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The generalized Korteweg-de Vries equation with time oscillating nonlinearity in scale critical Sobolev space

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Abstract. We consider the generalized Korteweg-de Vries (gKdV) equation with the time oscillating nonlinearity:

$$\partial_t u + \partial_x^3 u + g(\omega t)\partial_x(|u|^{p-1}u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Under the suitable assumption on g, we show that if the nonlinear term is mass critical or supercritical i.e., $p \geq 5$ and $u(0) \in \dot{H}^{s_p}$, where $s_p = 1/2 - 2/(p-1)$ is a scale critical exponent, then there exists a unique global solution to (gKdV) provided that $|\omega|$ is sufficiently large. We also obtain the behavior of the solution to (gKdV) as $|\omega| \to \infty$.

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Keyword. Generalized Korteweg-de Vries equation.

1. Introduction

We consider the generalized Korteweg-de Vries (gKdV) equation with the time oscillating nonlinearity:

$$\begin{cases} \partial_t u + \partial_x^3 u + g(\omega t)\partial_x(|u|^{p-1}u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $\phi : \mathbb{R} \to \mathbb{R}$ is a given function and $p > 1$. The function $g \in C^1(\mathbb{R}; \mathbb{R})$ is a given periodic function with period $L > 0$.

The class of equations (1.1) arises in several fields of physics. Equation (1.1) is a generalization of the notable Korteweg-de Vries equation which models long waves propagating in a channel [16] and the transitional KdV equation which describes the solitary waves propagating on the thermocline in a lake [14].

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For the case where g is constant, the local well-posedness of the initial value problem (1.1) in a scale subcritical Sobolev space $H^s(\mathbb{R})$, $s > s_p$ has been studied by many authors [2,9–13,20,24], where $s_p = 1/2 - 2/(p-1)$ is a scale critical exponent. A fundamental work on the local well-posedness is due to Kenig–Ponce–Vega [12]. They proved that (1.1) is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$ ($p = 2$), $s \geq 1/4$ ($p = 3$), $s \geq 1/12$ ($p = 4$) and $s \geq s_p$ ($p \geq 5$). Concerning the well-posedness of (1.1) in the scale critical $\dot{H}^{s_p}$ space, Kenig–Ponce–Vega [12] proved the local and (small data) global well-posedness when $p \geq 5$. Notice that Farah–Pastor [9] simplified Kenig–Ponce–Vega's proof. Later on, the above results were extended to a homogeneous Besov space $\dot{B}^{s_p}_{2,\infty}$ by Strunk [24] ($p \geq 5$).

Since the proof in [12] also works for the case where g is not constant, the local well-posedness for (1.1) with a non constant g follows from [12]. On the other hand, Nunes [22] showed the global existence for (1.1) with $p = 2$ in $H^1(\mathbb{R})$ under the assumption that $g \in C(\mathbb{R};\mathbb{R})$ and $g' \in L^1_{loc}(\mathbb{R})$. Notice that Nunes's proof is based on the almost conservation quantities for (1.1) and applicable for the arbitrary large data in H^1 for the defocusing case, i.e, g is strictly negative and for the focusing case, i.e, g is strictly positive with the mass sub-critical exponent $1 < p < 5$. Furthermore, in Martel–Merle [18], a family of solutions to (1.1) which blow up in finite time was constructed for the case where g is positive constant (focusing) and $p = 5$ (mass critical).

In this paper, we study the global existence and the behavior of solution u_ω to (1.1) as $|\omega| \to \infty$. We first review the known results on the nonlinear Schrödinger equation with the time oscillating nonlinearity:

$$\begin{cases} i\partial_t u + \Delta u + g(\omega t)(|u|^{p-1}u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $p > 1$, $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is an unknown function, $\phi : \mathbb{R}^n \to \mathbb{C}$ is a given function, and $g \in C^1(\mathbb{R};\mathbb{R})$ is a given periodic function with a period $L > 0$. Abdullaev–Caputo–Kraeukel–Malomed [1] and Konotop–Pacciani [15] investigated the effect of the time oscillation term $g(\omega t)$ in the global behavior of solution to (1.2) via the numerical methods. Cazenave–Scialom [4] proved that if $\phi \in H^1(\mathbb{R}^n)$ and $1 + 4/n \leq p < p^*$ with $p^* = \infty$ if $n = 1, 2$ and $p^* = 1 + 4/(n-2)$ if $n \geq 3$, then, the solution u_ω converges to the solution U to

$$\begin{cases} i\partial_t U + \Delta U + m(g)|U|^{p-1}U = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ U(0,x) = \phi(x), & x \in \mathbb{R}^n \end{cases} \tag{1.3}$$

as $|\omega| \to \infty$, where $m(g)$ is given by

$$m(g) = \frac{1}{L}\int_0^L g(s)ds. \tag{1.4}$$

Furthermore, Fang–Han [8] showed the similar result for the case where $\phi \in H^1(\mathbb{R}^n)$ and p is energy critical, i.e., $p = 1 + 4/(n-2)$ ($n \geq 3$). Damergi–Goubet [6] showed that the oscillations do not prevent the blow up for any $\omega \in \mathbb{R}$ for the case where $n = 2$, $p \geq 3$ and $g(\omega t) = \cos^2(\omega t)$.

Concerning the generalized KdV equation (1.1), Carvajal–Panthee–Scialom [3] have shown that if $\phi \in H^1(\mathbb{R})$ and p is mass critical, i.e., $p = 5$, then the solution u_ω to (1.1) converges to the solution U to

$$\begin{cases} \partial_t U + \partial_x^3 U + m(g)\partial_x(|U|^{p-1}U) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ U(0,x) = \phi(x), & x \in \mathbb{R} \end{cases} \tag{1.5}$$

as $|\omega| \to \infty$, where $m(g)$ is given by (1.4). Moreover, Panthee–Scialom [23] proved the similar result for the case where $\phi \in H^1(\mathbb{R})$ and p is mass super-critical, i.e., $p > 5$.

In the present paper, we shall show that if $\phi \in \dot{H}^{s_p}(\mathbb{R}), s_p = 1/2 - 2/(p-1)$, and p is mass critical or supercritical, i.e., $p \geq 5$, then the solution u_ω to (1.1) exists globally and converges to the solution U to (1.5) as $|\omega| \to \infty$. Since $H^1 \subset \dot{H}^{s_p}$, our result is an improvement of [3,23] for the case $p \geq 5$.

Before we state our main theorem, we introduce several notations. Throughout this paper, we denote $s_p := 1/2 - 2/(p-1)$. For $T \in (0,\infty]$, the norm $\|\cdot\|_{X_T}$ is defined by

$$\|f\|_{X_T} := \|f\|_{L_T^\infty \dot{H}^{s_p}} + \||\partial_x|^{s_p} f\|_{L_x^5 L_T^{10}} + \|f\|_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}}, \tag{1.6}$$

where $\|f\|_{L_T^p} = \|f\|_{L_{(0,T)}^p}$. The norm defined by (1.6) is used in Farah–Pastor [9]. Let $\{S(t)\}_{t\in\mathbb{R}}$ be a unitary group generated by $-\partial_x^3$:

$$(S(t)\phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi + it\xi^3} \hat{\phi}(\xi) d\xi.$$

Definition 1.1. (Solution) *Let $T \in (0,\infty]$. We say that u is a (mild) solution to (1.1) on $[0,T)$ in $\dot{H}^{s_p}$ if $u \in C([0,T); \dot{H}^{s_p}(\mathbb{R})) \cap X_T$ and satisfies*

$$u(t) = S(t)\phi - \int_0^t S(t-t')g(\omega t')\partial_x(|u|^{p-1}u)(t')dt'.$$

The solution to (1.5) is defined in a similar way.

For the solution u to (1.1), we define

$$T_{max} := \sup\{T \in (0,\infty] \mid \text{Solution } u \text{ to (1.1) can be extended to } [0,T)\}.$$

We say that u is a maximal solution to (1.1) if u is a solution to (1.1) on $[0, T_{max})$. Similarly, for the solution U to (1.5), we define

$$S_{max} := \sup\{T \in (0,\infty] \mid \text{Solution } U \text{ to (1.5) can be extended to } [0,T)\}.$$

We say that U is a maximal solution to (1.5) if U is a solution to (1.5) on $[0, S_{max})$.

The main result of this paper is as follows.

Theorem 1.2. *Assume $p \geq 5$ and $\phi \in \dot{H}^{s_p}(\mathbb{R})$. Let u_ω be a maximal solution to (1.1) and let U be a solution to (1.5) on maximal interval $[0, S_{max})$. Furthermore, we assume $S_{max} = \infty$ and*

$$\|U\|_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}} < \infty. \tag{1.7}$$

Then there exists $\omega_0 > 0$ *such that if* ω *satisfies* $|\omega| > \omega_0$*, then* u_ω *is global solution to* (1.1)*. Moreover, we have*

$$\|u_\omega - U\|_{X_\infty} \longrightarrow 0 \qquad as\ |\omega| \to \infty. \tag{1.8}$$

The proof of Theorem 1.2 is essentially based on the arguments due to Cazenave–Scialom [4] for the nonlinear Schrödinger equation (1.2) and Carvajal–Panthee–Scialom [3] and Panthee–Scialom [23] for the generalized KdV equation (1.1). Their proofs are based on the combination of the Strichartz estimate and the Gronwall type lemma. It is not likely that the Gronwall type lemma used in [4] works for the scaling critical space. Instead of the Gronwall type lemma, we divide the time interval into subintervals to obtain various estimates need to prove Theorem 1.2. Although this kind of technique is used for the energy critical nonlinear Schrödinger equation (see [8] for example), we meet some technical difficulties due to the presence of $L^p_x L^q_T$ type norms. We overcome this difficulty by modifying the argument used in Killip–Kwon–Shao–Visan [17, Theorem 3.1].

We give two examples of g and ϕ satisfying the assumptions of Theorem 1.2.

Example. Let $p \geq 5$ and $m(g) = 0$. In this case, Eq. (1.5) is "linear" (i.e., Airy equation). By the space-time estimates for the solution to the Airy equation (see Proposition 2.5 below), we see that for any $\phi \in \dot{H}^{s_p}(\mathbb{R})$, the solution U to (1.5) exists globally in time and satisfies the assumption (1.7) in Theorem 1.2.

Example. Let $p = 5$ and $m(g) < 0$. In this case, Dodson [7] showed that for any $\phi \in L^2(\mathbb{R})(= H^{s_5}(\mathbb{R}))$ there uniquely exists a global solution U to (1.5) satisfying the assumption (1.7) in Theorem 1.2.

Combing Theorem 1.2 and the above examples, we obtain the following:

Corollary 1.3. *Let* $p \geq 5$*. Suppose that* g *satisfies* $m(g) = 0$*. Then there exists* $\omega_0 > 0$ *such that if* ω *satisfies* $|\omega| > \omega_0$*, then for any* $\phi \in \dot{H}^{s_p}(\mathbb{R})$*, there exists a unique global solution* $u_\omega \in C([0,\infty); \dot{H}^{s_p}(\mathbb{R})) \cap X_\infty$ *to* (1.1)*.*

Corollary 1.4. *Let* $p = 5$*. Suppose that* g *satisfies* $m(g) < 0$*. Then there exists* $\omega_0 > 0$ *such that if* ω *satisfies* $|\omega| > \omega_0$*, then for any* $\phi \in L^2(\mathbb{R})$*, there exists a unique global solution* $u_\omega \in C([0,\infty); L^2(\mathbb{R})) \cap X_\infty$ *to* (1.1)*.*

The plan of this paper is as follows. In Sect. 2, we prove the linear estimates for the Airy equation and the nonlinear estimates for the generalized KdV equation, which are needed to prove Theorem 1.2. Section 3 is devoted to proving the global existence for (1.1). In Sect. 4, we prove Theorem 1.2. Finally, in Sect. 5, we consider the subdivision of the time interval used in the proof of Theorem 1.2.

Throughout this paper we use the following notations and function spaces. For $1 \leq p \leq \infty$, we denote the Hölder conjugate exponent of p by p'. Let $|\partial_x|^s = (-\partial_x^2)^{s/2}$ be a Riesz potential of order $-s$: $|\partial_x|^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$. For $1 \leq p, q \leq \infty$ and $T \in (0, \infty]$, we define the space-time norms:

$$\|f\|_{L_T^q L_x^p} = \|\|f(t,\cdot)\|_{L_x^p(\mathbb{R})}\|_{L_t^q(0,T)},$$
$$\|f\|_{L_x^p L_T^q} = \|\|f(\cdot,x)\|_{L_t^q(0,T)}\|_{L_x^p(\mathbb{R})}.$$

We denote $\|f\|_{L_\infty^q L_x^p}$ and $\|f\|_{L_x^p L_\infty^q}$ by $\|f\|_{L_t^q L_x^p}$ and $\|f\|_{L_x^p L_t^q}$, respectively.

2. Preliminaries

In this section, we prove the space-time estimates for a solution to the Airy equation and the nonlinear estimates for the generalized KdV equation.

2.1. Linear estimates

We first give the definition of admissible triple for the Airy equation.

Definition 2.1. (*Admissible triple*) Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$. We say that (p, q, α) is an admissible triple if (p, q, α) satisfies

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2} \quad \text{and} \quad \alpha = -\frac{1}{p} + \frac{2}{q}. \tag{2.1}$$

Lemma 2.2. *Let (p_1, q_1, α_1) and (p_2, q_2, α_2) be admissible triples. Then for any $T \in (0, \infty]$, there exist positive constants C_1 and C_2 such that the inequalities*

$$\|S(t)\phi\|_{L_T^\infty L_x^2} + \||\partial_x|^{\alpha_1} S(t)\phi\|_{L_x^{p_1} L_T^{q_1}} \le C_1 \|\phi\|_{L_x^2} \tag{2.2}$$

and

$$\left\| \int_0^t S(t-t')f(\cdot,t)dt' \right\|_{L_T^\infty L_x^2} + \left\| |\partial_x|^{\alpha_1} \int_0^t S(t-t')f(\cdot,t)dt' \right\|_{L_x^{p_1} L_T^{q_1}} \le C_2 \||\partial_x|^{-\alpha_2} f\|_{L_x^{p_2'} L_T^{q_2'}} \tag{2.3}$$

hold for any $\phi \in L^2$ and f satisfying $|\partial_x|^{-\alpha_2} f \in L_x^{p_2'} L_T^{q_2'}$, where C_1 depends only on p_1, q_1 and T, and C_2 depends only on p_1, p_2, q_1, q_2 and T.

Proof. The homogeneous estimate (2.2) follows from the combination of the Stein analytic interpolation, the Kato's smoothing effect [12, Theorem 3.5 (i)] and the Kenig–Ruiz estimate [12, Theorem 3.7 (i)]. The inhomogeneous estimates (2.3) is obtained by combining the homogeneous estimates (2.2) and the Christ–Kiselev's lemma [21, Lemma 2]. Since the proof is now standard, we omit the detail. □

Lemma 2.3. (Embedding) *Let $p \ge 5$ and let α_p, $\beta_p, \tilde{p}, \tilde{q}$ be given by*

$$\alpha_p = \frac{1}{10} - \frac{2}{5(p-1)}, \quad \beta_p = \frac{3}{10} - \frac{6}{5(p-1)},$$

$$\frac{1}{\tilde{p}} = \frac{2}{5(p-1)} + \frac{1}{10}, \quad \frac{1}{\tilde{q}} = \frac{3}{10} - \frac{4}{5(p-1)}.$$

Then for any $T \in (0, \infty]$, there exists a positive constant C such that the inequality

$$\|u\|_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} \le C \||\partial_x|^{\alpha_p} |\partial_t|^{\beta_p} u\|_{L_x^{\tilde{p}} L_T^{\tilde{q}}}. \tag{2.4}$$

holds for any u satisfying $|\partial_x|^{\alpha_p}|\partial_t|^{\beta_p}u \in L_x^{\tilde{p}}L_T^{\tilde{q}}$, where the constant C depends only on p and T.

Proof. See [12, Lemma 3.15]. □

Lemma 2.4. *Assume $p \geq 5$. Let $\alpha_p, \beta_p, \tilde{p}, \tilde{q}$ be given in Lemma 2.3 and let (p_2, q_2, α_2) be admissible triple. Then for any $T \in (0, \infty]$, there exist positive constants C_1 and C_2 such that the inequalities*

$$\||\partial_x|^{\alpha_p}|\partial_t|^{\beta_p}S(t)\phi\|_{L_x^{\tilde{p}}L_T^{\tilde{q}}} \leq C_1\|\phi\|_{\dot{H}^{s_p}} \tag{2.5}$$

and

$$\||\partial_x|^{\alpha_p}|\partial_t|^{\beta_p}\int_0^t S(t-t')f(t')dt'\|_{L_x^{\tilde{p}}L_T^{\tilde{q}}} \leq C_2\||\partial_x|^{s_p-\alpha_2}f\|_{L_x^{p_2'}L_T^{q_2'}} \tag{2.6}$$

hold for any for any $\phi \in \dot{H}^{s_p}(\mathbb{R})$ and g satisfying $|\partial_x|^{s_p-\alpha_2}f \in L_x^{p_2'}L_T^{q_2'}$, where C_1 depends only on p and T, and C_2 depends only on p, p_2, q_2 and T.

Proof. See [12, Lemma 3.14]. □

Combining Lemmas 2.2 and 2.4, we obtain the following:

Proposition 2.5. *Let $T \in (0, \infty]$ and let X_T be given by (1.6). Then there exists a positive constants C_1 and C_2 such that the inequalities*

$$\|S(t)\phi\|_{X_T} \leq C_1\|\phi\|_{\dot{H}^{s_p}(\mathbb{R})},$$
$$\left\|\int_0^t S(t-t')\partial_x f(t')dt'\right\|_{X_T} \leq C_2\||\partial_x|^{s_p}f\|_{L_x^1L_T^2}$$

hold for any $\phi \in \dot{H}^{s_p}(\mathbb{R})$ and f satisfying $|\partial_x|^{s_p}f \in L_x^1L_T^2$, where C_1 and C_2 depend only on p and T.

2.2. Nonlinear estimates

Lemma 2.6. *Let $\alpha \in (0,1), \alpha_1, \alpha_2 \in [0, \alpha]$ satisfy $\alpha = \alpha_1 + \alpha_2$ and let $p_1, p_2, q_1, q_2 \in (1, \infty)$ satisfy $1/p_1 + 1/p_2 = 1$ and $1/q_1 + 1/q_2 = 1/2$. Then for all $T \in (0, \infty]$, there exists a positive constant C such that the inequality*

$$\||\partial_x|^{\alpha}(fg) - f|\partial_x|^{\alpha}g - g|\partial_x|^{\alpha}f\|_{L_x^1L_T^2} \leq C\||\partial_x|^{\alpha_1}f\|_{L_x^{p_1}L_T^{q_1}}\||\partial_x|^{\alpha_2}g\|_{L_x^{p_2}L_T^{q_2}}$$

holds for any f and g satisfying $|\partial_x|^{\alpha_1}f \in L_x^{p_1}L_T^{q_1}$ and $|\partial_x|^{\alpha_2}g \in L_x^{p_2}L_T^{q_2}$, where C depends only on $\alpha_1, \alpha_2, p_1, p_2, q_1, q_2$ and T.

Proof. See [12, Theorem A.13]. □

Lemma 2.7. *Assume $\alpha \in (0,1)$. Let $p, q, p_1, p_2, q_2 \in (1, \infty)$ and $q_1 \in (1, \infty]$ satisfy $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. We also assume $F \in C^1(\mathbb{R}; \mathbb{R})$. Then for any $T \in (0, \infty]$, there exists a positive constant C such that the inequality*

$$\||\partial_x|^{\alpha}F(f)\|_{L_x^pL_T^q} \leq C\|F'(f)\|_{L_x^{p_1}L_T^{q_1}}\||\partial_x|^{\alpha}f\|_{L_x^{p_2}L_T^{q_2}} \tag{2.7}$$

holds for any f satisfying $F'(f) \in L_x^{p_1}L_T^{q_1}$ and $|\partial_x|^{\alpha}f \in L_x^{p_2}L_T^{q_2}$, where C depends only on $\alpha, p_1, p_2, q_1, q_2$ and T.

Proof. See [5, Proposition 3.1] and [12, Theorem A.6]. Notice that the alternative proof of the inequality (2.7) can be found in [19, Lemma 3.7]. □

Proposition 2.8. *Let $\alpha \in [0,1)$ and $p > 2$. Then for any $T \in (0,\infty]$, there exist positive constants C_1 and C_2 such that the inequalities*

$$\||\partial_x|^\alpha(|u|^{p-1}u)\|_{L^1_x L^2_T} \le C_1 \|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{4}(p-1)}} \||\partial_x|^\alpha u\|_{L^5_x L^{10}_T} \tag{2.8}$$

and

$$\begin{aligned} &\||\partial_x|^\alpha(|u|^{p-1}u - |v|^{p-1}v)\|_{L^1_x L^2_T} \\ &\le C_2 \Bigg\{ \left(\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} + \|v\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} \right) \||\partial_x|^\alpha(u-v)\|_{L^5_x L^{10}_T} \\ &\quad + \left(\|u\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} + \|v\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} \right) \\ &\quad \times \left(\||\partial_x|^\alpha u\|_{L^5_x L^{10}_T} + \||\partial_x|^\alpha v\|_{L^5_x L^{10}_T} \right) \|u-v\|_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} \Bigg\} \end{aligned} \tag{2.9}$$

hold, where the constants C_1 and C_2 depend on α, p and T.

Proof. The proof follows from an argument similar to that in [19, Lemma 3.4]. Hence we omit the detail. □

3. Proof of global existence

In this section, we prove propositions concerning the global existence and the convergence of solution to (1.1).

We consider

$$\begin{cases} \partial_t u + \partial_x^3 u + h(t)\partial_x(|u|^{p-1}u) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a given function.

Lemma 3.1. (Local existence) *Let $p \ge 5$. Then for any $\phi \in \dot{H}^{s_p}(\mathbb{R})$, there exist $T = T(\phi, h) > 0$ and a unique solution $u \in C([0,T); \dot{H}^{s_p}(\mathbb{R}))$ to (3.1) satisfying*

$$\|u\|_{X_T} \le C\|\phi\|_{\dot{H}^{s_p}(\mathbb{R})}, \tag{3.2}$$

where X_T is given by (1.6). Furthermore, let u be a maximal solution to (3.1) on $[0, T_{max})$. Then it holds one of the following:

(i) $T_{max} = \infty$,

(ii) $T_{max} < \infty$ *and* $\|u\|_{L_x^{\frac{5}{4}(p-1)} L_{T_{max}}^{\frac{5}{2}(p-1)}} = \infty$.

Proof. See Kenig–Ponce–Vega [12, Theorem 2.17]. □

Proposition 3.2. (Small data global existence I) *Assume $p \ge 5$ and $\phi \in \dot{H}^{s_p}(\mathbb{R})$. Then for any $A > 0$ satisfying $\|h\|_{L^\infty} \le A$ there exists $\varepsilon = \varepsilon(A)$ such that*

(i) *There exists $B>0$ such that if ϕ satisfies $\|S(t)\phi\|_{L_x^{5(p-1)/4}L_t^{5(p-1)/2}} \le \varepsilon$, then there exists a unique global solution u to (3.1) such that*

$$\|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}},$$
$$\|u\|_{X_\infty} \le B\|\phi\|_{\dot{H}^{s_p}(\mathbb{R})},$$

where X_∞ is given by (1.6).

(ii) *If u is a global solution to (3.1) and satisfies $\|u\|_{L_x^{5(p-1)/4}L_t^{5(p-1)/2}} \le 2\varepsilon$, then*

$$\|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}. \tag{3.3}$$

Proof. We first prove (i). By Lemma 3.1, there exists a unique solution $u \in C([0,T(\phi,h)); \dot{H}^{s_k}(\mathbb{R}))$ to (3.1). Let $u \in C([0,T_{max}); \dot{H}^{s_k}(\mathbb{R}))$ be a maximal solution to (3.1). Then u satisfies

$$\begin{aligned} u(t) &= S(t)\phi - \int_0^t S(t-t')h(t')\partial_x(|u|^{p-1}u)(t')dt' \\ &=: S(t)\phi + w(t). \end{aligned}$$

By Propositions 2.5 and 2.8, we obtain

$$\|w\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \le CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}}\||\partial_x|^{s_p}u\|_{L_x^5L_T^{10}}$$

for any $0<T<T_{\max}$. Since by Lemma 3.1 we have $\||\partial_x|^{s_p}u\|_{L_x^5L_T^{10}} < \infty$, we see

$$\|w\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \le CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}}. \tag{3.4}$$

Combining (3.4) and $\|S(t)\phi\|_{L_x^{5(p-1)/4}L_t^{5(p-1)/2}} \le \varepsilon$, we have

$$\|u\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \le \varepsilon + CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}}.$$

We now choose $\varepsilon=\varepsilon(A)$ so that $CA(2\varepsilon)^{p-2}<1/2$ and $\varepsilon<1/2$. Then by the continuity of the norm, we have

$$\|u\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} \le 2\varepsilon.$$

Therefore

$$\begin{aligned} \|u\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} &\le \|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} + CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} \\ &\le \|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} + CA(2\varepsilon)^{p-2}\|u\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{4}(p-1)}}. \end{aligned}$$

Since $CA(2\varepsilon)^{p-2}<1/2$, we obtain

$$\|u\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} \le 2\|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}}. \tag{3.5}$$

Propositions 2.5 and 2.8 imply

$$\begin{aligned}\|u\|_{X_T} &\le C\|\phi\|_{\dot H^{s_p}(\mathbb{R})} + CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}}\|u\|_{X_T}\\ &\le C\|\phi\|_{\dot H^{s_p}(\mathbb{R})} + CA(2\varepsilon)^{p-1}\|u\|_{X_T}\\ &\le C\|\phi\|_{\dot H^{s_p}(\mathbb{R})} + \frac{1}{2}\|u\|_{X_T}.\end{aligned}$$

Letting $T \to T_{\max}$, we obtain

$$\|u\|_{X_{T_{max}}} \le 2C\|\phi\|_{\dot H^{s_p}(\mathbb{R})}. \tag{3.6}$$

In particular, we have

$$\|u\|_{L_x^{\frac{5}{4}(p-1)}L_{T_{max}}^{\frac{5}{2}(p-1)}} \le 2C\|\phi\|_{\dot H^{s_p}(\mathbb{R})}.$$

Hence by the blow up criterion (Lemma 3.1), we see $T_{max} = \infty$. Combining this with (3.5) and (3.6), we obtain (i).

Next we prove (ii). Inequality (3.4) implies

$$\begin{aligned}\|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} &\le \|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} + \|w\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}\\ &\le \|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} + CA\|u\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}\\ &\le \|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}(1 + CA(2\varepsilon)^{p-2}).\end{aligned}$$

Since $CA(2\varepsilon)^{p-2} < 1$, we see

$$\|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\|u\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}.$$

Hence we have (ii). This completes the proof of Proposition 3.2. □

Proposition 3.3. (Small data global existence II) *Assume $p \ge 5$ and $\phi \in \dot H^{s_p}(\mathbb{R})$. Let $h \in L^\infty(\mathbb{R})$ satisfy $\|h\|_{L^\infty} \le A$. Let u be a solution to* (3.1) *on maximal interval $[0, T_{max})$. Suppose further that for $\varepsilon = \varepsilon(A) > 0$ and $B > 0$ given in Lemma* 3.2, *there exists $T \in (0, T_{max})$ such that if u satisfies $\|S(t)u(T)\|_{L_x^{5(p-1)/4}L_t^{5(p-1)/2}} \le \varepsilon$. Then $T_{max} = \infty$ and u satisfies*

$$\begin{aligned}\|u\|_{L_x^{\frac{5}{4}(p-1)}L_{(T,\infty)}^{\frac{5}{2}(p-1)}} &\le 2\varepsilon,\\ \|u\|_{X_{(T,\infty)}} &\le B\|u(T)\|_{\dot H^{s_p}(\mathbb{R})}.\end{aligned}$$

Proof. Let T be given in the statement of the proposition and let v be a solution to

$$\begin{cases}\partial_t v + \partial_x^3 v + h(t+T)\partial_x(|v|^{p-1}v) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R},\\ v(0,x) = u(T,x), & x \in \mathbb{R}.\end{cases}$$

Then v can be rewritten as the integral equation

$$v(t) = S(t)u(T) - \int_0^t S(t-t')h(t'+T)\partial_x(|v|^{p-1}v)(t')dt'. \tag{3.7}$$

Since

$$\|S(t)v(0)\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} = \|S(t)u(T)\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le \varepsilon, \tag{3.8}$$

Proposition 3.2 yields that v exists globally and satisfies

$$\|v\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\|S(t)u(T)\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\varepsilon, \tag{3.9}$$

$$\|v\|_{X_\infty} \le B\|u(T)\|_{\dot{H}^{s_p}(\mathbb{R})}. \tag{3.10}$$

Define $\tilde{u}$ by

$$\tilde{u}(t) := \begin{cases} u(t) & (0 \le t < T), \\ v(t-T) & (T \le t < \infty). \end{cases}$$

Then we see that $\tilde{u}$ satisfies the initial value problem (3.1) on $[0,\infty)$. By the uniqueness of (3.1) (Lemma 3.1), we see that $\tilde{u} = u$ and $T_{max} = \infty$. The inequalities (3.9) and (3.10) imply

$$\begin{aligned} \|u\|_{L_x^{\frac{5}{4}(p-1)}L_{(T,\infty)}^{\frac{5}{2}(p-1)}} &= \|v\|_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}} \le 2\varepsilon, \\ \|u\|_{X_{(T,\infty)}} &= \|v\|_{X_t} \le B\|u(T)\|_{\dot{H}^{s_p}(\mathbb{R})}. \end{aligned}$$

This completes the proof of Proposition 3.3. □

Lemma 3.4. *Let $p \ge 5$ and $T \in (0,\infty]$. Then for any f satisfying $|\partial_x|^{s_p} f \in L_x^1 L_T^2$, we have*

$$\int_0^t g(\omega t')S(t-t')\partial_x f(t')dt' \longrightarrow m(g)\int_0^t S(t-t')\partial_x f(t')dt' \quad in \quad X_T, \tag{3.11}$$

as $|\omega| \to \infty$.

Proof. Since the proof is now standard, we omit the detail (see [3, Lemma 3.1] for instance). □

Proposition 3.5. *Suppose $p \ge 5$ and $\phi \in \dot{H}^{s_p}(\mathbb{R})$. Let u_ω be a maximal solution to (1.1) and let U be a solution to (1.5) on maximal interval $[0, S_{max})$. Assume that for any T satisfying $0 < T < S_{max}$, u_ω exists on $[0,T)$ for $|\omega|$ large and*

$$\limsup_{|\omega|\to\infty} \left(\||\partial_x|^{s_p} u_\omega\|_{L_x^5 L_T^{10}} + \|u_\omega\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \right) < \infty \tag{3.12}$$

holds. Then we have

$$\|u_\omega - U\|_{X_T} \longrightarrow 0 \quad as \quad |\omega| \to \infty. \tag{3.13}$$

Proof. We first choose $A < \infty$ satisfying $\|g\|_{L^\infty} \le A$. Since u_ω and U satisfy (1.1) and (1.5),

$$\begin{aligned}
&(u_\omega - U)(t)\\
&= -\int_0^t g(\omega t')S(t-t')\partial_x(|u_\omega|^{p-1}u_\omega)dt'\\
&\quad + m(g)\int_0^t S(t-t')\partial_x(|U|^{p-1}U)dt'\\
&= -\int_0^t g(\omega t')S(t-t')\partial_x(|u_\omega|^{p-1}u_\omega - |U|^{p-1}U)dt'\\
&\quad - \int_0^t (g(\omega t') - m(g))\, S(t-t')\partial_x(|U|^{p-1}U)dt'\\
&=: I_1 + I_2. \qquad (3.14)
\end{aligned}$$

We show $\|I_2\|_{X_T} \to 0$. By Proposition 2.8 and Lemma 3.1,

$$\||\partial_x|^{s_p}(|U|^{p-1}U)\|_{L^1_x L^2_T} \le C\|U\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} \||\partial_x|^{s_p} U\|_{L^5_x L^{10}_T} < \infty. \qquad (3.15)$$

Hence by Lemma 3.4,

$$\|I_2\|_{X_T} =: C_\omega \longrightarrow 0 \quad as \quad |\omega| \to \infty. \qquad (3.16)$$

Next, we evaluate I_1. Propositions 2.5 and 2.8 imply

$$\begin{aligned}
&\|I_1\|_{X_T}\\
&\le CA\||\partial_x|^{s_p}(|u_\omega|^{p-1}u_\omega - |U|^{p-1}U)\|_{L^1_x L^2_T}\\
&\le CA\Bigg\{\left(\|u_\omega\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} + \|U\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}}\right) \||\partial_x|^{s_p}(u_\omega - U)\|_{L^5_x L^{10}_T}\\
&\qquad + \left(\|u_\omega\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}} + \|U\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}}\right)\\
&\qquad \times \left(\||\partial_x|^{s_p}u_\omega\|_{L^5_x L^{10}_T} + \||\partial_x|^{s_p}U\|_{L^5_x L^{10}_T}\right)\|u_\omega - U\|_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}}\Bigg\}.
\end{aligned}$$

Set

$$\begin{aligned}
M_T = &\||\partial_x|^{s_p}u_\omega\|_{L^5_x L^{10}_T} + \||\partial_x|^{s_p}U\|_{L^5_x L^{10}_T} + \|u_\omega\|_{L_x^{\frac{5}{4}(p-1)} L_T^{\frac{5}{2}(p-1)}}\\
&+ \|U\|_{L_x^{\frac{5}{4}(p-1)} L_T^{5(p-1)/2}}. \qquad (3.17)
\end{aligned}$$

Then we have

$$\|I_1\|_{X_T} \le CAM_T^{p-1}\|u_\omega - U\|_{X_T}. \qquad (3.18)$$

We split the time interval $[0, T]$ into subintervals $[t_i, t_{i+1}],\ i = 0, \ldots, J-1$ and $t_0 = 0, t_J = T$ so that for each intervals $[t_i, t_{i+1}]$,

$$CAM^{p-1}_{[t_i, t_{i+1}]} \le \tfrac{1}{2} \qquad (3.19)$$

hold, where C is the constant in (3.18), and

$$M_{[t_i,t_{i+1}]} = \||\partial_x|^{s_p} u_\omega\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}} + \||\partial_x|^{s_p} U\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}} + \|u_\omega\|_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}} + \|U\|_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}. \tag{3.20}$$

In Appendix, we show the existence of a subdivision satisfying (3.19). From (1.1) and (1.5),

$$\begin{aligned}(u_\omega - U)(t) &= S(t-t_i)(u_\omega(t_i) - U(t_i)) \\ &\quad - \int_{t_i}^t g(\omega t') S(t-t')\partial_x(|u_\omega|^{p-1}u_\omega - |U|^{p-1}U)dt' \\ &\quad - \int_{t_i}^t (g(\omega t') - m(g))\, S(t-t')\partial_x(|U|^{p-1}U)dt' \\ &=: S(t-t_i)(u_\omega(t_i) - U(t_i)) + I_{i,1} + I_{i,2}.\end{aligned} \tag{3.21}$$

For the first term on the right hand side of (3.21), we apply Proposition 2.5 to obtain

$$\|S(t-t_i)(u_\omega(t_i) - U(t_i))\|_{X_{[t_i,t_{i+1}]}} \le C\|u_\omega(t_i) - U(t_i)\|_{\dot H^{s_p}(\mathbb{R})}, \tag{3.22}$$

where $\|f\|_{X_{[t_i,t_{i+1}]}} = \|f\|_{L^\infty_{[t_i,t_{i+1}]}\dot H^{s_p}} + \||\partial_x|^{s_p} f\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}} + \|f\|_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}$. Propositions 2.5 and 2.8 imply

$$\begin{aligned}\|I_{i,1}\|_{X_{[t_i,t_{i+1}]}} &\le CA\Bigg\{\left(\|u_\omega\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}} + \|U\|^{p-1}_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}\right) \\ &\qquad \times \||\partial_x|^{s_p}(u_\omega - U)\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}} \\ &\qquad + \left(\|u_\omega\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}} + \|U\|^{p-2}_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}\right) \\ &\qquad \times \left(\||\partial_x|^{s_p} u_\omega\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}} + \||\partial_x|^{s_p} U\|_{L_x^5 L^{10}_{[t_i,t_{i+1}]}}\right) \\ &\qquad \times \|u_\omega - U\|_{L_x^{\frac{5}{4}(p-1)} L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}\Bigg\}. \\ &\le CAM^{p-1}_{[t_i,t_{i+1}]}\|u_\omega - U\|_{X_{[t_i,t_{i+1}]}} \\ &\le \frac{1}{2}\|u_\omega - U\|_{X_{[t_i,t_{i+1}]}}.\end{aligned} \tag{3.23}$$

Combining (3.16), (3.21), (3.22) and (3.23), we have

$$\|u_\omega - U\|_{X_{[t_i,t_{i+1}]}} \le C\|u_\omega(t_i) - U(t_i)\|_{\dot H^{s_p}(\mathbb{R})} + \frac{1}{2}\|u_\omega - U\|_{X_{[t_i,t_{i+1}]}} + C_\omega,$$

which implies

$$\|u_\omega - U\|_{X_{[t_i,t_{i+1}]}} \le 2C\|u_\omega(t_i) - U(t_i)\|_{\dot H^{s_p}(\mathbb{R})} + 2C_\omega. \tag{3.24}$$

Noting $u_\omega(0) - U(0) = 0$, we see from (3.24) with $i = 0$,

$$\|u_\omega - U\|_{X_{[0,t_1]}} \leq 2C_\omega.$$

In particular,

$$\|u_\omega(t_1) - U(t_1)\|_{\dot{H}^{s_p}(\mathbb{R})} \leq 2C_\omega. \tag{3.25}$$

By (3.24) with $i = 1$ and (3.25), we find

$$\|u_\omega - U\|_{X_{[t_1,t_2]}} \leq 2C_\omega + 4CC_\omega.$$

In particular, we obtain

$$\|u_\omega(t_2) - U(t_2)\|_{\dot{H}^{s_p}(\mathbb{R})} \leq 2C_\omega + 4CC_\omega.$$

Repeating this argument, we have

$$\|u_\omega - U\|_{X_T} \leq \sum_{i=0}^{J-1} \|u_\omega - U\|_{X_{[t_i,t_{i+1}]}} \leq 2\frac{(2C)^J - 1}{2C - 1} C_\omega.$$

By Appendix A, we see that $J \leq (4CA)^{5/2} M_T^{5(p-1)/2}$. Hence we have (3.13). □

Proposition 3.6. *Assume $p \geq 5$ and $\phi \in \dot{H}^{s_p}(\mathbb{R})$. Let u_ω be a maximal solution to (1.1) and let U be a solution to (1.5) on maximal interval $[0, S_{max})$. Then for any $T \in (0, S_{max})$, u_ω exists on $[0,T)$ for $|\omega|$ large. Furthermore,*

$$\|u_\omega - U\|_{X_T} \longrightarrow 0 \quad as \quad |\omega| \to \infty. \tag{3.26}$$

Proof. Let $A = \|g\|_{L^\infty}$ and fix $T \in (0, S_{max})$. We split the time interval $[0,T]$ into subintervals $[t_i, t_{i+1}]$, $i = 0, \ldots, J-1$ and $t_0 = 0$, $t_J = T$, where t_i are fixed later. For the interval $[t_i, t_{i+1}]$, we define $M^\omega_{[t_i,t_{i+1}]}$ by

$$M^\omega_{[t_i,t_{i+1}]} = \||\partial_x|^{s_p} u_\omega\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} + \|u_\omega\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_i,t_{i+1}]}^{\frac{5}{2}(p-1)}}. \tag{3.27}$$

By the Duhamel formula,

$$u_\omega(t) = S(t-t_i)u_\omega(t_i) - \int_{t_i}^t g(\omega t')S(t-t')\partial_x(|u_\omega|^{p-1}u_\omega)(t')dt' \tag{3.28}$$

for $t \in [t_i, t_{i+1}]$. Propositions 2.5 and 2.8 yield

$$\begin{aligned} M^\omega_{[t_i,t_{i+1}]} &\leq \|S(t-t_i)|\partial_x|^{s_p} u_\omega(t_i)\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} \\ &\quad + \|S(t-t_i)u_\omega(t_i)\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_i,t_{i+1}]}^{\frac{5}{2}(p-1)}} + CA(M^\omega_{[t_i,t_{i+1}]})^p \\ &\leq \|S(t-t_i)|\partial_x|^{s_p}(u_\omega(t_i) - U(t_i))\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} \\ &\quad + \|S(t-t_i)|\partial_x|^{s_p} U(t_i)\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} \\ &\quad + \|S(t-t_i)(u_\omega(t_i) - U(t_i))\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_i,t_{i+1}]}^{\frac{5}{2}(p-1)}} \\ &\quad + \|S(t-t_i)U(t_i)\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_i,t_{i+1}]}^{\frac{5}{2}(p-1)}} + CA(M^\omega_{[t_i,t_{i+1}]})^p. \end{aligned} \tag{3.29}$$

We now choose t_i so that for each i, the inequalities

$$M^\infty_{[t_i,t_{i+1}]} := \||\partial_x|^{s_p} U\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} + \|U\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_i,t_{i+1}]}^{\frac{5}{2}(p-1)}} < \eta \tag{3.30}$$

hold, where $\eta > 0$ is fixed later. The same argument as that in Appendix A yields that $J \le C\eta^{-5(p-1)/2}(M^\infty_{S_{\max}})^{5(p-1)/2}$, where $M^\infty_T := \||\partial_x|^{s_p}U\|_{L^5_x L^{10}_T} + \|U\|_{L_x^{5(p-1)/4}L_T^{5(p-1)/2}}$.

By an argument similar to the proof of (3.29),

$$\begin{aligned}&\|S(t-t_i)|\partial_x|^{s_p}U(t_i)\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} + \|S(t-t_i)U(t_i)\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}}\\ &\quad\le M^\infty_{[t_i,t_{i+1}]} + CA(M^\infty_{[t_i,t_{i+1}]})^p \le \eta + CA\eta^p.\end{aligned} \tag{3.31}$$

We choose $\varepsilon > 0$ so that $CA\eta^{p-1} < 1$. Then we see that the right hand side of (3.31) is bounded by 2η. Further, choosing η so small that $CA(6\eta)^{p-1} < 1/2$. Then we obtain

$$CA\left\{2\left(\|S(t-t_i)|\partial_x|^{s_p}U(t_i)\|_{L^5_x L^{10}_{[t_i,t_{i+1}]}} + \|S(t-t_i)U(t_i)\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[t_i,t_{i+1}]}} + \eta\right)\right\}^{p-1} < \frac{1}{2}. \tag{3.32}$$

Noting $u_\omega(t_0) - U(t_0) = 0$, we find

$$M^\omega_{[0,t_1]} \le \||\partial_x|^{s_p}S(t)\phi\|_{L^5_x L^{10}_{[0,t_1]}} + \|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[0,t_1]}} + CA(M^\omega_{[0,t_1]})^p. \tag{3.33}$$

By (3.32),

$$CA\left\{2\left(\||\partial_x|^{s_p}S(t)\phi\|_{L^5_x L^{10}_{[0,t_1]}} + \|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[0,t_1]}}\right)\right\}^{p-1} < \frac{1}{2}. \tag{3.34}$$

By (3.33), (3.34) and the continuity of the norm, we have

$$M^\omega_{[0,t_1]} \le 2\left(\||\partial_x|^{s_p}S(t)\phi\|_{L^5_x L^{10}_{[0,t_1]}} + \|S(t)\phi\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[0,t_1]}}\right) \le C\|\phi\|_{\dot H^{s_p}(\mathbb{R})}. \tag{3.35}$$

We show $T_{max} > t_1$ by contradiction argument. We assume that $T_{max} \le t_1$. Then by (3.35),

$$\|u_\omega\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{T_{max}}} \le M^\omega_{[0,t_1]} \le C\|\phi\|_{\dot H^{s_p}(\mathbb{R})}.$$

Hence by the blow up criterion for (1.1) (Lemma 3.1), we see $T_{max} = \infty$ which contradicts $T_{max} \le t_1$. Hence $T_{max} > t_1$. Furthermore, by Proposition 3.5, we obtain

$$\|u_\omega - U\|_{X_{[0,t_1]}} \longrightarrow 0 \quad as \quad |\omega| \to \infty. \tag{3.36}$$

Next we consider the case $i = 1$. By Proposition 2.5,

$$\begin{aligned}&\|S(t-t_1)|\partial_x|^{s_p}(u_\omega(t_1) - U(t_1))\|_{L^5_x L^{10}_{[t_1,t_2]}}\\ &\quad+\|S(t-t_1)(u_\omega(t_1) - U(t_1))\|_{L_x^{\frac{5}{4}(p-1)}L^{\frac{5}{2}(p-1)}_{[t_1,t_2]}}\\ &\le \|u_\omega(t_1) - U(t_1)\|_{\dot H^{s_p}(\mathbb{R})}\\ &\le \|u_\omega - U\|_{X_{[t_0,t_1]}}.\end{aligned} \tag{3.37}$$

By (3.29) with $i = 1$, (3.36) and (3.37), for $|\omega|$ sufficiently large,

$$M^{\omega}_{[t_1,t_2]} \leq \|S(t-t_1)|\partial_x|^{s_p}U(t_1)\|_{L^5_x L^{10}_{[t_1,t_2]}} + \|S(t-t_1)U(t_1)\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_1,t_2]}^{\frac{5}{2}(p-1)}} + CA(M^{\omega}_{[t_1,t_2]})^p + C_\omega,$$

where C_ω satisfies $C_\omega \to 0$ as $|\omega| \to \infty$. On the other hand by (3.32) with $i = 1$,

$$CA\left\{2\left(\|S(t-t_1)|\partial_x|^{s_p}U(t_1)\|_{L^5_x L^{10}_{[t_1,t_2]}} + \|S(t-t_1)U(t_1)\|_{L_x^{\frac{5}{4}(p-1)} L_{[t_1,t_2]}^{\frac{5}{2}(p-1)}} + C_\omega\right)\right\}^{p-1} < \frac{1}{2}$$

Hence by an argument similar to that in the case $i = 0$, we see that u_ω exists on $[t_0, t_2]$ for $|\omega|$ large, and

$$\|u_\omega - U\|_{X_{[t_1,t_2]}} \longrightarrow 0 \quad as \quad |\omega| \to \infty.$$

Since J is finite, we can repeat this argument J times. Hence we have that for $|\omega|$ sufficiently large, u_ω exists on $[0, T]$ and satisfies (3.26). This completes the proof of Proposition 3.6. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let $\varepsilon_0 = \varepsilon(A)$ be given in Proposition 3.2 and let $\varepsilon \in (0, \varepsilon_0)$. By the assumption $\|U\|_{L_x^{5(p-1)/4} L_t^{5(p-1)/2}} < \infty$, for $\varepsilon > 0$, we can choose $T > 0$ sufficiently large so that

$$\|U\|_{L_x^{\frac{5}{4}(p-1)} L_{(T,\infty)}^{\frac{5}{2}(p-1)}} \leq \frac{\varepsilon}{4} \tag{4.1}$$

holds. Let $\tilde{U}(t) := U(t + T)$. By (4.1),

$$\|\tilde{U}\|_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}} = \|U\|_{L_x^{\frac{5}{4}(p-1)} L_{(T,\infty)}^{\frac{5}{2}(p-1)}} \leq \frac{\varepsilon}{4}.$$

Hence by Proposition 3.2 (ii),

$$\begin{aligned} \|S(t)U(T)\|_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}} &= \|S(t)\tilde{U}(0)\|_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}} \\ &\leq 2\|\tilde{U}\|_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}} = 2\|U\|_{L_x^{\frac{5}{4}(p-1)} L_{(T,\infty)}^{\frac{5}{2}(p-1)}} \leq \frac{\varepsilon}{2}. \end{aligned} \tag{4.2}$$

Applying Proposition 3.3, we see

$$\|U\|_{X_{(T,\infty)}} \leq B\|U(T)\|_{\dot{H}^{s_p}(\mathbb{R})}. \tag{4.3}$$

Proposition 3.6 yields

$$\sup_{0<t\leq T} \|u_\omega(t) - U(t)\|_{\dot{H}^{s_p}(\mathbb{R})} \longrightarrow 0 \quad as \quad |\omega| \to \infty. \tag{4.4}$$

By Proposition 2.5, (4.2) and (4.4), we find that if $|\omega|$ is sufficiently large, then

$$\begin{aligned}
&\|S(t)u_\omega(T)\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \\
&\quad \le \|S(t)u_\omega(T)-S(t)U(T)\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} + \|S(t)U(T)\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \\
&\quad \le C\|u_\omega(T)-U(T)\|_{\dot H^{s_p}(\mathbb{R})} + \frac{\varepsilon}{2}. \\
&\quad \le \varepsilon.
\end{aligned}$$

Hence by Proposition 3.3, we see that if $|\omega|$ is sufficiently large, then u_ω exists globally and

$$\|u_\omega\|_{L_x^{\frac{5}{4}(p-1)}L_{(T,\infty)}^{\frac{5}{2}(p-1)}} \le 2\varepsilon, \tag{4.5}$$

$$\|u_\omega\|_{X_{(T,\infty)}} \le B\|u_\omega(T)\|_{\dot H^{s_p}(\mathbb{R})}. \tag{4.6}$$

Next we show (1.8). Set $M_0 := \sup_{0\le t\le T}\|U(t)\|_{\dot H^{s_p}(\mathbb{R})}$. By (4.4) and (4.6), we have for $\omega_0>0$ sufficiently large,

$$\begin{aligned}
&\sup_{|\omega|\ge\omega_0}\sup_{t\ge 0}\|u_\omega(t)\|_{\dot H^{s_p}(\mathbb{R})} \\
&\quad \le \sup_{|\omega|\ge\omega_0}\sup_{t\ge T}\|u_\omega(t)\|_{\dot H^{s_p}(\mathbb{R})} + \sup_{|\omega|\ge\omega_0}\sup_{0\le t\le T}\|u_\omega(t)-U(t)\|_{\dot H^{s_p}(\mathbb{R})} \\
&\qquad + \sup_{0\le t\le T}\|U(t)\|_{\dot H^{s_p}(\mathbb{R})} \\
&\quad \le B\sup_{|\omega|\ge\omega_0}\|u_\omega(T)\|_{\dot H^{s_p}(\mathbb{R})} + 1 + M_0 =: M_1.
\end{aligned} \tag{4.7}$$

By the Duhamel formula,

$$\begin{aligned}
u_\omega(T+t)-U(T+t) &= S(t)(u_\omega(T)-U(T)) \\
&\quad -\int_0^t S(t-T-t')g(\omega(T+t'))\partial_x(|u_\omega|^{p-1}u_\omega)(T+t')dt' \\
&\quad +m(g)\int_0^t S(t-T-t')\partial_x(|U|^{p-1}U)(T+t')dt' \\
&=: I_1+I_2+I_3.
\end{aligned} \tag{4.8}$$

We evaluate the X_∞ norm for I_i, $i=1,2,3$. For the term I_1, we apply Proposition 2.5 and (4.4) to conclude

$$\|I_1\|_{X_\infty} \le C\|u_\omega(T)-U(T)\|_{\dot H^{s_p}(\mathbb{R})} \longrightarrow 0 \quad as \quad |\omega|\to\infty. \tag{4.9}$$

By Propositions 2.5 and 2.8, and the inequality (4.6),

$$\begin{aligned}
\|I_2\|_{X_\infty} &\le CA\|u_\omega(T+\cdot)\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}\|u_\omega(T+\cdot)\|_{X_\infty} \\
&= CA\|u_\omega\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_{(T,\infty)}^{\frac{5}{2}(p-1)}}\|u_\omega\|_{X_{(T,\infty)}} \\
&\le CA(2\varepsilon)^{p-1}B\|u_\omega(T)\|_{\dot H^{s_p}(\mathbb{R})} \\
&\le CA(2\varepsilon)^{p-1}BM_1.
\end{aligned} \tag{4.10}$$

In a similar way, by Propositions 2.5 and 2.8 and the inequalities (4.1), (4.3) and (4.6),

$$\begin{aligned} \|I_3\|_{X_\infty} &\le CA\|U(T+\cdot)\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_t^{\frac{5}{2}(p-1)}}\|U(T+\cdot)\|_{X_\infty} \\ &= CA\|U\|^{p-1}_{L_x^{\frac{5}{4}(p-1)}L_{(T,\infty)}^{\frac{5}{2}(p-1)}}\|U\|_{X_{(T,\infty)}} \\ &\le CA\left(\frac{\varepsilon}{4}\right)^{p-1}B\|U(T)\|_{\dot{H}^{s_p}(\mathbb{R})} \\ &\le CA\left(\frac{\varepsilon}{4}\right)^{p-1}BM_0. \end{aligned} \tag{4.11}$$

Let $\delta > 0$ be an arbitrary number. Then by (4.9), for $|\omega|$ sufficiently large, we have

$$\|I_1\|_{X_\infty} < \frac{\delta}{2}. \tag{4.12}$$

Furthermore, we choose $\varepsilon > 0$ so that $CAB\varepsilon^{p-1}(M_0 + M_1) < \delta/2$. Then by (4.8), (4.10), (4.11) and (4.12),

$$\begin{aligned} \|u_\omega - U\|_{X_{(T,\infty)}} &= \|u_\omega(T+\cdot) - U(T+\cdot)\|_{X_\infty} \\ &\le \|I_1\|_{X_\infty} + \|I_2\|_{X_\infty} + \|I_3\|_{X_\infty} < \delta. \end{aligned} \tag{4.13}$$

On the other hand, by Proposition 3.6,

$$\|u_\omega - U\|_{X_T} \longrightarrow 0 \quad as \quad |\omega| \to \infty. \tag{4.14}$$

Collecting (4.13) and (4.14), we obtain (1.8). This completes the proof of Theorem 1.2.

5. Subdivision of time interval

In this section, we show that for any $T \in (0,\infty)$, there exists a positive integer J satisfying $J \le (4CA)^{5/2}M_T^{5(p-1)/2}$ and a sequence $0 = t_0 < t_1 < \cdots < t_J = T$ such that $[0,T] = \bigcup_{i=0}^{J-1} I_j,\ I_j = [t_i, t_{i+1}]$ and

$$\frac{1}{4} \le CAM^{p-1}_{[t_i,t_{i+1}]} \le \frac{1}{2} \quad \text{for any} \quad 0 \le i \le J-1,$$

where C is the constant in (3.19) and $M_{[t_i,t_{i+1}]}$ is defined by (3.20). We may assume $CAM^{p-1}_{[0,T]} > 1/2$ unless there is nothing to prove.

We first choose $t_1 \in [0,T]$ so that $t_0 < t_1$ and $CAM^{p-1}_{[t_0,t_1]} = 1/2$. Similarly, if $CAM^{p-1}_{[t_i,T]} > 1/2$, then we choose $t_{i+1} \in [0,T]$ so that $t_i < t_{i+1}$ and $CAM^{p-1}_{[t_i,t_{i+1}]} = 1/2$. We now show that $J \le (4CA)^{5/2}M_T^{5(p-1)/2}$ by the contradiction argument. Suppose $(4CA)^{5/2}M_T^{5(p-1)/2} < J \le \infty$. We choose an integer J' so that $J' = J$ if $J < \infty$ and $J' = (4CA)^{5/2}M_T^{5(p-1)/2} + 1$ if $J = \infty$.

For $0 \le i \le J'$, define

$$\begin{aligned} f_i^\omega(x) &:= \||\partial_x|^{s_p}u_\omega(\cdot,x)\|_{L_t^{10}[t_i,t_{i+1}]}, \quad f_i^\infty(x) := \||\partial_x|^{s_p}U(\cdot,x)\|_{L_t^{10}[t_i,t_{i+1}]}, \\ g_j^\omega(x) &:= \|u_\omega(\cdot,x)\|_{L_t^{\frac{5}{2}(p-1)}[t_i,t_{i+1}]}, \quad g_j^\infty(x) := \|U(\cdot,x)\|_{L_t^{\frac{5}{2}(p-1)}[t_i,t_{i+1}]}. \end{aligned}$$

Since

$$\|u_\omega\|_{L_x^{\frac{5}{4}(p-1)}L_T^{\frac{5}{2}(p-1)}} \geq \left\| \left(\|u_\omega(\cdot,x)\|^{\frac{5}{2}(p-1)}_{L_t^{\frac{5}{2}(p-1)}((0,t_{J'}))} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}} = \left\| \left(\sum_{i=0}^{J'-1} |g_i^\omega(x)|^{\frac{5}{2}(p-1)} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}},$$

we have

$$M_T \geq \left\| \left(\sum_{i=0}^{J'-1} |f_i^\omega(x)|^{10} \right)^{\frac{1}{10}} \right\|_{L_x^5} + \left\| \left(\sum_{i=0}^{J'-1} |g_i^\infty(x)|^{10} \right)^{\frac{1}{10}} \right\|_{L_x^5} + \left\| \left(\sum_{i=0}^{J'-1} |g_i^\omega(x)|^{\frac{5}{2}(p-1)} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}} + \left\| \left(\sum_{i=0}^{J'-1} |g_i^\infty(x)|^{\frac{5}{2}(p-1)} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}}. \tag{5.1}$$

By the Hölder inequality, we obtain

$$\left(\frac{1}{4CA}\right)^{\frac{1}{p-1}} J' \leq \sum_{i=0}^{J'-1} M_{[t_i,t_{i+1}]} \leq (J')^{\frac{4}{5}} \left(\sum_{i=0}^{J'-1} \||\partial_x|^{s_p} u^\omega\|^5_{L_x^5 L_t^{10}[t_i,t_{i+1}]} \right)^{\frac{1}{5}} + (J')^{\frac{4}{5}} \left(\sum_{i=0}^{J'-1} \||\partial_x|^{s_p} U\|^5_{L_x^5 L_t^{10}[t_i,t_{i+1}]} \right)^{\frac{1}{5}} + (J')^{\frac{5p-9}{5(p-1)}} \left(\sum_{i=0}^{J'-1} \|u^\omega\|^{\frac{5}{4}(p-1)}_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}[t_i,t_{i+1}]} \right)^{\frac{4}{5(p-1)}} + (J')^{\frac{5p-9}{5(p-1)}} \left(\sum_{i=0}^{J'-1} \|U\|^{\frac{5}{4}(p-1)}_{L_x^{\frac{5}{4}(p-1)} L_t^{\frac{5}{2}(p-1)}[t_i,t_{i+1}]} \right)^{\frac{4}{5(p-1)}}.$$

The right hand side of the above inequality can be rewritten as

$$(J')^{\frac{4}{5}} \left\| \left(\sum_{i=0}^{J'-1} |f_i^\omega(x)|^5 \right)^{\frac{1}{5}} \right\|_{L_x^5} + (J')^{\frac{4}{5}} \left\| \left(\sum_{i=0}^{J'-1} |f_i^\infty(x)|^5 \right)^{\frac{1}{5}} \right\|_{L_x^5} + (J')^{\frac{5p-9}{5(p-1)}} \left\| \left(\sum_{i=0}^{J'-1} |g_i^\omega(x)|^{\frac{5}{4}(p-1)} \right)^{\frac{4}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}} + (J')^{\frac{5p-9}{5(p-1)}} \left\| \left(\sum_{i=0}^{J'-1} |g_i^\infty(x)|^{\frac{5}{4}(p-1)} \right)^{\frac{4}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}}.$$

The Hölder inequality and (5.1) yield

$$\begin{aligned}\left(\frac{1}{4CA}\right)^{\frac{1}{p-1}} J' &\le (J')^{\frac{9}{10}} \left\| \left(\sum_{i=0}^{J'-1} |f_i^\omega(x)|^{10} \right)^{\frac{1}{10}} \right\|_{L_x^5} + (J')^{\frac{9}{10}} \left\| \left(\sum_{i=0}^{J'-1} |f_i^\infty(x)|^{10} \right)^{\frac{1}{10}} \right\|_{L_x^5} \\ &\quad + (J')^{\frac{5p-7}{5(p-1)}} \left\| \left(\sum_{i=0}^{J'-1} |g_i^\omega(x)|^{\frac{5}{2}(p-1)} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}} \\ &\quad + (J')^{\frac{5p-7}{5(p-1)}} \left\| \left(\sum_{i=0}^{J'-1} |g_i^\infty(x)|^{\frac{5}{2}(p-1)} \right)^{\frac{2}{5(p-1)}} \right\|_{L_x^{\frac{5}{4}(p-1)}} \\ &\le (J')^{\frac{5p-7}{5(p-1)}} M_T.\end{aligned}$$

Hence we obtain $J' \le (4CA)^{5/2} M_T^{5(p-1)/2}$. This contradicts the definition of J', which proves the claim.

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