du|^m \eta^{m^*} + (m^*)^{m^*-1}|D\eta|^{m^*},$$

we see that

$$\lambda = \lambda \int_{\mathbf{R}^N} \eta^{m^*}\, dx \le \int_{\mathbf{R}^N} f\eta^{m^*}\, dx + (m^*)^{m^*-1} \int_{\mathbf{R}^N} |D\eta|^{m^*}\, dx.$$

Fix any $\varepsilon > 0$, and observe that $f = \varepsilon + f - \varepsilon \le \varepsilon + (f-\varepsilon)_+$ in $\mathbf{R}^N$, where $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbf{R}$. Then

$$\lambda \le \varepsilon + \int_{\mathbf{R}^N} (f-\varepsilon)_+ \eta^{m^*}\, dx + (m^*)^{m^*-1} \int_{\mathbf{R}^N} |D\eta|^{m^*}\, dx.$$

Since $f(x) \to 0$ as $|x| \to \infty$, there exists a radius $R = R_\varepsilon > 0$ such that the support of $(f-\varepsilon)_+$ is contained in the ball B_R. In particular, setting $M := \sup_{\mathbf{R}^N} (f-\varepsilon)_+$, we obtain

$$\lambda \le \varepsilon + M \int_{|x|\le R} \eta^{m^*}\, dx + (m^*)^{m^*-1} \int_{\mathbf{R}^N} |D\eta|^{m^*}\, dx.$$

We now set $\eta_\delta(x) := \delta^{N/m^*} \eta(\delta x)$ for $\delta > 0$. Note that $\int_{\mathbf{R}^N} \eta_\delta(x)^{m^*}\, dx = 1$ for any $\delta > 0$. Then, plugging η_δ into the above η and using the change of variables $y = \delta x$, one can easily see that

$$\lambda \le \varepsilon + M \int_{|y|\le \delta R} \eta(y)^{m^*}\, dy + \delta^{m^*} (m^*)^{m^*-1} \int_{\mathbf{R}^N} |D\eta(y)|^{m^*}\, dy. \tag{2.3}$$

Sending $\delta \to 0$ and then $\varepsilon \to 0$, we obtain $\lambda \le 0$. Since $\lambda < \lambda_m$ is arbitrary, we conclude that $\lambda_m \le 0$. Hence, we have completed the proof. □

The following proposition states a stability of $\lambda_m(f)$ with respect to f.

Proposition 2.5. *Let $f, g \in C_0(\mathbf{R}^N)$. Then $|\lambda_m(f) - \lambda_m(g)| \le \max_{\mathbf{R}^N} |f-g|$. In particular, if $\{f_n\} \subset C_0(\mathbf{R}^N)$ converges as $n \to \infty$ to some $f \in C_0(\mathbf{R}^N)$ uniformly in $\mathbf{R}^N$, then $\lambda_m(f_n)$ converges to $\lambda_m(f)$ as $n \to \infty$. Moreover, if $\{u_n\}$ is a family of viscosity solutions of (2.1) with $f = f_n$ and $\lambda = \lambda_m(f_n)$, then, along a suitable subsequence, $\{u_n\}$ converges as $n \to \infty$ to a viscosity solution u of (2.1) with $\lambda = \lambda_m(f)$ locally uniformly in $\mathbf{R}^N$.*

Proof. Since $f \le g + \max_{\mathbf{R}^N}(f-g)_+$ in $\mathbf{R}^N$, we see, in view of Proposition 2.2 (i) and (iii), that $\lambda_m(f) - \lambda_m(g) \le \max_{\mathbf{R}^N}(f-g)_+$. Changing the role of f and g, we obtain the first claim. The second claim is obvious from the first one. In order to verify the last claim, we observe from Theorem 2.1, together with the normalization assumption $u_n(0) = 0$, that $\{u_n\}$ is pre-compact in $C(\mathbf{R}^N)$. Applying the Ascoli-Arzela theorem, we see that $\{u_n\}$ converges, along a suitable subsequence, to a function $u \in C^{0,\alpha}(\mathbf{R}^N)$ locally uniformly in $\mathbf{R}^N$. By the stability property of viscosity solutions, we conclude that u is a viscosity solution of (2.1) with $\lambda = \lambda_m(f)$. Hence, we have completed the proof. □

We now state the main result of this section.

Theorem 2.6. *For any $\lambda \le \lambda_m$, there exists a viscosity solution $u \in C^{0,\alpha}(\mathbf{R}^N)$ of (2.1). Moreover, if $f \in C^{0,1}(\mathbf{R}^N)$, then for any $\lambda \le \lambda_m$, there exists a classical solution $u \in C^2(\mathbf{R}^N)$ of (2.1).*

Proof. We first prove the latter claim. Let $f \in C^{0,1}(\mathbf{R}^N)$ and fix any $\lambda < \lambda_m$. Then, by virtue of Proposition 2.3, there exists a classical subsolution $u_- \in C^\infty(\mathbf{R}^N)$ of (2.1). Fix any $R > 0$ and consider the Dirichlet problem

$$\lambda - \Delta u + \frac{1}{m}|Du|^m - f = 0 \quad \text{in} \quad B_R, \qquad u = u_- \quad \text{on} \quad \partial B_R, \tag{2.4}$$

where $\partial B_R := \{x \in \mathbf{R}^N \mid |x| = R\}$. Then it is known (e.g. [16, Théorème I.1]) that there exists a unique classical solution $u_R \in C^{2,\gamma}(\overline{B}_R)$ of (2.4) for some $\gamma \in (0,1)$. We claim here that $\{u_R - u_R(0)\}_{R>0}$ is pre-compact in $C^2(\mathbf{R}^N)$. To justify this claim, it suffices to prove that, for any fixed $R_0 > 0$, there exist some $\delta \in (0,1)$ and $M > 0$ such that $|u_R - u_R(0)|_{2,\delta;B_{R_0}} \leq M$ for all $R > R_0 + 1$, where u_R denotes the solution of (2.4). In order to obtain such estimate, we first observe, in view of the so-called Bernstein method for elliptic equations with superlinear gradients, that $|Du_R|$ is bounded on B_{R_0} by a constant $M_1 > 0$ which may depend on $|f|_{0,1;B_{R_0+1}}$, but is independent of u_R for any $R > R_0 + 1$ (see, for instance, [10, Theorem A.1] for its proof). From the above estimate, one can also see that $|u_R - u_R(0)|$ is bounded on B_{R_0} for some $M_2 > 0$ depending only on R_0 and M_1. These uniform bounds lead to the Hölder estimate $|Du_R|_{0,\delta;B_{R_0}} \leq M_3$ for some $\delta \in (0,1)$ and $M_3 > 0$ not depending on u_R with $R > R_0 + 1$ (e.g, [15, Theorem 4.6.1]). Then, applying the standard interior estimate (e.g., [6, Theorem 4.6]) to the linear equation $-\Delta u_R = \tilde{f}$, where $\tilde{f} := f - \lambda - (1/m)|Du_R|^m$ is regarded as a given function in $C^{0,\delta}(\overline{B}_{R_0})$, we obtain $|u_R - u_R(0)|_{2,\delta;B_{R_0}} \leq M$ for some $M > 0$ not depending on u_R with $R > R_0 + 1$. Hence, in view of the Ascoli-Arzela theorem, we conclude that $\{u_R - u_R(0)\}_{R>0}$ is pre-compact in $C^2(\mathbf{R}^N)$.

We now let $R \to \infty$. Then, along a suitable subsequence $\{R_j\}$, we see that $\{u_{R_j}\}$ and their first and second derivatives converge as $j \to \infty$ to a function $u \in C^2(\mathbf{R}^N)$ and its corresponding derivatives, respectively, locally uniformly in $\mathbf{R}^N$. In particular, u is a classical solution of (2.1). In order to verify that (2.1) with $\lambda = \lambda_m$ has a classical solution, we choose any sequence $\{\lambda^{(n)}\}$ such that $\lambda^{(n)} \to \lambda_m$ as $n \to \infty$, and let $u^{(n)}$ denote the associated classical solution to (2.1) with $\lambda = \lambda^{(n)}$. Then one can see, similarly as above, that $\{u^{(n)} - u^{(n)}(0)\}$ is pre-compact in $C^2(\mathbf{R}^N)$. Passing to the limit as $n \to \infty$ along a suitable subsequence if necessary, we conclude that (2.1) with $\lambda = \lambda_m$ has a classical solution.

We next prove the former claim. Fix any $f \in C_0(\mathbf{R}^N)$ and choose a sequence $\{f_n\} \subset C^\infty(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$ which converges as $n \to \infty$ to f uniformly in $\mathbf{R}^N$. Let $\lambda^{(n)}$ be the generalized principal eigenvalue of (2.1) with f_n in place of f. Then, in view of Proposition 2.5, we observe that $\lambda^{(n)} \to \lambda_m$ as $n \to \infty$. Now, fix any $\lambda < \lambda_m$. We may assume without loss of generality that $\lambda < \lambda^{(n)}$ for any $n \geq 1$. For each $n \geq 1$, let $u^{(n)} \in C^2(\mathbf{R}^N)$ denote a classical solution of (2.1) with f_n in place of f. Then, by Theorem 2.1 and the stability of viscosity solutions, we conclude that, along a suitable subsequence, $\{u^{(n)}\}$ converges as $n \to \infty$ to a viscosity solution $u \in C^{0,\alpha}(\mathbf{R}^N)$ of (2.1) locally uniformly in $\mathbf{R}^N$. We can also construct a viscosity solution of (2.1) with $\lambda = \lambda_m$ similarly as in the previous case. Hence, we have completed the proof. $\square$

Theorem 2.6 implies that the following representation formula for λ_m holds:

$$\lambda_m = \max\{\lambda \in \mathbf{R} \,|\, (2.1) \text{ has a viscosity solution } u \in C^{0,\alpha}(\mathbf{R}^N)\}.$$

Furthermore, if $f \in C^{0,1}(\mathbf{R}^N)$, then

$$\lambda_m = \max\{\lambda \in \mathbf{R} \,|\, (2.1) \text{ has a classical solution } u \in C^2(\mathbf{R}^N)\}.$$

3. Convergence as $m \to \infty$

This section is devoted to the convergence of λ_m as $m \to \infty$. To be precise, we recall the limiting equation

$$\max\left\{\lambda - \Delta u - f, |Du| - 1\right\} = 0 \quad \text{in} \;\; \mathbf{R}^N, \quad u(0) = 0, \tag{3.1}$$

and redefine the generalized principal eigenvalue of (3.1) by

$$\lambda_\infty := \sup\{\lambda \in \mathbf{R} \,|\, (3.1) \text{ has a viscosity subsolution } u \in C^{0,1}(\mathbf{R}^N)\}. \tag{3.2}$$

The following result is crucial to our convergence result.

Proposition 3.1. *Let $\{m_k\} \subset \mathbf{R}$ be an increasing sequence such that $m_k \to \infty$ as $k \to \infty$. Let (λ_{m_k}, u_k) be a solution of (2.1) with $m = m_k$ for each k. Suppose that λ_k converges to some $\lambda \in \mathbf{R}$ as $k \to \infty$. Then, up to a subsequence, $\{u_k\}$ converges as $k \to \infty$ to a function $u \in C^{0,1}(\mathbf{R}^N)$ locally uniformly in $\mathbf{R}^N$. Moreover, (λ, u) is a solution of (3.1).*

Proof. In view of Theorem 2.1 (i), we see that there exist a subsequence of $\{u_k\}$, which we denote by $\{u_k\}$ again, and a function $u \in C(\mathbf{R}^N)$ with $u(0) = 0$ such that $u_k \to u$ as $k \to \infty$ locally uniformly in $\mathbf{R}^N$. Since the constant M_R in Theorem 2.1 (i) does not depend on any large $m > 2$, by sending $k \to \infty$ in the inequality $|u_k(x) - u_k(y)| \leq M_R|x-y|^{(m_k-2)/(m_k-1)}$ $(x, y \in B_R)$ and noting that $(m_k - 2)/(m_k - 1) \to 1$ as $k \to \infty$, we see that $|u(x) - u(y)| \leq M_R|x - y|$ for any $x, y \in B_R$. In particular, $u \in C^{0,1}(\mathbf{R}^N)$.

We now verify that u is a viscosity solution of (3.1). We first prove the subsolution property. Fix any $x_0 \in \mathbf{R}^N$ and let $\phi \in C^2(\mathbf{R}^N)$ be any function such that $\max_{\mathbf{R}^N}(u - \phi) = (u - \phi)(x_0)$. As is standard, one can assume that the maximum is strict, so that there exists a sequence $\{x_k\} \subset \mathbf{R}^N$ such that $u_k - \phi$ attains its local maximum at x_k and $x_k \to x_0$ as $k \to \infty$. Then, by the subsolution property of u_k, we see that

$$\lambda_{m_k} - \Delta\phi(x_k) + \frac{1}{m_k}|D\phi(x_k)|^{m_k} - f(x_k) \leq 0. \tag{3.3}$$

We now suppose that $|D\phi(x_0)| > 1$. Then there exists an $\eta > 0$ such that $|D\phi(x_k)| \geq 1 + \eta$ for all sufficiently large k. In particular, we have

$$\frac{1}{m_k}(1+\eta)^{m_k} \leq -\lambda_{m_k} + \Delta\phi(x_k) + f(x_k).$$

Sending $k \to \infty$, we get a contradiction since the right-hand side remains bounded, whereas the left-hand side goes to infinity as $k \to \infty$. Hence, we have $|D\phi(x_0)| \leq 1$. Furthermore, letting $k \to \infty$ in (3.3), we conclude that

$\lambda - \Delta\phi(x_0) - f(x_0) \le 0$, which implies that u is a viscosity subsolution of (3.1).

We next prove the supersolution property. Fix any $x_0 \in \mathbf{R}^N$ and let $\psi \in C^2(\mathbf{R}^N)$ be such that $\min_{\mathbf{R}^N}(u-\psi) = (u-\psi)(x_0)$. If $|D\psi(x_0)| \ge 1$, then there is nothing to prove, so we assume that $|D\psi(x_0)| < 1$. In particular, there exists some $\eta > 0$ such that $|D\psi(x_k)| \le 1-\eta$ for all sufficiently large k. Furthermore, there exists a sequence $\{x_k\} \subset \mathbf{R}^N$ such that $u_k - \psi$ attains its local minimum at x_k and $x_k \to x_0$ as $k \to \infty$. Then, by the supersolution property of u_k, we have

$$\lambda_{m_k} - \Delta\psi(x_k) + \frac{1}{m_k}|D\psi(x_k)|^{m_k} - f(x_k) \ge 0.$$

Letting $k \to \infty$ in the above inequality, we obtain $\lambda - \Delta\psi(x_0) - f(x_0) \ge 0$. Hence, we conclude that u is a viscosity supersolution of (3.1). □

We are now in position to state the main result of this section.

Theorem 3.2. *Let λ_m and λ_∞ be the generalized principal eigenvalues of (2.1) and (3.1), respectively. Then, λ_m converges to λ_∞ as $m \to \infty$. Moreover, Eq. (3.1) with $\lambda = \lambda_\infty$ has a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$.*

Proof. Set $\overline{\lambda} := \limsup_{m\to\infty} \lambda_m$. Note that $\overline{\lambda} \le 0$ in view of Proposition 2.4. Let (λ_{m_k}, u_{m_k}) be a sequence of solutions to (2.1) with $m = m_k$ such that $\lambda_{m_k} \to \overline{\lambda}$ as $k \to \infty$. Then, by taking a subsequence if necessary, we see from Proposition 3.1 that $\{u_{m_k}\}$ converges to a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$ of (3.1) locally uniformly in $\mathbf{R}^N$. In particular, by the definition of λ_∞, we have $\overline{\lambda} \le \lambda_\infty$.

To prove the reverse inequality, we set $\underline{\lambda} := \liminf_{m\to\infty} \lambda_m$. Fix any $\varepsilon > 0$ and let $u \in C^{0,1}(\mathbf{R}^N)$ be a viscosity subsolution of (3.1) with $\lambda = \lambda_\infty - \varepsilon$. Then, noting that $|Du| \le 1$ in $\mathbf{R}^N$ in the viscosity sense, we see that, for any $m > 2$, u is a viscosity subsolution of

$$\lambda_\infty - \varepsilon - \frac{1}{m} - \Delta u + \frac{1}{m}|Du|^m - f \le 0 \quad \text{in } \mathbf{R}^N.$$

This implies $\lambda_\infty - \varepsilon - 1/m \le \lambda_m$ for any $m > 2$, so that $\lambda_\infty - \varepsilon \le \underline{\lambda}$. Since $\varepsilon > 0$ is arbitrary, we obtain $\lambda_\infty \le \underline{\lambda} \le \overline{\lambda} \le \lambda_\infty$. Hence, we have completed the proof. □

The next result states that Proposition 2.3 remains valid for $m = \infty$.

Proposition 3.3. *Suppose that $f \in C^{0,1}(\mathbf{R}^N)$. Then, for any $\lambda < \lambda_\infty$, there exists a classical subsolution $u \in C^\infty(\mathbf{R}^N)$ of (3.1). In particular,*

$$\lambda_\infty = \sup\{\lambda \in \mathbf{R} \,|\, (3.1) \textit{ has a classical subsolution} u \in C^\infty(\mathbf{R}^N)\}.$$

Proof. Fix any $\lambda_0 < \lambda_\infty$, and let $\{\rho_\delta\}_{\delta>0} \subset C_c^\infty(\mathbf{R}^N)$ be such that $\rho_\delta \ge 0$ in $\mathbf{R}^N$, $\int_{\mathbf{R}^N} \rho_\delta(x)dx = 1$, and $\operatorname{supp}\rho_\delta \subset B_\delta$ for all $\delta > 0$. Let $\{\lambda_{m_k}\}$ be a sequence of generalized principal eigenvalues of (2.1) with $m = m_k$ such that $\lambda_{m_k} \to \lambda_\infty$ as $k \to \infty$. Such a sequence exists by virtue of Theorem 3.2. In what follows, we assume that $\lambda_0 < \lambda_{m_k}$ for all $k \ge 1$. Let $u^{(k)} \in C^2(\mathbf{R}^N)$

($k \geq 1$) be a classical solution of (2.1) with $m = m_k$ and $\lambda = \lambda_0$. Such a solution exists by virtue of Theorem 2.6. Taking a subsequence if necessary, one may also assume that $\{u^{(k)}\}$ converges as $k \to \infty$ to a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$ of (3.1) locally uniformly in $\mathbf{R}^N$.

Now we set $u_\delta^{(k)} := u^{(k)} * \rho_\delta$, $u_\delta := u * \rho_\delta$, and $f_\delta := f * \rho_\delta$, where $*$ stands for the usual convolution. We choose $\delta > 0$ so small that $\sup_{\mathbf{R}^N} |f_\delta - f| < \lambda_{m_k} - \lambda_0$ for all $k \geq 1$. Then, since $u^{(k)}$ is a classical solution of (2.1) with $m = m_k$ and $\lambda = \lambda_0$, we see that $u_\delta^{(k)}$ enjoys the inequality

$$\lambda_0 - \Delta u_\delta^{(k)} + \frac{1}{m_k} |Du_\delta^{(k)}|^{m_k} - f \leq 0 \quad \text{in} \quad \mathbf{R}^N$$

for all $k \geq 1$ and for any sufficiently small $\delta > 0$. This implies that $u_\delta^{(k)}$ is also a classical subsolution of

$$\lambda_0 - \Delta u_\delta^{(k)} - f \leq 0 \quad \text{in} \quad \mathbf{R}^N.$$

Letting $k \to \infty$ and noting the stability of viscosity solutions, we conclude that u_δ is a smooth viscosity subsolution, and therefore, a classical subsolution of the same equation. On the other hand, since $|Du| \leq 1$ a.e. in $\mathbf{R}^N$, which can be verified as in the proof of Proposition 3.1, we see that $|Du_\delta| \leq 1$ in $\mathbf{R}^N$. Hence, u_δ enjoys (3.1) with $\lambda = \lambda_0$ at every point $x \in \mathbf{R}^N$, and we have completed the proof. □

Remark 3.4. The first claim of Theorem 2.6 remains true for $m = \infty$. Namely, for any $\lambda \leq \lambda_\infty$, there exists a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$ of (3.1). To see this, fix any $\lambda_0 < \lambda_\infty$ and choose an m_0 so large that $\lambda_m > \lambda_0$ for any $m > m_0$. For $m > m_0$, let u_m be a viscosity solution of (2.1) with $\lambda = \lambda_0$. Then, by Proposition 3.1, we conclude that, along a subsequence, $\{u_m\}$ converges to a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$ of (3.1) with $\lambda = \lambda_0$. Since λ_0 is arbitrary, we conclude that (3.1) has a viscosity solution $u \in C^{0,1}(\mathbf{R}^N)$ for any $\lambda < \lambda_\infty$. The existence of a viscosity solution u to (3.1) with $\lambda = \lambda_\infty$ has been proved in Theorem 3.2. Hence, the first claim of Theorem 2.6 is also valid for $m = \infty$. We do not know if the second claim remains true for $m = \infty$.

4. Qualitative properties

In this section, we introduce real parameter β and consider the ergodic problem for $m > 2$:

$$\lambda - \Delta u + \frac{1}{m}|Du|^m - \beta f = 0 \quad \text{in} \quad \mathbf{R}^N, \quad u(0) = 0, \tag{4.1}$$

and its limiting equation as $m \to \infty$:

$$\max\{\lambda - \Delta u - \beta f, |Du| - 1\} = 0 \quad \text{in} \quad \mathbf{R}^N, \quad u(0) = 0. \tag{4.2}$$

In the rest of this paper, we impose the following assumption on f in addition to (A1):

(A2) $f \not\equiv 0$ and $|f(x)| \le C_0\langle x\rangle^{-m^*}$ in $\mathbf{R}^N$ for some $C_0 > 0$, where $\langle x\rangle := (1+|x|^2)^{1/2}$ and $m^* := m/(m-1)$ with the convention that $m^* := 1$ for $m = \infty$.

Let $\lambda_{m,\beta}$ and $\lambda_{\infty,\beta}$ be the generalized principal eigenvalues of (4.1) and (4.2), respectively. In view of Proposition 2.4 and Theorem 3.2, we observe that $\lambda_{m,\beta} \le 0$ for any $\beta \in \mathbf{R}$ and $2 < m \le \infty$. It is also easy to see that $\lambda_{m,0} = 0$ for any $2 < m \le \infty$. Furthermore, we have the following.

Proposition 4.1. *Let $2 < m \le \infty$. If $f_- := \max\{-f, 0\} \not\equiv 0$, then $\lambda_{m,\beta} \to -\infty$ as $\beta \to \infty$, and if $f_- \equiv 0$, then $\lambda_{m,\beta} = 0$ for any $\beta > 0$. Symmetrically, if $f_+ := \max\{f, 0\} \not\equiv 0$, then $\lambda_{m,\beta} \to -\infty$ as $\beta \to -\infty$, and if $f_+ \equiv 0$, then $\lambda_{m,\beta} = 0$ for any $\beta < 0$.*

Proof. We first consider the case where $2 < m < \infty$. In view of Proposition 2.5, we may assume that $f \in C^{0,1}(\mathbf{R}^N)$. Suppose that $f_- \not\equiv 0$, and choose any $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\eta \ge 0$ in $\mathbf{R}^N$, $\int_{\mathbf{R}^N} \eta(x)^{m^*} dx = 1$, and $\operatorname{supp}\eta \subset \operatorname{supp} f_-$. Then, taking a classical solution $u \in C^2(\mathbf{R}^N)$ of (4.1) with $\lambda = \lambda_{m,\beta}$, multiplying both sides of (4.1) by η, and applying integration by parts, we see as in the proof of Proposition 2.4 that

$$\lambda_{m,\beta} \le -\beta \int_{\mathbf{R}^N} f_-(x)\eta(x)^{m^*}\, dx + \frac{1}{m^*}\int_{\mathbf{R}^N} |D\eta(x)|^{m^*} dx. \tag{4.3}$$

Since the integral of $f_-\eta^{m^*}$ over $\mathbf{R}^N$ is strictly positive, we conclude that $\lambda_{m,\beta} \to -\infty$ as $\beta \to \infty$. We now take the limit as $m \to \infty$ in (4.3). Then, since $m^* \to 1$ as $m \to \infty$, we see from Theorem 3.2 that the claim is also valid for $m = \infty$.

We now suppose that $f_- \equiv 0$. Then, for any $\beta > 0$, the pair $(\lambda, u) = (0, 0)$ is a subsolution of (4.1) and (4.2). This implies that $\lambda_{m,\beta} = 0$ for any $2 < m \le \infty$ and $\beta > 0$. By choosing $-f$ and $-\beta$ in place of f and β, respectively, we see that the latter claim of this proposition is also valid. Hence, we have completed the proof. □

From Propositions 2.2 (ii), 2.4, and 4.1, for each $2 < m \le \infty$, one can define β_-, β_+ by

$$\beta_+ := \sup\{\beta \in \mathbf{R} \mid \lambda_{m,\beta} = 0\}, \qquad \beta_- := \inf\{\beta \in \mathbf{R} \mid \lambda_{m,\beta} = 0\}.$$

Obviously, $-\infty \le \beta_- \le 0 \le \beta_+ \le \infty$, and β_+ (resp. β_-) is finite if and only if $f_- \not\equiv 0$ (resp. $f_+ \not\equiv 0$). Moreover, since $f \not\equiv 0$, either β_+ or β_- is finite. As is mentioned in the introduction, we wish to know whether $0 < |\beta_\pm|$ $(< \infty)$. The main result of this section can be stated as follows.

Theorem 4.2. *Let β_+ be defined as above, and let $f_- \not\equiv 0$.*

(i) *Suppose that $N \ge 2$ and $2 < m \le \infty$. Then $\beta_+ > 0$.*
(ii) *Suppose that $N = 1$ and $2 < m < \infty$. Then $\beta_+ = 0$.*
(iii) *Suppose that $N = 1$ and $m = \infty$. Then $\beta_+ > 0$ provided $f_- \in L^1(\mathbf{R})$.*

Changing (β, f) into $(-\beta, -f)$, one has the following symmetrical result as a corollary of Theorem 4.2.

Corollary 4.3. *Let β_- be defined as above, and let $f_+ \not\equiv 0$.*

(i) *Suppose that $N \geq 2$ and $2 < m \leq \infty$. Then $\beta_- < 0$.*
(ii) *Suppose that $N = 1$ and $2 < m < \infty$. Then $\beta_- = 0$.*
(iii) *Suppose that $N = 1$ and $m = \infty$. Then $\beta_- < 0$ provided $f_+ \in L^1(\mathbf{R})$.*

Remark 4.4. If $N \geq 2$ and f is sign-changing, then $\beta_- < 0 < \beta_+$ for any $2 < m \leq \infty$. From the ergodic stochastic control point of view, this implies that there exist two different critical points β_+ and β_- at which the controller changes his/her optimal strategy. We remark that, if f is nonnegative or nonpositive in $\mathbf{R}^N$, then there is only one such critical point.

In the rest of this section, we prove (i)–(iii) of Theorem 4.2 one by one. The key to the proof of claim (i) is the following estimate.

Proposition 4.5. *Let $N \geq 2$ and $2 < m < \infty$. Set*

$$\beta_0 := \frac{(N - m^*)^{m^*}}{m^* C_0} > 0,$$

where $m^* := m/(m-1)$ and $C_0 > 0$ is the constant in (A2). Then, for any $|\beta| \leq \beta_0$, there exists a subsolution $u \in C^\infty(\mathbf{R}^N)$ of (4.1) with $\lambda = 0$.

Proof. We define $u : \mathbf{R}^N \to \mathbf{R}$ by $u(x) := (K/\alpha)\langle x\rangle^\alpha$, where $\alpha = (m-2)/(m-1)$ and $K > 0$ is some constant that will be specified later. Then, by direct computations, we see that $Du = K\langle x\rangle^{-m^*}x$ and $\Delta u = KN\langle x\rangle^{-m^*} - Km^*\langle x\rangle^{-m^*-2}|x|^2$. Thus,

$$\begin{aligned}
-\Delta u + \frac{1}{m}|Du|^m &= \langle x\rangle^{-m^*}\Big\{-KN + Km^*|x|^2\langle x\rangle^{-2}\Big\} + \frac{K^m}{m}|x|^m\langle x\rangle^{-mm^*} \\
&= \langle x\rangle^{-m^*}\Big\{-KN + Km^*|x|^2\langle x\rangle^{-2} + \frac{K^m}{m}|x|^m\langle x\rangle^{-m}\Big\} \\
&\leq \langle x\rangle^{-m^*}\Big\{-(N - m^*)K + \frac{K^m}{m}\Big\}.
\end{aligned}$$

Since the function $K \mapsto f(K) := (K^m/m) - (N - m^*)K$ attains its minimum $-(1/m^*)(N - m^*)^{m^*}$ at $K = (N - m^*)^{1/(m-1)} =: K_m$, we choose $K = K_m$ in the definition of u to obtain

$$-\Delta u + \frac{1}{m}|Du|^m + \beta f \leq \langle x\rangle^{-m^*}\Big\{|\beta|C_0 - \frac{1}{m^*}(N - m^*)^{m^*}\Big\} \quad \text{in } \mathbf{R}^N.$$

This implies that u is a subsolution of (2.1) with $\lambda = 0$ provided $|\beta| \leq \beta_0$. Hence, we have completed the proof. □

As a corollary of this proposition, one can prove claim (i) of Theorem 4.2.

Proof of Theorem 4.2 (i). Let β_0 be the constant taken from Proposition 4.5. Then, it is obvious that $\beta_+ \geq \beta_0 > 0$ for any $2 < m < \infty$. Moreover, since $m^* \to 1$ as $m \to \infty$, we see that $\beta_+ \geq \beta_0 \geq (N-1)/(2C_0) > 0$ for any large m. Hence, letting $m \to \infty$ and noting that $\lambda_{m,\beta}$ converges to $\lambda_{\infty,\beta}$ as $m \to \infty$ for any $\beta \in \mathbf{R}$, we conclude that $\lambda_{\infty,\beta} = 0$ for any $\beta \leq (N-1)/(2C_0)$. This yields that $\beta_+ > 0$ for $N \geq 2$ and $m = \infty$. Hence, we have completed the proof. □

Remark 4.6. In the case where $N \geq 2$ and $2 < m < \infty$, the positivity $\beta_+ > 0$ has been observed in [11, Proposition 2.4] when $f \in C^{0,1}(\mathbf{R}^N)$. The new ingredient here is that we have an explicit lower bound of β_+, uniform in m, which leads to the positivity of β_+ not only for $2 < m < \infty$ but also for $m = \infty$. Recall that $\beta_+ = 0$ for $N = m = 2$ (see [10]). This exhibits a striking contrast between quadratic and superquadratic cases.

In what follows, we concentrate on the case where $N = 1$, in which case the ergodic problem (4.1) takes the form

$$\lambda - u'' + \frac{1}{m}|u'|^m - \beta f = 0 \quad \text{in } \mathbf{R}, \quad u(0) = 0. \tag{4.4}$$

We first prove claim (ii) of Theorem 4.2.

Proof of Theorem 4.2 (ii). We may assume without loss of generality that $f \in C^{0,1}(\mathbf{R}^N)$. We prove that $\lambda_{m,\beta} < 0$ for any $\beta > 0$. We argue by contradiction assuming that $\lambda_{m,\beta} = 0$ for some $\beta > 0$. Let $C > 0$ be such that $C^m = \max_{\mathbf{R}^N}(\beta f)_-$, and let $u \in C^2(\mathbf{R})$ be a classical solution of (4.4) with $\lambda = 0$. Then, we see that

$$-u'' + \frac{1}{m}|u'|^m = \beta f \geq -C^m \quad \text{in } \mathbf{R}.$$

By changing the variable such as $s = u'(y)/C$ and using the inequality above, we have

$$\begin{aligned}\int_{u'(0)/C}^{u'(x)/C} \frac{m}{|s|^m + m}\, ds &= \int_0^x \frac{m}{|u'(y)/C|^m + m}\Big(\frac{u'(y)}{C}\Big)'\, dy \\ &\leq \frac{1}{C}\int_0^x \frac{|u'(y)|^m + mC^m}{|u'(y)/C|^m + m}\, dy = C^{m-1}\int_0^x dy = C^{m-1}x\end{aligned}$$

for all $x \in \mathbf{R}$. In particular, we obtain

$$C^{m-1}x \geq \int_{u'(0)/C}^{u'(x)/C} \frac{m}{|s|^m + m}\, ds \geq -\int_{\mathbf{R}} \frac{m}{|s|^m + m}\, ds > -\infty$$

for all $x \in \mathbf{R}$. Sending $x \to -\infty$, we get a contradiction. Hence, $\lambda_{m,\beta} < 0$ for all $\beta > 0$. □

We finally prove claim (iii) of Theorem 4.2. Let $N = 1$ and $m = \infty$. In this case, (4.2) can be written as

$$\max\{\lambda - u'' - \beta f, |u'| - 1\} = 0 \quad \text{in } \mathbf{R}, \quad u(0) = 0. \tag{4.5}$$

Proposition 4.7. *Let $N = 1$ and $m = \infty$. Suppose that $f_- \not\equiv 0$, and set*

$$L := \int_{\mathbf{R}} f_-(u)du, \quad K := \sup\Big\{ \int_x^y -f(u)du \Big| x, y \in \mathbf{R}, x < y\Big\}.$$

Then $2/L \leq \beta_+ \leq 2/K$, where $2/L := 0$ for $L = \infty$ and $2/K := 0$ for $K = \infty$.

Proof. We first show that $2/L \leq \beta_+$. We may assume $L < \infty$, otherwise the inequality is obvious. Notice here that $L > 0$ by assumption. We set $\beta_0 := 2/L$

and construct a classical subsolution $u \in C^2(\mathbf{R})$ of (4.5) with $\lambda = 0$ and $\beta = \beta_0$. Let us consider the linear equation

$$-u'' + \beta_0 f_- = 0 \quad \text{in } \mathbf{R}, \quad u(0) = 0. \tag{4.6}$$

Then, for any $C \in \mathbf{R}$, the function $u \in C^2(\mathbf{R})$ defined by

$$u(x) = \beta_0 \int_0^x F(y)dy + Cx, \quad F(y) := \int_0^y f_-(u)du, \tag{4.7}$$

is a classical solution to (4.6). We now choose

$$C := \frac{1}{L}\left(\int_{-\infty}^0 f_-(u)du - \int_0^\infty f_-(u)du\right).$$

Then, noting that $x \mapsto F(x)$ is nondecreasing in $\mathbf{R}^N$ and $u'(x) = \beta_0 F(x) + C$ for all $x \in \mathbf{R}$, we have

$$u'(x) \le \frac{2}{L}\int_0^\infty f_-(u)du + C = 1, \quad u'(x) \ge -\frac{2}{L}\int_{-\infty}^0 f_-(u)du + C = -1$$

for all $x \in \mathbf{R}$. Hence, u with the above C is a subsolution of (4.5) with $\lambda = 0$ and $\beta = \beta_0$, which implies that $\beta_+ \ge 2/L$.

We next show that $\beta_+ \le 2/K$. Recall that $K > 0$ by assumption. We argue by contradiction assuming that $\beta_+ > 2/K$. Fix any β such that $2/K < \beta < \beta_+$. Then, $\lambda_{\infty,\beta} = 0$ by the definition of β_+. Fix an arbitrary $\delta > 0$. Then, in view of Proposition 3.3, there exists a classical subsolution $u \in C^\infty(\mathbf{R})$ of (4.5) with $\lambda = -\delta$. In particular, we have

$$-\delta - u'' - \beta f \le 0, \quad |u'| \le 1 \quad \text{in } \mathbf{R}.$$

This yields that, for any $x, y \in \mathbf{R}$ with $x < y$,

$$\beta \int_x^y -f(s)ds \le \int_x^y (u''(s) + \delta)ds = u'(y) - u'(x) + \delta(y-x) \le 2 + \delta(y-x).$$

Letting $\delta \to 0$ and taking the supremum over all $x, y \in \mathbf{R}$ such that $x < y$, we obtain $\beta K \le 2$, which is a contradiction. Hence, we have completed the proof. □

Claim (iii) of Theorem 4.2 is a direct consequence of the above proposition.

Remark 4.8. Suppose that $f_+ \equiv 0$, that is, $f \le 0$ in $\mathbf{R}$. Then $L = K = \int_{\mathbf{R}} |f(u)|du$, so that $\beta_+ = 2/\int_{\mathbf{R}} |f(u)|du$. This implies that $\beta_+ > 0$ if and only if $f \in L^1(\mathbf{R})$.

Remark 4.9. As far as the uniqueness for u, up to an additive constant, is concerned, equation (1.3) with $\lambda = \lambda_\infty$ may have multiple solutions in general. Indeed, let $N = 1$ and $f(x) := -(1-|x|)_+$ in (1.3). Then, in view of Remark 4.8, it is not difficult to observe that $\lambda_\infty = 0$. Furthermore, we define $u : \mathbf{R} \to \mathbf{R}$ by

$$u(x) := \int_0^x F(y)dy + Cx, \quad F(y) := \int_0^y (1-|u|)_+ du,$$

where $C \in \mathbf{R}$ is a constant. Then, similarly as in the proof of Proposition 4.7, we see that u is a classical solution of (1.3) for any $C \in [-1/2, 1/2]$. In particular, uniqueness for u does not hold without any growth condition as $|x| \to \infty$. We remark here that, if $N = 1$ and f is convex and superlinear with respect to x, then, up to an additive constant, there exists only one viscosity solution u of (1.3) which satisfies $u(x)/|x| \to 1$ as $|x| \to \infty$ (see [8, Proposition 5.1]). At this stage, we do not know any uniqueness result for (1.3) under our assumptions (A1)–(A2).

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