Remarks on non homogeneous elliptic Kirchhoff equations

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Abstract. Extending the results in (Ambrosetti and Arcoya, Positive solutions of elliptic Kirchhoff equations (to appear)), the existence of solutions for some classes of nonlinear elliptic, possibly degenerate, non homogeneous Kirchhoff equations are proved. We handle smooth nonlinearities as well as the case in which the nonlinearity has an upward jump discontinuity.

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1. Introduction

In this paper we continue the study carried out in a recent paper [1], in which we have considered Kirchhoff boundary value problems

$$-(1 + \gamma G'(\|u\|^2)) \Delta u = F'(u), \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\gamma \geq 0$, $\|u\|^2 = \int_{\Omega} |\nabla u|^2$, G is a $C^1(\mathbb{R}^+)$ function, with $G(t) \sim t^q$ and F is a C^1 function with subcritical growth. In [1] it is shown that standard Nonlinear Functional Analysis, such as global bifurcation and critical point theory, can be used to obtain several existence results.

Here we deal with the non-homogeneous Kirchhoff boundary value problems as

$$\begin{cases} -(1 + \gamma G'(\|u\|^2)) \Delta u = F'(u) + h(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where $h \in L^2(\Omega)$.

Solutions of (1) are found as stationary points of an appropriate functional J , see Sect. 2 below. We consider two cases.

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First, in Sect. 2 we deal with $F \in C^1(\mathbb{R})$ such that, roughly, $|F(u)| \sim |u|^p$, $1 \leq p < 2^*$. The kind of results we obtain depend upon the growth of $G(\|u\|^2) \sim \|u\|^{2q}$ and that of $F(u) \sim |u|^p$: in Theorem 2.1 we take $2q > p$, in Theorem 2.4, $2q < p$ and in Theorem 2.6, $2q = p$. In particular, to highlight the rôle of the Kirchhoff coefficient G , we mention that when $2q > p$, we can prove the existence of solutions to (1) for all $h \in L^2(\Omega)$ and all $2 < p < 2^*$, while for the usual superlinear, subcritical Dirichlet problem

$$-\Delta u = F'(u) + h(x), \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

solutions are known to exist for a smaller range of p , see [7], or for almost every $h \in L^2(\Omega)$, see [5]. Roughly, this essentially depends upon the fact that, if $2q > p$, the term $G(\|u\|^2)$ makes the functional J coercive.

In Sect. 3 we extend [1] in another direction: we discuss the case in which F is continuous but F' has an upward jump discontinuity. Taking $2q > p$, it turns out that J has a global minimum that solves (1) almost everywhere in Ω and hence is not sensitive to the jump of F' .

The previous existence results hold true also in the degenerate case, namely when the Eq. (1) becomes (without loss of generality, we can take here $\gamma = 1$)

$$\begin{cases} -G'(\|u\|^2)\Delta u = F'(u) + h(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \quad (2)$$

We thank with pleasure Lucio Boccardo for the suggestion to investigate this degenerate equation.

2. Main existence results

Throughout all this section, $F \in C^1(\mathbb{R})$ and $G \in C^1(\mathbb{R}^+)$. Moreover, we first assume that

(F1) $\exists a_1, a_2 > 0$ such that $|F(u)| \leq a_1 + a_2|u|^p$ with $1 \leq p < 2^*$, where

$$2 < p < 2^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n \geq 3, \\ +\infty, & \text{if } n = 1, 2; \end{cases}$$

(G0) $G(0) = G'(0) = 0$ and $G'(t) \geq 0$, $\forall t \geq 0$;

(G1) there exist $k > 0$ and $q \geq 1$ such that $G(t) \geq kt^q$, $\forall t \geq 0$.

We will work in the Sobolev space $E = H_0^1(\Omega)$ endowed with scalar product $\langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v$ and norm $\|u\|^2 = \langle u, u \rangle = \int_{\Omega} |\nabla u|^2$.

Solutions of (1) are the stationary points $u \in E$ of the C^1 functional

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\gamma G(\|u\|^2) - \mathcal{F}(u) - \int_{\Omega} hu,$$

where

$$\mathcal{F}(u) = \int_{\Omega} F(u).$$

In order to study J we will consider separately the cases $p < 2q$, $p > 2q$ and $p = 2q$.

Theorem 2.1. *Suppose that $(F1), (G0), (G1)$ hold and let $p < 2q$. Then (1) has a solution for all $h \in L^2(\Omega)$ and all $\gamma > 0$.*

Proof. Assumptions $(F1), (G1)$ and Sobolev embedding imply

$$J(u) \geq \frac{1}{2}\gamma k \|u\|^{2q} - c_1 - c_2 \|u\|^p - c_3 \|u\|. \quad (3)$$

It follows that J is coercive provided that $p < 2q$ and $\gamma > 0$. Since $p < 2^*$, the functional $\mathcal{F}$ is weakly continuous. Moreover, taking into account that $G'(t) \geq 0$, we infer that $G(\|u\|^2)$ is weakly lower semicontinuous (in short w.l.s.c.). In conclusion, J is coercive and w.l.s.c. and thus has a global minimum that gives rise to a solution of (1). $\square$

Remark 2.2. In the case of the degenerate Eq. (2), the functional J has to be substituted by

$$\tilde{J}(u) = \frac{1}{2}G(\|u\|^2) - \mathcal{F}(u) - \int_{\Omega} hu.$$

It is clear that, likewise J , $\tilde{J}$ is coercive, w.l.s.c. (because $G(t)$ is increasing) and hence it has a negative minimum. Thus (2) has a solution, extending Theorem 2.1.

Below we discuss two examples to underline the rôle played by the Kirchhoff coefficient G . See also the Example 2.8 later on.

Example 2.3. (a) Take $\gamma = 1$, $G(t) = \frac{1}{q}t^q$, $F(u) = \frac{1}{p}|u|^p$ and consider the problem

$$-(1 + \|u\|^{2q-2})\Delta u = |u|^{p-2}u + h(x), \quad u|_{\partial\Omega} = 0.$$

Using Theorem 2.1 we find that it has a solution for all $p \in (2, 2^*)$ with $2q > p$ and all $h \in L^2(\Omega)$.

On the other hand, let us point out that the existence of solutions to the problem

$$-\Delta u = |u|^{p-2}u + h(x), \quad u|_{\partial\Omega} = 0$$

is known provided $2 < p < 2^*/2$, see [6, 7]. Moreover, we recall that if $2 < p < 2^*$ then a solution exists for h in a dense subset of $L^2(\Omega)$, see [5].

(b) Let λ_k denote the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and let $b(u)$ be a bounded continuous function. It is well known, see e.g. [2, Chapter 9], that if the problem at resonance

$$-\Delta u = \lambda_k u + b(u) + h(x), \quad u|_{\partial\Omega} = 0 \quad (4)$$

has a solution, then

$$\inf_{u \in \mathbb{R}} b(u) \cdot \int_{\Omega} \phi \leq - \int_{\Omega} h\phi \leq \sup_{u \in \mathbb{R}} b(u) \cdot \int_{\Omega} \phi,$$

for every eigenfunction ϕ associated to λ_k . Conversely if, for example, $b(u)$ is monotone and

$$\inf_{u \in \mathbb{R}} b(u) \cdot \int_{\Omega} \phi < - \int_{\Omega} h\phi < \sup_{u \in \mathbb{R}} b(u) \cdot \int_{\Omega} \phi,$$

then (4) has a solution. Notice that $F(u) = \frac{1}{2}\lambda_k u^2 + \int_0^u b(s)ds$ satisfies (F1) with $p = 2$, and hence, according to Theorem 2.1, the equation

$$-(1 + G'(\|u\|^2))\Delta u = \lambda_k u + b(u) + h(x), \quad u|_{\partial\Omega} = 0,$$

possesses a solution for all $h \in L^2(\Omega)$, provided $G(t) \sim t^q$ and $q > 1$.

Next, we consider the case in which $p > 2q$. It is known that the Dirichlet problem

$$-\Delta u = F'(u) + h(x), \quad u|_{\partial\Omega} = 0,$$

has at least two solutions provided $\|h\|_{L^2}$ is small enough and F satisfies (F1) and

$$(F2) \quad 0 < pF(u) \leq uF'(u), \quad \forall u \neq 0.$$

To extend this multiplicity result to Kirchhoff Eq. (1), the following assumptions on G are in order:

$$(G1') \quad \exists K > 0, q \geq 1 \text{ such that } G(t) \leq K t^q, \quad \forall t \geq 0.$$

$$(G2) \quad pG(t) \geq 2G'(t)t, \text{ for all } t \geq 0.$$

Let us point out explicitly that (G2) is consistent with $p > 2q$. For example, this is the case if $G(t) = \frac{1}{q}t^q$. Moreover, we anticipate that (F2) and (G2) can be slightly improved, see Remark 2.5.

Theorem 2.4. *Suppose that $F, G \in C^1$ satisfy (F1), (F2), (G0), (G1'), (G2) and let $p > 2q$. Then there exists $\epsilon > 0$ such that (1) has at least two solutions provided $\|h\|_{L^2} < \epsilon$, for all $\gamma \geq 0$.*

Proof. The arguments are similar to the ones carried out in [1, Sect. 4].

Step 1. Using in particular (F2) and (G2), one shows that J satisfies the Palais-Smale condition at level c , $(PS)_c$: every sequence u_j such that (i) $J(u_j) \rightarrow c$ and (ii) $J'(u_j) \rightarrow 0$, has a strongly convergent subsequence. This is essentially Lemma 4.3 of [1]. To be selfcontained, we outline the proof. From (i) we infer

$$\|u_j\|^2 + \gamma G(\|u_j\|^2) \leq c + 2 \int_{\Omega} F(u_j) + 2 \int_{\Omega} h u_j. \quad (5)$$

From (ii) we have $|\langle J'(u_j), u_j \rangle| = o(1)\|u_j\|$, which yields

$$\int_{\Omega} F'(u_j) u_j \leq \|u_j\|^2 + \gamma G'(\|u_j\|^2) \|u_j\|^2 - \int_{\Omega} h u_j + o(1)\|u_j\|. \quad (6)$$

Using (F2) and (6), inequality (5) becomes

$$\begin{aligned} \|u_j\|^2 + \gamma G(\|u_j\|^2) &\leq c + \frac{2}{p} \int_{\Omega} F'(u_j) u_j + 2 \int_{\Omega} h u_j \\ &\leq c + \frac{2}{p} \left(\|u_j\|^2 + \gamma G'(\|u_j\|^2) \|u_j\|^2 - \int_{\Omega} h u_j + o(1)\|u_j\| \right) + 2 \int_{\Omega} h u_j. \end{aligned} \quad (7)$$

It follows, for some $c_1 > 0$,

$$\left(1 - \frac{2}{p}\right) \|u_j\|^2 + \gamma \left(G(\|u_j\|^2) - \frac{2}{p} G'(\|u_j\|^2) \|u_j\|^2\right) \leq c + c_1 \|u_j\| + o(1)\|u_j\|.$$

Using (G2) we infer

$$\left(1 - \frac{2}{p}\right) \|u_j\|^2 \leq c + c_1 \|u_j\| + o(1) \|u_j\|.$$

Since $p > 2$, this implies that $\|u_j\|$ is bounded. Let R be defined on E by setting $\langle R(u), v \rangle = \int_{\Omega} [F'(u) + h]v$, $\forall u, v \in E$. Then

$$u_j + \gamma G'(\|u_j\|^2) u_j = J'(u_j) + R(u_j)$$

and hence

$$u_j = \frac{J'(u_j) + R(u_j)}{1 + \gamma G'(\|u_j\|^2)}. \quad (8)$$

Since $\|u_j\|$ is bounded then, up to a subsequence, $\|u_j\| \rightarrow r$ and thus $1 + \gamma G'(\|u_j\|^2) \rightarrow 1 + \gamma G'(r^2) \geq 1$. Taking into account that $J'(u_j) \rightarrow 0$ and that R is weakly continuous (recall that the growth of F is subcritical), it follows from (8) that u_j has a strongly convergent subsequence.

Step 2. Let J_0 denote the functional J with $h = 0$. Since $G \geq 0$, for $\gamma \geq 0$ one has

$$J_0(u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \gamma G(\|u\|^2) - \mathcal{F}(u) \geq \frac{1}{2} \|u\|^2 - \mathcal{F}(u).$$

Since the functional $\frac{1}{2} \|u\|^2 - \mathcal{F}(u)$ has a strict local minimum at $u = 0$, the same holds for J_0 for all $\gamma \geq 0$. Hence, there exist $\delta, r > 0$ such that $J_0(v) \geq \delta$, provided that $\|v\| = r$. Therefore, there is $\epsilon > 0$ such that if $\|h\|_{L^2(\Omega)} < \epsilon$ then $J(v) \geq \delta/2$ when $\|v\| = r$. As a consequence, since (PS) holds, $J(u)$ has a strict local minimum u_1 in the interior of the ball centered at zero with radius r , giving rise to a first solution of (1).

Step 3. Fixed $v \in E$, $\|v\| = 1$, $v > 0$, (G1') implies

$$J(tv) \leq \frac{t^2}{2} + \gamma K t^{2q} - \int_{\Omega} F(tv) - t \int_{\Omega} h v.$$

On the other hand, from (F2) it follows that there is $c_2 > 0$ such that $F(u) \geq c_2 u^p$ for all $u \geq 0$. Thus

$$J(tv) \leq \frac{t^2}{2} + \gamma K t^{2q} - c_2 t^p \int_{\Omega} v^p - t \int_{\Omega} h v. \quad (9)$$

Since $p > 2q$, we have $\lim_{t \rightarrow +\infty} J(u) = -\infty$ and J has the Mountain-Pass geometry, namely: J has a local minimum at u_1 and there exists $z \in E$ such that $J(z) < J(u_1)$.

Now, applying the Mountain-Pass theorem, we find a second critical point u_2 of J such that $J(u_2) > J(u_1)$. Thus such a u_2 gives rise to another solution u_2 of (1), different from u_1 . $\square$

Remark 2.5. We can easily check that the preceding proof continues to hold if in (F2) and (G2) we replace p with any $\sigma \in (2q, p]$.

We finally consider the case in which $p = 2q$.

Theorem 2.6. *Let $F \in C^1(\mathbb{R})$ and $G \in C^1(\mathbb{R}^+)$.*

(a) *If (F1), (G0) and (G1) hold, with $p = 2q$, then (1) has a solution for all $h \in L^2(\Omega)$ and all $\gamma \gg 1$;*

(b) if $(F1), (F2), (G0), (G1')$ and $(G2)$ hold, with $p = 2q$, then there exists $\epsilon > 0$ such that (1) has at least two solutions provided $\|h\|_{L^2} < \epsilon$ and $0 \leq \gamma \ll 1$.

Proof. We limit ourselves to show the modifications to the arguments carried out in the preceding theorems.

(a) If $p = 2q$ inequality (3) becomes

$$J(u) \geq \frac{1}{2} \gamma k \|u\|^{2q} - c_1 - c_2 \|u\|^{2q} - c_3 \|u\|.$$

Therefore J is still coercive provided γ is sufficiently large. Then, for $\gamma \gg 1$, the existence of a solution follows as in Theorem 2.1.

(b) Steps 1 and 2 of the proof of Theorem 2.4 hold for all $\gamma \geq 0$. As for Step 3, concerning the Mountain-Pass geometry, it suffices to notice that from (9) (with $p = 2q$) it follows that, if $\gamma \ll 1$, $J(tv) \rightarrow -\infty$ as $t \rightarrow +\infty$ once more. $\square$

Remark 2.7. Item (a) can be sharpened in the presence of specific F and G . For example, if $G(t) = \frac{1}{q} t^q$ and $F(u) = \frac{1}{2q} |u|^{2q}$, one finds

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|^2 + \gamma \frac{1}{2q} \|u\|^{2q} - \frac{1}{2q} \int_{\Omega} |u|^{2q} - \int_{\Omega} hu \\ &\geq \frac{1}{2} \|u\|^2 + \gamma \frac{1}{2q} \|u\|^{2q} - \frac{S}{2q} \|u\|^{2q} - \int_{\Omega} hu, \end{aligned}$$

where S is the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{2q}(\Omega)$. If $\gamma \geq S$, J is coercive and (1) has a solution.

Example 2.8. Completing Example 2.3-(a), we can handle problems like

$$(1 + \gamma \|u\|^{2q}) \Delta u = |u|^{p-2} u + h(x), \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad (10)$$

with $2q \leq p$.

If $2q < p$, Theorem 2.4 implies that (10) has two solutions for all $p \in (2, 2^*)$ and all $\|h\|_{L^2}$ small.

If $2q = p$, we use Theorem 2.6 to find that (10) has a solution either for $\gamma \gg 1$ and all $h \in L^2(\Omega)$, or for $0 \leq \gamma \ll 1$ and $\|h\|_{L^2}$ small enough.

Remark 2.9. Theorem 2.4 can be extended to the degenerate equation (2). To be short, we will limit ourselves to consider the model case in which $F(u) = \frac{1}{p} |u|^p$ and $G(t) = \frac{1}{q} t^q$ with $1 \leq q < \frac{p}{2}$, so that

$$\tilde{J}(u) = \frac{1}{2q} \|u\|^{2q} - \frac{1}{p} \int_{\Omega} |u|^p dx - \int_{\Omega} h u dx.$$

When $\|h\|_{L^2} \ll 1$, both J and $\tilde{J}$ have the mountain-pass geometry and their mountain-pass level is positive. Therefore, as far as the (PS) condition is concerned, we can consider a sequence u_j such that $\tilde{J}(u_j) \rightarrow c > 0$, and

$\tilde{J}'(u_j) \rightarrow 0$. Then, as in (7), we find

$$\begin{aligned} \frac{1}{2q} \|u_j\|^{2q} &\leq c_0 + \frac{1}{p} \int_{\Omega} |u_j|^p + \int_{\Omega} h u_j \\ &\leq c_0 + \frac{1}{p} \left[\|u_j\|^{2q} - \int_{\Omega} h u_j + o(1) \|u_j\| \right] + \int_{\Omega} h u_j, \end{aligned}$$

whence

$$\left[\frac{1}{2q} - \frac{1}{p} \right] \|u_j\|^{2q} \leq c_0 + \left[1 - \frac{1}{p} \right] \int_{\Omega} h u_j + o(1) \|u_j\| \leq c_0 + c_1 \|u_j\|.$$

Since $2 \leq 2q < p$, it follows that u_j is bounded.

In this case, (8) becomes

$$u_j = \frac{\tilde{J}'(u_j) + R(u_j)}{\|u_j\|^{2q-1}}. \quad (11)$$

Since $\|u_j\|$ is bounded, we can assume, as before, that $\|u_j\| \rightarrow r$. Moreover, $r > 0$ because, otherwise, $u_j \rightarrow 0$, in contradiction to the fact that $\tilde{J}(u_j) \rightarrow c > 0$. Then from (11) we infer that u_j has a strongly convergent subsequence, so that $\tilde{J}$ satisfies the $(PS)_c$ condition.

It follows that $\tilde{J}$ possesses, as J , a mountain-pass critical point that gives rise to a positive solution of (2).

The cases covered by Theorem 2.6, namely when $2q = p$, $G(t) = \frac{\gamma}{q} t^q$ and either (a) $\gamma \gg 1$ or (b) $0 < \gamma \ll 1$ and $\|h\|_{L^2}$ is small enough, can be handled in a quite similar way.

3. Discontinuous nonlinearities

In this section we consider the case in which F' has a jump discontinuity. Elliptic problems with discontinuous nonlinearities arise in applications and have been widely investigated, see e.g. [3, 4, 9, 10] and references therein. Here we extend some of those results to Kirchhoff equations.

To be specific, here we limit ourselves to consider the case in which F' has an upward jump discontinuity at a point $u = a$. More precisely, we suppose that $F \in C(\mathbb{R})$ is convex, satisfies the growth condition (F1) and the following condition:

(F3) $\exists a \in \mathbb{R}$ such that $F \in C^1(\mathbb{R} - \{a\})$ and the limits $F'(a \pm) = \lim_{u \rightarrow a \pm} F'(u)$ exist, are finite, with $F'(a-) < F'(a+)$.

A function F satisfying the preceding conditions has a subdifferential $\partial F(u)$ (recall that, by definition, $\partial F(u)$ is the set $\{\xi \in \mathbb{R} : F(v) - F(u) \geq \xi \cdot (v - u), \forall v \in \mathbb{R}\}$), which is given by

$$\partial F(u) = \begin{cases} \{F'(u)\} & \text{if } u \neq a, \\ [F'(a-), F'(a+)] & \text{if } u = a. \end{cases}$$

A model nonlinearity that satisfies (F3) with $a = 0$ is the function

$$F(u) = \begin{cases} \alpha u & \text{if } u \leq 0, \\ \beta u & \text{if } u > 0, \end{cases}$$

with $\alpha < \beta$, so that

$$\partial F(u) = H(u) = \begin{cases} \alpha & \text{if } u < 0, \\ \beta & \text{if } u > 0, \\ [\alpha, \beta] & \text{if } u = 0, \end{cases}$$

is a Heaviside like function.

If F satisfies ((F1) and) (F3), then the functional $\mathcal{F}(u) = \int_{\Omega} F(u)$ is not Fréchet differentiable (because F' is discontinuous), but merely locally Lipschitz continuous with subdifferential given by $\partial \mathcal{F}(u) = \partial F(u)$, see e.g. [9, Section 2]).

As a consequence, also the functional (below the value of γ plays no rôle and we will take $\gamma = 1$)

$$J(u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} G(\|u\|^2) - \mathcal{F}(u) - \int_{\Omega} hu, \quad u \in E,$$

is only locally Lipschitz continuous. The subdifferential of $J(u)$ is given by

$$\partial J(u) = A(u) - \partial \mathcal{F}(u) - h = A(u) - \partial F(u) - h,$$

where A is defined by setting

$$\langle A(u), v \rangle = (1 + G'(\|u\|^2)) \langle u, v \rangle.$$

Recall that if $u \in E$ is a (local or global) minimum of J then $0 \in \partial J(u) = A(u) - h - \partial F(u)$ and thus

$$A(u) - h \in \partial F(u) \quad \text{a.e. in } \Omega,$$

or, equivalently

$$-(1 + G'(\|u\|^2)) \Delta u(x) - h(x) \in \partial F(u(x)), \quad \text{a.e. in } \Omega. \quad (12)$$

The next lemma has an interest in itself.

Lemma 3.1. *If $\bar{u} \in E$ is a local minimum of J , then*

$$\Omega_a = \{x \in \Omega : \bar{u}(x) = a\}$$

has null Lebesgue measure (possibly $\Omega_a = \emptyset$), so that $\bar{u}$ satisfies (1) for almost all $x \in \Omega$.

Proof. The arguments are close to those of [4]. By the well known Theorem by Morrey - Stampacchia, one has that $\Delta \bar{u} = 0$ a.e. in Ω_a , and hence (12) yields $-h(x) \in \partial F(a)$ a.e. in Ω_a , namely

$$F'(a-) \leq -h(x) \leq F'(a+) \quad \text{a.e. in } \Omega_a. \quad (13)$$

Let us take a function $\psi \in E \cap C(\bar{\Omega})$, $\psi > 0$ in Ω . Since $\bar{u}$ is a local minimum of J , there exists $\delta > 0$ such that

$$J(\bar{u} + \varepsilon \psi) \geq J(\bar{u}), \quad \forall |\varepsilon| \leq \delta.$$

Therefore, by the Lebesgue Dominated Convergence Theorem, and using that $\psi > 0$ in Ω , it follows

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{J(\bar{u} + \varepsilon\psi) - J(\bar{u})}{\varepsilon} = \langle A(\bar{u}), \psi \rangle - \int_{\Omega} F'(\bar{u}+) \psi - \int_{\Omega} h \psi \\ &= \int_{\Omega} \{-(1 + G'(\|\bar{u}\|^2) \Delta \bar{u} - F'(\bar{u}+) - h) \} \psi \\ &= \int_{\Omega - \Omega_a} \{-(1 + G'(\|\bar{u}\|^2) \Delta \bar{u} - F'(\bar{u}) - h) \} \psi - \int_{\Omega_a} \{F'(a+) + h\} \psi. \end{aligned}$$

Similarly, taking $\varepsilon < 0$ and repeating the previous arguments, we find

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \rightarrow 0^-} \frac{J(\bar{u} + \varepsilon\psi) - J(\bar{u})}{\varepsilon} = \int_{\Omega} \{-(1 + G'(\|\bar{u}\|^2) \Delta \bar{u} - F'(\bar{u}-) - h) \} \psi \\ &= \int_{\Omega - \Omega_a} \{-(1 + G'(\|\bar{u}\|^2) \Delta \bar{u} - F'(\bar{u}) - h) \} \psi - \int_{\Omega_a} \{F'(a-) + h\} \psi. \end{aligned}$$

Recalling that $\partial F(u(x)) = \{F'(u(x))\}$ provided $x \in \Omega - \Omega_a$, (12) yields $-(1 + G'(\|\bar{u}\|^2) \Delta \bar{u} = F'(\bar{u}) + h(x)$ a.e. in $\Omega - \Omega_a$ and thus we infer

$$\int_{\Omega_a} \{F'(a+) + h(x)\} \psi \leq 0, \quad (14)$$

$$\int_{\Omega_a} \{F'(a-) + h(x)\} \psi \geq 0. \quad (15)$$

Since, see (13), $F'(a+) + h(x) \geq 0$ a.e. in Ω_a and $\psi > 0$ in Ω , then (14) implies that $meas(\Omega_a) = 0$, unless

$$F'(a+) + h(x) = 0, \quad \text{a.e. in } \Omega_a.$$

But if this equation holds, (15) becomes

$$\int_{\Omega_a} \{F'(a-) - F'(a+)\} \psi \geq 0,$$

which yields $meas(\Omega_a) = 0$, because $F'(a-) < F'(a+)$ and $\psi > 0$ in Ω . The proof is completed. $\square$

Remark 3.2. If $-h(x) \notin [F'(a-), F'(a+)]$ for almost every $x \in \Omega$, Lemma 3.1 can be extended to every critical point $u \in E$ of J , namely any u such that $0 \in \partial J(u)$. Actually, from (13), it follows that

$$meas(\{x \in \Omega : u(x) = a\}) = 0.$$

As a consequence, u solves (1) almost everywhere in Ω .

Lemma 3.1 allows to find the following existence result that extends Theorem 2.1 to discontinuous nonlinearities. The cases of Theorems 2.4 and 2.6 are shortly discussed in Remark 3.4 - (c) below.

Theorem 3.3. *Suppose that $F \in C(\mathbb{R})$ is convex and satisfies (F1), (F3) and let $G \in C^1(\mathbb{R}^+)$ satisfies (G0) and (G1), with $2q > p$. Then for all $h \in L^2(\Omega)$, problem (12) has a solution $\bar{u} \in E$. Moreover, the set $\Omega_a = \{x \in \Omega : \bar{u}(x) = a\}$ has null Lebesgue measure and hence $\bar{u}$ solves (1) almost everywhere in Ω .*

Proof. If $(F1)$, $(F3)$ hold and G satisfies $(G0)$ and $(G1)$ with $2q > p$, the functional J is coercive and w.l.s.c. and hence J has a minimum $\bar{u}$ in E which gives rise to a solution of (12). We can now apply Lemma 3.1 to infer that $meas(\Omega_a) = 0$, completing the proof. $\square$

Remark 3.4.

- (a) In a way similar to that discussed in Remarks 2.2 and 2.9, one checks that Theorem 3.3 can be extended to degenerate equations (2) with discontinuous nonlinearities.
- (b) We can allow F to have a finite number of upward discontinuity points $a_1, \dots, a_k$, each satisfying $(F3)$. Moreover, we can consider discontinuous functions F which are not convex. In such a case, it suffices to use the generalized subgradient introduced by Clarke [8] and Chang [9].
- (c) We shortly indicate how we can handle also cases in which $2q \leq p$ covered by Theorems 2.4 and 2.6. It is understood that F satisfies the assumptions in those Theorems, as well as $(F3)$.

If $2q < p$ we can show, as Theorem 2.4, that J possesses a minimum $u_1 \in E$ and a Mountain Pass critical point u_2 , provided $\|h\|_{L^2} < \epsilon$.

Similarly, if $2q = p$ we can find, as in Theorem 2.6, a minimum $u_1 \in E$ of J provided that either $\gamma \gg 1$ or $0 \leq \gamma \ll 1$ and $\|h\|_{L^2} < \epsilon$. Moreover, under the latter conditions, J possesses a Mountain Pass critical point u_2 .

In both cases, according to Lemma 3.1, the minima u_1 solve (1) almost everywhere in Ω . As for the Mountain Pass critical points u_2 , we can use Remark 3.2, provided $-h(x) \notin [F'(a-), F'(a+)]$ a.e. in Ω . For example, if $h(x) > 0$ in Ω and $F'(a-) \geq 0$, or if $h(x) < 0$ in Ω and $F'(a+) \leq 0$, it follows that also u_2 solves (1) almost everywhere in Ω .

References

- [1] Ambrosetti, A., Arcoya, D.: Positive solutions of elliptic Kirchhoff equations. Adv. Nonlin. Stud. (to appear)
- [2] Ambrosetti, A., Arcoya, D.: An introduction to nonlinear functional analysis and elliptic problems. PNLDE 82, Birkhäuser (2011)
- [3] Ambrosetti, A., Turner, R.E.L.: Some discontinuous variational problems. Differ. Integral Equ. 1, 341–349 (1988)
- [4] Arcoya, D., Calahorrano, M.: Some discontinuous problems with a quasilinear operator. J. Math. Anal. Appl. 187, 1059–1072 (1994)
- [5] Bahri, A.: Topological results on a certain class of functionals and application. J. Funct. Anal. 41, 397–427 (1981)
- [6] Bahri, A., Berestycki, H.: A perturbation method in critical point theory and applications. Trans. A.M.S. 267, 1–32 (1981)

- [7] Bahri, A., Lions, P.L.: Morse index of some min-max critical points. I. Application to multiplicity results. *Commun. Pure Appl. Math.* **41**, 1027–1037 (1988)
- [8] Clarke, F.H.: A new approach to Lagrange multipliers. *Math. Oper. Res.* **1**, 165–174 (1976)
- [9] Chang, K.C.: Variational methods for nondifferentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.* **80**, 102–129 (1981)
- [10] Stuart, C., Toland, J.: A variational method for boundary value problems with discontinuous nonlinearities. *J. Lond. Math. Soc.* **21**(3), 319–328 (1980)

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