Global existence of small solutions for the fourth-order nonlinear Schrödinger equation

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Abstract. We consider the Cauchy problem for the fourth-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = f(u), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n = 1, 2$. We prove global existence of small solutions under the growth condition of $f(u)$ satisfying $|\partial_u^j f(u)| \leq C|u|^{p-j}$, where $p > 1 + \frac{4}{n}, 0 \leq j \leq 3$.

Mathematics Subject Classification. 35Q55, 35Q35, 35Q51.

Keywords. Fourth-order nonlinear Schrödinger equation, Global existence, Non gauge invariant.

1. Introduction

We consider the Cauchy problem for the fourth-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = f(u), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where the space dimension $n = 1, 2$, and the nonlinearity $f(u)$ satisfies the estimate

$$\left| \frac{d^j}{du^j} f(u) \right| \leq C|u|^{p-j}, \quad (1.2)$$

with $p > 1 + \frac{4}{n}, 0 \leq j \leq 3$. A typical example of the nonlinearity $f(u)$ is given by $|u|^p$. The class of the fourth-order nonlinear Schrödinger equations (1.1) describes deep water wave dynamics (see [2]). Local and global existence of solutions for more general equations including (1.1) have been studied by many authors (see, e.g., [4, 10–12, 15]). In the case of the gauge invariant nonlinearity $|u|^{p-1}u$ with $p > 1 + \frac{4}{n}$, the small data global existence (SDGE) to (1.1) was shown in [8] in the one dimensional case $n = 1$. We are interested in (SDGE) to (1.1) with more general nonlinearities and dimensions. As far as we know,

the critical power of the nonlinearity for (SDGE) is $p = 1 + \frac{4}{n}$. Previously in [8, 9], we have applied the operator

$$\begin{aligned}\mathcal{J} &= U(t)xU(-t) = e^{-\frac{it}{4}\Delta^2}xe^{\frac{it}{4}\Delta^2} \\ &= \left(e^{-\frac{it}{4}\Delta^2}x_1e^{\frac{it}{4}\Delta^2}, \dots, e^{-\frac{it}{4}\Delta^2}x_ne^{\frac{it}{4}\Delta^2}\right) \\ &= (x_1 - it\Delta\partial_1, \dots, x_n - it\Delta\partial_n) = x - it\Delta\nabla\end{aligned}$$

to show (SDGE) of (1.1) with the gauge invariant nonlinearity for $n = 1$. A similar operator

$$e^{\frac{it}{2}\Delta}xe^{-\frac{it}{2}\Delta} = (x_1 + it\partial_1, \dots, x_n + it\partial_n) = x + it\nabla$$

was widely used for the study of the large time asymptotic behavior of solutions to the nonlinear Schrödinger equations with a gauge invariant nonlinearity such that

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = |u|^{p-1}u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where it works well. However it is known from [7] that the operator $e^{\frac{it}{2}\Delta}xe^{-\frac{it}{2}\Delta}$ does not work well for the nonlinearity of the form $|u|^p$. In [14], the $\mathbf{L}^{p+1} \cap \mathbf{L}^{\frac{p+1}{p}}$ time decay estimate of evolution operator $e^{\frac{it}{2}\Delta}$ was applied to (1.3) to obtain (SDGE), when the initial data are small in $\mathbf{L}^{\frac{p+1}{p}}$ and the order p satisfies $\frac{n}{2} \left(\frac{p-1}{p+1} \right) p > 1$, namely $p > p_{2,s}(n)$, where

$$p_{2,s}(n) = \frac{1}{2} \left(1 + \frac{2}{n} + \sqrt{\left(1 + \frac{2}{n} \right)^2 + 4 \left(\frac{2}{n} \right)} \right)$$

is the so-called Strauss exponent (see [13]). In the case of the fourth-order nonlinear Schrödinger equation (1.1), the Strauss exponent $p_{2,s}(n)$ is replaced by

$$p_{4,s}(n) = \frac{1}{2} \left(1 + \frac{4}{n} + \sqrt{\left(1 + \frac{4}{n} \right)^2 + 4 \left(\frac{4}{n} \right)} \right).$$

Applying the method of Strauss, (SDGE) for (1.1) can be obtained if $p > p_{4,s}(n)$. Note that $p_{4,s}(n) = p_{2,s}\left(\frac{n}{2}\right)$, also

$$\begin{aligned}p_{4,s}(1) &= \frac{1}{2} \left(5 + \sqrt{41} \right) \approx 5.70156, \\ p_{4,s}(2) &= p_{2,s}(1) = \frac{1}{2} \left(3 + \sqrt{17} \right) \approx 3.56155.\end{aligned}$$

Therefore we are interested in the case when

$$1 + \frac{4}{n} < p \leq p_{4,s}(n).$$

The operator $\mathcal{J} = e^{-\frac{it}{4}\Delta^2}xe^{\frac{it}{4}\Delta^2}$ is useful for obtaining the time decay estimates of solutions $e^{\frac{it}{4}\Delta^2}u_0$ to the linear problem. However, it is not clear if $\mathcal{J}$ acts well on the nonlinear term. In the present paper, our aim is to show

that the operator $\mathcal{J}$ works well not only on the gauge invariant nonlinearity $|u|^{p-1}u$, but also for $|u|^p$.

To state our results precisely we introduce *Notation and Function Spaces*. We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}^n} |\phi(x)|^p dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess. sup}_{x \in \mathbb{R}^n} |\phi(x)|$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{ \varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x \rangle^s \langle i\partial_x \rangle^m \phi\|_{\mathbf{L}^p} < \infty \right\},$$

$m, s \in \mathbb{R}$, $1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1 + |x|^2}$, $\langle i\partial_x \rangle = \sqrt{1 - \Delta}$. Also define the homogeneous Sobolev space

$$\dot{\mathbf{H}}_p^m = \left\{ \varphi \in \mathbf{S}'; \|\phi\|_{\dot{\mathbf{H}}_p^m} = \sum_{|\alpha|=m} \|\partial_x^\alpha \phi\|_{\mathbf{L}^p} < \infty \right\},$$

We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$, $\dot{\mathbf{H}}^m = \dot{\mathbf{H}}_2^m$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I}; \mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C .

In order to obtain the result for the two dimensional case, we use the dilation operator

$$\mathcal{P} = x \cdot \nabla + 4t\partial_t = \sum_{j=1}^n x_j \partial_j + 4t\partial_t$$

which is related to the operator $\mathcal{J}$ by the identity

$$\begin{aligned} \mathcal{J} \cdot \nabla &= \sum_{j=1}^n \mathcal{J}_j \partial_j = \sum_{j=1}^n (x_j \partial_j - it\Delta \partial_j^2) \\ &= x \cdot \nabla - it\Delta^2 = x \cdot \nabla + 4t\partial_t + 4it \left(i\partial_t - \frac{1}{4}\Delta^2 \right) \\ &= \mathcal{P} + 4it\mathcal{L}, \end{aligned}$$

where $\mathcal{L} = i\partial_t - \frac{1}{4}\Delta^2$ is the linear part of equation (1.1). We have the commutation relations $[\mathcal{J}, \mathcal{L}] = 0$, and $[\mathcal{L}, \mathcal{P}] = 4\mathcal{L}$. We introduce the function space

$$\mathbf{Z}_\infty = \{U(-t)v \in \mathbf{C}([0, \infty); \mathbf{Z}); \|v\|_{\mathbf{Z}_\infty} < \infty\},$$

where

$$\begin{aligned} \|v\|_{\mathbf{Z}_\infty} &= \sup_{t \in [0, \infty)} \|U(-t)v(t)\|_{\mathbf{Z}} \\ \mathbf{Z} &= \mathbf{H}^{0, \frac{1}{2}+\delta} \cap \mathbf{H}^{0,1} \cap \mathbf{H}^{\frac{1}{2}+\delta}, \\ \|\phi\|_{\mathbf{Z}} &= \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\delta}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} + \|\phi\|_{\mathbf{H}^{\frac{1}{2}+\delta}} \end{aligned}$$

for $n = 1$ and

$$\begin{aligned} \|v\|_{\mathbf{Z}_\infty} &= \sup_{t \in [0, \infty)} \|U(-t)v(t)\|_{\mathbf{Z}} + \sup_{t \in [0, \infty)} \|\mathcal{P}v(t)\|_{\mathbf{L}^2}, \\ \mathbf{Z} &= \mathbf{H}^{1,1} \cap \mathbf{H}^{1+\delta}, \end{aligned}$$

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{1,1}} + \|\phi\|_{\mathbf{H}^{1+\delta}}$$

for $n = 2$, where $\delta > 0$ is small enough, $U(-t) = e^{\frac{it}{4}\Delta^2}$.

Our main result is the following.

Theorem 1.1. *Assume that $p > 1 + \frac{4}{n}$, and the initial data are such that $\|u_0\|_{\mathbf{Z}} < \infty$. Then there exists an $\varepsilon_1 > 0$ such that (1.1) has a unique global solution $u \in \mathbf{Z}_\infty$ satisfying $\|u\|_{\mathbf{Z}_\infty} \leq 2\tilde{\varepsilon}$ for any $\tilde{\varepsilon}$ such that $\|u_0\|_{\mathbf{Z}} \leq \tilde{\varepsilon} \leq \varepsilon_1$.*

In view of the proof of the above theorem we have

Corollary 1.1. *Let u be the solution constructed in Theorem 1.1. Then the time decay estimates are fulfilled*

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^r} &\leq Ct^{-\frac{n}{4}(1-\frac{1}{r})}, \quad 3 < r < \infty, \\ \|u(t)\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{1}{4}}, \quad \|\partial_x u(t)\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}}, \quad n = 1, \\ \|u(t)\|_{\dot{\mathbf{H}}_r^1} &\leq Ct^{-\frac{1}{4}-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})}, \quad 6 < r < \infty, \quad 1 > \frac{1}{q} - \frac{1}{r} > \frac{5}{6}, \quad n = 2. \end{aligned}$$

Corollary 1.2. *Let u be the solution constructed in Theorem 1.1. Then there exists a unique scattering state u^+ such that $u^+ \in \mathbf{H}^{0, \frac{1}{2}+\delta} \cap \mathbf{H}^{\frac{1}{2}+\delta}$,*

$$\lim_{t \rightarrow \infty} \|U(-t)u(t) - u^+\|_{\mathbf{H}^{0, \frac{1}{2}+\delta} \cap \mathbf{H}^{\frac{1}{2}+\delta}} = 0$$

for $n = 1$ and $u^+ \in \mathbf{Z}$

$$\lim_{t \rightarrow \infty} \|U(-t)u(t) - u^+\|_{\mathbf{Z}} = 0$$

for $n = 2$.

Our proof is based on the estimate of the operator $\mathcal{J} = U(t)xU(-t) = x - it\partial_x^3$ in the $\mathbf{L}^2$ -norm. The main ingredient of the proof is that, when acting by $\mathcal{J}$ on the nonlinearity $f(u)$, we obtain the worst term $it\partial_x^3$, which can be expressed again in terms of $\mathcal{J}$. Then the derivatives of the first and the second order give us gain of the time decay of solutions of order $t^{-\frac{n}{4}}$ and $t^{-\frac{n}{2}}$, respectively according to the properties of the linear evolutions group $U(t)$. Also the multiplication of the nonlinearity $f(u)$ by x can be decomposed as follows $xf(u) = O\left(\left(|x|^{\frac{1}{3}}u\right)^3|u|^{p-3}\right)$, so we need to estimate the loss of the time decay, when multiplying the solution u by $|x|^{\frac{1}{3}}$.

We organize the rest of our paper as follows. Sections 2 and 4 are devoted to the estimates for the solution of the linear problem in $\mathbf{L}^p$ and weighted $\mathbf{L}^\infty$ (Lemmas 2.1–4.3) and the estimates of $\mathcal{J}f(u)$ (Lemmas 2.3, 4.4) in the one dimensional and two dimensional cases, respectively. We prove the main result in Sects. 3 and 5 in the one dimensional and two dimensional cases, respectively.

2. Preliminary estimates for $n = 1$

We first estimate the solutions of linear problem in the following lemma.

Lemma 2.1. *Let $\mu > 0$ be small. Then the following estimates*

$$\|U(t)\phi\|_{\mathbf{L}^q} \leq Ct^{-\frac{1}{4}(1-\frac{1}{q})} \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}$$

if $3 < q \leq \infty$,

$$\|\partial_x U(t)\phi\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}$$

and

$$\left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \partial_x^2 U(t)\phi \right\|_{\mathbf{L}^\infty} \leq Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t \rangle^{\frac{1}{8}} \left(\|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} \right)$$

are true for all $t > 0$.

Proof. We have

$$\begin{aligned} U(t)\phi &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + \frac{it}{4}|\xi|^4} \widehat{\phi}(\xi) d\xi \\ &= \int_{\mathbb{R}} A(x-y, t) \phi(y) dy, \end{aligned}$$

where the kernel

$$A(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + \frac{it}{4}|\xi|^4} d\xi.$$

By the estimate of the kernel $|\partial_x^j A(x, t)| \leq Ct^{-\frac{1+j}{4}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1-j}{3}}$ given in paper [1], we have

$$|\partial_x^j U(t)\phi| \leq Ct^{-\frac{1+j}{4}} \int_{\mathbb{R}} \left\langle (x-y)t^{-\frac{1}{4}} \right\rangle^{-\frac{1-j}{3}} |\phi(y)| dy. \quad (2.1)$$

Therefore using the Young's inequality

$$\|f * g\|_{\mathbf{L}^s} \leq C \|f\|_{\mathbf{L}^r} \|\phi\|_{\mathbf{L}^q},$$

where $\frac{1}{s} = \frac{1}{r} + \frac{1}{q} - 1$ and the convolution $(f * g)(x) = \int f(x-y)g(y)dy$, we get

$$\|U(t)\phi\|_{\mathbf{L}^q} \leq Ct^{-\frac{1}{4}(1-\frac{1}{q})} \|\phi\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{4}(1-\frac{1}{q})} \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}$$

if $3 < q \leq \infty$, and

$$\|\partial_x U(t)\phi\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}.$$

Then the first estimate of the lemma follows by the Sobolev embedding theorem. We consider the third estimate to find that

$$\begin{aligned} &\left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} |\partial_x^2 U(t)\phi| \\ &\leq Ct^{-\frac{3}{4}} \int_{\mathbb{R}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \left\langle (x-y)t^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} \left\langle yt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \left\langle yt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} |\phi(y)| dy \\ &\leq Ct^{-\frac{3}{4}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^1} \leq Ct^{-\frac{3}{4}} \left\| x^{\frac{1}{2}+\mu} \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{2}-\mu} \phi \right\|_{\mathbf{L}^1} \end{aligned}$$

$$\begin{aligned}
&\leq Ct^{-\frac{3}{4}} \left(\|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + t^{\frac{\mu}{4}-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} \right) \\
&\leq Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t \rangle^{\frac{1}{8}} \left(\|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} \right).
\end{aligned}$$

□

In order to estimate the action of $\mathcal{J}$ on the nonlinearity we need the following lemmas.

Lemma 2.2. *Let $p > \frac{9}{2}$. Then there exists small $\mu > 0$ such that*

$$\left\| |x|^{\frac{1}{3}} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}-\frac{\mu}{12}} \langle t \rangle^{\frac{\mu}{12}} \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}$$

is true for all $t > 0$.

Proof. By (2.1) and the Hölder's inequality with $\frac{1}{r} + \frac{1}{s} = 1$

$$\begin{aligned}
\left| |x|^{\frac{1}{3}} U(t) \phi \right| &\leq Ct^{-\frac{1}{4}} \int |x-y|^{\frac{1}{3}} \left\langle (x-y)t^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} |\phi(y)| dy \\
&\quad + Ct^{-\frac{1}{4}} \int \left\langle (x-y)t^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} |y|^{\frac{1}{3}} |\phi(y)| dy \\
&\leq Ct^{-\frac{1}{4}+\frac{1}{12}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{4}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \right\|_{\mathbf{L}^s} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^r} \\
&\leq Ct^{-\frac{1}{6}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{4r}} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^r}
\end{aligned}$$

if $s > 3$, i.e. $1 \leq r < \frac{3}{2}$. We choose $r = \frac{3}{2+\mu}$, where $\mu > 0$ is small enough. Then we obtain

$$\left\| |x|^{\frac{1}{3}} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{6}-\frac{\mu}{12}} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^{\frac{3}{2+\mu}}}.$$

Since $\|\phi\|_{\mathbf{L}^1} \leq C \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}$ and

$$\left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^{\frac{3}{2+\mu}}} \leq \left\| \langle x \rangle^{\frac{1}{6}+\mu} |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^{-\frac{1}{6}-\mu} \right\|_{\mathbf{L}^{\frac{6}{1+2\mu}}} \leq C \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}},$$

we have

$$\left\| |x|^{\frac{1}{3}} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq C \left(t^{-\frac{1}{6}} + t^{-\frac{1}{6}-\frac{\mu}{12}} \right) \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}.$$

□

Next we estimate the action of the operator $\mathcal{J} = U(t)xU(-t) = x - it\partial_x^3$ on the nonlinearity $f(u)$. Define the norms

$$\begin{aligned}
\|u\|_{\mathbf{V}} &= \|U(-t)u\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + \|u\|_{\mathbf{H}^{\frac{1}{2}+\mu}}, \\
\|u\|_{\mathbf{W}} &= \|U(-t)u\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + \langle t \rangle^{-\frac{1}{8}} \|U(-t)u\|_{\mathbf{H}^{0,1}}.
\end{aligned}$$

Lemma 2.3. *Let $p > \frac{9}{2}$. Then there exists a small $\mu > 0$ such that*

$$\|\mathcal{J}f(u)\|_{\mathbf{L}^2} \leq Ct^{-\frac{5}{8}} \langle t \rangle^{\frac{\mu}{4}-\frac{1}{4}(p-4)} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}}$$

is valid for all $t > 0$.

Proof. By a direct computation, we find

$$|\mathcal{J}f(u)| \leq Ct|u|^{p-3}|\partial_x u|^3 + Ct|u|^{p-2}|\partial_x u||\partial_x^2 u| + C|x||u|^p + C|u|^{p-1}|\mathcal{J}u|. \quad (2.2)$$

By the second and third estimates of Lemma 2.1 with $\phi = U(-t)u$, we have

$$\|\partial_x u\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}}\|u\|_{\mathbf{V}}$$

and

$$\begin{aligned} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \partial_x^2 u \right\|_{\mathbf{L}^\infty} &\leq Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t \rangle^{\frac{1}{8}} \left(\|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} \right) \\ &\leq Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t \rangle^{\frac{1}{8}} \|u\|_{\mathbf{W}}. \end{aligned}$$

Hence the first term of the right-hand side of (2.2) is estimated as

$$t \left\| |u|^{p-3} (\partial_x u)^3 \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}} \|u\|_{\mathbf{V}}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3}$$

and for the second term by Lemma 2.2 we get

$$\begin{aligned} &t \left\| |u|^{p-2} (\partial_x u) \partial_x^2 u \right\|_{\mathbf{L}^2} \\ &\leq Ct \|\partial_x u\|_{\mathbf{L}^\infty} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \partial_x^2 u \right\|_{\mathbf{L}^\infty} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} |u|^{p-3} \right\|_{\mathbf{L}^2} \\ &\leq Ct^{\frac{\mu}{4}-\frac{3}{8}} \langle t \rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \\ &\quad + Ct^{\frac{\mu}{4}-\frac{3}{8}-\frac{1}{12}} \langle t \rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \left\| |x|^{\frac{1}{3}} u \right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\leq Ct^{\frac{\mu}{4}-\frac{3}{8}} \langle t \rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \\ &\quad + Ct^{\frac{\mu}{6}-\frac{5}{8}} \langle t \rangle^{\frac{1}{8}+\frac{\mu}{12}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{V}}^2 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3}. \end{aligned}$$

By the first estimate of Lemma 2.1 we get

$$\|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \leq C \langle t \rangle^{-\frac{1}{4}(p-\frac{5}{2})} \|u\|_{\mathbf{V}}^{p-2}$$

and

$$\|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \leq C \langle t \rangle^{-\frac{1}{4}(p-\frac{7}{2})} \|u\|_{\mathbf{V}}^{p-3}$$

if $2(p-3) > 3$. Therefore

$$t \left\| |u|^{p-3} (\partial_x u)^3 \right\|_{\mathbf{L}^2} + t \left\| |u|^{p-2} (\partial_x u) \partial_x^2 u \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{5}{8}} \langle t \rangle^{\frac{\mu}{4}-\frac{1}{4}(p-4)} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}}.$$

By Lemma 2.2 we also have for the third summand of the right-hand side of (2.2)

$$\begin{aligned} \|x|u|^p\|_{\mathbf{L}^2} &\leq \left\| |x|^{\frac{1}{3}} u \right\|_{\mathbf{L}^\infty}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\leq Ct^{-\frac{1}{2}-\frac{\mu}{4}} \langle t \rangle^{\frac{\mu}{4}} \|U(-t)u\|_{\mathbf{H}^{0, \frac{1}{2}+\mu}}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\leq Ct^{-\frac{1}{2}-\frac{\mu}{4}} \langle t \rangle^{-\frac{1}{4}(p-\frac{7}{2}-\mu)} \|u\|_{\mathbf{V}}^p. \end{aligned}$$

The last term of the right-hand side of (2.2) is estimated by Lemma 2.1 as

$$\left\| |u|^{p-1} \mathcal{J}u \right\|_{\mathbf{L}^2} \leq \|u\|_{\mathbf{L}^\infty}^{p-1} \|\mathcal{J}u\|_{\mathbf{L}^2} \leq C \langle t \rangle^{-\frac{1}{4}(p-\frac{3}{2})} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}}.$$

Therefore we have the desired estimate. $\square$

3. Proof of Theorem 1.1 for $n = 1$

We consider the linearized equation

$$i\partial_t u - \frac{1}{4}\Delta^2 u = f(v), u(0) = u_0, \quad (3.1)$$

where

$$\begin{aligned} v &\in \mathbf{Z}_{\infty, \rho} = \{v \in \mathbf{Z}_\infty; \|v\|_{\mathbf{Z}_\infty} \leq \rho\}, \\ \|v\|_{\mathbf{Z}_\infty} &= \sup_{t \in [0, \infty)} \|U(-t)v(t)\|_{\mathbf{Z}}, \\ \|\phi\|_{\mathbf{Z}} &= \|\phi\|_{\mathbf{H}^{0, \frac{1}{2}+\delta}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}} + \|\phi\|_{\mathbf{H}^{\frac{1}{2}+\delta}}. \end{aligned}$$

We have the integral equation associated with (3.1)

$$u(t) = Sv(t) = U(t)u_0 - i \int_0^t U(t-s)f(v)ds. \quad (3.2)$$

By Lemma 4 in [6], we get

$$\begin{aligned} \sup_{t \in [0, T]} \|u(t)\|_{\mathbf{H}^{\frac{1}{2}+\delta}} &\leq \|u_0\|_{\mathbf{H}^{\frac{1}{2}+\delta}} + C \int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} \|v(s)\|_{\mathbf{H}^{\frac{1}{2}+\delta}} ds \leq \|u_0\|_{\mathbf{H}^{\frac{1}{2}+\delta}} \\ &\quad + C\rho^p. \end{aligned} \quad (3.3)$$

By Lemma 2.3 and (3.2) we have

$$\begin{aligned} \|\mathcal{J}u(t)\|_{\mathbf{L}^2} &\leq \|xu_0\|_{\mathbf{L}^2} + C\rho^p \int_0^t s^{-\frac{5}{8}} \langle s \rangle^{\frac{\mu}{4} - \frac{1}{4}(p-4)} ds \\ &\leq \|xu_0\|_{\mathbf{L}^2} + C\rho^p t^{\frac{1}{8}} \end{aligned} \quad (3.4)$$

and also

$$\left\| |x|^{\frac{1}{2}+\delta} U(-t)u(t) \right\|_{\mathbf{L}^2} \leq \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C \int_0^\infty \left\| |x|^{\frac{1}{2}+\delta} U(-s)f(v) \right\|_{\mathbf{L}^2} ds. \quad (3.5)$$

By Hölder's inequality using Lemma 2.3 and the first estimate of Lemma 2.1 we find

$$\begin{aligned} \left\| |x|^{\frac{1}{2}+\delta} U(-s)f(v) \right\|_{\mathbf{L}^2} &\leq \|xU(-s)f(v)\|_{\mathbf{L}^2}^{\frac{1}{2}+\delta} \|f(v)\|_{\mathbf{L}^2}^{\frac{1}{2}-\delta} \\ &\leq C\rho^p s^{-\frac{5}{16} - \frac{5}{8}\delta} \langle s \rangle^{-\frac{1}{4}(p-\frac{9}{4}) + \frac{7}{8}\delta + \frac{1}{4}\mu\delta + \frac{1}{8}\mu}. \end{aligned} \quad (3.6)$$

Therefore applying (3.6) to (3.5) we find

$$\begin{aligned} \left\| |x|^{\frac{1}{2}+\delta} U(-t) u(t) \right\|_{\mathbf{L}^2} &\leq \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C\rho^p \\ &\quad + C\rho^p \int_1^\infty s^{-\frac{1}{4}(p-1)+\frac{1}{4}(\mu+\delta+\mu\delta)} ds \\ &\leq \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C\rho^p \end{aligned} \quad (3.7)$$

if $p > 5 + \mu + \delta + \mu\delta$. Thus (3.3), (3.4) and (3.7) show that the mapping S defined by $u = Sv$ transforms $\mathbf{Z}_{\infty, \rho}$ into itself if the data are sufficiently small in $\mathbf{Z}_\infty$. We also have

$$\sup_{t \in [0, \infty)} \|Sv_1(t) - Sv_2(t)\|_{\mathbf{L}^2} \leq C\rho^{p-1} \sup_{t \in [0, \infty)} \|v_1(t) - v_2(t)\|_{\mathbf{L}^2}. \quad (3.8)$$

Therefore the contraction mapping shows that there exists a unique global solution u such that

$$U(-t)u \in \mathbf{L}^\infty\left(0, \infty; \mathbf{H}^{0, \frac{1}{2}+\delta} \cap \mathbf{H}^{\frac{1}{2}+\delta}\right), \langle t \rangle^{\frac{1}{8}} U(-t)u \in \mathbf{L}^\infty\left(0, \infty; \mathbf{H}^{0,1}\right)$$

and

$$u \in \mathbf{C}\left([0, \infty); \mathbf{L}^2\right).$$

The continuity in time of solutions $U(-t)u$ in $\mathbf{H}^{0,1} \cap \mathbf{H}^{\frac{1}{2}+\delta}$ follows from (3.3) and (3.4).

4. Preliminary estimates for $n = 2$

We have

$$\begin{aligned} |\partial_x|^\alpha U(t)\phi &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi + \frac{it}{4}|\xi|^4} |\xi|^\alpha \widehat{\phi}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} A_\alpha(x-y, t) \phi(y) dy, \end{aligned}$$

where $0 \leq \alpha < 4 - n$ and the kernel

$$A_\alpha(x, t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi + \frac{it}{4}|\xi|^4} |\xi|^\alpha d\xi.$$

The estimate of the kernel $A_\alpha(x, t)$ with an integer α was obtained in paper [1]. We are interested here in fractional α . By the scaling $xt^{-\frac{1}{4}} = y$, $\xi t^{\frac{1}{4}} = \eta$ and then changing the variable of integration $\eta = z|y|^{\frac{1}{3}}$, we obtain

$$\begin{aligned} A_\alpha(x, t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi + \frac{it}{4}|\xi|^4} |\xi|^\alpha d\xi \\ &= t^{-\frac{n+\alpha}{4}} \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{iy \cdot \eta + \frac{i}{4}|\eta|^4} |\eta|^\alpha d\eta \\ &= t^{-\frac{n+\alpha}{4}} |y|^{\frac{n+\alpha}{3}} \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{i|y|^{\frac{4}{3}}\left(\frac{y}{|y|} \cdot z + \frac{1}{4}|z|^4\right)} |z|^\alpha dz \\ &= t^{-\frac{n+\alpha}{4}} |y|^{\frac{n+\alpha}{3}} \widetilde{A}_\alpha(y), \end{aligned}$$

where

$$\widetilde{A}_\alpha(y) = \int_{\mathbb{R}^n} e^{i|y|^{\frac{4}{3}}(\frac{y}{|y|} \cdot z + \frac{1}{4}|z|^4)} |z|^\alpha dz.$$

It is easy to see that $|\widetilde{A}_\alpha(y)| < \infty$ for $|y| \leq 1$ since $0 \leq \alpha < 4 - n$. To estimate $\widetilde{A}_\alpha(y)$ for large y we apply the stationary phase method (see [3], p. 110)

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) e^{i\lambda S(x)} dx &= e^{i\lambda S(x_0)} \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} |\det S''(x_0)|^{-\frac{1}{2}} e^{i\frac{\pi}{4} \text{sgn} S''(x_0)} f(x_0) \\ &\quad + O(\lambda^{-\frac{n}{2}-1}), \end{aligned}$$

where $\lambda = |y|^{\frac{4}{3}}$, $S(x) = \frac{y}{|y|} \cdot x + \frac{1}{4}|x|^4$ and the stationary point $x_0 = (x_{0,1}, \dots, x_{0,n})$ is defined by the equation

$$\begin{aligned} \nabla S(x_0) &= (\partial_1 S(x_0), \dots, \partial_n S(x_0)) \\ &= \left(\frac{y_1}{|y|} + |x_0|^2 x_{0,1}, \dots, \frac{y_n}{|y|} + |x_0|^2 x_{0,n} \right) \\ &= (0, \dots, 0). \end{aligned}$$

We have $x_0 = -\frac{y}{|y|}$ and for $n = 2$

$$\det S''(x_0) = \begin{vmatrix} |x_0|^2 + 2x_{0,1}^2 & 2x_{0,1}x_{0,2} \\ 2x_{0,1}x_{0,2} & |x_0|^2 + 2x_{0,2}^2 \end{vmatrix} = 3|x_0|^4 = 3,$$

$\text{sgn} S''(x_0)$ is the difference between the number of positive and negative eigenvalues of $S''(x_0)$ and $\text{sgn} S''(x_0) = 2$, $f(x_0) = |x_0|^\alpha = 1$. Then

$$\widetilde{A}_\alpha(y) = \frac{1}{\sqrt{3}} 2\pi i |y|^{-\frac{4}{3}} e^{-\frac{3}{4}i|y|^{\frac{4}{3}}} + O(|y|^{-\frac{8}{3}})$$

for $|y| \rightarrow \infty$. Thus we get the estimate

$$|A_\alpha(x, t)| \leq Ct^{-\frac{2+\alpha}{4}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2-\alpha}{3}}$$

for $n = 2$.

In the two dimensional case we would like to avoid the use of the norm $\|\phi\|_{\mathbf{L}^1}$ since we restrict to the weighted Sobolev space $\|\phi\|_{\mathbf{H}^{1,1}}$.

Lemma 4.1. *Let $\mu > 0$ be small. Then the following estimates are true for all $t > 0$*

$$\|U(t)\phi\|_{\mathbf{L}^q} \leq C \langle t \rangle^{-\frac{1}{2}(1-\frac{1}{q}-\frac{\mu}{6})} (\|\phi\|_{\mathbf{H}^{0,1}} + \|\phi\|_{\mathbf{H}^{1+\mu}})$$

if $\frac{6}{2-\mu} < q \leq \infty$,

$$\|U(t)\phi\|_{\dot{\mathbf{H}}_q^1} \leq Ct^{-\frac{3}{8}} \langle t \rangle^{-\frac{1}{2}(\frac{3}{4}-\frac{1}{q}-\frac{\mu}{6})} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{6}{1-\mu} < q \leq \infty$, and

$$\|U(t)\phi\|_{\dot{\mathbf{H}}_q^2} \leq Ct^{-\frac{1}{2}(2-\frac{1}{q}-\frac{7\mu}{12})} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{12}{\mu} < q \leq \infty$.

Proof. As in the proof of Lemma 2.1, applying the Young's inequality with $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{s}$, we find

$$\begin{aligned} \left\| |\partial_x|^j U(t) \phi \right\|_{\mathbf{L}^q} &\leq C t^{-\frac{2+j}{4}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y) t^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} |\phi(y)| dy \right\|_{\mathbf{L}^q} \\ &\leq C t^{-\frac{2+j}{4}} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} \right\|_{\mathbf{L}^r} \|\phi\|_{\mathbf{L}^s} \leq C t^{-\frac{2+j}{4} - \frac{1}{2r}} \|\phi\|_{\mathbf{L}^s}, \end{aligned}$$

since

$$\left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} \right\|_{\mathbf{L}^r}^r = t^{\frac{1}{2}} \int_{\mathbb{R}^2} \langle x \rangle^{-\frac{2-j}{3}r} dx \leq C t^{\frac{1}{2}}$$

if $0 \leq 1 + \frac{1}{q} - \frac{1}{s} = \frac{1}{r} < \frac{2-j}{6}$, $j = 0, 1$. Hence taking $\frac{1}{s} = 1 - \frac{\mu}{6}$, we obtain

$$\|U(t) \phi\|_{\mathbf{H}_q^j} \leq C t^{-\frac{1}{2}(\frac{2+j}{2} - \frac{1}{q} - \frac{\mu}{6})} \|\phi\|_{\mathbf{H}^{0,1}},$$

if $\frac{6}{2-j-\mu} < q \leq \infty$, $j = 0, 1$. For the case of $0 < t < 1$, we apply the Sobolev embedding theorem $\|U(t) \phi\|_{\mathbf{L}^q} \leq C \|U(t) \phi\|_{\mathbf{H}^{1+\mu}} \leq C \|\phi\|_{\mathbf{H}^{1+\mu}}$ for $1 \leq q \leq \infty$, to get the first estimate of the lemma. For the second estimate in the case of $0 < t < 1$ we write

$$\begin{aligned} \|\nabla U(t) \phi\|_{\mathbf{L}^q} &\leq C t^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y) t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |\nabla \phi(y)| dy \right\|_{\mathbf{L}^q} \\ &\leq C t^{-\frac{1}{2}} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} \right\|_{\mathbf{L}^r} \|\nabla \phi\|_{\mathbf{L}^s} \leq C t^{-\frac{1}{2} + \frac{1}{2r}} \|\nabla \phi\|_{\mathbf{L}^s} \leq C t^{-\frac{3}{8}} \|\phi\|_{\mathbf{H}^{1,1}}, \end{aligned}$$

if $0 \leq 1 + \frac{1}{q} - \frac{1}{s} = \frac{1}{r} < \frac{1}{3}$, taking $r = 4$, $\frac{1}{s} = \frac{3}{4} + \frac{1}{q}$ we get the second estimate of the lemma. To prove the third estimate of the lemma, we apply the Young's inequality with $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{s}$, $\alpha = 2 - \mu$, to get

$$\begin{aligned} \left\| |\partial_x|^2 U(t) \phi \right\|_{\mathbf{L}^q} &= \left\| |\partial_x|^{2-\mu} U(t) |\partial_x|^\mu \phi \right\|_{\mathbf{L}^q} \\ &\leq C t^{\frac{\mu}{4}-1} \left\| \int_{\mathbb{R}^2} \left\langle (x-y) t^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} |\partial_y|^\mu \phi(y) dy \right\|_{\mathbf{L}^q} \\ &\leq C t^{\frac{\mu}{4}-1} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} \right\|_{\mathbf{L}^r} \|\partial_x|^\mu \phi\|_{\mathbf{L}^s}. \end{aligned}$$

Then using $\left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} \right\|_{\mathbf{L}^r} \leq C t^{\frac{1}{2r}}$ for $r > \frac{6}{\mu}$, and choosing $\frac{1}{s} = 1 - \frac{\mu}{12}$, we obtain

$$\left\| |\partial_x|^2 U(t) \phi \right\|_{\mathbf{L}^q} \leq C t^{-1 + \frac{1}{2q} + \frac{7\mu}{24}} \|\partial_x|^\mu \phi\|_{\mathbf{L}^s} \leq C t^{-\frac{1}{2}(2 - \frac{1}{q} - \frac{7\mu}{12})} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{12}{\mu} < q \leq \infty$. □

In order to estimate the action of $\mathcal{J}$ on the nonlinearity we need the following lemmas.

Lemma 4.2. *Let $\mu > 0$ be small. Then the estimate*

$$\left\| |x|^{\frac{2}{3}-\mu} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{3}-\frac{\mu}{12}} \|\phi\|_{\mathbf{H}^{0,1}}$$

is true for all $t > 0$.

Proof. Applying the Young's inequality with $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'}$, we find

$$\begin{aligned} \left\| |x|^a U(t) \phi \right\|_{\mathbf{L}^\infty} &\leq C t^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} |x-y|^a \left\langle (x-y) t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |\phi(y)| dy \right\|_{\mathbf{L}^\infty} \\ &\quad + C t^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y) t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |y|^a |\phi(y)| dy \right\|_{\mathbf{L}^\infty} \\ &\leq C t^{-\frac{1}{2}+\frac{a}{4}} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{a-\frac{2}{3}} \right\|_{\mathbf{L}^{r'}} \|\phi\|_{\mathbf{L}^r} \\ &\quad + C t^{-\frac{1}{2}} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} \right\|_{\mathbf{L}^{s'}} \left\| |x|^a \phi \right\|_{\mathbf{L}^s} \\ &\leq C t^{-\frac{1}{2}+\frac{a}{4}+\frac{1}{2r'}} \|\phi\|_{\mathbf{L}^r} + C t^{-\frac{1}{2}+\frac{1}{2s'}} \left\| |x|^a \phi \right\|_{\mathbf{L}^s} \end{aligned}$$

if $(\frac{2}{3}-a)r' > 2$, $\frac{2}{3}s' > 2$, i.e. $1 \leq r < \frac{6}{4+3a}$, $1 \leq s < \frac{3}{2}$. Therefore we obtain

$$\left\| |x|^a U(t) \phi \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{2r}+\frac{a}{4}} \|\phi\|_{\mathbf{L}^r} + C t^{-\frac{1}{2s}} \left\| |x|^a \phi \right\|_{\mathbf{L}^s}$$

for $0 \leq a < \frac{2}{3}$, $1 \leq r < \frac{6}{4+3a}$, $1 \leq s < \frac{3}{2}$. Taking $a = \frac{2}{3} - \mu$, $\frac{1}{r} = 1 - \frac{\mu}{3}$, $\frac{1}{s} = \frac{2}{3} + \frac{\mu}{6}$ we get

$$\left\| |x|^{\frac{2}{3}-\mu} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{3}-\frac{\mu}{12}} \|\phi\|_{\mathbf{L}^{\frac{3}{3-\mu}}} + C t^{-\frac{1}{3}-\frac{\mu}{12}} \left\| |x|^{\frac{2}{3}-\mu} \phi \right\|_{\mathbf{L}^{\frac{6}{4+\mu}}}.$$

Since

$$\|\phi\|_{\mathbf{L}^{\frac{3}{3-\mu}}} \leq \|\langle x \rangle \phi\|_{\mathbf{L}^2} \left\| \langle x \rangle^{-1} \right\|_{\mathbf{L}^{\frac{6}{3-2\mu}}} \leq C \|\phi\|_{\mathbf{H}^{0,1}}$$

and

$$\left\| |x|^{\frac{2}{3}-\mu} \phi \right\|_{\mathbf{L}^{\frac{6}{4+\mu}}} \leq \left\| \langle x \rangle^{\frac{1}{3}+\mu} |x|^{\frac{2}{3}-\mu} \phi \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^{-\frac{1}{3}-\mu} \right\|_{\mathbf{L}^{\frac{6}{1+\mu}}} \leq C \|\phi\|_{\mathbf{H}^{0,1}},$$

we find the estimate of the lemma

$$\left\| |x|^{\frac{2}{3}-\mu} U(t) \phi \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{3}-\frac{\mu}{12}} \|\phi\|_{\mathbf{H}^{0,1}}.$$

□

Define the norm

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{1,1}} + \|\phi\|_{\mathbf{H}^{1+\mu}}.$$

Lemma 4.3. *Let $p > 3$. Then there exists a small $\mu > 0$ such that the estimate*

$$\|x|U(t)\phi|^p\|_{\mathbf{L}^2} \leq C t^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|\phi\|_{\mathbf{Z}}^p$$

holds for all $t > 0$.

Proof. We use Lemma 4.2 to find

$$\begin{aligned} \| |x| |U(t)\phi|^p \|_{\mathbf{L}^2} &= \left\| \left| |x|^{\frac{2}{3}-\mu} |U(t)\phi| \right|^{\frac{3}{2-3\mu}} |U(t)\phi|^{p-\frac{3}{2-3\mu}} \right\|_{\mathbf{L}^2} \\ &\leq C \left\| |x|^{\frac{2}{3}-\mu} |U(t)\phi| \right\|_{\mathbf{L}^\infty}^{\frac{3}{2-3\mu}} \|U(t)\phi\|_{\mathbf{L}^{2(p-\frac{3}{2-3\mu})}}^{p-\frac{3}{2-3\mu}} \\ &\leq C t^{-(1+\frac{\mu}{4})\frac{1}{2-3\mu}} \|\phi\|_{\mathbf{H}^{0,1}}^{\frac{3}{2-3\mu}} \|U(t)\phi\|_{\mathbf{L}^{2(p-\frac{3}{2-3\mu})}}^{p-\frac{3}{2-3\mu}}. \end{aligned}$$

For a small $\mu > 0$, we find that $2\left(p - \frac{3}{2-3\mu}\right) > \frac{6}{2-\mu}$, hence the first estimate of Lemma 4.1 says

$$\|U(t)\phi\|_{\mathbf{L}^{2(p-\frac{3}{2-3\mu})}}^{p-\frac{3}{2-3\mu}} \leq C \langle t \rangle^{-\frac{1}{2}(p-\frac{3}{2-3\mu}-\frac{1}{2}-\frac{\mu}{6}(p-\frac{3}{2-3\mu}))} \|\phi\|_{\mathbf{Z}}^{p-\frac{3}{2-3\mu}}.$$

Therefore we get

$$\begin{aligned} \| |x| |U(t)\phi|^p \|_{\mathbf{L}^2} &\leq C t^{-(1+\frac{\mu}{4})\frac{1}{2-3\mu}} \langle t \rangle^{-\frac{1}{2}(p-\frac{3}{2-3\mu}-\frac{1}{2}-\frac{\mu}{6}(p-\frac{3}{2-3\mu}))} \|\phi\|_{\mathbf{Z}}^p \\ &\leq C t^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|\phi\|_{\mathbf{Z}}^p \end{aligned}$$

since $p > 3$ and $\mu > 0$ is small enough. $\square$

Next we estimate the action of the operator $\mathcal{J} = U(t)xU(-t) = x - it\partial_x^3$ on the nonlinearity $f(u)$.

Lemma 4.4. *Let $p > 3$. Then there exists a small $\mu > 0$ such that the estimate*

$$\|\mathcal{J}f(u)\|_{\mathbf{L}^2} \leq C t^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|U(-t)u\|_{\mathbf{Z}}^p$$

is true for all $t > 0$.

Proof. As in (2.2) we start with

$$\begin{aligned} \|\mathcal{J}f(u)\|_{\mathbf{L}^2} &\leq C t \sum_{|\beta|=1} I_\beta + C t \sum_{|\alpha|=2, |\beta|=1} I_{\alpha,\beta} \\ &\quad + C \|x|u|^p\|_{\mathbf{L}^2} + C \| |u|^{p-1} \mathcal{J}u \|_{\mathbf{L}^2}, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} I_\beta &= t \left\| |u|^{p-3} |\partial_x^\beta u|^3 \right\|_{\mathbf{L}^2}, \\ I_{\alpha,\beta} &= t \left\| |u|^{p-2} (\partial_x^\beta u) \partial_x^\alpha u \right\|_{\mathbf{L}^2}. \end{aligned}$$

By the Hölder's inequality with $\sum_{j=1}^2 \frac{1}{r_j} = \frac{1}{2}$ we get

$$I_\beta \leq t \left\| \partial_x^\beta u \right\|_{\mathbf{L}^{3r_1}}^3 \|u\|_{\mathbf{L}^{r_2(p-3)}}^{p-3}.$$

Then taking $\frac{1}{r_1} = \frac{1}{2} - \mu$, $\frac{1}{r_2} = \mu$, and applying the first and the second estimates of Lemma 4.1 with $\phi = U(-t)u$ we find

$$\|u\|_{\mathbf{L}^{r_2(p-3)}}^{p-3} \leq C \langle t \rangle^{-\frac{1}{2}(p-3-\frac{\mu}{6}(p+3))} \|U(-t)u\|_{\mathbf{Z}}^{p-3}$$

and

$$\left\| \partial_x^\beta u \right\|_{\mathbf{L}^{3r_1}}^3 \leq C t^{-\frac{9}{8}} \langle t \rangle^{-\frac{1}{4}(\frac{7}{2}+\mu)} \|U(-t)u\|_{\mathbf{Z}}^3.$$

Hence

$$\begin{aligned} I_\beta &\leq Ct^{-\frac{1}{8}} \langle t \rangle^{-\frac{1}{2}(p-\frac{5}{4})+\frac{p}{12}\mu} \|U(-t)u\|_{\mathbf{Z}}^p \\ &\leq Ct^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|U(-t)u\|_{\mathbf{Z}}^p \end{aligned} \quad (4.2)$$

if $p > 3$ and $\mu > 0$ is small. Similarly, by the Hölder's inequality with $\sum_{j=1}^3 \frac{1}{q_j} = \frac{1}{2}$ we get

$$I_{\alpha,\beta} \leq Ct \|\partial_x^\alpha u\|_{\mathbf{L}^{q_1}} \|\partial_x^\beta u\|_{\mathbf{L}^{q_2}} \|u\|_{\mathbf{L}^{q_3(p-2)}}^{p-2}.$$

Then taking $\frac{1}{q_1} = \mu$, $\frac{1}{q_2} = \frac{1}{6} - \mu$, $\frac{1}{q_3} = \frac{1}{3}$, and applying Lemma 4.1 with $\phi = U(-t)u$ we find

$$\begin{aligned} \|u\|_{\mathbf{L}^{q_3(p-2)}}^{p-2} &\leq C \langle t \rangle^{-\frac{1}{2}(p-2-\frac{1}{3}-\frac{\mu}{6}(p-2))} \|U(-t)u\|_{\mathbf{Z}}^{p-2}, \\ \|\partial_x^\beta u\|_{\mathbf{L}^{q_2}} &\leq Ct^{-\frac{3}{8}} \langle t \rangle^{-\frac{1}{2}(\frac{3}{4}-\frac{1}{6}+\frac{5\mu}{6})} \|U(-t)u\|_{\mathbf{Z}} \end{aligned}$$

and

$$\|\partial_x^\alpha u\|_{\mathbf{L}^{q_1}} \leq Ct^{-1+\frac{19}{24}\mu} \|U(-t)u\|_{\mathbf{Z}}.$$

Hence

$$\begin{aligned} I_{\alpha,\beta} &\leq Ct^{-\frac{3}{8}+\frac{19}{24}\mu} \langle t \rangle^{-\frac{1}{2}(p-\frac{7}{4})+\frac{1}{12}\mu(p-7)} \|U(-t)u\|_{\mathbf{Z}}^p \\ &\leq Ct^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|U(-t)u\|_{\mathbf{Z}}^p. \end{aligned} \quad (4.3)$$

By Lemma 4.3 we find

$$\| |x| |u|^p \|_{\mathbf{L}^2} \leq Ct^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|U(-t)u\|_{\mathbf{Z}}^p. \quad (4.4)$$

Also the last term of the right-hand side of (4.1) is estimated by Lemma 4.1 as

$$\begin{aligned} \| |u|^{p-1} \mathcal{J}u \|_{\mathbf{L}^2} &\leq \|u\|_{\mathbf{L}^\infty}^{p-1} \|\mathcal{J}u\|_{\mathbf{L}^2} \\ &\leq C \langle t \rangle^{-\frac{p-1}{2}(1-\frac{\mu}{6})} \|U(-t)u\|_{\mathbf{Z}}^p \leq Ct^{-\frac{3}{4}} \langle t \rangle^{-\frac{1}{4}-\mu} \|U(-t)u\|_{\mathbf{Z}}^p. \end{aligned} \quad (4.5)$$

Therefore we have the desired estimate from (4.2), (4.3), (4.4) and (4.5). $\square$

5. Proof of Theorem 1.1 for $n = 2$

We consider the linearized equation (3.1), where

$$v \in \mathbf{Z}_{\infty,\rho} = \{v \in \mathbf{Z}_\infty; \|v\|_{\mathbf{Z}_\infty} \leq \rho\},$$

with the norm

$$\begin{aligned} \|v\|_{\mathbf{Z}_\infty} &= \sup_{t \in [0, \infty)} \|U(-t)v(t)\|_{\mathbf{Z}} + \sup_{t \in [0, \infty)} \|\mathcal{P}v(t)\|_{\mathbf{L}^2}, \\ \|\phi\|_{\mathbf{Z}} &= \|\phi\|_{\mathbf{H}^{1,1}} + \|\phi\|_{\mathbf{H}^{1+\delta}}. \end{aligned}$$

We consider the integral equation (3.2) associated with (3.1). We use Lemma 4.4 in (3.2) to obtain

$$\|\mathcal{J}u(t)\|_{\mathbf{L}^2} \leq \|xu_0\|_{\mathbf{L}^2} + C\rho^p. \quad (5.1)$$

By Lemma 4 in [6], we get

$$\begin{aligned} \|u(t)\|_{\mathbf{H}^{1+\delta}} &\leq \|u_0\|_{\mathbf{H}^{1+\delta}} + C \int_0^t \|f(v(s))\|_{\mathbf{H}^{1+\delta}} ds \\ &\leq \|u_0\|_{\mathbf{H}^{1+\delta}} + C \int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} \|v(s)\|_{\mathbf{H}^{1+\delta}} ds. \end{aligned} \quad (5.2)$$

We also have by the first estimate of Lemma 4.1

$$\|v(s)\|_{\mathbf{L}^\infty} \leq C \langle s \rangle^{-\frac{1}{2}(1-\frac{\mu}{6})} \rho.$$

Therefore

$$\|u(t)\|_{\mathbf{H}^{1+\delta}} \leq \|u_0\|_{\mathbf{H}^{1+\delta}} + C\rho^p. \quad (5.3)$$

By the identity

$$\mathcal{J} \cdot \nabla u = \mathcal{P}u + 4it\mathcal{L}u = \mathcal{P}u + 4itf(v),$$

we get

$$\|\mathcal{J} \cdot \nabla u\|_{\mathbf{L}^2} \leq \|\mathcal{P}u\|_{\mathbf{L}^2} + Ct\|f(v)\|_{\mathbf{L}^2} \leq \|\mathcal{P}u\|_{\mathbf{L}^2} + C\rho^p. \quad (5.4)$$

Applying $\mathcal{P}$ to both sides of (3.2), we obtain

$$\mathcal{L}\mathcal{P}u = (4 + \mathcal{P})f(v)$$

and by the energy estimate

$$\begin{aligned} \|\mathcal{P}u(t)\|_{\mathbf{L}^2} &\leq \|x \cdot \nabla u_0\|_{\mathbf{L}^2} + C \int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} (\|\mathcal{P}v(s)\|_{\mathbf{L}^2} + \|v(s)\|_{\mathbf{L}^2}) ds \\ &\leq \|x \cdot \nabla u_0\|_{\mathbf{L}^2} + C\rho^p. \end{aligned} \quad (5.5)$$

We apply (5.5) to (5.4) to get

$$\|\mathcal{J} \cdot \nabla u(t)\|_{\mathbf{L}^2} \leq \|x \cdot \nabla u_0\|_{\mathbf{L}^2} + C\rho^p. \quad (5.6)$$

Applying $\Omega_{j,k} = x_j \partial_k - x_k \partial_j$ to both sides of (3.2), we obtain

$$L\Omega_{j,k}u = \Omega_{j,k}f(v)$$

and by the energy method

$$\begin{aligned} \|\Omega_{j,k}u(t)\|_{\mathbf{L}^2} &\leq \|\Omega_{j,k}u_0\|_{\mathbf{L}^2} + C \int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} \|\Omega_{j,k}v(s)\|_{\mathbf{L}^2} ds \\ &\leq \|\Omega_{j,k}u_0\|_{\mathbf{L}^2} + C\rho^p. \end{aligned} \quad (5.7)$$

As in [5], we get by integration by parts

$$\begin{aligned}
 \sum_{j,k=1}^2 \|\mathcal{J}_j \partial_k u\|_{\mathbf{L}^2}^2 &= \sum_{j,k=1}^2 (\mathcal{J}_j \partial_k u, \mathcal{J}_j \partial_k u) \\
 &= \sum_{j,k=1}^2 (x_j \partial_k u, x_j \partial_k u) + \sum_{j,k=1}^2 (x_j \partial_k u, -it \Delta \partial_j \partial_k u) \\
 &\quad + \sum_{j,k=1}^2 (-it \Delta \partial_j \partial_k u, x_j \partial_k u) + \sum_{j,k=1}^2 (it \Delta \partial_j \partial_k u, it \Delta \partial_j \partial_k u) \\
 &= \frac{1}{2} \sum_{j,k=1}^2 (\Omega_{j,k} u, \Omega_{j,k} u) + ((x \cdot \nabla) u, (x \cdot \nabla) u) \\
 &\quad + (-it \Delta \nabla \cdot \nabla u, (x \cdot \nabla) u) + ((x \cdot \nabla) u, -it \Delta \nabla \cdot \nabla u) \\
 &\quad + (it \Delta \nabla \cdot \nabla u, it \Delta \nabla \cdot \nabla u) \\
 &= \frac{1}{2} \sum_{j,k=1}^2 \|\Omega_{j,k} u\|_{\mathbf{L}^2}^2 + \|\mathcal{J} \cdot \nabla u\|_{\mathbf{L}^2}^2,
 \end{aligned}$$

where we have denoted the inner product by $(\cdot, \cdot)$. By (5.1), (5.3), (5.6) and (5.7), we have

$$\|u\|_{\mathbf{Z}_\infty} \leq \varepsilon + C\rho^p. \quad (5.8)$$

Hence the mapping S defined by $u = Sv$ transforms $\mathbf{Z}_{\infty, \rho}$ into itself. In the same way as in the proof of (3.8), we have (3.8) for $n = 2$. Therefore the contraction mapping shows that there exists a unique global solution u such that

$$U(-t)u \in \mathbf{L}^\infty(0, \infty; \mathbf{H}^{1,1} \cap \mathbf{H}^{1+\delta}),$$

and

$$u \in \mathbf{C}([0, \infty); \mathbf{L}^2).$$

The continuity in time of solutions $U(-t)u$ in $\mathbf{H}^{1,1} \cap \mathbf{H}^{1+\delta}$ follows from (5.2), (5.4) and (5.5).

Acknowledgements

We would like to thank the referee for useful comments on the scattering problem. The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers JP25220702, JP15H03630. The work of P.I.N. is partially supported by CONACYT and PAPIIT project IN100616.

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Received: 22 June 2016.

Accepted: 31 October 2016.