Symmetry breaking of an elliptic equation in expanding annuli

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Abstract. We obtain non-radial bifurcation from radial solutions of a semilinear elliptic equation in expanding annuli of $\mathbb{R}^N$. To obtain the main results, we use a blow-up argument via the Morse index of the regular entire solutions of the equation

$$-\Delta u = \lambda u^p \quad \text{in } \mathbb{R}^N. \quad (0.1)$$

The main results of this paper can be seen as applications of the properties of regular entire solutions of (0.1).

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1. Introduction

We study symmetry breaking of radial solutions to the equation

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } D_R, \\ u > 0 & \text{in } D_R, \\ u = 0 & \text{on } \partial D_R, \end{cases} \quad (1.1)$$

where $D_R = \{x \in \mathbb{R}^N : R < |x| < R + 1\}$ with $N \geq 2$,

$$R \geq [(N-1)^{\frac{1}{N-2}} - 1]^{-1} \quad \text{for } N \geq 3; \quad R \geq [e - 1]^{-1} \quad \text{for } N = 2, \quad (1.2)$$

λ is a positive parameter, f satisfies the conditions:

- (A-1) $f \in C^2(\mathbb{R}^1)$ and $f(u) > 0$ for $u \geq 0$,
- (A-2) $f'(u) > 0$ and $f''(u) > 0$ for all $u \geq 0$,

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(A-3) $\lim_{u \rightarrow \infty} \frac{f(u)}{u^p} = 1$ with $p \in (1, p_c(N)) \setminus \{\frac{N+2}{N-2}\}$, where

$$p_c(N) := \begin{cases} \infty & \text{if } 2 \leq N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11. \end{cases}$$

Note that

$$1 < \frac{N+2}{N-2} < p_c(N) \quad \text{for } N \geq 3$$

and $p_c(N)$ is a root of the equation

$$\frac{2p}{p-1} \left[N - 2 - \frac{2}{p-1} \right] = \frac{(N-2)^2}{4}.$$

A typical example for f satisfying (A-1)–(A-3) is $f(u) = |u+1|^{p-1}(u+1)$.

We will obtain bifurcations of nonradial solutions from the radial solutions of (1.1). It is well-known that if $u \in C^2(\mathbb{R}^N)$ is a nonnegative solution of the equation

$$-\Delta u = \lambda u^p \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

then $u \equiv 0$ in $\mathbb{R}^N$ provided that $1 < p < \frac{N+2}{N-2}$; if $u \in C^2(\mathbb{R}^N)$ is a stable nonnegative solution of (1.3), then $u \equiv 0$ in $\mathbb{R}^N$ provided that $p = \frac{N+2}{N-2}$ (see [10, 12]) (note that (1.3) admits solutions with finite Morse index provided $p = \frac{N+2}{N-2}$ (see [10])); and if $u \in C^2(\mathbb{R}^N)$ is a positive solution of (1.3), then $i(\lambda, u) = \infty$ provided that $\frac{N+2}{N-2} < p < p_c(N)$ (see [8, 9, 11, 12]), where $i(\lambda, u)$ is the Morse index at (λ, u) . (Note that under a scaling $v = \lambda^{1/(p-1)}u$, then $-\Delta v = v^p$.)

Let u be a solution to the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded or unbounded domain, $\lambda > 0$. Denote by Q_u the bilinear form associated to u , i.e.

$$Q_u(\phi, \psi) = \int_{\Omega} \nabla \phi \cdot \nabla \psi dx - \int_{\Omega} \lambda f'(u) \phi \psi dx, \quad \phi, \psi \in H_0^1(\Omega).$$

The Morse index at (λ, u) , denoted by $i(\lambda, u)$, is the maximal dimension of a subspace $X \subset C_0^1(\Omega)$ such that

$$Q_u(\psi, \psi) < 0 \quad \forall \psi \in X \setminus \{0\}.$$

This is equivalent to say that $i(\lambda, u)$ is the number of negative eigenvalues of $-\Delta - \lambda f'(u)I$ computed with their multiplicity. If $\Omega = \{x : a < |x| < b\}$ with $b > a > 0$ and u is a radial solution of (1.4), the radial Morse index at (λ, u) : $i_{rad}(\lambda, u)$ is the number of negative eigenvalues of the problem

$$\begin{cases} \psi'' + \frac{N-1}{r} \psi' + \lambda f'(u) \psi = -\mu \psi, & r \in (a, b), \\ \psi(a) = \psi(b) = 0. \end{cases}$$

Problem (1.1) arises from many branches of mathematics and applied mathematics. It was studied by many authors, for example, [1–3, 5, 6, 13–19, 21, 22] and the references therein.

It is known from Theorem 2.11 of [16] that, for any R satisfying (1.2), there exists $\lambda_R^* > 0$ such that (1.1) has exactly two positive radial solutions $0 < \underline{u}_R^\lambda < \bar{u}_R^\lambda$ for $\lambda \in (0, \lambda_R^*)$, exactly one u_R^* at $\lambda = \lambda_R^*$, and none for $\lambda > \lambda_R^*$. It is well-known that $\underline{u}_R^\lambda$ ($0 < \lambda < \lambda_R^*$) and u_R^* are *stable*. Moreover, u_R^* is the unique solution of (1.1) at which the first eigenvalue of the linearized problem is 0. Note that $\bar{u}_R^\lambda$ is radially nondegenerate for $\lambda \in (0, \lambda_R^*)$. To see this, we notice that if μ is an eigenvalue of the problem

$$\begin{cases} -\phi''(r) - \frac{N-1}{r}\phi'(r) - \lambda f'(\bar{u}_R^\lambda)\phi = \mu\phi, & R < r < R+1, \\ \phi(R) = \phi(R+1) = 0, \end{cases}$$

then μ is simple. By the bifurcation theory on the critical point of odd multiplicity, [24, 25], if there is some $\mu = 0$, it is actually a secondary bifurcation point, i.e. there is a new radial solution branch of (1.1) coming from $(\lambda, \bar{u}_R^\lambda)$ (note that the nonlinearity of (1.1) is convex). This contradicts the fact that (1.1) has exactly two radial solutions for $\lambda \in (0, \lambda_R^*)$. Meanwhile, it is known from [4] that $i(\lambda, \underline{u}_R^\lambda) = i_{rad}(\lambda, \underline{u}_R^\lambda) = 0$ for $\lambda \in (0, \lambda_R^*)$, $i(\lambda, \bar{u}_R^\lambda) \geq 1$ and $i_{rad}(\lambda, \bar{u}_R^\lambda) = 1$ for $\lambda \in (0, \lambda_R^*)$.

Arguments similar to those in the proof of Theorem 1.1 of [16] imply that the similar result is also true for the problem

$$\begin{cases} -u'' = \lambda f(u), & \text{in } (0, 1), \\ u(0) = 0 = u(1), \end{cases} \quad (1.5)$$

i.e. there exists a $\lambda_0^* \in (0, \infty)$ such that problem (1.5) admits no solutions for $\lambda > \lambda_0^*$, one and only one solution u_0^* for $\lambda = \lambda_0^*$ and exactly two solutions $0 < \underline{u}_0^\lambda < \bar{u}_0^\lambda$ for $0 < \lambda < \lambda_0^*$. It is well-known that $\underline{u}_0^\lambda$ ($0 < \lambda < \lambda_0^*$) and u_0^* are *stable* and the Morse index of $\bar{u}_0^\lambda$ ($0 < \lambda < \lambda_0^*$) is not less than 1. Moreover, u_0^* is the unique solution of (1.5) at which the first eigenvalue of the linearized problem is 0. (Note that when $N = 1$, i.e. equation (1.5), $\rho(s)$ and s_0 as defined in [16] are $\rho(s) \equiv 1$ and $s_0 = 0$. We easily see that (2.26) in [16] holds).

In this paper, we are interested in studying the asymptotic behaviors of the radial solutions and bifurcation of nonradial solutions from a radial solution of (1.1) when R varies. To do these, we first need to know the asymptotic behavior of λ_R^* as $R \rightarrow \infty$. The main results of this paper are the following theorems.

Theorem 1.1. *Let f satisfy (A-1)–(A-3). Then*

$$\lim_{R \rightarrow +\infty} \lambda_R^* = \lambda_0^*. \quad (1.6)$$

Moreover, for $\lambda \in (0, \lambda_0^)$, $\underline{u}_R^\lambda(R+t) \rightarrow \underline{u}_0^\lambda(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$ and $u_R^*(R+t) \rightarrow u_0^*(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$.*

Theorem 1.2. *Let f satisfy (A-1)–(A-3). Then for any fixed $\lambda \in (0, \lambda_0^*)$, there exist $R_* > 1$ and a sequence $\{R_j\}$ with $R_j \geq R_*$ and $R_j \rightarrow \infty$ as $j \rightarrow \infty$ such that the Morse index $i(\lambda, \bar{u}_{R_j}^\lambda) \rightarrow \infty$ as $j \rightarrow \infty$.*

Theorem 1.3. *Let f satisfy (A-1)–(A-3). Then for any fixed $\lambda \in (0, \lambda_0^*)$, there exists a sequence $\{R_k\}$ with $R_k \geq R_*$ and $R_k \rightarrow \infty$ as $k \rightarrow \infty$ such that a nonradial bifurcation occurs at $(\lambda, R_k, \bar{u}_{R_k}^\lambda)$.*

Bifurcation and symmetry breaking of positive radial solutions of the equations with “positive exponent” in annular domains have been studied by many authors, see, for example, [1, 14–18] and the references therein. Lin ([16]) obtained non-radial bifurcation from radial solutions of the problem as (1.1) on a fixed annular domain. He also obtained symmetry breaking type results for a problem as (1.1) in varying annuli but the growth rate of $f(u)$ is subcritical or supercritical in a special range (see [17]). Bartsch et al. [1] obtained symmetry breaking type results for the Dirichlet problem of $-\Delta u = u^p$ in varying annuli without any extra restriction on $p > 1$, but they used the fact that the radial solution of the problem is unique. Gladiali et al. [14] studied the Dirichlet problem of $-\Delta u = u^p + \lambda u$ on expanding annuli with any $p > 1$, they also used the fact that the radial solution is unique. We will obtain symmetry breaking type results for (1.1) in varying annuli with a better assumption on the growth rate of the nonlinearity $f(u)$, but our problem admits multiple radial solutions for each fixed $\lambda > 0$. It is easily seen that our assumption on the growth rate of the nonlinearity $f(u)$ is “sharp” in low dimensional case ($2 \leq N \leq 10$). We think that our assumption on the growth rate of the nonlinearity $f(u)$ to obtain the symmetry breaking type results is also “sharp” even for higher dimensional case ($N \geq 11$), but we can not give the proof here. To obtain our main results, we use a blow-up argument via the Morse index of the regular entire solutions of the Eq. (1.3).

2. Preliminaries

In this section we obtain some results on the spectrum of the linearized operator associated to a radial solution u of the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \quad (2.1)$$

where $D = \{x \in \mathbb{R}^N : \alpha < |x| < \beta\}$ ($0 < \alpha < \beta < \infty$), $\lambda > 0$. Note that u depends on λ . The linearized operator at u is defined as

$$L_u = -\Delta - \lambda f'(u)I.$$

We recall that a solution u to (2.1) is degenerate if the equation

$$\begin{cases} -\Delta z - \lambda f'(u)z = 0 & \text{in } D, \\ z = 0 & \text{on } \partial D \end{cases} \quad (2.2)$$

admits nontrivial solutions.

In order to study the spectrum of the operator L_u , as in [1], we introduce the operators $\tilde{L} : H^2(D) \cap H_0^1(D) \rightarrow L^2(D)$,

$$\tilde{L}_u := |x|^2 \left(-\Delta - \lambda f'(u)I \right), \quad x \in D \quad (2.3)$$

and $\hat{L} : H^2(\alpha, \beta) \cap H_0^1(\alpha, \beta) \rightarrow L^2(\alpha, \beta)$,

$$\hat{L}_u(w) := r^2 \left(-w'' - \frac{N-1}{r}w' - \lambda f'(u)w \right) \quad r \in (\alpha, \beta). \quad (2.4)$$

Note that the eigenvalues of $\tilde{L}_u$ are given by

$$\tilde{\mu}_i = \inf_{W \subset H_0^1(D), \dim W = i} \max_{w \in W, w \neq 0} \frac{\int_D |\nabla w|^2 - \int_D \lambda f'(u) w^2}{\int_D |x|^{-2} w^2} \quad (2.5)$$

and the eigenvalues $\hat{\mu}_i$ of $\hat{L}_u$ are obtained in the same way just replacing the space $H_0^1(D)$ by $H_0^1(\alpha, \beta)$.

The proofs of the following lemmas can be obtained by arguments similar to those in the proofs of Lemmas 2.1, 2.2 and 2.3 in [14].

Lemma 2.1. *Let u be a radial solution to (2.1). Then the Morse index $i(\lambda, u)$ of u is equal to the number of negative eigenvalues $\tilde{\mu}_u$ of the operator $\tilde{L}_u$.*

Denoting by $\sigma(\cdot)$ the spectrum of a linear operator, we have:

Lemma 2.2. *Let u be a radial solution to (2.1), then*

$$\sigma(\tilde{L}_u) = \sigma(\hat{L}_u) + \sigma(-\Delta_{S^{N-1}}).$$

Let us denote by λ_k , $k = 0, 1, 2, \dots$ the eigenvalues of the operator $-\Delta_{S^{N-1}}$ and by ϕ_k the corresponding eigenfunctions. It is known that $\lambda_k = k(k + N - 2)$. Let us denote by $\hat{\mu}_i$ the eigenvalues of $\hat{L}_u$ and by w_i the corresponding eigenfunctions with $\|w_i\|_\infty = 1$.

Lemma 2.3. *Let u be a radial solution to (2.1) which is nondegenerate in the space $H_{0,rad}^1(D)$. Then problem (2.2) has a nontrivial solution if and only if there exists $k \geq 1$ such that*

$$\hat{\mu}_1 + \lambda_k = 0. \quad (2.6)$$

Moreover the solutions z to (2.2) can be written as

$$z(x) = w_1(|x|)\phi_k\left(\frac{x}{|x|}\right)$$

where $w_1(r)$ is the first positive eigenfunction of $\hat{L}_u$ and $\phi_k(\theta)$ is the eigenfunction of $-\Delta_{S^{N-1}}$ relative to the eigenvalue λ_k .

3. Asymptotic behaviors and Morse index of the radial solutions

In this section, we consider the asymptotic behaviors and Morse index of the radial solutions of (1.1). Note that the domains D_R are expanding annuli when R varies. We will present the proof of Theorems 1.1 and 1.2 in this section.

Lemma 3.1. *Let $0 < a \ll 1$ be sufficiently small and $A_a = \{x : 1 < |x| < 1 + a\} \subset \mathbb{R}^N$. Then there is $0 < \lambda_a^* < \infty$ such that for $\lambda \in (0, \lambda_a^*)$, the problem*

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } A_a, \\ u = 0 & \text{on } \partial A_a \end{cases} \quad (3.1)$$

admits exactly two radial solutions $0 < \underline{u}_\lambda^a < \overline{u}_\lambda^a$; (3.1) admits exactly one radial solution u_λ^a for $\lambda = \lambda_a^$; (3.1) does not admit a radial solution for $\lambda > \lambda_a^*$. Moreover, λ_a^* is a decreasing function with respect to a and $\lim_{a \rightarrow 0^+} \lambda_a^* = \infty$. There exists $C > 0$ such that*

$$\lambda_a^* \geq Ca^{-2}. \quad (3.2)$$

Proof. We only need to show that λ_a^* is a decreasing function with respect to a and (3.2), the other claims are standard and known from Theorem 2.11 of [16].

For any $a_1, a_2 > 0$, $a_2 > a_1$, we see that $A_{a_1} \subset A_{a_2}$ and $u_*^{a_2}$ is a supersolution to the problem

$$\begin{cases} -\Delta u = \lambda_{a_2}^* f(u) & \text{in } A_{a_1} \\ u = 0 & \text{on } \partial A_{a_1}. \end{cases} \quad (3.3)$$

This implies that (3.1) with $a = a_1$ admits a minimal radial solution for $\lambda = \lambda_{a_2}^*$. Note that $u \equiv 0$ is a subsolution to (3.3) (note $f(0) > 0$). Therefore, $\lambda_{a_2}^* \leq \lambda_{a_1}^*$. This shows that λ_a^* is a decreasing function with respect to a and $\lim_{a \rightarrow 0} \lambda_a^*$ exists, maybe $+\infty$.

We now show (3.2). For any sufficiently small $a > 0$, we can choose a proper annulus A'_a which contains A_a with $\text{meas}(A'_a) \rightarrow 0$ as $a \rightarrow 0$ such that the first eigenfunction φ_1^a with $\|\varphi_1^a\|_\infty = 1$ of the problem

$$\begin{cases} -\Delta \varphi = \sigma \varphi & \text{in } A'_a, \\ \varphi = 0 & \text{on } \partial A'_a \end{cases} \quad (3.4)$$

satisfies $\frac{1}{2} < m_a \leq \varphi_1^a \leq 1$ on A_a . Indeed, for any sufficiently small $a > 0$, there is a constant $1 \leq M \leq \frac{1}{2a}$, which is independent of a and will be determined later, such that $1 - aM \geq 1/2$ (note that M can be chosen to be sufficiently large if a is sufficiently small). Set $A'_a = \{x : 1 - aM < |x| < 1 + aM\}$. Then $\varphi_1 := \varphi_1^a$ is a positive radial solution of (3.4) and

$$\begin{aligned} -\varphi_1'' - \frac{N-1}{r} \varphi_1' &= \sigma_1 \varphi_1, \quad 1 - aM < r < 1 + aM, \\ \varphi_1(1 - aM) &= \varphi_1(1 + aM) = 0 \end{aligned} \quad (3.5)$$

where $\sigma_1 := \sigma_1^a$ is the first eigenvalue of (3.4) corresponding to the first eigenfunction φ_1 . Set $r = 1 + a\rho$ and $\tilde{\varphi}_1(\rho) = \varphi_1(r)$. We have that

$$-\tilde{\varphi}_1'' - \frac{(N-1)a}{1+a\rho} \tilde{\varphi}_1' = \sigma_1 a^2 \tilde{\varphi}_1, \quad -M < \rho < M, \quad \tilde{\varphi}_1(-M) = \tilde{\varphi}_1(M) = 0. \quad (3.6)$$

By comparing σ_1 with the first eigenvalue of the eigenvalue problem on the ball $B_{aM}(x_0)$

$$-\Delta \varphi = \sigma \varphi \quad \text{in } B_{aM}(x_0), \quad \varphi = 0 \quad \text{on } \partial B_{aM}(x_0),$$

where $x_0 = (1, 0, \dots, 0)$, we find that there is a constant $C := C(M) > 0$ independent of a such that

$$\sigma_1 a^2 \leq C,$$

since $B_{aM}(x_0) \subset A'_a$. We can now apply the L^q estimate and imbeddings to conclude that there is a sequence of a , say a_n , such that $\tilde{\varphi}_1^{a_n} \rightarrow \psi$ in $C^1([-M, M])$, where $\psi(\rho)$ is the positive function satisfying

$$-\psi'' = \mu \psi \quad \text{in } (-M, M), \quad \psi(-M) = \psi(M) = 0$$

with $\|\psi\|_\infty = 1$ and some constant $\mu \in [0, C]$. Clearly $\mu = \left(\frac{\pi}{2M}\right)^2$ and $\psi(\rho) = \cos\left(\frac{\pi\rho}{2M}\right)$. By a compact and uniqueness argument we have

$$\varphi_1^a(1 + a\rho) \rightarrow \cos\left(\frac{\pi\rho}{2M}\right) \text{ in } C^1([-M, M]) \quad \text{and} \quad \sigma_1^a a^2 \rightarrow \left(\frac{\pi}{2M}\right)^2 \text{ as } a \rightarrow 0^+.$$

Therefore for sufficiently small $a > 0$ and $1 - \frac{aM}{3} < r < 1 + \frac{aM}{3}$, we have

$$\varphi_1^a(r) > \frac{1}{2} \quad \text{and} \quad \frac{\pi^2}{8M^2 a^2} < \sigma_1^a < \frac{\pi^2}{2M^2 a^2}. \quad (3.7)$$

Fix an $M \geq 6$. We have $A_a \subset A'_a$ and $\varphi_1^a > \frac{1}{2}$ on A_a . Then we see that φ_1^a is a supersolution to (3.1) with

$$\lambda_a = \sigma_1 \min_{\frac{1}{2} \leq s \leq 1} \frac{s}{f(s)}. \quad (3.8)$$

This implies that

$$\lambda_a \leq \lambda_a^*. \quad (3.9)$$

Since $\sigma_1 := \sigma_1^a \rightarrow \infty$ as $a \rightarrow 0$, we easily see $\lambda_a \rightarrow \infty$ as $a \rightarrow 0$ and hence $\lambda_a^* \rightarrow \infty$ as $a \rightarrow 0$. Moreover, (3.2) can be obtained from (3.7)–(3.9). $\square$

Lemma 3.2. *Let $D_R = \{x \in \mathbb{R}^N : R < |x| < R + 1\}$ with R satisfying (1.2). There exists $\lambda_R^* > 0$ such that problem*

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } D_R, \\ u = 0 & \text{on } \partial D_R \end{cases} \quad (3.10)$$

admits no radial solutions for $\lambda > \lambda_R^$, one and only one radial solution u_R^* for $\lambda = \lambda_R^*$ and exactly two radial solutions $0 < \underline{u}_R^\lambda < \bar{u}_R^\lambda$ for $0 < \lambda < \lambda_R^*$. Moreover,*

$$\lambda_R^* \geq C \quad (3.11)$$

where $C > 0$ is given in (3.2).

Proof. We only need to show (3.11). Making the transformation:

$$v(\rho) = u(r), \quad \rho = \frac{r}{R},$$

we see that v satisfies the problem

$$\begin{cases} -\Delta v = \lambda R^2 f(v) & \text{in } A_{\frac{1}{R}}, \\ v = 0 & \text{on } \partial A_{\frac{1}{R}}. \end{cases} \quad (3.12)$$

It follows from Lemma 3.1 that

$$\lambda_R^* R^2 = \lambda_{\frac{1}{R}}^*. \quad (3.13)$$

Both (3.13) and (3.2) imply (3.11). $\square$

Let R^* be a large number which satisfies (1.2). For each $R \geq R^*$, we now study the linearized problem at the solution $(\lambda, \bar{u}_R^\lambda)$ of (3.10):

$$\begin{cases} \Delta h + \lambda f'(\bar{u}_R^\lambda) h = -\mu h & \text{in } D_R, \\ h = 0 & \text{on } \partial D_R. \end{cases} \quad (3.14)$$

It is known from Crandall and Rabinowitz [4] that $i(\lambda, \bar{u}_R^\lambda) \geq 1$ and $i_{rad}(\lambda, \bar{u}_R^\lambda) \geq 1$ for $\lambda \in (0, \lambda_R^*)$, since $f(s)$ is convex.

Let $\Lambda^* = \inf_{R \geq R^*} \lambda_R^*$. It is known from (3.11) that $\Lambda^* \geq C$.

Proof of Theorem 1.1 Let (σ_R^1, ϕ_R^1) be the first eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta \phi = \sigma_R^1 \phi & \text{in } D_R, \\ \phi = 0 & \text{on } \partial D_R. \end{cases}$$

It is known from Lemma A.1 of [19] that $\sigma_R^1 \rightarrow \pi^2$ as $R \rightarrow \infty$. We know from the conditions of $f(s)$ that there exists a constant $A > 0$ such that $f(s) \geq As$ for $s \in [0, \infty)$. Then multiplying ϕ_R^1 on both the sides of the equation in (3.10), we see that

$$\sigma_R^1 \int_{D_R} u \phi_R^1 dx \geq \lambda A \int_{D_R} u \phi_R^1 dx.$$

This implies that

$$\lambda_R^* \leq \frac{1}{A} \sigma_R^1. \quad (3.15)$$

Both (3.15) and (3.11) imply that there is $R_* > R^*$ such that for $R \geq R_*$,

$$C \leq \lambda_R^* \leq \frac{10}{A} \pi^2. \quad (3.16)$$

For any $\lambda \in (0, \min_{R \geq R_*} \lambda_R^*)$, we know that $i(\lambda, \underline{u}_R) = 0$ for all $R \geq R_*$, since $\underline{u}_R$ is the minimal positive solution of (3.10) and hence is stable. Then we claim that there is $C := C(\lambda) > 0$, which is independent of R , such that for all $R \geq R_*$,

$$\underline{u}_R + |\nabla \underline{u}_R|^{\frac{2}{p+1}} \leq C \text{dist}^{-\frac{2}{p-1}}(x, \partial D_R) \quad \forall x \in D_R. \quad (3.17)$$

Note that we write $\underline{u}_R^\lambda$ as $\underline{u}_R$.

Assume that (3.17) fails. Then there exist sequences $\{R_k\}$, $\{\underline{u}_k\} (\equiv \{\underline{u}_{R_k}\})$, $y_k \in D_{R_k}$, such that $\underline{u}_k$ solves the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } D_{R_k}, \\ u = 0 & \text{on } \partial D_{R_k} \end{cases} \quad (3.18)$$

and the functions

$$M_k(x) := \underline{u}_k^{\frac{p-1}{2}} + |\nabla \underline{u}_k|^{\frac{p-1}{p+1}}, \quad k = 1, 2, \dots$$

satisfy

$$M_k(y_k) > 2k \text{dist}^{-1}(y_k, \partial D_{R_k}).$$

By a doubling lemma (see Lemma 5.1) of [23] we see that there exists $x_k \in D_{R_k}$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k \text{dist}^{-1}(x_k, \partial D_{R_k})$$

and

$$M_k(z) \leq 2M_k(x_k), \quad |z - x_k| \leq kM_k^{-1}(x_k).$$

Now we rescale $\underline{u}_k$ by setting

$$w_k(y) := \tau_k^{2/(p-1)} \underline{u}_k(x_k + \tau_k y), \quad |y| \leq k \quad \text{with} \quad \tau_k = M_k^{-1}(x_k).$$

Then w_k solves

$$-\Delta_y w_k(y) = f_k(w_k(y)) := \tau_k^{\frac{2p}{p-1}} f(\tau_k^{-\frac{2}{p-1}} w_k(y)), \quad |y| \leq k,$$

$$[w_k^{\frac{p-1}{2}} + |\nabla w_k|^{\frac{p-1}{p+1}}](0) = \tau_k M_k(x_k) = 1$$

and

$$[w_k^{\frac{p-1}{2}} + |\nabla w_k|^{\frac{p-1}{p+1}}](y) \leq 2, \quad |y| \leq k.$$

Arguments similar to those in the proof of Theorem 2.1 of [23] imply that some subsequence of $\{w_k\}$ converges in $C_{loc}^1(\mathbb{R}^N)$ to a function w and w is a positive solution of

$$-\Delta w = \lambda w^p, \quad y \in \mathbb{R}^N. \quad (3.19)$$

Moreover, w and $|\nabla w|$ are bounded in $\mathbb{R}^N$. This is a contradiction provided $1 < p < \frac{N+2}{N-2}$. On the other hand, since $i(\lambda, \underline{u}_k) = 0$ for all k , we see that $i(\lambda, w_k) = i(\lambda, \underline{u}_k) = 0$ for all k and hence $i(\lambda, w) = 0$ (see the proof of step 1 in the proof of Theorem 1.2 below). This implies that w is a regular stable positive solution of (3.19) and contradictions for $p = \frac{N+2}{N-2}$ and $\frac{N+2}{N-2} < p < p_c(N)$. It is known from [9, 10, 12] that regular stable positive solution of (3.19) does not exist for $p = \frac{N+2}{N-2}$; it is also known from [8, 9, 11, 12] that $i(\lambda, w) = \infty$ provided $\frac{N+2}{N-2} < p < p_c(N)$. Therefore, (3.17) holds.

We now use the blow-up argument again to obtain an upper bound for $\underline{u}_R$. We have that there exists $C := C(\lambda) > 0$ independent of R such that for all $R > R_*$ that

$$\underline{u}_R \leq C \quad \text{in} \quad D_R. \quad (3.20)$$

On the contrary, there is a sequence $\{R_k\}$ with $R_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\underline{u}_{R_k}(x_k) \rightarrow \infty$ as $k \rightarrow \infty$, where $\underline{u}_{R_k}(x_k) = \max_{D_{R_k}} \underline{u}_{R_k}$. Denote $C_k = \underline{u}_{R_k}(x_k) (\rightarrow \infty \text{ as } k \rightarrow \infty)$. Setting

$$y = C_k^{\frac{p-1}{2}}(x - \eta_k), \quad \tilde{u}_k(y) = \frac{\underline{u}_{R_k}(x)}{C_k},$$

where $\eta_k \in \partial D_{R_k}$ such that $\text{dist}(x_k, \eta_k) = \text{dist}(x_k, \partial D_{R_k})$, we see that $\tilde{u}_k$ satisfies the problem

$$\begin{cases} -\Delta_y \tilde{u}_k = \lambda \frac{f(C_k \tilde{u}_k)}{C_k^p} & \text{in } \Omega_k, \\ \tilde{u}_k = 0 & \text{on } \partial \Omega_k, \end{cases}$$

where $\Omega_k = \{y = C_k^{\frac{p-1}{2}}(x - \eta_k), x \in D_{R_k}\}$. Moreover, it is known from (3.17) that there is $C > 0$ independent of k such that

$$C_k^{\frac{p-1}{2}}(x_k - \eta_k) \leq C.$$

Using elliptic L^p estimates and standard imbeddings, we deduce that some subsequence of $\{\tilde{u}_k\}$ converges in $C_{loc}^1(\Gamma)$ to a nonnegative (classical) solution $\tilde{u}$ of

$$\begin{cases} -\Delta \tilde{u} = \lambda \tilde{u}^p, & \text{in } \Gamma, \\ \tilde{u} = 0, & \text{on } \partial\Gamma, \end{cases} \quad (3.21)$$

where Γ is a half space of $\mathbb{R}^N$, say $\Gamma = \{y = (y_1, \dots, y_N), y_N > 0\}$. Moreover, $\tilde{u}(\tilde{y}) = \max_{y \in \Gamma} \tilde{u}(y) = 1$ and $\text{dist}(\tilde{y}, 0) \leq C$, where $\tilde{y} = \lim_{k \rightarrow \infty} C_k^{\frac{p-1}{2}}(x_k - \eta_k) \in \Gamma$. This is impossible since it is known from Theorem 2 of [7] that $\frac{\partial \tilde{u}(y)}{\partial y_N} > 0$ for $y \in \bar{\Gamma}$. This implies that (3.20) holds.

We now show that

$$\underline{u}_R(R+t) \rightarrow \underline{u}_0(t) \quad \text{uniformly for } t \in [0, 1] \quad \text{as } R \rightarrow \infty, \quad (3.22)$$

where $\underline{u}_0$ is the minimal solution of the problem

$$\begin{cases} -u_0'' = \lambda f(u_0) & \text{in } (0, 1), \\ u_0(0) = u_0(1) = 0. \end{cases} \quad (3.23)$$

Set $\tilde{u}_R(t) = \underline{u}_R(t+R)$. We see that for all $R \geq R_*$,

$$\begin{aligned} \lambda \int_0^1 f(\tilde{u}_R(t)) dt &= \lambda \int_R^{R+1} f(\underline{u}_R) dr \\ &\leq \frac{\lambda}{R^{N-1}} \int_R^{R+1} r^{N-1} f(\underline{u}_R) dr \\ &\leq C \end{aligned} \quad (3.24)$$

where $C > 0$ is independent of R (we use (3.20) here).

Multiplying (1.1) by $\underline{u}_R$ and integrating, we see from (3.24) that

$$\int_R^{R+1} r^{N-1} (\underline{u}_R')^2 dr = \lambda \int_R^{R+1} r^{N-1} f(\underline{u}_R) dr \leq CR^{N-1}.$$

This implies that

$$R^{N-1} \left(\int_0^1 (\tilde{u}_R')^2 dt \right) \leq \int_R^{R+1} r^{N-1} (\underline{u}_R')^2 dr \leq CR^{N-1}$$

and hence

$$\int_0^1 (\tilde{u}_R')^2 dt \leq C. \quad (3.25)$$

Observe that $\tilde{u}_R$ satisfies the problem

$$\begin{cases} -\tilde{u}_R'' - \frac{N-1}{t+R} \tilde{u}_R' = \lambda f(\tilde{u}_R) & \text{in } (0, 1), \\ \tilde{u}_R(0) = \tilde{u}_R(1) = 0. \end{cases}$$

Therefore, since by (3.25) $\tilde{u}_R$ is bounded in $H_0^1(0, 1)$, it is also bounded in $C^1(0, 1)$. Consequently there is a sequence $\{R_k\}$ such that $\tilde{u}_{R_k} \rightarrow \hat{u}_0$ uniformly for $t \in [0, 1]$ as $k \rightarrow \infty$, where $\hat{u}_0$ satisfies (3.23). Let us show $\hat{u}_0 = \underline{u}_0$. We know that

$$i(\lambda, \underline{u}_{R_k}) = i_{rad}(\lambda, \underline{u}_{R_k}) = i(\lambda, \tilde{u}_{R_k}) = i_{rad}(\lambda, \tilde{u}_{R_k}) = 0 \quad \text{for all } R_k \geq R_*.$$

Then $i(\lambda, \hat{u}_0) = 0$ (note that $i(\lambda, \hat{u}_0) = i_{rad}(\lambda, \hat{u}_0)$) and hence $\hat{u}_0 \equiv \underline{u}_0$. By a compact and uniqueness argument we have that $\underline{u}_R(R+t) \rightarrow \underline{u}_0(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$. This also implies $\min_{R \geq R_*} \lambda_R^* \leq \lambda_0^*$.

We now show (1.6). On the contrary, there is a sequence $\{R_k\}$ with $R_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\lambda_{R_k}^* \rightarrow \hat{\lambda} \neq \lambda_0^*$ as $k \rightarrow \infty$. It follows from (3.16) that $0 < \hat{\lambda} < \infty$. Arguments similar to those in the proof of (3.22) imply that $u_{R_k}^*(R_k+t) \rightarrow \hat{u}(t)$ uniformly for $t \in [0, 1]$ as $k \rightarrow \infty$, where $\hat{u}(t)$ solves the ODE:

$$\begin{cases} -\hat{u}'' = \hat{\lambda} f(\hat{u}) & \text{in } (0, 1), \\ \hat{u}(0) = \hat{u}(1) = 0. \end{cases}$$

On the other hand, we know that for all k , the first eigenvalues $\mu_1(u_{R_k}^*)$ of the problem

$$\begin{cases} \phi''(r) + \frac{N-1}{r} \phi'(r) + \lambda_{R_k}^* f'(u_{R_k}^*) \phi = -\mu \phi & \text{in } (R_k, R_k+1), \\ \phi(R_k) = \phi(R_k+1) = 0 \end{cases}$$

equal to 0 for all k . This implies that for any k , there is a $\phi_k(r) > 0$ for $r \in (R_k, R_k+1)$ and $\|\phi_k\|_{L^\infty(R_k, R_k+1)} = 1$ such that

$$\begin{cases} \phi_k''(r) + \frac{N-1}{r} \phi_k'(r) + \lambda_{R_k}^* f'(u_{R_k}^*) \phi_k = 0 & \text{in } (R_k, R_k+1), \\ \phi_k(R_k) = \phi_k(R_k+1) = 0. \end{cases}$$

Set

$$\tilde{\phi}_k(t) = \phi_k(r), \quad r = R_k + t.$$

We see that $\tilde{\phi}_k > 0$ in $(0, 1)$ and $\|\tilde{\phi}_k\|_{L^\infty(0,1)} = 1$, which satisfies

$$\begin{cases} \tilde{\phi}_k''(t) + \frac{N-1}{R_k+t} \tilde{\phi}_k'(t) + \lambda_{R_k}^* f'(u_{R_k}^*(R_k+t)) \tilde{\phi}_k(t) = 0 & \text{in } (0, 1), \\ \tilde{\phi}_k(0) = \tilde{\phi}_k(1) = 0. \end{cases}$$

The regularity arguments as the above imply that there is a subsequence of $\{\tilde{\phi}_k\}$ (still denoted by $\{\tilde{\phi}_k\}$) such that

$$\tilde{\phi}_k \rightarrow \tilde{\phi} \text{ in } C^1(0, 1) \text{ as } k \rightarrow \infty$$

and $\tilde{\phi} > 0$ in $(0, 1)$ with $\|\tilde{\phi}\|_{L^\infty(0,1)} = 1$, which satisfies the problem

$$\begin{cases} \tilde{\phi}''(t) + \hat{\lambda} f'(\hat{u}) \tilde{\phi} = 0 & \text{in } (0, 1), \\ \tilde{\phi}(0) = \tilde{\phi}(1) = 0. \end{cases}$$

This implies that $\hat{\lambda} = \lambda_0^*$ and $\hat{u} \equiv u_0^*$ and contradicts our assumption $\hat{\lambda} \neq \lambda_0^*$. Therefore, (1.6) holds and for any $\lambda \in (0, \lambda_0^*)$, $\underline{u}_R(R+t) \rightarrow \underline{u}_0(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$, $u_R^*(R+t) \rightarrow u_0^*(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$. It follows from (3.13) that

$$\lim_{R \rightarrow \infty} \frac{\lambda_{\frac{1}{R}}^*}{R^2} = \lambda_0^*. \quad (3.26)$$

This also implies that

$$\lim_{a \rightarrow 0^+} a^2 \lambda_a^* = \lambda_0^*, \quad (3.27)$$

where λ_a^* is as in Lemma 3.1. $\square$

Proof of Theorem 1.2 Since $\lambda \in (0, \lambda_0^*)$ and $\lim_{R \rightarrow \infty} \lambda_R^* = \lambda_0^*$, we can choose R_* (given above) such that $\lambda \in (0, \lambda_{R_*}^*)$ provided $R \geq R_*$.

Suppose that the conclusion of Theorem 1.2 does not hold, we see that there exists an integer $\theta \geq 1$ independent of R such that

$$i(\lambda, \bar{u}_R^\lambda) \leq \theta \quad \forall R \geq R_*. \quad (3.28)$$

For convenience, we write $\bar{u}_R^\lambda$ as $\bar{u}_R$ in the following.

The proof can be divided into several steps:

Step 1. We show that there is $C > 0$ independent of R such that for any $R \geq R_*$,

$$\bar{u}_R + |\nabla \bar{u}_R|^{\frac{2}{p+1}} \leq C \text{dist}^{-\frac{2}{p-1}}(x, \partial D_R) \quad \forall x \in D_R. \quad (3.29)$$

The proof of (3.29) is similar to that of (3.17) provided that (3.28) holds. Indeed, we use the blow-up argument as in the proof of (3.17). If we define

$$\tilde{M}_k(x) := \bar{u}_k^{\frac{p-1}{2}} + |\nabla \bar{u}_k|^{\frac{p-1}{p+1}}, \quad k = 1, 2, \dots$$

and

$$\tilde{w}_k(y) := \tilde{\tau}_k^{2/(p-1)} \bar{u}_k(x_k + \tilde{\tau}_k y), \quad |y| \leq k \quad \text{with} \quad \tilde{\tau}_k = \tilde{M}_k^{-1}(x_k),$$

we see that $\tilde{w}_k(y)$ satisfies the equation

$$-\Delta_y \tilde{w}_k(y) = f_k(\tilde{w}_k(y)) := \tilde{\tau}_k^{\frac{2p}{p-1}} f(\tilde{\tau}_k^{-\frac{2}{p-1}} \tilde{w}_k(y)), \quad |y| \leq k,$$

$$[\tilde{w}_k^{\frac{p-1}{2}} + |\nabla \tilde{w}_k|^{\frac{p-1}{p+1}}](0) = \tilde{\tau}_k \tilde{M}_k(x_k) = 1$$

and

$$[\tilde{w}_k^{\frac{p-1}{2}} + |\nabla \tilde{w}_k|^{\frac{p-1}{p+1}}](y) \leq 2, \quad |y| \leq k.$$

On the other hand, we see that the first eigenvalue

$$\begin{aligned} \mu_1^{(k)} &= \inf_{\varphi \in H_0^1(D_{R_k}), \varphi \not\equiv 0} \frac{\int_{D_{R_k}} [|\nabla_x \varphi|^2 - \lambda f'(\bar{u}_k) \varphi^2] dx}{\int_{D_{R_k}} \varphi^2 dx} \\ &= \inf_{\psi \in H_0^1(\tilde{D}_{R_k}), \psi \not\equiv 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - \lambda f'_k(\tilde{w}_k(y)) \psi^2] dy}{\tilde{\tau}_k^2 \int_{\tilde{D}_{R_k}} \psi^2 dy}, \end{aligned}$$

where $\tilde{D}_{R_k} = \{y : x_k + \tilde{\tau}_k y \in D_{R_k}\}$, $\psi(y) := \varphi(x)$ and $x = x_k + \tilde{\tau}_k y$. This implies that

$$\tilde{\mu}_1^{(k)} = \tilde{\tau}_k^2 \mu_1^{(k)},$$

where

$$\tilde{\mu}_1^{(k)} = \inf_{\psi \in H_0^1(\tilde{D}_{R_k}), \psi \not\equiv 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - \lambda f'_k(\tilde{w}_k) \psi^2] dy}{\int_{\tilde{D}_{R_k}} \psi^2 dy}.$$

By the fact that

$$\mu_j^{(k)} = \inf_{\mathcal{A} \subset H_0^1(D_{R_k}), \dim \mathcal{A}=j} \max_{\varphi \in \mathcal{A}, \varphi \neq 0} \frac{\int_{D_{R_k}} [|\nabla_x \varphi|^2 - \lambda f'(\bar{u}_k) \varphi^2] dx}{\int_{D_{R_k}} \varphi^2 dx},$$

we also see that

$$\tilde{\mu}_j^{(k)} = \tilde{\tau}_k^2 \mu_j^{(k)}, \quad (3.30)$$

where

$$\tilde{\mu}_j^{(k)} = \inf_{\mathcal{B} \subset H_0^1(\tilde{D}_{R_k}), \dim \mathcal{B}=j} \max_{\psi \in \mathcal{B}, \psi \neq 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - \lambda f'_k(\tilde{w}_k) \psi^2] dy}{\int_{\tilde{D}_{R_k}} \psi^2 dy}.$$

This and (3.28) imply that

$$i(\lambda, \tilde{w}_k) \leq \theta. \quad (3.31)$$

Therefore, the maximal dimension of all subspaces X of $C_0^1(\tilde{D}_{R_k})$ such that

$$\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - \lambda f'_k(\tilde{w}_k) \psi^2] dy < 0 \quad \forall \psi \in X \setminus \{0\} \quad (3.32)$$

is not bigger than θ . Note that $\tilde{\tau}_k^2 \rightarrow 0$ as $k \rightarrow \infty$.

On the other hand, arguments similar to those in the proof of (3.17) imply that $\{\tilde{w}_k\}$ converges in $C_{loc}^1(\mathbb{R}^N)$ to a (classical) solution $\tilde{w}$ of the equation

$$-\Delta \tilde{w} = \lambda \tilde{w}^p \quad \text{in } \mathbb{R}^N. \quad (3.33)$$

Moreover $[\tilde{w}^{\frac{p-1}{2}} + |\nabla \tilde{w}|^{\frac{p-1}{p+1}}](0) = 1$. This is impossible for $1 < p < \frac{N+2}{N-2}$. For $\frac{N+2}{N-2} < p < p_c(N)$, it follows from (3.32) that the maximal dimension of all subspaces X of $C_0^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} [|\nabla_y \psi|^2 - \lambda p \tilde{w}^{p-1} \psi^2] dy < 0 \quad \forall \psi \in X \setminus \{0\}$$

is not bigger than θ and thus

$$i(\lambda, \tilde{w}) \leq \theta. \quad (3.34)$$

Note that for each $y \in \mathbb{R}^N$, $f'_k(\tilde{w}_k(y)) \rightarrow p \tilde{w}^{p-1}(y)$. It is known from [8–10, 12] that $i(\lambda, \tilde{w}) = \infty$ provided $\frac{N+2}{N-2} < p < p_c(N)$. This contradicts to (3.34). These contradictions imply that (3.29) holds. (We can not obtain (3.29) for $p = \frac{N+2}{N-2}$, since in this case, (1.3) admits solutions with finite Morse index.) Then the blow-up argument as in the proof of (3.20) implies that there is $C := C(\lambda)$ independent of R such that for $R \geq R_*$,

$$\bar{u}_R \leq C \quad \text{in } D_R \quad (3.35)$$

provided that (3.28) holds.

Step 2. We show that $\bar{u}_R(R+t) \rightarrow \bar{u}_0^\lambda(t)$ uniformly for $t \in [0, 1]$ as $R \rightarrow \infty$, where $\bar{u}_0^\lambda$ is the solution on the upper solution branch of the problem

$$\begin{cases} -u_0'' = \lambda f(u_0) & \text{in } (0, 1), \\ u_0(0) = u_0(1) = 0. \end{cases} \quad (3.36)$$

The proof of this step is similar to that of (3.22). To guarantee that the limit is $\bar{u}_0^\lambda$, we need to notice that $i_{rad}(\lambda, \bar{u}_R) \geq 1$ for all $R \geq R_*$.

Step 3. We show that if $\hat{\mu}_1(R)$ is the first eigenvalue of the operator $\hat{L}_R$ defined by $\hat{L}_R : H^2(R, R+1) \cap H_0^1(R, R+1) \rightarrow L^2(R, R+1)$,

$$\hat{L}_R(w) = r^2 \left(-w'' - \frac{1}{r}w' - \lambda f'(\bar{u}_R)w \right) \quad r \in (R, R+1).$$

Then

$$\hat{\mu}_1(R) = \sigma_1 R^2 + o(R^2) \quad \text{as } R \rightarrow \infty \quad (3.37)$$

where $\sigma_1 < 0$ is the first eigenvalue for the problem

$$\begin{cases} -w'' - \lambda f'(\bar{u}_0^\lambda)w = \sigma w & \text{in } (0, 1) \\ w(0) = w(1) = 0 \end{cases} \quad (3.38)$$

where $\bar{u}_0^\lambda$ is as in step 2.

The proof of this step is standard (see Lemma 3.2 of [1] and Proposition 3.2 of [14]). The eigenvalue $\hat{\mu}_1(R)$ can be characterized as

$$\hat{\mu}_1(R) = \inf_{w \in H_0^1(R, R+1), w \neq 0} \frac{\int_R^{R+1} r^{N-1} (w')^2 dr - \int_R^{R+1} \lambda r^{N-1} f'(\bar{u}_R) w^2 dr}{\int_R^{R+1} r^{N-3} w^2 dr}. \quad (3.39)$$

To estimate $\hat{\mu}_1(R)$ let us consider a function $\hat{\phi} \in C_0^\infty(0, 1)$, $\hat{\phi} \geq 0$ and set $\phi(r) = \hat{\phi}(r - R)$. Then

$$\begin{aligned} \hat{\mu}_1(R) &\leq \frac{\int_R^{R+1} r^{N-1} (\phi')^2 dr - \int_R^{R+1} \lambda r^{N-1} f'(\bar{u}_R) \phi^2 dr}{\int_R^{R+1} r^{N-3} \phi^2 dr} \\ &= \frac{\int_0^1 (R+t)^{N-1} (\hat{\phi}')^2 dt - \int_0^1 \lambda (R+t)^{N-1} f'(\bar{u}_R) \hat{\phi}^2 dt}{\int_0^1 (R+t)^{N-3} \hat{\phi}^2 dt} \\ &\leq CR^2. \end{aligned} \quad (3.40)$$

In order to get the reverse inequality let us denote by $w_{1,R}$ the eigenfunction associated to $\hat{\mu}_1(R)$ with $\|w_{1,R}\|_{L^\infty(D_R)} = 1$. Inserting $w_{1,R}$ into (3.39) we have, for a positive constant $C > 0$,

$$\begin{aligned} \hat{\mu}_1(R) &= \frac{\int_R^{R+1} r^{N-1} (w'_{1,R})^2 dr - \int_R^{R+1} \lambda r^{N-1} f'(\bar{u}_R) w_{1,R}^2 dr}{\int_R^{R+1} r^{N-3} w_{1,R}^2 dr} \\ &\geq \frac{-\lambda K \int_R^{R+1} r^{N-1} w_{1,R}^2 dr}{\int_R^{R+1} r^{N-3} w_{1,R}^2 dr} \\ &\geq -CR^2. \end{aligned} \quad (3.41)$$

since we know from step 1 and the conditions of f that $0 < f'(\bar{u}_R) \leq K$.

Now we define $\tilde{w}_{1,R}(t) = w_{1,R}(t + R)$ in $(0, 1)$, we see that $\tilde{w}_{1,R}$ satisfies the problem

$$\begin{cases} -\tilde{w}''_{1,R} - \frac{N-1}{t+R} \tilde{w}'_{1,R} - \lambda f'(\tilde{u}_R) \tilde{w}_{1,R} = \hat{\mu}_1(R) \frac{\tilde{w}_{1,R}}{(t+R)^2} & \text{in } (0, 1) \\ \tilde{w}_{1,R}(0) = \tilde{w}_{1,R}(1) = 0. \end{cases} \quad (3.42)$$

It is known from (3.40) and (3.41) that $\left| \frac{\hat{\mu}_1(R)}{R^2} \right| \leq C$ for R sufficiently large. Since $\|\tilde{w}_{1,R}\|_\infty = 1$, the regularity argument as the above implies that there is a sequence $\{R_k\}$ with $R_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\tilde{w}_{1,R_k} \rightarrow \xi_1 \geq 0$ uniformly in $[0, 1]$ as $k \rightarrow \infty$, and ξ_1 with $\|\xi_1\|_{L^\infty(0,1)} = 1$ is a solution to the problem

$$\begin{cases} -\xi_1'' - \lambda f'(\bar{u}_0^\lambda) \xi_1 = \sigma_1 \xi_1 & \text{in } (0, 1) \\ \xi_1(0) = \xi_1(1) = 0 \end{cases} \quad (3.43)$$

where $\bar{u}_0^\lambda$ is as in step 2 and $\sigma_1 = \lim_{k \rightarrow \infty} \frac{\hat{\mu}_1(R_k)}{R_k^2}$. The strong maximum principle implies that $\xi_1 > 0$ in $(0, 1)$. Hence, σ_1 is the first eigenvalue of the problem (3.38) and ξ_1 is the corresponding eigenfunction. The uniqueness of the first eigenvalue and corresponding eigenfunction of (3.38) implies that

$$\lim_{R \rightarrow \infty} \frac{\hat{\mu}_1(R)}{R^2} = \sigma_1 < 0, \quad \tilde{w}_{1,R} \rightarrow \xi_1 \quad \text{uniformly in } [0, 1] \text{ as } R \rightarrow \infty.$$

Then we have, with a change of variable, from (3.41):

$$\hat{\mu}_1(R) = \frac{\int_0^1 (t+R)^{N-1} (\tilde{w}'_{1,R})^2 dt - \int_0^1 \lambda(t+R)^{N-1} f'(\tilde{u}_R) \tilde{w}_{1,R}^2 dt}{\int_0^1 (t+R)^{N-3} \tilde{w}_{1,R}^2 dt}$$

and, passing to the limit as $R \rightarrow \infty$,

$$\frac{\hat{\mu}_1(R)}{R^2} = \frac{\int_0^1 (\xi_1')^2 dt - \lambda \int_0^1 f'(\bar{u}_0^\lambda) \xi_1^2 dt}{\int_0^1 \xi_1^2 dt} + o(1).$$

From this and (3.43) we obtain (3.37).

Step 4. We complete the proof of this theorem.

It is known from Lemma 2.2 that $\tilde{\mu}(R, k) = \hat{\mu}_1(R) + \lambda_k (= \hat{\mu}_1(R) + k(k + N - 2))$, $k = 0, 1, 2, \dots$ are eigenvalues of the operator $\tilde{L}_{\bar{u}_R^\lambda}$. Note that $\bar{u}_R^\lambda$ is nondegenerate in the space $H_{0,rad}^1(D_R)$. Since step 3 implies that $\hat{\mu}_1(R) \rightarrow -\infty$ as $R \rightarrow \infty$, we see that there are infinitely many negative $\tilde{\mu}(R, k)$ as $R \rightarrow \infty$. Therefore, Lemma 2.1 implies that

$$i(\lambda, \bar{u}_R) \rightarrow +\infty \quad \text{as } R \rightarrow \infty.$$

This contradicts to (3.28) and completes the proof of this theorem. $\square$

Corollary 3.3. *There exists a sequence $\{R_k\}$ with $R_k \geq R_*$ and $R_k \rightarrow +\infty$ as $k \rightarrow \infty$ such that for each k , $\bar{u}_{R_k}^\lambda$ is degenerate.*

Proof. It is known from Theorem 1.2 that there exists a sequence $\{R_j\}$ with $R_j \geq R_*$ and $R_j \rightarrow \infty$ such that $i(\lambda, \bar{u}_{R_j}^\lambda) \rightarrow \infty$ as $j \rightarrow \infty$. We can choose a subsequence of $\{R_j\}$ (still denoted by $\{R_j\}$) such that for any j ,

$$i(\lambda, \bar{u}_{R_j}^\lambda) < i(\lambda, \bar{u}_{R_{j+1}}^\lambda).$$

By the continuity of the eigenvalues of $L_{\bar{u}_R^\lambda}$ with respect to R , we easily see that for each j , there exists $k = k(j)$ such that $R_k \in [R_j, R_{j+1}]$ and $\bar{u}_{R_k}^\lambda$ is degenerate. It is known from Lemma 2.3 that R_k can be determined from $\hat{\mu}_1(R_k) + \lambda_k = 0$. This implies that the conclusion of this corollary holds. $\square$

Remark 3.4. The conclusions of Theorem 1.1 hold for the case $N \geq 3$ and $p = \frac{N+2}{N-2}$. We notice that the conclusions of Theorem 1.2 also hold for the case $p = \frac{N+2}{N-2}$ if we have that there is $C > 0$ independent of R such that

$$\|\bar{u}_R\|_\infty \leq C. \quad (3.44)$$

To obtain (3.44), we need

$$\tilde{\tau}_k^2 \mu_1^{(k)} \rightarrow 0 \text{ as } k \rightarrow \infty \quad (3.45)$$

in the proof of Theorem 1.2. It is known from (3.30) that $\tilde{\mu}_j^{(k)} = \tilde{\tau}_k^2 \mu_j^{(k)}$. Note that $\tilde{\tau}_k^2 \rightarrow 0$ as $k \rightarrow \infty$. If we have (3.45), we will see that the $\tilde{w}$ in the proof of Theorem 1.2 is stable. This is impossible, since $\tilde{w}$ is a positive entire solution of (1.3) with $p = \frac{N+2}{N-2}$. (We know that the equation (1.3) does not admit a stable solution.) This implies that we can obtain (3.29) and hence we can obtain (3.44).

Remark 3.5. We still do not know the related results for the case $N \geq 11$ and $p \geq p_c(N)$, since we can not obtain the uniform boundedness of $\{\|\underline{u}_R\|_\infty\}$ and $\{\|\bar{u}_R\|_\infty\}$ for large R . The blow-up argument as in the proof of Theorems 1.1 and 1.2 can not be used to discuss the case $p \geq p_c(N)$, since in this case, the Eq. (1.3) admits positive stable entire solutions.

4. Bifurcation results

In this section we obtain the nonradial bifurcation from the radial solution $\bar{u}_R$ by showing that there is a change in the Leray-Schauder degree of certain associated maps as R crosses R_k . Where R_k is given in Corollary 3.3. The main idea is similar to those in [14].

Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$, and let $D := \{x \in \mathbb{R}^N, \alpha < |x| < \beta\}$. For any $R \in (1, \infty)$, let

$$h_R: D_R \rightarrow D$$

be the diffeomorphism that maps D_R into D , i.e.

$$h_R(r, \theta) = (\alpha + (\beta - \alpha)(r - R), \theta).$$

This map h_R induces the map

$$h_R^*: C_0^{1,\sigma}(\overline{D_R}) \rightarrow C_0^{1,\sigma}(\overline{D})$$

defined by $h_R^*(u)(x) = u(h_R^{-1}(x))$ for $x \in D$.

Then our equation (1.1) in D_R becomes

$$\begin{cases} L_R w = \lambda f(w) & \text{in } D, \\ w = 0 & \text{on } \partial D \end{cases} \quad (4.1)$$

where $w(x) = h_R^*(u)(x)$, $L_R(w) = h_R^* \left(-\Delta \left((h_R^*)^{-1}(w) \right) \right)$. Finally we can rewrite (4.1) as $w = T_R(w)$, where $T_R(w) = L_R^{-1}(\lambda f(w))$. It is easy to see that, for any $R > R_*$, T_R is well defined $C_0^{1,\sigma}(\overline{D})$ into $C_0^{1,\sigma}(\overline{D})$ and is a compact operator.

If $\mathcal{F}$ is an open bounded set in $C_0^{1,\sigma}(\overline{D})$ such that $I - T_R \neq 0$ on $\partial\mathcal{F}$, then the Leray-Schauder degree for the map $I - T_R$, i.e. $\deg(I - T_R, \mathcal{F}, 0)$ is well-defined.

Applying a result of [20] (see Proposition 2, p. 243) we have that

$$\deg(I - T_R, \mathcal{F}, 0) = \deg\left(I - P_R, (h_R^*)^{-1}(\mathcal{F}), 0\right) \quad (4.2)$$

where $P_R(v) = (-\Delta)^{-1}(\lambda f(v))$ and it is defined in $C_0^{1,\sigma}(\overline{D_R})$.

The operator P_R is differentiable at $\overline{u}_R$, and as we have seen before the operator $I - P'_R(\overline{u}_R)$ is invertible if $R \neq R_k$ and near R_k , $R_k > R_*$ is as in Corollary 3.3. Hence we have that, for any $R \neq R_k$ and near R_k ,

$$\deg(I - P_R, (h_R^*)^{-1}(\mathcal{F}), 0) = \deg(I - P'_R(\overline{u}_R), (h_R^*)^{-1}(\mathcal{F}), 0) = (-1)^{i(\lambda, \overline{u}_R)} \quad (4.3)$$

if $(h_R^*)^{-1}(\mathcal{F})$ is a neighborhood of $\overline{u}_R$ in $C_0^{1,\sigma}(\overline{D_R})$ such that $\overline{u}_R$ is the only solution of $(I - P_R)(u) = 0$ in the closure of $(h_R^*)^{-1}(\mathcal{F})$.

Proof of Theorem 1.3 We consider the subspace $\mathcal{H}_R$ of $C^{1,\sigma}(\overline{D_R})$ given by

$$\begin{aligned} \mathcal{H}_R &:= \{u \in C^{1,\sigma}(\overline{D_R}), \quad u(x_1, \dots, x_N) \\ &= u(g(x_1, \dots, x_{N-1}), x_N) \text{ for any } g \in O(N-1)\}, \end{aligned}$$

where $O(N-1)$ is the orthogonal group in $\mathbb{R}^{N-1}$.

It follows from Proposition 5.2 of [26] that for any i the eigenspace V_i of the operator $-\Delta_{S^{N-1}}$, spanned by the eigenfunctions corresponding to the eigenvalue λ_i , which are $O(N-1)$ invariant, is one dimensional. Thus, the Morse index $i(\lambda, \overline{u}_R)$ in $\mathcal{H}_R$ grows by 1 when R cross R_k . Then $i(\lambda, \overline{u}_{R_k+\epsilon}) = i(\lambda, \overline{u}_{R_k-\epsilon}) + 1$ if ϵ is small enough. Then for any positive, small enough ϵ , letting $w_R = h_R^*(\overline{u}_R)$, we get from (4.2) and (4.3)

$$\deg(I - T_{R_k-\epsilon}, \mathcal{F}_{R_k-\epsilon}, 0) = -\deg(I - T_{R_k+\epsilon}, \mathcal{F}_{R_k+\epsilon}, 0)$$

if $\mathcal{F}_R$ is a neighborhood of w_R in the space of functions which are $O(N-1)$ invariant in D . This implies that there is a change in the degree at the point (λ, R_k, w_{R_k}) and then bifurcation must occur. Moreover the bifurcating solutions are nonradial, since for each R and λ (1.1) admits exactly two radial solutions $\underline{u}_R$ and $\overline{u}_R$. This completes the proof. $\square$

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