Calderón–Zygmund estimates for a class of obstacle problems with nonstandard growth

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Abstract. We prove Calderón–Zygmund type estimates for obstacle problems with so-called $L^{p(\cdot)} \log L$ -growth. We also find suitable conditions on the variable exponent $p(\cdot)$ and the coefficients of the obstacle problems to obtain desired estimates.

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1. Introduction

The aim of this paper is to establish optimal higher integrability results for solutions to a class of obstacle problems with non-standard growth conditions. These have been the object of intensive investigation over the last years, after the pioneering papers of Marcellini [36, 37]; see for instance also [1–3, 7, 16, 17, 25, 26, 33, 34]. Specifically, we consider obstacle problems involving the following energy functional:

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} \Phi(x, |Dv|) dx, \quad \text{where} \quad \Phi(x, t) := |t|^{p(x)} \log(e + t). \quad (1.1)$$

Here, Ω is a open subset of $\mathbb{R}^n$ ($n \geq 2$) and $p(\cdot) : \Omega \rightarrow \mathbb{R}$ satisfies

$$1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty \quad (1.2)$$

for some γ_1 and γ_2 . We always assume that $p(\cdot)$ is at least uniformly continuous, hence there exists a modulus of continuity of $p(\cdot)$, say $\omega : [0, \infty) \rightarrow [0, \infty)$, such that $\omega(0) = 0$ and it is nondecreasing and concave and satisfies

$$|p(x) - p(y)| \leq \omega(|x - y|) \quad \text{for all } x, y \in \Omega.$$

For the function $\Phi : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ given in (1.1), consider the Musielak–Orlicz space $L^{\Phi}(\Omega)$ and its Sobolev space $W^{1,\Phi}(\Omega)$, see the next section for

the definitions of those spaces. Then we fix the function $\psi \in W^{1,\Phi}(\Omega)$, which is called the *obstacle*, and denote an admissible set by

$$\mathcal{K}_{\psi}^{\Phi}(\Omega) := \{f \in W^{1,\Phi}(\Omega) : f \geq \psi \text{ a.e. in } \Omega\}.$$

Furthermore, suppose that $F \in L^{\Phi}(\Omega)$ and $\mathbf{a} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$\begin{cases} |\mathbf{a}(x, \xi)| + |D_{\xi} \mathbf{a}(x, \xi)| |\xi| \leq \Lambda \Phi(x, |\xi|) |\xi|^{-1} = \Lambda |\xi|^{p(x)-1} \log(e + |\xi|), \\ D_{\xi} \mathbf{a}(x, \xi) \eta \cdot \eta \geq \nu \Phi(x, |\xi|) |\xi|^{-2} |\eta|^2 = \nu |\xi|^{p(x)-2} \log(e + |\xi|) |\eta|^2 \end{cases} \quad (1.3)$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^n$, and for some $0 < \nu \leq \Lambda$. Note that the prototype of $\mathbf{a}$ is

$$\mathbf{a}(x, \xi) = a(x) D_{\xi} [\Phi(x, |\xi|)], \quad \text{where } 0 < \nu \leq a(\cdot) \leq \Lambda. \quad (1.4)$$

In this setting, we establish a (local) nonlinear Calderón-Zygmund theory for $u \in \mathcal{K}_{\psi}^{\Phi}(\Omega)$ satisfying the variational inequality

$$\int_{\Omega} \mathbf{a}(x, Du) \cdot D(\varphi - u) \, dx \geq \int_{\Omega} |F|^{p(x)-2} \log(e + |F|) F \cdot D(\varphi - u) \, dx \quad (1.5)$$

for all $\varphi \in \mathcal{K}_{\psi}^{\Phi}(\Omega)$ with $\varphi - u$ has a compact support in Ω . Precisely, we prove the following implication:

$$\Phi(\cdot, |D\psi|), \Phi(\cdot, |F|) \in L_{loc}^q(\Omega) \implies \Phi(\cdot, |Du|) \in L_{loc}^q(\Omega), \quad q \geq 1. \quad (1.6)$$

under the suitable condition on the variable exponent $p(\cdot)$ and the nonlinearity $\mathbf{a}$, by obtaining corresponding estimates. Note that we will prove the existence and uniqueness of $u \in \mathcal{K}_{\psi}^{\Phi}(\Omega)$ satisfying the variational inequality (1.5) in Appendix, see Sect. 5.

In recent years, concerns about non-autonomous functionals have been rapidly increasing. Those are the functionals

$$\mathcal{F}(v) := \int_{\Omega} F(x, Dv) \, dx$$

which have ellipticity or growth properties strongly depending on the space variable x . Zhikov have introduced several examples of non-autonomous functionals in the context of Lavrentiev's Phenomenon and the homogenization, see [44]. The one of them is the $p(x)$ -Laplace equation for which there have been great research activities in the last two decades. Another example, whose phenomenology makes it strictly connected to the problems we are considering here, is the so-called *double phase problem* which is described by the functional $\mathcal{F}$ with $F(x, \xi) \approx |\xi|^p + a(x)|\xi|^q$ or, its borderline case, $\mathcal{F}(x, \xi) \approx |\xi|^p + a(x)|\xi|^p \log(e + |\xi|)$. Recently, significant regularity results have been obtained by Baroni, Colombo and Mingione in [6, 7, 16–18]. The functional $F(x, \xi) = \Phi(x, |\xi|)$ we consider in this paper is also an example of non-autonomous functionals. This functional was first considered by Giannetti and Passarelli di Napoli in [27], and then has been studied in [39–41]. The main point of this functional is that it has the properties of both functionals with variable growth and ones with so-called *Orlicz* growth. Note that if the variable exponent $p(\cdot)$ in Φ is constant, that is, $p(\cdot) \equiv p$, then $\Phi(x, t) = t^p \log(e + t)$ is the typical example of Orlicz functions. The non-autonomous functionals in fact belong to a class of so-called (p, q) -growth functionals, which has been first

introduced by Marcellini in [36, 37] and satisfies $|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q$. This has been also under spotlight in recent years, see [19, 33, 34]. We also point out that Harjulehto, Hästö and Klén considered a class of functionals which contains the functionals with $p(\cdot)$ -growth, the double phase problems and the functional Φ in (1.1), see [29].

The nonlinear Calderón–Zygmund theory is the study of the integrability of the gradient of solutions to equations with nonlinear growth, which was first established by Iwaniec for the elliptic p -Laplace equations [31]. Since then, DiBenedetto and Manfredi considered the elliptic p -Laplace systems [20], and Acerbi and Mingione considered the $p(x)$ -Laplace systems and the parabolic p -Laplace systems [3, 4]. For related results, we refer to [5, 8, 12, 13, 15, 22, 32] and references therein. Recently, the author of this paper obtained the global version of the implication (1.6) without the term $\Phi(\cdot, |D\psi|)$ for the following equation

$$\operatorname{div} D_\xi[\Phi(x, \xi)] = \operatorname{div} (|F|^{p(x)-2} \log(e + |F|) F) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.7)$$

under suitable conditions on the variable exponent $p(\cdot)$ and Ω , see [39]. On the other hand, the nonlinear Calderón–Zygmund theory for the obstacle problems was first established by Bögelein et al. [9]. They considered obstacle problems of parabolic p -Laplace type. We refer to, for instance, [10, 11, 23, 24, 42] for related results for the obstacle problems of p -Laplace or $p(x)$ -Laplace type.

We point out that the implication (1.6) is not holds for any nonlinearity $\mathbf{a}$ satisfying (1.3). In fact, it does not hold even for any $p(\cdot)$ -Laplace equation with a plain continuous function $p(\cdot)$, see [30], or any linear equation with a coefficient matrix which satisfies the uniformly ellipticity condition and the boundedness condition only, see [14]. Therefore, we need extra conditions for $p(\cdot)$ and $\mathbf{a}$. Let us introduce our main assumptions to obtain the implication (1.6).

Assumption Let $\delta > 0$ and $R \in (0, 1)$.

(AP) $p(\cdot)$ satisfies that

$$\sup_{0 < r \leq R} \omega(r) \log \left(\frac{1}{r} \right) \leq \delta. \quad (1.8)$$

(AA) $\mathbf{a}$ satisfies that

$$\sup_{0 < r \leq R} \sup_{y \in \Omega} \int_{B_r(y)} \mathcal{O}(\mathbf{a}, B_r(y))(x) \, dx \leq \delta, \quad (1.9)$$

where

$$\begin{aligned} & \mathcal{O}(x, B_r(y))(x) \\ &:= \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{\mathbf{a}(x, \xi)}{|\xi|^{p(x)-1} \log(e + |\xi|)} - \int_{B_r(y)} \frac{\mathbf{a}(x, \xi)}{|\xi|^{p(x)-1} \log(e + |\xi|)} \, dx \right| \leq 2\Lambda. \end{aligned} \quad (1.10)$$

There are several remarks for the above assumption. In the main result, Theorem 1.1, we will choose sufficiently small δ which depends on q in (1.6), while $R \in (0, 1)$ will be arbitrary given. If $p(\cdot)$ satisfies (1.8), we say $p(\cdot)$ is *log-Hölder continuous* in Ω . In the prototype case (1.4), the condition **(AA)** means that the coefficient function $a(\cdot)$ is a (locally) BMO function with $[a]_{BMO} \leq \delta$. Finally, those assumptions have been naturally arisen in the problems of $p(x)$ -Laplace type, see [10, 13].

Now we state our main result in this paper. Let

$$M := \int_{\Omega} [\Phi(x, |Du|) + \Phi(x, |F|) + \Phi(x, |D\psi|) + 1] dx + 1. \quad (1.11)$$

Theorem 1.1. *Let $\psi \in W^{1,\Phi}(\Omega)$, $F \in L^{\Phi}(\Omega, \mathbb{R}^n)$ and $q \geq 1$. There exists small $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, q) \in (0, 1)$, for which if $p(\cdot)$ and $\mathbf{a}$ satisfy (1.8) and (1.9) for some $R \in (0, 1)$ and $u \in \mathcal{K}_{\psi}^{\Phi}(\Omega)$ satisfies variational inequality (1.5), then we have the implication (1.6). Moreover, there exists $\eta_0 = \eta_0(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) \in (0, 1)$ such that for any $r \in (0, \eta_0 M^{-1})$ and $B_{2r} \subset \Omega$, we have*

$$\begin{aligned} \int_{B_r} \Phi(x, |Du|)^q dx &\leq c \left(\int_{B_{2r}} \Phi(x, |Du|) dx \right)^q \\ &\quad + c \int_{B_{2r}} [\Phi(x, |F|)^q + \Phi(x, |D\psi|)^q + 1] dx \end{aligned} \quad (1.12)$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, q) \geq 1$.

We would like mention the technique of the proof. As mentioned earlier, in [39] the author proved a similar result for the Eq. (1.7) by using the so-called maximal operator approach due to Caffarelli and Peral [15]. However, in this paper, we prove Theorem 1.1 in a different way introduced by Acerbi and Mingione [4]. Note that the key of this method is the so-called *stopping time argument* established in [38] first. Moreover, the nonlinearity $\mathbf{a}$ considering here not only has the same behavior to $D_{\xi}[\Phi(x, |\xi|)]$ but also contains the coefficient part, see the prototype (1.4), hence we have to consider the effect of coefficient in comparison estimates. At this point, we point out that the structure of $\mathbf{a}$ is more complicated than the prototype (1.4). Because, for the prototype (1.4) we can control coefficient $a(\cdot)$ and the variable exponent $p(\cdot)$ independently, while for the nonlinearity $\mathbf{a}$ we cannot distinguish the coefficient part and the growth part. This yields more delicate analyses when we derive comparison estimates.

This paper is now organized as follows. Section 1 contains the introduction with main theorem. In Sect. 2, we present notations, basic properties and auxiliary lemmas. In Sect. 3, we derive comparison estimates. In Sect. 4, we prove our main theorem, Theorem 1.1. Finally, in Sect. 5, we prove the existence and uniqueness of the solution to the obstacle problems

2. Preliminaries

2.1. Notations and properties

We start with introducing basic notations. For $y \in \mathbb{R}^n$ and $r > 0$, $B_r(y)$ is a ball in $\mathbb{R}^n$ centered y with radius r . We shall write $B_r = B_r(x)$ if the center is not important. For $f \in L^1_{loc}(\mathbb{R}^n)$ and a bounded open set U in $\mathbb{R}^n$, $(f)_U$ is denoted by the integral average of f in U such that $(f)_U = \int_U f dx = \frac{1}{|U|} \int_U f dx$.

We recall $\Phi : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ denoted in (1.1) and, for $1 < p < \infty$, define $\Phi : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ by $\Phi_p(t) := t^p \log(e + t)$. Note that $\Phi(x, t) = \Phi_{p(x)}(t)$. We further denote the conjugate of Φ_p , $\Phi_p^* : [0, \infty) \rightarrow [0, \infty)$, by

$$\Phi_p^*(s) := \sup_{t \geq 0} (st - \Phi_p(t)).$$

We then state elementary properties of Φ_p and Φ_p^* . we refer to [40] for their proofs.

Proposition 2.1. *Let $1 < p < \infty$, $t, s > 0$ and $\theta > 1$ and $0 < \delta < 1$.*

- (1) $\Phi_p(\theta t) \leq \theta^{p+1} \Phi_p(t)$ and $\Phi_p(\delta t) \leq \delta^p \Phi_p(t)$.
- (2) $\Phi_p^*(\theta s) \leq \theta^{\frac{p}{p-1}} \Phi_p^*(s)$ and $\Phi_p^*(\delta s) \leq \delta^{\frac{p+1}{p}} \Phi_p^*(s)$.
- (3) $\Phi_p(t + s) \leq \frac{1}{2}(\Phi_p(2t) + \Phi_p(2s)) \leq 2^p(\Phi_p(t) + \Phi_p(s))$.
- (4) (*Young's inequality*) For any $\kappa \in (0, 1]$, we have

$$st \leq \Phi_p(\kappa^{\frac{1}{p}} t) + \Phi_p^*(\kappa^{-\frac{1}{p}} s) \leq \kappa \Phi_p(t) + \kappa^{-\frac{1}{p-1}} \Phi_p^*(s)$$

and

$$st \leq \Phi_p^*(\kappa^{\frac{p-1}{p}} s) + \Phi_p(\kappa^{-\frac{p-1}{p}} t) \leq \kappa \Phi_p^*(s) + \kappa^{-\frac{p^2-1}{p}} \Phi_p(t). \quad (2.1)$$

- (5) There exists $c = c(p) > 0$ such that

$$\Phi_p^*(\Phi_p(|\xi|)|\xi|^{-1}) \leq c \Phi_p(|\xi|). \quad (2.2)$$

We remark that in the above proposition, if $1 < \gamma_1 < p < \gamma_2 < \infty$, then the constants c depending on p can be replaced by other constants depending on γ_1, γ_2 instead of p .

For Φ denoted in (1.1) and $N \geq 1$, we denote the Musielak–Orlicz space $L^\Phi(\Omega, \mathbb{R}^N)$ by the set of measurable functions $\mathbf{f} : \Omega \rightarrow \mathbb{R}^N$ satisfying $\int_\Omega \Phi(x, |\mathbf{f}|) dx < \infty$ with the norm

$$\|\mathbf{f}\|_{L^\Phi(\Omega, \mathbb{R}^N)} := \inf \left\{ t > 0 : \int_\Omega \Phi \left(x, \frac{|\mathbf{f}|}{t} \right) dx \leq 1 \right\}.$$

If $N = 1$, we simply write $L^\Phi(\Omega) = L^\Phi(\Omega, \mathbb{R}^1)$. In addition, we denote its Sobolev spaces by

$$W^{1,\Phi}(\Omega) := \{f \in W^{1,1}(\Omega) : f, |Df| \in L^\Phi(\Omega)\}.$$

with the norm $\|u\|_{W^{1,\Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \|Du\|_{L^\Phi(\Omega, \mathbb{R}^n)}$, and $W_0^{1,\Phi}(\Omega)$ by the closure of the set of all functions in $W^{1,\Phi}(\Omega)$ which have compact supports in Ω . We also denote $L^{\Phi^*}(\Omega)$, $L^{\Phi_p}(\Omega)$ and $W^{1,\Phi_p}(\Omega)$ in the same way above by replacing Φ with Φ^* or Φ_p . Note that a Hölder type inequality holds such that

$$\int_\Omega fg dx \leq 2\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\Phi^*}(\Omega)},$$

see [21, Lemma 2.6.5]. Moreover, if $\|f\|_{L^\Phi(\Omega)} \geq 1$, one can see that

$$\|f\|_{L^\Phi(\Omega)}^{\gamma_1} \leq \int_{\Omega} \Phi(x, |f|) dx. \quad (2.3)$$

Indeed, from [21, Lemma 2.1.14], we see that $\|f\|_{L^\Phi(\Omega)} \geq 1$ implies $\int_{\Omega} \Phi(x, |f|) dx \geq 1$ and that (2.3) is equivalent to

$$\int_{\Omega} \Phi \left(x, \left[\int_{\Omega} \Phi(x, |f|) dx \right]^{-\frac{1}{\gamma_1}} |f| \right) dx \leq 1,$$

which is proved by the second inequality in (1) of Proposition 2.1 such that

$$\begin{aligned} \int_{\Omega} \Phi \left(x, \left[\int_{\Omega} \Phi(x, |f|) dx \right]^{-\frac{1}{\gamma_1}} |f| \right) dx &\leq \int_{\Omega} \left[\int_{\Omega} \Phi(x, |f|) dx \right]^{-\frac{p(x)}{\gamma_1}} \Phi(x, |f|) dx \\ &\leq \int_{\Omega} \left[\int_{\Omega} \Phi(x, |f|) dx \right]^{-1} \Phi(x, |f|) dx \\ &= 1. \end{aligned}$$

Finally, we investigate the monotonicity and the coercivity of the nonlinearity $\mathbf{a}$. From the second condition in (1.3), we have the following monotonicity condition:

$$(\mathbf{a}(x, \xi_1) - \mathbf{a}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \tilde{\nu}(|\xi_1| + |\xi_2|)^{p(x)-2} \log(e + |\xi_1| + |\xi_2|) |\xi_1 - \xi_2|^2. \quad (2.4)$$

In particular, if $p(x) \geq 2$,

$$(\mathbf{a}(x, \xi_1) - \mathbf{a}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \tilde{\nu} \Phi(x, |\xi_1 - \xi_2|). \quad (2.5)$$

Moreover, letting $\xi_1 = \xi$ and $\xi_2 = 0$ in (2.4), we have

$$\mathbf{a}(x, \xi) \cdot \xi \geq \tilde{\nu} \Phi(x, |\xi|). \quad (2.6)$$

2.2. Auxiliary lemmas

We first state the local self improving property to the gradient of weak solutions to the following homogeneous equation

$$\operatorname{div} \mathbf{a}(x, Dw) = 0 \quad \text{in } B_{2r} \quad (2.7)$$

The proof is almost same to, rather simpler than, the one of [39, Theorem 3.7]. Therefore, we omit to prove the next lemma here.

Lemma 2.2. *Let $M_1 \geq 1$. Suppose that $\mathbf{a} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (1.3) and $p(\cdot) : \Omega \rightarrow \mathbb{R}$ does (1.2) and (1.8) for some $\delta > 0$ and $R \in (0, 1)$. Then there exist $\sigma_0 = \sigma_0(n, \nu, \Lambda, \gamma_1, \gamma_2, \delta) \in (0, 1)$ and $\eta_1 = \eta_1(n, \gamma_1, \gamma_2, \omega(\cdot)) \in (0, 1)$ such that for any $B_{2r} \subset \subset \Omega$ with $r \leq \frac{\eta_1 R}{M_1}$, if $w \in W^{1, \Phi}(B_{2r})$ is a weak solution to (2.7) with*

$$\int_{B_{2r}} [\Phi(x, |Dw|) + 1] dx + 1 \leq M_1,$$

then $\Phi(x, |Dw|) \in L^{1+\sigma_0}(B_r)$ with the following estimates: for any $0 < \sigma \leq \sigma_0$,

$$\int_{B_r} \Phi(x, |Dw|)^{1+\sigma} dx \leq c \left\{ \left(\int_{B_{2r}} \Phi(x, |Dw|) dx \right)^{1+\sigma} + 1 \right\}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \delta) \geq 1$. Moreover, if $\delta \in (0, 1)$, one can find σ_0 and c independent of δ .

Without loss of generality, we always assume that

$$\sigma_0 \leq 4(\gamma_1 - 1). \quad (2.8)$$

Remark 2.3. The previous lemma still holds true under milder conditions on the nonlinearity $\mathbf{a}$. Precisely, even if $\mathbf{a}$ only satisfies that

$$|\mathbf{a}(x, \xi)| \leq \Lambda |\xi|^{p(x)-1} \log(e + |\xi|), \quad (2.9)$$

instead of (1.3), and (2.6), we can obtain the same result in the previous lemma.

The next lemma gives the gradient bound of weak solutions to homogeneous equations related to the function Φ_p , which is the special case of the result in [35, Lemma 5.1].

Lemma 2.4. *Let $1 < p < \infty$. Suppose $v \in W^{1, \Phi_p}(B_{2r})$ be a weak solution to*

$$\operatorname{div} \bar{\mathbf{b}}(Dv) = 0 \quad \text{in } B_{2r},$$

where $\bar{\mathbf{b}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$\begin{cases} |\bar{\mathbf{b}}(\xi)| + |D_\xi \bar{\mathbf{b}}(\xi)| |\xi| \leq \Lambda \Phi_p(|\xi|) |\xi|^{-1} = \Lambda |\xi|^{p-1} \log(e + |\xi|), \\ D_\xi \bar{\mathbf{b}}(\xi) \eta \cdot \eta \geq \nu \Phi_p(|\xi|) |\xi|^{-2} |\eta|^2 = \nu |\xi|^{p-2} \log(e + |\xi|) |\eta|^2 \end{cases}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^n$. Then we have

$$\sup_{B_r} \Phi_p(|Dv|) \leq c \int_{B_{2r}} \Phi_p(|Dv|) dx$$

for some $c = c(n, \nu, \Lambda, p) > 0$.

We end this section introducing a technical lemma, see [28, Lemma 4.3].

Lemma 2.5. *Let ϕ be a bounded nonnegative function on $[r_1, r_2]$. Suppose that for any s_1, s_2 with $0 < r_1 \leq s_1 < s_2 \leq r_2$,*

$$\phi(s_1) \leq \eta \phi(s_2) + \frac{C_1}{(s_2 - s_1)^\theta} + C_2,$$

where $\theta, C_1, C_2 \geq 0$ and $\eta \in (0, 1)$. Then for any $r_1 \leq s_1 < s_2 \leq r_2$, it holds that

$$\phi(r_1) \leq c \left[\frac{C_1}{(s_2 - s_1)^\theta} + C_2 \right]$$

for some $c = c(\eta, \theta) > 0$.

3. Comparison estimates

In this section, we shall obtain two comparison estimates, which will be obtained by comparing between the variational inequality (1.5) and homogeneous equations related to the functions Φ and Φ_p , respectively, see (3.2) and (3.14). Note that, by Lemma 2.4, weak solutions to (3.14) have the bounded gradients.

Let us start with a comparison principle.

Lemma 3.1. *Let $\psi, w \in W^{1,\Phi}(U)$, $U \subset \Omega$, satisfy*

$$\begin{cases} -\operatorname{div} \mathbf{a}(x, D\psi) \leq -\operatorname{div} \mathbf{a}(x, Dw) & \text{in } U, \\ \psi \leq w & \text{on } \partial U, \end{cases}$$

in the weak sense, that is, $(\psi - w)^+ \in W_0^{1,\Phi}(\Omega)$ and

$$\int_U (\mathbf{a}(x, D\psi) - \mathbf{a}(x, Dw)) \cdot D\varphi \, dx \leq 0$$

for all $\varphi \in W_0^{1,\Phi}(U)$ with $\varphi \geq 0$. Then we have $\psi \leq w$ a.e. in U .

Proof. By taking $\varphi = (\psi - w)_+$ in the above inequality and using (2.4), we have

$$\int_{\{x \in U : \psi(x) > w(x)\}} (|D\psi| + |Dw|)^{p(x)} \log(e + |D\psi| + |Dw|) |D\psi - Dw|^2 \, dx \leq 0,$$

which implies that $\psi \leq w$ a.e. in U . $\square$

Let $u \in \mathcal{K}_\psi^\Phi(\Omega)$ satisfy the variational inequality to (1.5) and $r > 0$ be a sufficiently small number to be determined later, see (3.8) and (3.12). Fix $B_{4r} \Subset \Omega$. We then consider two comparison maps;

$$\begin{cases} -\operatorname{div} \mathbf{a}(x, Dk) = -\operatorname{div} \mathbf{a}(x, D\psi) & \text{in } B_{4r}, \\ k = u & \text{on } \partial B_{4r}, \end{cases} \quad (3.1)$$

$$\begin{cases} -\operatorname{div} \mathbf{a}(x, Dw) = 0 & \text{in } B_{4r}, \\ w = k & \text{on } \partial B_{4r}. \end{cases} \quad (3.2)$$

Note that the existence and the uniqueness of weak solutions to the above equations can be proved in a similar way in the proof of Theorem 5.1.

Lemma 3.2. *Let k and w in $W^{1,\Phi}(B_{4r})$ be the weak solutions to (3.1) and (3.2), respectively. Then we have*

$$\int_{B_{4r}} \Phi(x, |Dk|) \, dx \leq c \int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] \, dx, \quad (3.3)$$

$$\begin{aligned} \int_{B_{4r}} \Phi(x, |Dw|) \, dx &\leq c \int_{B_{4r}} \Phi(x, |Dk|) \, dx \\ &\leq c \int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] \, dx, \end{aligned} \quad (3.4)$$

for any $\kappa \in (0, 1)$,

$$\begin{aligned} \int_{B_{4r}} \Phi(x, |Du - Dw|) dx &\leq \kappa \int_{B_{4r}} \Phi(x, |Du|) dx \\ &\quad + c(\kappa) \int_{B_{4r}} [\Phi(x, |F|) + \Phi(x, D\psi)] dx, \end{aligned} \quad (3.5)$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$ and $c(\kappa) = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \kappa) \geq 1$.

Proof. By taking $k - u$ as a test function in the weak formulation of (3.1), and using (2.6), (1.3) and (2.1), we have

$$\begin{aligned} \int_{B_{4r}} \Phi(x, Dk) dx &\leq c \int_{B_{4r}} \mathbf{a}(x, Dk) \cdot Dk dx \\ &= c \int_{B_{4r}} [\mathbf{a}(x, Dk) \cdot Du + \mathbf{a}(x, D\psi) \cdot (Dk - Du)] dx \\ &\leq \frac{1}{2} \int_{B_{4r}} \Phi(x, Dk) dx + c \int_{B_{4r}} [\Phi(x, Du) + \Phi(x, D\psi)] dx, \end{aligned}$$

which implies (3.3). Similarly, by taking $w - k$ as a test function in the weak formulation of (3.2) and using (3.3), we obtain (3.4).

It remains to show (3.5). Since $\psi \leq k$ a.e. in B_{4r} by Lemma 3.1 and $u = k$ on ∂B_{4r} , extending k by u in $\Omega \setminus B_{4r}$, we have $k \in \mathcal{K}_\psi^\Phi(\Omega)$. Therefore by taking k and $u - k$ as test functions on (1.5) and (3.1), respectively, and using (2.4) and (2.1), we have

$$\begin{aligned} &\int_{B_{4r}} (|Du| + |Dk|)^{p(x)-2} \log(e + |Du| + |Dk|) |Du - Dk|^2 dx \\ &\leq \frac{1}{\tilde{\nu}} \int_{B_{4r}} (\mathbf{a}(x, Du) - \mathbf{a}(x, Dk)) \cdot (Du - Dk) dx \\ &\leq \frac{1}{\tilde{\nu}} \int_{B_{4r}} \left(|F|^{p(x)-2} \log(e + |F|) F + \mathbf{a}(x, D\psi) \right) \cdot (Du - Dk) dx \\ &\leq c \int_{B_{4r}} (\Phi(x, |F|) |F|^{-1} + \Phi(x, |D\psi|) |D\psi|^{-1}) \cdot (Du - Dk) dx \\ &\leq \frac{1}{2} \int_{B_{4r}} \Phi(x, |Du - Dk|) dx + c \int_{B_{4r}} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx. \end{aligned}$$

At this point, if $p(x) < 2$, we have from Young's inequality that for any $\kappa \in (0, 1)$,

$$\begin{aligned} \Phi(x, |Du - Dk|) &= (|Du| + |Dk|)^{\frac{(2-p(x))p(x)}{2}} \\ &\quad \times (|Du| + |Dk|)^{\frac{(p(x)-2)p(x)}{2}} |Du - Dk|^{p(x)} \log(e + |Du - Dk|) \\ &\leq \kappa 2^{\gamma_2} (1 + \log 2) (\Phi(x, |Du|) + \Phi(x, |Dk|)) \\ &\quad + c(\kappa) (|Du| + |Dk|)^{p(x)-2} \log(e + |Du| + |Dk|) |Du - Dk|^2 \end{aligned} \quad (3.6)$$

for some $c(\kappa) = c(n, \gamma_1, \gamma_2, \kappa) \geq 1$. Note by (2.5) that (3.6) also holds for $p(x) \geq 2$. Therefore, combining two previous inequalities and using (3.3), we have for any $\kappa \in (0, 1)$,

$$\begin{aligned} \int_{B_{4r}} \Phi(x, |Du - Dk|) dx &\leq \frac{\kappa}{2} \Phi(x, |Du|) dx + c(\kappa) \int_{B_{4r}} [\Phi(x, |F|) \\ &\quad + \Phi(x, |D\psi|)] dx. \end{aligned}$$

In a similar way above, by testing $k - w$ in (3.1) and (3.2), we have

$$\int_{B_{4r}} \Phi(x, |Dk - Dw|) dx \leq \frac{\kappa}{2} \int_{B_{4r}} \Phi(x, |Du|) dx + c(\kappa) \int_{B_{4r}} \Phi(x, |D\psi|) dx.$$

Therefore, we obtain (3.5). $\square$

Note that from (3.3) and (3.4),

$$\int_{B_{4r}} [\Phi(x, |Dw|) + 1] dx + 1 \leq c_0 M \quad (3.7)$$

for some $c_0 = c_0(n, \nu, \Lambda, \gamma_1, \gamma_2) \geq 1$, where M is denoted in (1.11).

Suppose $r > 0$ satisfies that

$$r \leq \frac{\eta_1 R}{2c_0 M} \quad \text{and} \quad \omega(4r) \leq \min \left\{ \frac{\sigma_0}{4}, 1 \right\}, \quad (3.8)$$

where $\sigma_0, \eta_1 \in (0, 1)$ are determined in Lemma 2.2, and set

$$p_2 := \sup_{B_{2r}} p(\cdot) \quad \text{and} \quad p_1 := \inf_{B_{2r}} p(\cdot).$$

Then, in view of Lemma 2.2 with (r, M_1) replaced by $(2r, c_0 M)$, we have $\Phi(x, |Dw|) \in L^{1+\sigma_0}(B_{2r})$ with the estimates

$$\int_{B_{2r}} \Phi(x, |Dw|)^{1+\sigma} dx \leq c \left(\int_{B_{4r}} \Phi(x, |Dw|) dx \right)^{1+\sigma} + c.$$

for all $\sigma \in (0, \sigma_0]$. In particular, since $p_2 - p_1 \leq \omega(4r) \leq \sigma_0/4$ by (3.8) and (2.8), using the previous inequality when $\sigma = \omega(4r)$ and (3.4) we have

$$\begin{aligned} \int_{B_{2r}} \Phi_{p_2}(|Dw|) dx &\leq c \left\{ \left(\int_{B_{4r}} \Phi(x, |Dw|) dx \right)^{1+\omega(4r)} + 1 \right\} \\ &\leq c \left(\int_{B_{4r}} \Phi(x, |Dw|) dx + 1 \right) \\ &\leq c \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx + 1 \right). \quad (3.9) \end{aligned}$$

In the last estimate above we have used the fact from (3.7), (3.8) and (1.8) that

$$\left(\int_{B_{4r}} \Phi(x, |Dw|) dx \right)^{\omega(4r)} \leq c(4r)^{-(n+1)\omega(4r)} \leq ce^{(n+1)\delta} \leq c.$$

Similarly, since $p_2(1 + \sigma_0/4) \leq p(x)(1 + \sigma_0/4) + \omega(4r)(1 + \sigma_0/4) \leq p(x)(1 + \sigma + \omega(4r))$, we have

$$\begin{aligned} & \int_{B_{2r}} \Phi_{p_2}(|Dw|)^{1+\sigma_0/4} dx \\ & \leq c \left\{ \left(\int_{B_{4r}} \Phi(x, |Dw|) dx \right)^{1+\sigma_0/4+\omega(4r)} + 1 \right\} \\ & \leq c \left\{ \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx \right)^{1+\sigma_0/4} + 1 \right\}. \end{aligned} \quad (3.10)$$

We will construct another comparison map. Consider $\mathbf{b} : B_{2r} \times \mathbb{R}^n$ denoted by

$$\mathbf{b}(x, \xi) := |\xi|^{p_2-p(x)} \mathbf{a}(x, \xi).$$

Then, we have

$$D_\xi \mathbf{b}(x, \xi) = (p_2 - p(x)) |\xi|^{p_2-p(x)-2} \xi \otimes \mathbf{a}(x, \xi) + |\xi|^{p_2-p(x)} D_\xi \mathbf{a}(x, \xi),$$

hence by (1.3) it follows that

$$\begin{cases} |\mathbf{b}(x, \xi)| + |D_\xi \mathbf{b}(x, \xi)| |\xi| \leq 3\Lambda \Phi_{p_2}(|\xi|) |\xi|^{-1} = 3\Lambda |\xi|^{p_2-1} \log(e + |\xi|), \\ D_\xi \mathbf{b}(x, \xi) \eta \cdot \eta \geq \frac{\nu}{2} \Phi_{p_2}(|\xi|) |\xi|^{-2} |\eta|^2 = \frac{\nu}{2} |\xi|^{p_2-2} \log(e + |\xi|) |\eta|^2 \end{cases} \quad (3.11)$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^n$, provided that

$$\omega(4r) \leq \min \left\{ 1, \frac{\nu}{2n^2} \right\}. \quad (3.12)$$

Further we denote $\bar{\mathbf{b}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\bar{\mathbf{b}}(\xi) := \int_{B_{2r}} \mathbf{b}(x, \xi) dx.$$

Then we see that (3.11) with $\mathbf{b}$ replaced by $\bar{\mathbf{b}}$ also holds and, by (1.10),

$$|\mathbf{b}(x, \xi) - \bar{\mathbf{b}}(\xi)| \leq \mathcal{O}(\mathbf{a}, B_{2r})(x) |\xi|^{p_2-1} \log(e + |\xi|) \quad (3.13)$$

for all $x \in B_{2r}$ and $\xi \in \mathbb{R}^n$. Now we consider the following equation:

$$\begin{cases} -\operatorname{div} \bar{\mathbf{b}}(Dv) = 0 \text{ in } B_{2r}, \\ v = w \text{ on } \partial B_{2r}. \end{cases} \quad (3.14)$$

Then, making use of the same method estimating (3.3) along with (3.9) and (3.4), we have

$$\begin{aligned} \int_{B_{2r}} \Phi_{p_2}(|Dv|) dx & \leq c \int_{B_{2r}} \Phi_{p_2}(|Dw|) dx \\ & \leq c \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx + 1 \right). \end{aligned} \quad (3.15)$$

In addition, by Lemma 2.4,

$$\begin{aligned}
\sup_{B_r} \Phi_{p_2}(|Dv|) &\leq c \int_{B_{2r}} \Phi_{p_2}(|Dv|) dx \\
&\leq c \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx + 1 \right) \quad (3.16)
\end{aligned}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$.

Finally, we derive the comparison estimate between w and v .

Lemma 3.3. *Let $w \in W^{1,\Phi}(B_{4r}) \cap W^{1,\Phi_{p_2}}(B_{2r})$ and $v \in W^{1,\Phi_{p_2}}(B_{4r})$ be weak solutions to (3.2) and (3.14), respectively, where r satisfies (3.8) and (3.12). Then we have for any $\kappa \in (0, 1)$,*

$$\begin{aligned}
&\int_{B_{2r}} \Phi_{p_2}(|Dw - Dv|) dx \\
&\leq \left(\kappa + c(\kappa) \delta^{\frac{\sigma_0/4}{1+\sigma_0/4}} \right) \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx + 1 \right) \quad (3.17)
\end{aligned}$$

for some $c(\kappa) \geq 1$ depending on $n, \nu, \Lambda, \gamma_1, \gamma_2, \kappa$.

Proof. By testing $w - v \in W_0^{1,\Phi_{p_2}}(B_{2r}) \subset W_0^{1,\Phi}(B_{2r})$, we have

$$\begin{aligned}
I_1 &:= \int_{B_{2r}} (\bar{\mathbf{b}}(Dw) - \bar{\mathbf{b}}(Dv)) \cdot (Dw - Dv) dx \\
&\leq \int_{B_{2r}} (\mathbf{b}(x, Dw) - \mathbf{a}(x, Dw)) \cdot (Dw - Dv) dx \\
&\quad + \int_{B_{2r}} (\bar{\mathbf{b}}(Dw) - \mathbf{b}(x, Dw)) \cdot (Dw - Dv) dx =: I_2 + I_3.
\end{aligned}$$

Using a similar technique estimating (3.6) along with (3.15), (3.9) and (3.4), we have for any $\kappa \in (0, 1)$,

$$\int_{B_{2r}} \Phi_{p_2}(|Dw - Dv|) dx \leq \frac{\kappa}{2} \left(\int_{B_{4r}} [\Phi(x, |Du|) + \Phi(x, |D\psi|)] dx \right) + c_1(\kappa) I_1 \quad (3.18)$$

for some $c_1(\kappa) = c_1(n, \nu, \Lambda, \gamma_1, \gamma_2, \kappa) \geq 1$. We next estimate I_2 . Note that applying the mean value theorem to the map $t \in [0, 1] \mapsto |\xi|^{t(p_2 - p(x))}$ we have

$$|\xi|^{p_2 - p(x)} - 1 = (p_2 - p(x)) |\xi|^{t_\xi(p_2 - p(x))} \log |\xi| \quad \text{for some } t_\xi \in (0, 1),$$

which implies

$$|\mathbf{b}(x, Dw) - \mathbf{a}(x, Dw)| \leq c\omega(4r) |Dw|^{t_{Dw}(p_2 - p(x)) + p(x) - 1} \log(e + |Dw|) \log |Dw|.$$

Since the previous quantity is bounded when $|Dw| \leq 1$, we obtain

$$|\mathbf{b}(x, Dw) - \mathbf{a}(x, Dw)| \leq c\omega(4r) (|Dw|^{p_2 - 1} \log^2(e + |Dw|) + 1),$$

and so

$$|I_2| \leq c\omega(4r) \int_{B_{2r}} (|Dw|^{p_2 - 1} \log^2(e + |Dw|) + 1) |Dw - Dv| dx.$$

At this stage, we follow the same computations in the proof of [39, Lemma 4.3], see p.25 - p.26 in there. Then we obtain

$$|I_2| \leq c\omega(4r) \log\left(\frac{1}{4r}\right) \left(\int_{B_{4r}} [\Phi_{p_2}(|Dw|) + \Phi_{p_2}(|D\psi|)] dx + 1 \right). \quad (3.19)$$

Finally, we estimate I_3 . By (3.13), (2.1), (2.2), (1.10) and Hölder's inequality, we have for any $\tilde{\kappa} \in (0, 1)$

$$\begin{aligned} |I_3| &\leq \int_{B_{2r}} \mathcal{O}(\mathbf{a}, B_{2r}) |Dw|^{p_2-1} \log(e + |Dw|) |Dw - Dv| dx \\ &\leq \int_{B_{2r}} \mathcal{O}(\mathbf{a}, B_{2r}) \left[c(\tilde{\kappa}) \Phi_{p_2}(|Dw|) + \frac{\tilde{\kappa}}{2\Lambda} \Phi_{p_2}(|Dw - Dv|) \right] dx \\ &\leq \tilde{\kappa} \int_{B_{2r}} \Phi_{p_2}(|Dw - Dv|) dx \\ &\quad + c(\tilde{\kappa}) \left(\int_{B_{2r}} \mathcal{O}(\mathbf{a}, B_{2r}) dx \right)^{\frac{\sigma_0/4}{1+\sigma_0/4}} \left(\int_{B_{2r}} \Phi_{p_2}(|Dw|)^{1+\sigma_0/4} dx \right)^{\frac{1}{1+\sigma_0/4}}. \end{aligned} \quad (3.20)$$

Consequently, Combining (3.18)–(3.20), choosing $\tilde{\kappa} = \frac{1}{2c_1(\kappa)}$ and using (3.9), (3.10), (3.15) and the definition of the (δ, R) -vanishing condition for $p(\cdot)$ and $\mathbf{a}$, we obtain (3.17). $\square$

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. In fact, it suffices to show the estimate (1.12). $R \in (0, 1)$ be an arbitrary given number, while $\delta > 0$ will be determined later. With this R, δ , we assume that $p(\cdot)$ and $\mathbf{a}$ satisfies (1.8) and (1.9), respectively.

Let $r > 0$ satisfy that

$$r \leq \frac{2\eta_1 R}{c_0 M} \quad \text{and} \quad \omega(r) \leq \min \left\{ \frac{\sigma_0}{4}, \frac{\nu}{2n^2}, 1 \right\}, \quad (4.1)$$

and fix $B_{2r} \Subset \Omega$. Define

$$\lambda_0 := \int_{B_{2r}} \left[\Phi(x, |Du|) + \frac{1}{\delta} \{ \Phi(x, |F|) + \Phi(x, |D\psi|) \} \right] dx + 1 \quad (4.2)$$

and, for $1 \leq s \leq 2$ and $\lambda > 0$, the super-level set of $\Phi(\cdot, |Du(\cdot)|)$ by

$$E(s, \lambda) := \{x \in B_{sr} : \Phi(x, |Du(x)|) > \lambda\}.$$

Lemma 4.1. *Fix any $1 \leq s_1 < s_2 \leq 2$. If*

$$\lambda > A_1 \lambda_0 \geq 1, \quad \text{where} \quad A_1 := \left(\frac{40}{s_2 - s_1} \right)^n. \quad (4.3)$$

Then there exist mutually disjoint balls $B_{r_i}(x_i)$, $i = 1, 2, 3, \dots$, where $x_i \in E(s_1, \lambda)$ and $r_i \in \left(0, \frac{(s_2 - s_1)r}{20}\right]$, such that for each i

$$\int_{B_{r_i}(x_i)} \left[\Phi(x, |Du|) + \frac{1}{\delta} \{ \Phi(x, |F|) + \Phi(x, |D\psi|) \} \right] dx = \lambda, \quad (4.4)$$

for any $\rho \in (r_i, (s_2 - s_1)r)$,

$$\int_{B_\rho(x_i)} \left[\Phi(x, |Du|) + \frac{1}{\delta} \{ \Phi(x, |F|) + \Phi(x, |D\psi|) \} \right] dx < \lambda, \quad (4.5)$$

and

$$E(s_1, \lambda) \setminus N \subset \bigcup_{i=1}^{\infty} B_{5r_i}(x_i) \quad (4.6)$$

for some a measure zero set N .

Proof. For $\rho > 0$ and $x \in E(s_1, \lambda)$, let us set

$$I(x, \rho) := \int_{B_\rho(x)} \left[\Phi(x, |Du|) + \frac{1}{\delta} \{ \Phi(x, |F|) + \Phi(x, |D\psi|) \} \right] dx.$$

Then, by (4.2),

$$I(x, \rho) \leq \left(\frac{40}{s_2 - s_1} \right)^n \lambda_0 < \lambda \quad \text{for all } \rho \in \left(\frac{(s_2 - s_1)r}{20}, (s_2 - s_1)r \right).$$

On the other hand, by the definition of $E(s_1, \lambda)$ and Lebesgue's differentiation theorem, we have

$$\lim_{\rho \rightarrow 0} I(x, \rho) > \lambda \quad \text{for a.e. } x \in E(s_1, \lambda).$$

Since I is continuous for ρ , for a.e. $x \in E(s_1, \lambda)$ there exists $r_x \in \left(0, \frac{(s_2 - s_1)r}{20}\right)$ such that

$$I(x, r_x) = \lambda \quad \text{and} \quad I(x, \rho) < \lambda \quad \text{for all } \rho \in (r_x, (s_2 - s_1)r).$$

Therefore, by Vitali's covering lemma, we can obtain the desired result. $\square$

Under the results of the above lemma, setting

$$B^i = B_{5r_i}(x_i), \quad jB^i := B_{5jr_i}(x_i) \quad (j = 2, 3, 4), \quad p_{i2} := \sup_{2B^i} p(\cdot).$$

we have from (4.5) that

$$\int_{4B^i} \Phi(x, |Du|) dx \leq \lambda \quad \text{and} \quad \int_{4B^i} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx \leq \delta\lambda.$$

Then, in view of Lemma 3.2, Lemma 3.3 and Lemma (3.16), we have the following result. Note that from (4.1) and the fact that $5r_i \leq r/4$, the all restrictions of $r > 0$ in the previous section, see (3.8) and (3.12), are satisfied when $r = 5r_i$.

Lemma 4.2. *For any $\epsilon \in (0, 1)$ there exists small $\delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \epsilon) \in (0, 1)$ such that the following holds: under the above setting, for each $i = 1, 2, 3, \dots$, there exist $w_i \in W^{1, \Phi}(4B^i) \cap W^{1, \Phi_{p_{i2}}}(2B^i)$ and $v_i \in W^{1, \Phi_{p_{i2}}}(2B^i) \cap W^{1, \infty}(B^i)$ such that*

$$\int_{B^i} \Phi(x, |Du - Dw_i|) dx + \int_{B^i} \Phi_{p_{i2}}(|Dw_i - Dv_i|) dx \leq \epsilon \lambda \quad (4.7)$$

and

$$\sup_{x \in B^i} \Phi(x, |Dv_i|) \leq c_L \lambda \quad (4.8)$$

for some $c_L = c_L(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof of the estimate (1.12). We first estimate

$$\int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx, \quad \text{where } A_2 := 2 \cdot 4^{\gamma_2} (c_L + 2). \quad (4.9)$$

From (4.6) we see $E(s_1, \lambda) \setminus N \subset \bigcup_{i=1}^{\infty} B^i$, hence

$$\int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx \leq \sum_{i=1}^{\infty} \int_{E(s_1, A_2 \lambda) \cap B^i} \Phi(x, |Du|) dx.$$

By (3) of Proposition 2.1 and (4.8), we have for $x \in E(s_1, A_2 \lambda) \cap B^i$

$$\begin{aligned} \Phi(x, |Du|) &\leq 4^{\gamma_2} (\Phi(x, |Du - Dw_i|) + \Phi_{p_{i2}}(|Dw_i - Dv_i|) + \Phi_{p_{i2}}(|Dv_i|) + 2) \\ &\leq 4^{\gamma_2} (\Phi(x, |Du - Dw_i|) + \Phi_{p_{i2}}(|Dw_i - Dv_i|) + (c_L + 2)\lambda) \\ &\leq 4^{\gamma_2} (\Phi(x, |Du - Dw_i|) + \Phi_{p_{i2}}(|Dw_i - Dv_i|)) + \frac{1}{2} \Phi(x, |Du|), \end{aligned}$$

which together with (4.7) imply

$$\int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx \leq 2 \cdot 4^{\gamma_2} \epsilon \lambda \sum_{i=1}^{\infty} |B^i| = 2 \cdot 4^{\gamma_2} 5^n \epsilon \lambda \sum_{i=1}^{\infty} |B_{r_i}(x_i)|. \quad (4.10)$$

Note that by (4.4) we have,

$$\frac{\lambda}{2} \leq \int_{B_{r_i}(x_i)} \Phi(x, |Du|) dx \quad \text{or} \quad \frac{\delta \lambda}{2} \leq \int_{B_{r_i}(x_i)} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx.$$

Then, the first case yields

$$|B_{r_i}(x_i)| \leq \frac{4}{\lambda} \int_{\{x \in B_{r_i}(x_i) : \Phi(x, |Du|) > \frac{\lambda}{4}\}} \Phi(x, |Du|) dx,$$

and the second case does

$$|B_{r_i}(x_i)| \leq \frac{4}{\delta \lambda} \int_{\{x \in B_{r_i}(x_i) : \Phi(x, |F|) + \Phi(x, |D\psi|) > \frac{\delta \lambda}{4}\}} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx.$$

Inserting these two estimates into (4.10) and using the fact that $B_{r_i}(x_i)$, $i = 1, 2, \dots$, are mutually disjoint and contained in $B_{s_2 r}$, we finally obtain

$$\begin{aligned}
\int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx &\leq c_1 \epsilon \left\{ \int_{\{x \in B_{s_2 r} : \Phi(x, |Du|) > \frac{\lambda}{4}\}} \Phi(x, |Du|) dx \right. \\
&\quad \left. + \frac{1}{\delta} \int_{\{x \in B_{s_2 r} : \Phi(x, |F|) + \Phi(x, |Du|) > \frac{\delta \lambda}{4}\}} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx \right\}
\end{aligned} \tag{4.11}$$

for some $c_1 = c_1(n, \nu, \Lambda, \gamma_1, \gamma_2) \geq 1$.

we next estimate

$$\int_{B_{s_1 r}} [\Phi(\cdot, |Du|)]_N^{q-1} \Phi(x, |Du|) dx$$

where $q > 1$, $N \in \mathbb{N}$ and

$$[\Phi(\cdot, |Du|)]_N(x) := \min \{\Phi(x, |Du(x)|), N\}.$$

Applying Fubini's theorem, we have for sufficiently large N ,

$$\begin{aligned}
&\int_{B_{s_1 r}} [\Phi(\cdot, |Du|)]_N^{q-1} \Phi(x, |Du|) dx \\
&= (q-1) A_2^{q-1} \int_0^{N/A_2} \lambda^{q-2} \int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx d\lambda \\
&= (q-1) A_2^{q-1} \int_0^{A_1 \lambda_0} \lambda^{q-2} \int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx d\lambda \\
&\quad + (q-1) A_2^{q-1} \int_{A_2 \lambda_0}^{N/A_2} \lambda^{q-2} \int_{E(s_1, A_2 \lambda)} \Phi(x, |Du|) dx d\lambda =: I_4 + I_5.
\end{aligned}$$

where A_1 and A_2 are denoted in (4.3) and (4.9), respectively. Then for I_4 , we have

$$|I_4| \leq (A_1 A_2 \lambda_0)^{q-1} \int_{B_{s_1 r}} \Phi(x, |Du|) dx.$$

As for I_5 , using (4.11) and Fubini's theorem, we have

$$\begin{aligned}
|I_5| &\leq (q-1) A_2^{q-1} c_1 \epsilon \int_0^{4N} \lambda^{q-2} \left\{ \int_{\{x \in B_{s_2 r} : \Phi(x, |Du|) > \frac{\lambda}{4}\}} \Phi(x, |Du|) dx \right. \\
&\quad \left. + \frac{1}{\delta} \int_{\{x \in B_{s_2 r} : \Phi(x, |F|) + \Phi(x, |Du|) > \frac{\delta \lambda}{4}\}} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx \right\} d\lambda \\
&\leq (q-1) (4A_2)^{q-1} c_1 \epsilon \int_0^N \lambda^{q-2} \int_{\{x \in B_{s_2 r} : \Phi(x, |Du|) > \lambda\}} \Phi(x, |Du|) dx d\lambda \\
&\quad + \frac{(q-1) A_2^{q-1} c_1 \epsilon}{\delta^q} \\
&\quad \times \int_0^\infty \lambda^{q-2} \int_{\{x \in B_{s_2 r} : \Phi(x, |F|) + \Phi(x, |Du|) > \lambda\}} [\Phi(x, |F|) + \Phi(x, |D\psi|)] dx d\lambda \\
&\leq (4A_2)^{q-1} c_1 \epsilon \int_{B_{s_2 r}} [\Phi(x, |Du|)]_N^{q-1} \Phi(x, |Du|) dx
\end{aligned}$$

$$+ \frac{A_2^{q-1} c_1 \epsilon}{\delta^q} \int_{B_{s_2 r}} [\Phi(x, |F|) + \Phi(x, |D\psi|)]^q dx.$$

Therefore, choosing $\epsilon \in (0, 1)$ such that $(4A_2)^{q-1} c_1 \epsilon = \frac{1}{2}$, hence δ is determined as a constant depending on $n, \nu, \Lambda, \gamma_1, \gamma_2$, and inserting the estimates for I_4 and I_5 into (4.12), we have

$$\begin{aligned} \int_{B_{s_1 r}} [\Phi(x, |Du|)]_N^{q-1} \Phi(x, |Du|) dx &\leq \frac{1}{2} \int_{B_{s_2 r}} [\Phi(x, |Du|)]_N^{q-1} \Phi(x, |Du|) dx \\ &+ \frac{c\lambda_0^{q-1}}{(s_2 - s_1)^{(q-1)n}} \int_{B_{2r}} \Phi(x, |Du|) dx + c \int_{B_{2r}} [\Phi(x, |F|) + \Phi(x, |D\psi|)]^q dx. \end{aligned}$$

Since $1 \leq s_1 < s_2 \leq 2$ and the constants c are independent of s_1 and s_2 , in view of Lemma 2.5, we have

$$\begin{aligned} \int_{B_r} [\Phi(x, |Du|)]_N^{q-1} \Phi(x, |Du|) dx &\leq c\lambda_0^{q-1} \int_{B_{2r}} \Phi(x, |Du|) dx \\ &+ c \int_{B_{2r}} [\Phi(x, |F|) + \Phi(x, |D\psi|)]^q dx. \end{aligned}$$

Applying (4.2) and using Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned} \int_{B_r} [\Phi(x, |Du|)]_N^{q-1} \Phi(x, |Du|) dx &\leq c \left(\int_{B_{2r}} \Phi(x, |Du|) dx \right)^q \\ &+ c \int_{B_{2r}} [\Phi(x, |F|)^q + \Phi(x, |D\psi|)^q + 1] dx. \end{aligned}$$

Finally, by Lebesgue's monotone convergence theorem, letting $N \rightarrow \infty$, we obtain (1.12). $\square$

5. Appendix

We prove the existence and the uniqueness of $u \in \mathcal{K}_\psi^\Phi(\Omega)$ satisfying (1.5). For $f, g \in W^{1,\Phi}(\Omega)$, $f = g$ (resp. $f \leq g$) on $\partial\Omega$ means $f - g$ (resp. $(f - g)_+$) $\in W_0^{1,\Phi}(\Omega)$.

Theorem 5.1. *Suppose that $p(\cdot)$ satisfies (1.8) for some $\delta > 0$ and $R \in (0, 1)$ and the nonlinearity $\mathbf{a} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (2.4) and (2.9). If $\psi, g \in W^{1,\Phi}(\Omega)$ with $\psi \leq g$ and $F \in L^\Phi(\Omega, \mathbb{R}^n)$, then there exists a unique $u \in \mathcal{K}_\psi^\Phi(\Omega)$ with $u = g$ on $\partial\Omega$ satisfying the variational inequality (1.5).*

Proof. Note that the uniqueness directly follows from the monotonicity (2.4). Hence we prove the existence. By Poincaré's inequality for $W_0^{1,\Phi}(\Omega)$, see [39, Theorem 3.5], we set $\|f\|_{W_0^{1,\Phi}(\Omega)} := \|Df\|_{L^\Phi(\Omega, \mathbb{R}^n)}$ in this proof, without loss of generality. We further define $\mathbf{A} : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))'$, where $(W_0^{1,\Phi}(\Omega))'$ is the dual space of $W_0^{1,\Phi}(\Omega)$, by

$$\mathbf{A}f(\varphi) = \int_{\Omega} \mathbf{a}(x, Df + Dg) \cdot D\varphi dx,$$

and $\mathbf{F} \in (W_0^{1,\Phi}(\Omega))'$ by

$$\mathbf{F}(\varphi) = \int_{\Omega} |F|^{p(x)-2} \log(e + |F|) F \cdot D\varphi \, dx.$$

In view of Theorem 2.3 of Chapter II in [43], if for some $r > 0$ there holds

$$\mathbf{A}\varphi(\varphi - (\psi - g)^+) > \mathbf{F}(\varphi - (\psi - g)^+) \quad (5.1)$$

for all $\varphi \in \mathcal{K}_{\psi-g}^{\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$ with $\|\varphi\|_{W_0^{1,\Phi}(\Omega)} > r$, then there exists $v \in \mathcal{K}_{\psi-g}^{\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$ such that

$$\mathbf{A}\varphi(\varphi - v) > \mathbf{F}(\varphi - v)$$

for all $\varphi \in \mathcal{K}_{\psi-g}^{\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$. Therefore, $u := v + g \in \mathcal{K}_{\psi}^{\Phi}(\Omega)$ satisfies the variational inequality (1.5).

Hence we prove (5.1). It suffices to show that

$$\frac{\mathbf{A}\varphi(\varphi - (\psi - g)^+)}{\|\varphi - (\psi - g)^+\|_{W_0^{1,\Phi}(\Omega)}} \rightarrow \infty \quad \text{as} \quad \|\varphi\|_{W_0^{1,\Phi}(\Omega)} \rightarrow \infty. \quad (5.2)$$

By (1.3) and (2.6) and using (2.1) and (2.2), we have

$$\begin{aligned} \frac{\mathbf{A}\varphi(\varphi - (\psi - g)^+)}{\|\varphi - (\psi - g)^+\|_{W_0^{1,\Phi}(\Omega)}} &\geq \frac{\int_{\Omega} \mathbf{a}(x, D[\varphi + g]) \cdot (D[\varphi - (\psi - g)^+]) \, dx}{\|Dg\|_{L^{\Phi}(\Omega, \mathbb{R}^n)} + \|D(\psi - g)^+\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}} \\ &\geq \frac{\frac{\tilde{\nu}}{2} \int_{\Omega} \Phi(x, |D[\varphi + g]|) \, dx - c \int_{\Omega} \Phi(x, |D[g - (\psi - g)^+]|) \, dx}{\|D\varphi\|_{L^{\Phi}(\Omega, \mathbb{R}^n)} + \|D(\psi - g)^+\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}}. \end{aligned}$$

for some constant $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$. Finally, using the relation (2.3) we have

$$\begin{aligned} &\frac{\mathbf{A}f(\varphi - (\psi - g)^+)}{\|\varphi - (\psi - g)^+\|_{W_0^{1,\Phi}(\Omega)}} \\ &\geq \frac{\frac{\tilde{\nu}}{2} \left\{ \|D\varphi\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}^{\gamma_1} - \|Dg\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}^{\gamma_1} \right\} - c \int_{\Omega} \Phi(x, |D(\psi - g)^+|) \, dx}{\|D\varphi\|_{L^{\Phi}(\Omega, \mathbb{R}^n)} + \|D(\psi - g)^+\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}}, \end{aligned}$$

if $\|Dg\|_{L^{\Phi}(\Omega)} \geq 1$. Consequently, since $\gamma_1 > 1$, we have (5.2). $\square$

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