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Hölder estimates for viscosity solutions of equations of fractional p-Laplace type

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Abstract. We prove Hölder estimates for viscosity solutions of a class of possibly degenerate and singular equations modelled by the fractional p-Laplace equation

$$\mathrm{PV} \int_{\mathbb{R}^n} \frac{|u(x) - u(x+y)|^{p-2}(u(x) - u(x+y))}{|y|^{n+sp}} \, dy = 0,$$

where $s \in (0,1)$ and $p > 2$ or $1/(1-s) < p < 2$. Our results also apply for inhomogeneous equations with more general kernels, when p and s are allowed to vary with x, without any regularity assumption on p and s. This complements and extends some of the recently obtained Hölder estimates for weak solutions.

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1. Introduction

We study the local Hölder regularity for viscosity solutions of possibly degenerate and singular non-local equations of the form

$$\mathrm{PV} \int_{\mathbb{R}^n} |u(x) - u(x+y)|^{p-2}(u(x) - u(x+y))K(x,y) \, dy = f(x), \qquad (1.1)$$

where f is bounded and $K(x,y)$ essentially behaves like $|y|^{-n-sp}$. Here PV stands for the *principal value*. We also allow p and s to depend on x.

This type of equations is one possible non-local counterpart of equations of p-Laplace type and arises for instance as the Euler-Lagrange equation of functionals in fractional Sobolev spaces. Solutions can also be constructed directly via Perron's method, which has been done (for p and s constant) in

[27] and [30], and in a slightly different setting in [22]. For the general case of variable exponents s and p, the full details of the existence theory are yet to be written down.

In the case $K(y) = |y|^{-n-sp}$, when properly rescaled, solutions converge to solutions of the p-Laplace equation

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) = 0$$

as the parameter s tends to 1, see [22].

The main difference of the paper compared to earlier results (such as [33]) is the nonlinearity and the possible degeneracy or singularity of the operator. The difficulties mainly appear when estimating the operator acting on $u + k\beta$ from above in the proof of Proposition 2. Here β is a certain help function, see Section 3.

Throughout the paper, we denote by B_r, the ball of radius r centered at the origin. We also assume that the operator L is given by

$$Lu(x) := \text{PV} \int_{\mathbb{R}^n} |u(x) - u(x+y)|^{p(x)-2}(u(x) - u(x+y))K(x,y)\,dy \quad (1.2)$$

where $K(x,y) = K(x,-y)$ and

$$\frac{\lambda}{|y|^{n+s(x)p(x)}} \leq K(x,y) \leq \frac{\Lambda}{|y|^{n+s(x)p(x)}}, \quad \text{for all } y \in B_2, x \in B_2,$$

$$0 \leq K(x,y) \leq \frac{M}{|y|^{n+\gamma}}, \quad \text{for all } y \in \mathbb{R}^n \setminus B_{\frac{1}{4}}, x \in B_2, \quad (1.3)$$

where $0 < s_0 < s(x) < s_1 < 1$ and $1 < p_0 < p(x) < p_1 < \infty$. In the case $p(x) < 2$ we require additionally that there is $\tau > 0$ such that

$$p(x)(1 - s(x)) - 1 > \tau.$$

Our main result is that bounded viscosity solutions (see Section 2) of the equation

$$Lu = f,$$

with f bounded, are locally Hölder continuous, see the theorem below.

Theorem 1. *Assume K satisfies (1.3) and that L is as in (1.2). Let $f \in C(B_2) \cap L^\infty(B_2)$ and let $u \in L^\infty(\mathbb{R}^n)$ be a viscosity solution of*

$$Lu = f \quad in \ B_2.$$

Then u is Hölder continuous in B_1 and in particular there exist α and C depending on $\lambda, \Lambda, M, p_0, p_1, s_0, s_1, \gamma$ and τ such that

$$\|u\|_{C^\alpha(B_1)} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \max\left(\|f\|_{L^\infty(B_2)}^{\frac{1}{p_0-1}}, \|f\|_{L^\infty(B_2)}^{\frac{1}{p_1-1}}\right)\right).$$

In particular, Theorem 1 applies for the fractional p-Laplace equation

$$\text{PV} \int_{\mathbb{R}^n} \frac{|u(x) - u(x+y)|^{p-2}(u(x) - u(x+y))}{|y|^{n+sp}}\,dy = 0.$$

Remark 1. The result above applies to bounded solutions. It might be possible to treat unbounded solutions by truncation and treating the resulting error term as a right hand side. In the setting of [10], the local boundedness is proved, given that a certain non-local "tail" can be controlled.

Remark 2. It might seem odd that the two conditions above on K are supposed to be satisfied in overlapping regions, B_2 and $B_{\frac{1}{4}}$. This is only for notational convenience. It would be sufficient to have the first condition satisfied in B_ρ for some $\rho > 0$ and the second one satisfied outside B_R for some large R as long as we ask K to be bounded in $B_R \setminus B_\rho$.

1.1. Known results

Equations similar to (1.1) were, to the author's knowledge, introduced in [22]. There existence and uniqueness is established in the case of constant p and s, even though in a slightly different setting. It is also shown that the solutions converge to solutions of the p-Laplace equation, as $s \to 1$. In [27] and [26], Perron's method is successfully applied to more general equations of this type. In [30] the equivalence of different notions of solutions is studied. Similar equations were also studied in [7], where the focus lies in the asymptotic behavior as $p \to \infty$. Related equations have also been suggested to be used in image processing and machine learning, see [15] and [16].

Recently, in [9,10,28], Hölder estimates, a Harnack inequality and regularity for the case with measure data were obtained for weak solutions of a very general class of equations of this type (with s and p constant). The difference between these results and the ones in the present paper can be seen as the difference between equations in divergence form and those in non-divergence form in the non-local setting. In other words, their results are more in the flavour of Di Giorgi–Nash–Moser (cf. [14,31,32]) while the results in this paper are more in the flavour of Krylov–Safonov (cf. [29]). Very recently, the sharp regularity up to the boundary was obtained in [20] and [21].

In the case $p = 2$, when (1.1) reduces to

$$\mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, dy = f(x), \tag{1.4}$$

a similar development has already taken place. In [33], a surprisingly simple proof of Hölder estimates for viscosity solutions were given for a very general class of equations corresponding to equations of non-divergence form, where s is allowed to depend on x. In [25], Hölder estimates were obtained for weak solutions for a class of equations corresponding to equations of divergence form, including equations of the form (1.4).

Related is also [1] and [2], where a different type of degenerate (or singular) non-local equation is studied. Hölder estimates and some higher regularity theory are established. It is also proved that these equations approach the p-Laplace equation in the local limit.

Very recently, a new type of quasilinear nonlocal equations was introduced in [6], with some similarities to the approach in [2] and [1]. These equations do also approximate an associated local equation of p-Laplace type and they relate to an associated Lévy process.

1.2. Comments on the equation

Let us very briefly point out the difference between the class of equations considered in [10] and [9], and the class of equations considered here when p and s are constant (see also [33] for a similar discussion). There, weak solutions are considered, in the sense that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))G(x,y)\,dxdy=0 \quad (1.5)$$

for any $\phi\in C_0^\infty(B_2)$, where $G(x,y)$ behaves like $|x-y|^{-n-sp}$. These solutions arise for instance as minimizers of functionals of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x)-u(y)|^pG(x,y)\,dxdy.$$

In the most favorable of situations, we are allowed to change the order of integration and write (1.5) as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x)-u(y)|^{p-2}(u(x)-u(y))(G(x,y)+G(y,x))\phi(x)\,dxdy=0,$$

and conclude

$$\text{PV}\int_{\mathbb{R}^n}|u(x)-u(y)|^{p-2}(u(x)-u(y))(G(x,y)+G(y,x))\,dy=0.$$

The change of variables $y=z+x$ yields

$$\text{PV}\int_{\mathbb{R}^n}|u(x)-u(z+x)|^{p-2}(u(x)-u(z+x))(G(x,z+x)+G(z+x,x))\,dz=0,$$

or

$$\text{PV}\int_{\mathbb{R}^n}|u(x)-u(z+x)|^{p-2}(u(x)-u(z+x))K(x,z)\,dz=0,$$

where $K(x,z)=G(x,z+x)+G(z+x,x)$. Then necessarily $K(x,z-x)=K(z,x-z)$. Moreover, we are not always allowed to perform the transformations above. Hence, the two types of equations overlap but neither is contained in the other. In other words, the results in [9] and [10] do not always apply to the equations considered in this paper, and vice versa, the results in this paper do not always apply to the equations studied therein. See also [26], for the equivalence of different notions of solutions.

Another important remark is that the estimates obtained in this paper are, as in [33], not uniform as s approaches 1. However, it seems very unlikely that a result like Theorem 1 should be true, without the upper bound on s, since this would imply a very general Hölder regularity result for equations being a "discontinuous mix" of degenerate non-local equations and degenerate local equations with discontinuous exponents. To the best of my knowledge, such a result is not even known in the case of $p=2$. The case of p-Laplacian

with different and discontinuous coefficients is related to Lavrentiev's phenomenon, see [17,34,35].

For fully nonlinear equations of fractional Laplace type, uniform estimates as $s \to 1$ have been obtained (see for instance [3] and [8]), but they are more involved, and they follow the same strategy as the estimates for fully nonlinear (local) equations. The reader may also consult [18] for an overview of the theory in the local setting.

In our case, when s and $p > 2$ are constant, if $\phi \in C_0^2$ and $p > 2$ then

$$(1-s) \, \mathrm{PV} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(x+y)|^{p-2}(\phi(x) - \phi(x+y))}{|y|^{n+sp}} \, dy \to -C_{p,n} \Delta_p \phi,$$

as $s \to 1$. If we instead have a kernel of the form

$$G\left(\frac{y}{|y|}\right) \frac{1}{|y|^{n+sp}},$$

then

$$(1-s) \, \mathrm{PV} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(x+y)|^{p-2}(\phi(x) - \phi(x+y))G(\frac{y}{|y|})}{|y|^{n+sp}} \, dy$$
$$\to -C_{p,n} |\nabla \phi|^{p-2} a_{ij}(\nabla \phi) D_{ij}^2 \phi,$$

as $s \to 1$, where the matrix $(a_{ij})(\nabla \phi)$ is positive definite and can be given explicitly as integrals over the sphere in terms of G. This type of degenerate (or singular) equations of non-divergence form, remained fairly unstudied until quite recently. Starting with [4], these equations have attracted an increasing amount of attention; See [5,11–13,19,23,24] for related results.

2. Viscosity solutions

In this section, we introduce the notion of viscosity solutions (as in [8]) and prove that viscosity solutions can be treated almost as classical solutions.

Definition 1. Let D be an open set and let L be as in (1.2). A function $u \in L^\infty(\mathbb{R}^n)$ which is upper semicontinuous in D is a subsolution of

$$Lu \leq C \quad \text{in } D$$

if the following holds: whenever $x_0 \in D$ and $\phi \in C^2(B_r(x_0))$ for some $r > 0$ are such that

$$\phi(x_0) = u(x_0), \quad \phi(x) \geq u(x) \quad \text{for } x \in B_r(x_0) \subset D$$

then we have

$$L\phi_r(x_0) \leq C,$$

where

$$\phi_r = \begin{cases} \phi & \text{in } B_r(x_0), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

A supersolution is defined similarly and a solution is a function which is both a sub- and a supersolution.

The following result verifies that whenever we can touch a subsolution from above with a C^2 function, we can treat the subsolution as classical subsolution. The proof is almost identical to the one of Lemma 3.3 in [8].

Proposition 1. *Assume the hypotheses of Theorem 1. Suppose $Lu \leq C$ in B_1 in the viscosity sense and that $x_0 \in B_1$ and $\phi \in C^2(B_r(x_0))$ is such that*

$$\phi(x_0) = u(x_0), \quad \phi(x) \geq u(x) \quad in \ B_r(x_0) \subset B_1,$$

for some $r > 0$. Then Lu is defined pointwise at x_0 and $Lu(x_0) \leq C$.

Proof. For $0 < r' \leq r$, let

$$\phi_{r'} = \begin{cases} \phi & \text{in } B_{r'}(x_0), \\ u & \text{in } \mathbb{R}^n \setminus B_{r'}(x_0). \end{cases}$$

Since u is a viscosity subsolution, $L\phi_{r'}(x_0) \leq C$. Now introduce the notation

$$\delta(\phi_{r'}, x, y) = \frac{1}{2}|\phi_{r'}(x) - \phi_{r'}(x+y)|^{p(x)-2}(\phi_{r'}(x) - \phi_{r'}(x+y))$$
$$+ \frac{1}{2}|\phi_{r'}(x) - \phi_{r'}(x-y)|^{p(x)-2}(\phi_{r'}(x) - \phi_{r'}(x-y)),$$

$$\delta^{\pm}(\phi_{r'}, x, y) = \max(\pm\delta(\phi_{r'}, x, y), 0).$$

By simply interchanging $y \to -y$ we have

$$\int_{\mathbb{R}^n} \delta(\phi_{r'}, x_0, y)K(x_0, y)\, dy \leq C, \tag{2.1}$$

since one can easily see that the integral is well defined since $\phi_{r'}$ is C^2 near x_0. Moreover,

$$\delta(\phi_{s_2}, x_0, y) \leq \delta(\phi_{s_1}, x_0, y) \leq \delta(u, x_0, y) \quad \text{for } s_1 < s_2 < r,$$

so that

$$\delta^-(u, x_0, y) \leq |\delta(\phi_r, x_0, y)|.$$

Since $|\delta(\phi_r, x_0, y)K(x_0, y)|$ is integrable, so is $\delta^-(u, x_0, y)K(x_0, y)$. In addition, by (2.1)

$$\int_{\mathbb{R}^n} \delta^+(\phi_{r'}, x_0, y)K(x_0, y)\, dy \leq \int_{\mathbb{R}^n} \delta^-(\phi_{r'}, x_0, y)K(x_0, y)\, dy + C.$$

Thus, for $s_1 < s_2$

$$\int_{\mathbb{R}^n} \delta^+(\phi_{s_1}, x_0, y)K(x_0, y)\, dy \leq \int_{\mathbb{R}^n} \delta^-(\phi_{s_1}, x_0, y)K(x_0, y)\, dy + C$$
$$\leq \int_{\mathbb{R}^n} \delta^-(\phi_{s_2}, x_0, y)K(x_0, y)\, dy + C < \infty. \tag{2.2}$$

Since $\delta^+(\phi_{r'}, x_0, y) \nearrow \delta^+(u, x_0, y)$ as $r' \to 0$, the monotone convergence theorem implies

$$\int_{\mathbb{R}^n} \delta^+(\phi_{r'}, x_0, y)K(x_0, y)\, dy \to \int_{\mathbb{R}^n} \delta^+(u, x_0, y)K(x_0, y)\, dy,$$

as $r' \to 0$. By letting $s_1 \to 0$ in (2.2) we obtain

$$\int_{\mathbb{R}^n} \delta^+(u, x_0, y) K(x_0, y) \, dy \le \int_{\mathbb{R}^n} \delta^-(\phi_{r'}, x_0, y) K(x_0, y) \, dy + C < \infty, \quad (2.3)$$

for any $0 < r' < r$. We conclude that $\delta^+(u, x_0, y) K(x_0, y)$ is integrable. By (2.2) and the bounded convergence theorem, we can pass to the limit in the right hand side of (2.3) and obtain

$$\int_{\mathbb{R}^n} \delta(u, x_0, y) K(x_0, y) \, dy = \lim_{r' \to 0} \int_{\mathbb{R}^n} \delta(\phi_{r'}, x_0, y) K(x_0, y) \, dy \le C.$$

This implies that $Lu(x_0)$ exists in the pointwise sense and $Lu(x_0) \le C$. $\square$

3. Hölder regularity

In this section we give the proof of our main theorem. This is based on Lemma 1, sometimes referred to as the oscillation lemma.

By abuse of notation, we introduce the function

$$\beta(x) = \beta(|x|) = \left((1 - |x|^2)^+\right)^2.$$

The exact form of β is not important, we could have chosen any radial function which is C^2 and zero outside B_1 and non-increasing along rays from the origin.

Below we prove that a kernel $K(x, y)$ satisfying (1.3) satisfies certain inequalities that might look strange at a first glance, but they are exactly the ones that will appear in the proof of our key lemma later.

Proposition 2. *Assume K satisfies (1.3). Then for any $\delta > 0$ there are $1/2 \ge k > 0$ and $\eta > 0$ such that for $p(x) \in (2, \infty)$*

$$2^{p(x)-2} k^{p(x)-1} \, \mathrm{PV} \int_{x+y \in B_1} |\beta(x) - \beta(x+y)|^{p(x)-2} (\beta(x) - \beta(y+x)) K(x, y) \, dy$$

$$+ 2^{p(x)-2} \int_{y \in \mathbb{R}^n \setminus B_{\frac{1}{4}}} |k\beta(x) + 2((|8y|^\eta - 1)|^{p(x)-1} K(x, y) \, dy$$

$$+ 2^{p(x)} \int_{y \in \mathbb{R}^n \setminus B_{\frac{1}{4}}} ((|8y|^\eta - 1)|^{p(x)-1} K(x, y) \, dy$$

$$< 2^{1-p(x)} \inf_{A \subset B_2, |A| > \delta} \int_A K(x, y) \, dy \qquad (3.1)$$

and for $p(x) \in (1/(1-s), 2)$

$$(3^{p(x)-1} + 2^{p(x)-1}) k^{p(x)-1} \int_{\mathbb{R}^n} |\beta(x) - \beta(x+y)|^{p(x)-1} K(x, y) \, dy$$

$$+ 2^{p(x)} \int_{\mathbb{R}^n \setminus B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p(x)-1} K(x, y) \, dy < 2^{1-p(x)} \inf_{A \subset B_2, |A| > \delta} \int_A K(x, y) \, dy,$$

$$(3.2)$$

for any $x \in B_{3/4}$. Here k and η depend on $\lambda, \Lambda, M, p_0, p_1, s_0, s_1, \gamma, \tau$ and δ.

Proof. In order to simplify the notation we write $p = p(x)$ and $s = s(x)$ throughout the proof. The proof is split into two different cases.

Case 1: $p \geq 2$

The first term in the left hand side of (3.1) reads

$$2^{p-2}k^{p-1} \operatorname{PV} \int_{x+y\in B_1} |\beta(x) - \beta(x+y)|^{p-2}(\beta(x) - \beta(y+x))K(x,y)\,dy$$

$$= 2^{p-2}k^{p-1} \operatorname{PV} \int_{x+y\in B_1, y\notin B_{\frac{1}{4}}} |\beta(x) - \beta(x+y)|^{p-2}(\beta(x)$$

$$- \beta(y+x))K(x,y)\,dy$$

$$+ 2^{p-2}k^{p-1} \operatorname{PV} \int_{y\in B_{\frac{1}{4}}} |\beta(x) - \beta(x+y)|^{p-2}(\beta(x) - \beta(y+x))K(x,y)\,dy$$

$$= I_1 + I_2.$$

Since $|\beta(x) - \beta(x+y)| \leq 1$, we can, using the upper bound on K outside $B_{1/4}$, obtain

$$|I_1| \leq |2k|^{p-1} \int_{\mathbb{R}^n\setminus B_{\frac{1}{4}}} K(x,y)\,dy \leq |2k|^{p-1}M \int_{\mathbb{R}^n\setminus B_{\frac{1}{4}}} \frac{dy}{|y|^{n+\gamma}}, \qquad (3.3)$$

which is finite and converges to zero as $k \to 0$.

For I_2 we proceed as follows

$$I_2 = 2^{p-2}k^{p-1} \operatorname{PV} \int_{y\in B_{\frac{1}{4}}} |\beta(x) - \beta(x+y)|^{p-2}(\beta(x) - \beta(y+x))K(x,y)\,dy$$

$$= 2^{p-3}k^{p-1} \operatorname{PV} \int_{y\in B_{\frac{1}{4}}} |\beta(x) - \beta(x+y)|^{p-2}(\beta(x) - \beta(y+x))K(x,y)\,dy$$

$$+ 2^{p-3}k^{p-1} \operatorname{PV} \int_{y\in B_{\frac{1}{4}}} |\beta(x) - \beta(-y+x)|^{p-2}(\beta(x)$$

$$- \beta(-y+x))K(x,y)\,dy.$$

Introducing the notation

$$F = -(\beta(x) - \beta(x-y)), \quad G = (\beta(x) - \beta(x-y)) + (\beta(x) - \beta(x+y)),$$

I_2 can be written as

$$2^{p-3}k^{p-1} \int_{y\in B_{\frac{1}{4}}} \left(|F+G|^{p-2}(F+G) - |F|^{p-2}F \right) K(x,y)\,dy$$

$$\leq 2^{p-3}k^{p-1}(p-1) \int_{y\in B_{\frac{1}{4}}} |G|(|F| + |G|)^{p-2}K(x,y)\,dy,$$

by Lemma 2. Since β is uniformly C^2, $|F| + |G| \leq C|y|$ and $|G| \leq C|y|^2$, for some fixed constant $C > 0$. Invoking the upper bound on K in B_2 yields the estimate

$$I_2 \le C^{p-1} 2^{p-3} k^{p-1} (p-1) \Lambda \int_{y \in B_{\frac{1}{4}}} |y|^{p-n-sp} \, dy$$

$$\le \frac{C^{p-1} 2^{p-3} k^{p-1} (p-1) \Lambda \left(\frac{1}{4}\right)^{p(1-s)}}{p(1-s)}. \tag{3.4}$$

Clearly the left hand side of (3.4) goes to zero as $k \to 0$.

For the rest of the terms in the left hand side we observe first that if $\eta < \gamma/(p-1)$ then from the upper bound on K outside $B_{1/4}$

$$\int_{\mathbb{R}^n \setminus B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} K(x,y) \, dy \le M \int_{\mathbb{R}^n \setminus B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} \frac{dy}{|y|^{n+\gamma}}, \tag{3.5}$$

which is uniformly bounded and tends to zero as $\eta \to 0$, by the dominated convergence theorem.

In addition, since $|\beta| \le 1$ we have

$$\int_{\mathbb{R}^n \setminus B_{\frac{1}{4}}} |k\beta(x)|^{p-1} K(x,y) \, dy \le k^{p-1} M \int_{\mathbb{R}^n \setminus B_{\frac{1}{4}}} \frac{dy}{|y|^{n+\gamma}}, \tag{3.6}$$

which is finite and converges to zero as $k \to 0$, where we again have used the upper bound on K outside $B_{1/4}$.

Thus, if we choose η and k small enough (depending on Λ, M, p_0, p_1, s_0, s_1 and γ) we can make all the terms in the left hand side as small as desired.

Now we turn our attention to the right hand side. We have, due to the lower bound on K in B_2

$$2^{1-p} \inf_{A \subset B_2, |A| > \delta} \int_A K(x,y) \, dy \ge \frac{2^{1-p} \lambda \delta}{2^{n+sp}}.$$

Then it is clear that we can choose η and k, depending only on λ, Λ, M, p_0, p_1, s_0, s_1, γ and δ, so that the left hand side is larger than the right hand side.

Case 2: $1/(1-s) < p < 2$

The only difference from the case $p > 2$ is the first term in the left hand side. We need to show that for k small enough, the term

$$(3^{p-1} + 2^{p-1}) k^{p-1} \int_{\mathbb{R}^n} |\beta(x) - \beta(x+y)|^{p-1} K(x,y) \, dy,$$

is small. We split the integral into two parts, one in B_1 and one in $\mathbb{R}^n \setminus B_1$. We have $|\beta(x) - \beta(x+y)| \le C|y|$ for $y \in B_1$. Hence,

$$(3^{p-1} + 2^{p-1}) k^{p-1} \int_{B_1} |\beta(x) - \beta(x+y)|^{p-1} K(x,y) \, dy$$

$$\le \Lambda C^{p-1} (3^{p-1} + 2^{p-1}) k^{p-1} \int_{B_1} |y|^{p-1-n-sp} \, dy$$

$$\le \Lambda C^{p-1} (3^{p-1} + 2^{p-1}) k^{p-1} \frac{1}{p(1-s)-1}, \tag{3.7}$$

where we have used the upper bound on K in B_2. For the part outside B_1 we have

$$(3^{p-1} + 2^{p-1})k^{p-1} \int_{\mathbb{R}^n \setminus B_1} |\beta(x) - \beta(x+y)|^{p-1} K(x,y) \, dy$$

$$\leq M(3^{p-1} + 2^{p-1})k^{p-1} \int_{\mathbb{R}^n \setminus B_1} \frac{dy}{|y|^{n+\gamma}}$$

$$\leq M(3^{p-1} + 2^{p-1})k^{p-1}\gamma^{-1}, \tag{3.8}$$

from the upper bound on K outside $B_{1/4}$ and the fact that $|\beta| \leq 1$. By choosing k small (depending on Λ, M, p_0, p_1, s_0, s_1, γ and τ) we can make both of these terms as small as desired. Hence, the result follows as in the case $p \geq 2$. $\qquad\square$

The lemma below is the core of this paper. The proof is an adaptation of the proof of Lemma 4.1 in [33].

Lemma 1. *Assume K satisfies* (1.3) *and that L is as in* (1.2). *Suppose*

$$Lu \leq \varepsilon \quad in \ B_1,$$
$$u \leq 1 \quad in \ B_1,$$
$$u(x) \leq 2|2x|^\eta - 1 \quad in \ \mathbb{R}^n \setminus B_1,$$
$$|B_1 \cap \{u \leq 0\}| > \delta,$$

where η is as in Proposition 2 *and*

$$\varepsilon = \inf_{x \in B_{\frac{3}{4}}} \min(2, 2^{p(x)-1}) \int_{y \notin B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p(x)-1} K(x,y) \, dy.$$

Then $u \leq 1 - \theta$ in $B_{1/2}$, where $\theta = \theta(\lambda, \Lambda, M, p_0, p_1, s_0, s_1, \gamma, \tau, \delta) > 0$.

Proof. In order to simplify the notation we write $p = p(x)$ and $s = s(x)$ throughout the proof. We argue by contradiction. Let

$$\theta = k \left(\beta(1/2) - \beta(3/4) \right) = \frac{95k}{256},$$

where k is as in Proposition 2. If there is $x_0 \in B_{1/2}$ such that $u(x_0) > 1 - \theta$, then

$$u(x_0) + k\beta(1/2) > 1 + k\beta(3/4).$$

Moreover, for any $y \in B_1 \setminus B_{3/4}$ there holds

$$u(x_0) + k\beta(x_0) > u(x_0) + k\beta(1/2) > 1 + k\beta(3/4) \geq u(y) + k\beta(y).$$

Hence, the maximum of $u + k\beta$ in B_1 is attained inside $B_{3/4}$ and it is strictly larger than 1. Suppose that the maximum is attained at the point x.

The rest of the proof is devoted to estimating $L(u + k\beta)(x)$ from above and from below in order to obtain a contradiction with Proposition 2. At this point, we remark that $-k\beta + (u + k\beta)(x)$ touches u from above at x. Hence, by Proposition 1, $Lu(x) \leq \varepsilon$ in the pointwise sense.

We first estimate $L(u + k\beta)(x)$ from below. We split the integrals into two parts and write

$$L(u + k\beta)(x) = \text{PV} \int_{x+y \in B_1} + \int_{x+y \notin B_1}$$

$$= \lim_{r \to 0} \int_{x+y \in B_1, y \notin B_r} + \int_{x+y \notin B_1} = \lim_{r \to 0} I_r + I_2,$$

where there is no need for the principal value in the second integral, since $x \in B_{3/4}$. Using that $u(x) + k\beta(x) > 1$ is the maximum of $u + k\beta$ in B_1 we see that the integrand in I_r is non-negative and we have the estimate

$$I_r \geq \int_{A_0 \cap B_r^c} (1 - k\beta(x + y))^{p-1} K(x, y)\, dy,$$

where

$$A_0 = \{x + y \in B_1, \quad u(x + y) \leq 0\}.$$

Since $\beta \leq 1$ and $k \leq 1/2$ we conclude

$$\liminf_{r \to 0} I_r \geq \frac{1}{2^{p-1}} \inf_{A_0 \subset B_2, |A_0| > \delta} \int_{A_0} K(x, y)\, dy.$$

Now we estimate I_2 from below. Using that $u(x) + k\beta(x) > 1$ and $u(z) \leq 2|2z|^\eta - 1$ for $z \in \mathbb{R}^n \setminus B_1$ and $\beta = 0$ in $\mathbb{R}^n \setminus B_1$, we have

$$I_2 \geq \int_{x+y \notin B_1} 2^{p-1} \left|1 - |2(x+y)|^\eta\right|^{p-2} (1 - |2(x+y)|^\eta) K(x, y)\, dy$$

$$\geq 2^{p-1} \int_{y \notin B_{\frac{1}{4}}} \left|1 - \left|2\left(|y| + \frac{3}{4}\right)\right|^\eta\right|^{p-2} \left(1 - \left|2\left(|y| + \frac{3}{4}\right)\right|^\eta\right) K(x, y)\, dy$$

$$\geq -2^{p-1} \int_{y \notin B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} K(x, y)\, dy.$$

Adding the two estimates together we can summarize

$$L(u + k\beta)(x)$$

$$\geq \frac{1}{2^{p-1}} \inf_{A_0 \subset B_2, |A_0| > \delta} \int_{A_0} K(x, y)\, dy - 2^{p-1} \int_{y \notin B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} K(x, y)\, dy.$$

$$(3.9)$$

The next step is to estimate $L(u + k\beta)(x)$ from above. This part of the proof is split into two cases: $p \geq 2$ and $p < 2$.

Case 1: $p \geq 2$

Again we split the integral defining $L(u + k\beta)(x)$ into two parts

$$L(u + k\beta)(x) = \text{PV} \int_{x+y \in B_1} + \int_{x+y \notin B_1} := I_1 + I_2,$$

where again, there is no need for the principal value in the second integral. We first treat I_1 by noting that when $x + y \in B_1$, we know

$$u(x) + k\beta(x) - u(x + y) - k\beta(x + y) \geq 0,$$

recalling that $u + k\beta$ attains its maximum (in B_1) at x.

From Lemma 4

$$|u(x) - u(x + y) + k\beta(x) - k\beta(x + y)|^{p-2}(u(x) - u(x + y) + k\beta(x)$$
$$- k\beta(x + y)) \le 2^{p-2}|u(x) - u(x + y)|^{p-2}(u(x) - u(x + y))$$
$$+ 2^{p-2}|k\beta(x) - k\beta(x + y)|^{p-2}(k\beta(x) - k\beta(x + y)).$$

Hence,

$$I_1 \le 2^{p-2}\,\mathrm{PV} \int_{x+y\in B_1} |u(x) - u(x + y)|^{p-2}(u(x) - u(x + y))K(x, y)\,dy$$

$$+ 2^{p-2}k^{p-1}\,\mathrm{PV} \int_{x+y\in B_1} |\beta(x) - \beta(x + y)|^{p-2}(\beta(x) - \beta(x + y))K(x, y)\,dy.$$

Now we turn our attention to I_2. We note that when $x + y \notin B_1$, we cannot apply Lemma 4 directly, but we still have from the hypothesis

$$u(x) + k\beta(x) > 1, \quad u(x + y) + k\beta(x + y) \le 2|2(x + y)|^\eta - 1.$$

In other words,

$$u(x) - u(x + y) + k\beta(x) - k\beta(x + y) > 2(1 - |2(x + y)|^\eta).$$

By adding the term $2(|2(x + y)|^\eta - 1) > 0$ to the expression, we increase the integrand, and we also make the integrand non-negative so that we can, once more, apply Lemma 4. It follows that

$$I_2 \le \int_{x+y\notin B_1} |u(x) - u(x + y) + k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1)|^{p-2}$$
$$\times (u(x) - u(x + y) + k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1))K(x, y)\,dy$$

$$\le 2^{p-2} \int_{x+y\notin B_1} |u(x) - u(x + y)|^{p-2}(u(x) - u(x + y))K(x, y)\,dy$$

$$+ 2^{p-2} \int_{x+y\notin B_1} |k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1)|^{p-2}$$
$$\times (k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1))K(x, y)\,dy.$$

Adding the estimates for I_1 and I_2 together we arrive at

$$L(u + k\beta)(x) \le 2^{p-2}Lu(x)$$

$$+ 2^{p-2}k^{p-1}\,\mathrm{PV} \int_{x+y\in B_1} |\beta(x) - \beta(x + y)|^{p-2}(\beta(x) - \beta(x + y))K(x, y)\,dy$$

$$+ 2^{p-2} \int_{x+y\notin B_1} |k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1)|^{p-2}$$
$$\times (k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1))K(x, y)\,dy$$

$$\le 2^{p-2}k^{p-1}\,\mathrm{PV} \int_{x+y\in B_1} |\beta(x) - \beta(x + y)|^{p-2}(\beta(x) - \beta(x + y))K(x, y)\,dy$$

$$+ 2^{p-2} \int_{x+y\notin B_1} |k\beta(x) - k\beta(x + y) + 2(|2(x + y)|^\eta - 1)|^{p-1}K(x, y)\,dy,$$

$$+ 2^{p-1} \int_{y \notin B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} K(x,y) \, dy \tag{3.10}$$

since $Lu(x) \le \varepsilon$.

Case 2: $\frac{1}{1-s} < p < 2$

From Lemma 3

$$\begin{aligned}
|u(x) &- u(x+y) + k\beta(x) - k\beta(x+y)|^{p-2}(u(x) - u(x+y) \\
&+ k\beta(x) - k\beta(x+y)) \\
&\le |u(x) - u(x+y)|^{p-2}(u(x) - u(x+y)) + (3^{p-1} + 2^{p-1})k^{p-1}|\beta(x) \\
&- \beta(x+y)|^{p-1}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
L(u + k\beta)(x) &\le Lu(x) + k^{p-1}(3^{p-1} + 2^{p-1}) \int_{\mathbb{R}^n} |\beta(x) \\
&- \beta(x+y)|^{p-1} K(x,y) \, dy \\
&\le k^{p-1}(3^{p-1} + 2^{p-1}) \int_{\mathbb{R}^n} |\beta(x) - \beta(x+y)|^{p-1} K(x,y) \, dy \\
&+ 2^{p-1} \int_{y \notin B_{\frac{1}{4}}} (|8y|^\eta - 1)^{p-1} K(x,y) \, dy, \tag{3.11}
\end{aligned}$$

again since $Lu(x) \le \varepsilon$.

Finally, we arrive at a contradiction by observing that (3.9) combined with either (3.10) or (3.11) results in a contradiction with (3.1) or (3.2) in Proposition 2. □

Once the lemma above is established, the proof of the Hölder regularity is standard. We follow the lines of the proof of Theorem 5.1 in [33].

Proof of Theorem 1 We first rescale u by the factor

$$\left(2\|u\|_{L^\infty(\mathbb{R}^n)} + 2^{\frac{p_1-1}{p_0-1}} \max \left\{ \left(\frac{\|f\|_{L^\infty(B_2)}}{\varepsilon} \right)^{\frac{1}{p_0-1}}, \left(\frac{\|f\|_{L^\infty(B_2)}}{\varepsilon} \right)^{\frac{1}{p_1-1}} \right\} \right)^{-1},$$

where ε is chosen as in Lemma 1. Then one readily verifies that

$$Lu = \tilde{f} \quad \text{in } B_2, \quad \|\tilde{f}\|_{L^\infty(B_2)} \le \frac{\varepsilon}{2^{p_1-1}}, \quad \mathrm{osc}_{\mathbb{R}^n} u \le 1.$$

We will now by induction find a_j and b_j such that

$$b_j \le u \le a_j \quad \text{in } B_{2^{-j}(x_0)}, \quad |a_j - b_j| \le 2^{-j\alpha}, \tag{3.12}$$

where we require from α that

$$\frac{2-\theta}{2} \le 2^{-\alpha}, \alpha \le \eta \quad \text{and} \quad \alpha \le \frac{s_0 p_0}{p_1 - 1},$$

where θ and η are from Lemma 1 with $\delta = |B_1|/2$. Clearly, (3.12) is satisfied for $j \le 0$ with the choice $b_j = \inf_{\mathbb{R}^n} u$ and $a_j = b_j + 1$. Now, given that (3.12)

holds for $j \leq k$ we construct a_{k+1} and b_{k+1}. Define

$$v(x) = 2^{\alpha k+1}(u(2^{-k}x + x_0) - m), \quad \text{with } m = \frac{a_k + b_k}{2}.$$

Then

$$\text{PV} \int_{\mathbb{R}^n} |v(x) - v(x+y)|^{p(x)-2}(v(x) - v(x+y))K_{x_0,2^{-k}}(x,y)\,dy$$
$$= 2^{(\alpha k+1)(p(2^{-k}x+x_0)-1)-k(s(2^{-k}x+x_0)p(2^{-k}x+x_0))}\tilde{f} \quad \text{in } B_1,$$

and

$$|v| \leq 1 \quad \text{in } B_1.$$

Note that

$$K_{x_0,2^{-k}}(x,y) = 2^{-k(n+s(2^{-k}x+x_0)p(2^{-k}x+x_0))}K(2^{-k}x + x_0, 2^{-k}y)$$

satisfies the same assumptions as K. From our choice of α it also follows that

$$\left| 2^{(\alpha k+1)(p(2^{-k}x+x_0)-1)-k(s(2^{-k}x+x_0)p(2^{-k}x+x_0))}\tilde{f} \right| \leq \varepsilon \quad \text{in } B_1.$$

We observe that for $|y| > 1$ such that $2^\ell \leq |y| \leq 2^{\ell+1}$ we have

$$v(y) = 2^{\alpha k+1}(u(2^{-k}y + x_0) - m) \leq 2^{\alpha k+1}(a_{k-\ell-1} - m)$$
$$\leq 2^{\alpha k+1}(a_{k-\ell-1} - b_{k-\ell-1} + b_k - m)$$
$$\leq 2^{\alpha k+1}(2^{-\alpha(k-\ell-1)} - \frac{1}{2}2^{-k\alpha})$$
$$\leq 2^{1+\alpha(\ell+1)} - 1 \leq 2|2y|^\alpha - 1$$
$$\leq 2|2y|^\eta - 1,$$

where we have used that (3.12) holds for $j \leq k$. Supposing now that

$$|\{v \leq 0\} \cap B_1| \geq |B_1|/2$$

(if not we would apply the same procedure to $-v$), we see that v satisfies all the assumptions of Lemma 1 with $\delta = |B_1|/2$. Lemma 1 implies

$$v(x) \leq 1 - \theta \quad \text{in } B_{\frac{1}{2}}.$$

Scaling back to u this yields

$$u(x) \leq 2^{-1-\alpha k}(1-\theta) + m \leq 2^{-1-k\alpha}(1-\theta) + \frac{a_k + b_k}{2}$$
$$\leq b_k + 2^{-1-\alpha k}(1-\theta) + 2^{-1-\alpha k}$$
$$\leq b_k + 2^{-\alpha(k+1)}$$

by our choice of α. Thus the choice $b_{k+1} = b_k$ and $a_{k+1} = b_k + 2^{-\alpha(k+1)}$ settles (3.12) for the step $j = k+1$. Hence, we arrive at the estimate

$$\text{osc}_{B_r(x_0)}\, u \leq 2^\alpha r^\alpha, \quad r > 0.$$

Recalling our rescaling factor in the beginning and rescaling back to our original u yields

$\operatorname{osc}_{B_r(x_0)} u$

$$\leq 2^\alpha \left(2\|u\|_{L^\infty(\mathbb{R}^n)} + 2^{\frac{p_1-1}{p_0-1}} \max \left\{ \left(\frac{\|f\|_{L^\infty(B_2)}}{\varepsilon} \right)^{\frac{1}{p_0-1}}, \left(\frac{\|f\|_{L^\infty(B_2)}}{\varepsilon} \right)^{\frac{1}{p_1-1}} \right\} \right) r^\alpha$$

$$\leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \max \left(\|f\|_{L^\infty(B_2)}^{\frac{1}{p_0-1}}, \|f\|_{L^\infty(B_2)}^{\frac{1}{p_1-1}} \right) \right) r^\alpha,$$

which is the desired result.

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4. Appendix

Here we give the proof of a couple of auxiliary inequalities. In what follows, $a, b \in \mathbb{R}$.

Lemma 2. *Let $p \geq 2$. Then*

$$\left| |a+b|^{p-2}(a+b) - |a|^{p-2}a \right| \leq (p-1)|b|(|a|+|b|)^{p-2}.$$

Proof. We have

$$\begin{aligned}
\left| |a+b|^{p-2}(a+b) - |a|^{p-2}a \right| &\leq \int_0^{|b|} \left| \frac{d}{ds}(|a+s|^{p-2}(a+s)) \right| ds \\
&= \int_0^{|b|} (p-1)|a+s|^{p-2} \, ds \\
&\leq (p-1)|b|(|a|+|b|)^{p-2}.
\end{aligned}$$

$\square$

Lemma 3. *Let $p \in (1,2)$. Then*

$$\left| |a+b|^{p-2}(a+b) - |a|^{p-2}a \right| \leq (3^{p-1} + 2^{p-1})|b|^{p-1}.$$

Proof. We split the proof into two cases.

Case 1: $|a| \leq 2|b|$. Then

$$\left| |a+b|^{p-2}(a+b) - |a|^{p-2}a \right| \leq |a+b|^{p-1} + |a|^{p-1} \leq (3^{p-1} + 2^{p-1})|b|^{p-1}.$$

Case 2: $|a| > 2|b|$**.** Then for $|s| \leq |b|$

$$|a + s| \geq |a| - |s| > 2|b| - |b| = |b|,$$

so that

$$\left||a + b|^{p-2}(a + b) - |a|^{p-2}a\right| \leq \int_0^{|b|} (p - 1)|a + s|^{p-2}\, ds \leq (p - 1)|b|^{p-1}.$$

Since $p - 1 \leq 3^{p-1} + 2^{p-1}$, this concludes the proof. $\qquad\square$

Lemma 4. *Let $p \geq 2$ and assume $a + b \geq 0$. Then*

$$|a + b|^{p-2}(a + b) \leq 2^{p-2}(|a|^{p-2}a + |b|^{p-2}b).$$

Proof. The inequality is trivial for $p = 2$ so we assume $p > 2$. Since $a + b \geq 0$, $|a|^{p-2}a + |b|^{p-2}b \geq 0$. Without loss of generality we can assume $a > 0$ and define $t = b/a$. The statement of the lemma is then equivalent to

$$|1 + t|^{p-2}(1 + t) \leq 2^{p-2}(1 + |t|^{p-2}t), \quad \text{for } t \geq -1.$$

This is trivially true for $t = -1$. Hence we are lead to study the function

$$f(t) := \frac{|1 + t|^{p-2}(1 + t)}{1 + |t|^{p-2}t}, \quad \text{for } t > -1.$$

We find that f has critical points at $t = 1$ and $t = 0$. In addition,

$$f(1) = 2^{p-2}, \lim_{t \searrow -1} f(t) = 0, f(0) = 1, \lim_{|t| \to \infty} f(t) = 1.$$

We conclude that $f(t) \leq 2^{p-2}$ for all $t \geq -1$, and the result follows. $\qquad\square$

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