Existence of infinitely many solutions for the (p, q) -Laplace equation

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Abstract. In this paper, we study the (p, q) -Laplace equation in a bounded domain under the Dirichlet boundary condition. We give a sufficient condition of the nonlinear term for the existence of a sequence of solutions converging to zero or diverging to infinity. Moreover, we give a priori estimates of the C^1 -norms of solutions under a suitable condition on the nonlinear term.

Mathematical Subject Classification. Primary 35J20; Secondary 35J25, 35J66.

Keywords. (p, q) -Laplace equation, superlinear, sublinear, infinitely many solutions, a priori estimate, variational method.

1. Introduction

We study the (p, q) -Laplace equation

$$-\Delta_p u - \Delta_q u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Delta_r u := \operatorname{div}(|\nabla u|^{r-2} \nabla u)$ is the r -Laplacian with $r = p, q$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $f(x, u)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}$. Throughout the paper, we assume that $1 < q \leq p$. When $q = p$, the left hand side of (1.1) is considered as the p -Laplacian after dividing both sides by 2 and replacing $(1/2)f(x, u)$ by $f(x, u)$. Therefore our result is valid for the p -Laplace equation as well as the (p, q) -Laplace equation. We shall investigate the existence or non-existence of sequences of solutions whose $C^1(\overline{\Omega})$ norms converge to zero or diverge to infinity. A solution of the (p, q) -Laplace equation does not necessarily belong to $C^2(\overline{\Omega})$ but belongs to $C^{1,\theta}(\overline{\Omega})$ with some $\theta \in (0, 1)$ (see [24, 25]). Here $C^{1,\theta}(\overline{\Omega})$ denotes the set of all $C^1(\overline{\Omega})$ functions whose first derivatives are Hölder continuous with exponent θ .

To explain our purpose, we consider the Emden–Fowler equation

$$-\Delta u = |u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.2)$$

Since the left hand side is the Laplacian, any solution has a $C^2(\overline{\Omega})$ regularity. By the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz [3], it is proved that if $1 < r < 2$, then (1.2) has a sequence of solutions whose $C^2(\overline{\Omega})$ norms converge to zero (see [1, Example 9] or [21]); if $2 < r < 2^*$, then (1.2) has a sequence of solutions whose $C^2(\overline{\Omega})$ norms diverge to infinity (see [3, 32, 34]). Here 2^* is the critical exponent defined by

$$2^* := 2N/(N-2) \quad \text{when } N \geq 3, \quad 2^* := \infty \quad \text{when } N = 1, 2.$$

If $r \geq 2^*$ and Ω is star-shaped, then (1.2) has no non-trivial solutions because of the Pohozaev identity [31].

The existence result can be extended to the p -Laplace equation. Azorero and Alonso [4] proved the existence of infinitely many solutions for the p -Laplace equation with a critical exponent

$$-\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{s-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

when $\lambda > 0$ is small enough, $p < N$ and $1 < s < p^*$, where p^* is a critical exponent defined by

$$p^* := Np/(N-p) \quad \text{when } p < N, \quad p^* := \infty \quad \text{when } p \geq N. \quad (1.3)$$

See also [7, 11, 27, 37] for the existence of multiple solutions.

We are interested in the problem whether these results can be extended to the (p, q) -Laplace equation (1.1). There are many contributions which have studied (1.1). In particular, many authors studied the equation

$$-\Delta_p u - \Delta_q u = a(x)|u|^{r-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.4)$$

or

$$-\Delta_p u - \Delta_q u = a(x)|u|^{r-2}u + b(x)|u|^{s-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.5)$$

where $1 < s < r$, $a(x)$ and $b(x)$ are positive continuous functions on $\overline{\Omega}$. Hsu and Lin [19] proved the existence of multiple positive solutions for (1.5) when $p < N$ and $r = p^*$ is a critical exponent given by (1.3). Cherfils and Il'yasov [10] and Yin and Yang [39] proved the existence and non-existence of positive solutions for (1.5) with $r = p$ and $s = q$. See also [13, 15, 28, 30, 35, 38]. For the case where $\Omega = \mathbb{R}^N$, the existence of non-trivial solutions is proved in [6, 8, 9, 14, 17, 23, 29, 36].

As mentioned above, many authors dealt with (1.4) or (1.5) with $a(x) > 0$ and $b(x) > 0$, that is, the nonlinear term is a power of u or a sum of two powers with positive coefficients. The purpose of the paper is to study a more general nonlinear function $f(x, u)$, which is subcritical but may change its sign and which is not necessarily a power nonlinearity (see Assumptions 2.1 and 2.5). We shall elucidate what properties of $f(x, u)$ yield infinitely many solutions. Our aim is to find a weak sufficient condition of $f(x, u)$ for the existence of a sequence of solutions converging to zero in $C^1(\overline{\Omega})$ or diverging to infinity. Another purpose is to give a priori lower estimates $\|u\|_{C^1} \geq c$ for all non-trivial

solutions u of (1.1) under a suitable condition on $f(x, u)$, where $\|u\|_{C^1}$ denotes the $C^1(\bar{\Omega})$ -norm of u .

We organize this paper into five sections. In Sect. 2, we state the main result, which gives the existence of infinitely many solutions and provides a priori estimates of solutions. In Sect. 3, we prove the existence of a sequence of solutions diverging to infinity by using the symmetric mountain pass lemma. In Sect. 4, we prove the existence of a sequence of solutions converging to zero. In Sect. 5, we prove a priori estimates of solutions.

2. Main result

In this section, we state the main results consisting of three theorems: the first one gives a sequence of solutions diverging to infinity, the second one provides a sequence of solutions converging to zero and the third one gives a priori lower estimates of solutions. To state the first theorem, we denote by $\Lambda(p)$ the first eigenvalue of the p -Laplacian:

$$-\Delta_p u = \Lambda(p)|u|^{p-2}u, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (2.1)$$

It is known that $\Lambda(p) > 0$, an eigenfunction corresponding to $\Lambda(p)$ has no zeros in Ω , and therefore it can be chosen as a positive function. Moreover the first eigenspace is one-dimensional. We introduce the assumption below.

Assumption 2.1. Assume that $1 < q \leq p$ and $f(x, u)$ is a continuous function on $\bar{\Omega} \times \mathbb{R}$ which satisfies the following conditions.

- (i) $f(x, u)$ is odd with respect to u , i.e., $f(x, -u) = -f(x, u)$.
- (ii) There exist constants $C > 0$ and $r \in (p, p^*)$ with p^* defined by (1.3) such that

$$|f(x, u)| \leq C(|u|^{r-1} + 1) \quad \text{for } (x, u) \in \Omega \times \mathbb{R}. \quad (2.2)$$

- (iii) There exist $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset \Omega$ and

$$\inf_{x \in B(x_0, \varepsilon)} \frac{F(x, u)}{|u|^p} \rightarrow \infty \quad \text{as } |u| \rightarrow \infty,$$

where $B(x_0, \varepsilon)$ denotes a ball centered at x_0 with radius ε and $F(x, u)$ is defined by

$$F(x, u) := \int_0^u f(x, s) ds. \quad (2.3)$$

- (iv) There exists an $\alpha > p$ such that

$$\limsup_{|u| \rightarrow \infty} \left(\sup_{x \in \Omega} \frac{\alpha F(x, u) - u f(x, u)}{|u|^p} \right) < \frac{\Lambda(p)(\alpha - p)}{p},$$

where $\Lambda(p) > 0$ is the first eigenvalue of the p -Laplacian defined by (2.1).

We call $f(x, u)$ *locally p -superlinear* if it satisfies (iii). This local condition mainly yields a sequence of solutions diverging to infinity in $C^1(\bar{\Omega})$. Indeed, we have the first main theorem.

Theorem 2.2. *Under Assumption 2.1, there exists a sequence u_n of solutions for (1.1) such that $\|u_n\|_{C^1} \rightarrow \infty$ as $n \rightarrow \infty$, where $\|\cdot\|_{C^1}$ denotes the $C^1(\bar{\Omega})$ -norm.*

Remark 2.3. The condition (iv) in Assumption 2.1 seems complicated, but it guarantees the Palais–Smale condition (see Assumption 4.1 for its definition) for the Lagrangian functional. This condition (iv) is related to the Ambrosetti–Rabinowitz condition below (see [3] or [32, p. 9]): there exists an $\alpha > p$ such that

$$0 < \alpha F(x, u) \leq uf(x, u) \quad \text{for } x \in \Omega \text{ and } |u| \text{ large enough,}$$

or a slightly weaker condition,

$$\alpha F(x, u) \leq uf(x, u) \quad \text{for } x \in \Omega \text{ and } |u| \text{ large enough.} \quad (2.4)$$

It is clear that (2.4) implies (iv). We emphasize that our assumption (iv) is strictly weaker than (2.4) and hence Theorem 2.2 covers a wide class of nonlinear functions $f(x, u)$. To prove this claim, we shall give an example of $f(x, u)$ which satisfies (i)–(iv) in Assumption 2.1 but does not satisfy (2.4). Define

$$f(x, u) := a(x)|u|^{r-2}u + b(x)|u|^{s-2}u,$$

where $a(x)$ and $b(x)$ are continuous functions on $\bar{\Omega}$ which satisfy

$$a(x_1) > 0 > a(x_2), \quad b(x_3) > 0 \quad \text{at some } x_1, x_2, x_3 \in \Omega, \quad (2.5)$$

and r and s satisfy either (A) or (B) below:

- (A) $1 < s < p < r < p^*$,
- (B) $s = p < r < p^*$ and $\max_{x \in \bar{\Omega}} b(x) < \Lambda(p)$.

We shall prove that $f(x, u)$ does not satisfy (2.4) for any $\alpha \in (p, \infty)$. Define

$$G(x, u) := \alpha F(x, u) - uf(x, u) = \frac{\alpha - r}{r}a(x)|u|^r + \frac{\alpha - s}{s}b(x)|u|^s.$$

Let x_i with $i = 1, 2, 3$ be as in (2.5). When $\alpha > r$, we find that

$$G(x_1, u) = \frac{\alpha - r}{r}a(x_1)|u|^r + \frac{\alpha - s}{s}b(x_1)|u|^s \rightarrow \infty \quad \text{as } |u| \rightarrow \infty.$$

Hence (2.4) is not fulfilled. When $\alpha < r$, we obtain

$$G(x_2, u) = \frac{\alpha - r}{r}a(x_2)|u|^r + \frac{\alpha - s}{s}b(x_2)|u|^s \rightarrow \infty \quad \text{as } |u| \rightarrow \infty.$$

When $\alpha = r$, it holds that

$$G(x_3, u) = \frac{r - s}{s}b(x_3)|u|^s \rightarrow \infty \quad \text{as } |u| \rightarrow \infty.$$

Consequently, (2.4) does not hold for any $\alpha > p$.

We shall prove that $f(x, u)$ satisfies (iv) in Assumption 2.1. Choose $\alpha := r$. Then we have

$$G(x, u) = \frac{r - s}{s}b(x)|u|^s,$$

which implies that

$$\max_{x \in \bar{\Omega}} \frac{\alpha F(x, u) - u f(x, u)}{|u|^p} = \frac{r-s}{s} |u|^{s-p} \max_{x \in \bar{\Omega}} b(x). \quad (2.6)$$

Assume Case (A). Since $1 < s < p$, the right hand side of (2.6) converges to zero as $|u| \rightarrow \infty$. Assume Case (B). Since $s = p$, $\alpha = r$ and $\max_{x \in \bar{\Omega}} b(x) < \Lambda(p)$, the right hand side of (2.6) is less than $\Lambda(p)(\alpha - p)/p$. Thus $f(x, u)$ satisfies (iv) in Assumption 2.1. Therefore (iv) is strictly weaker than (2.4). Furthermore, $f(x, u)$ satisfies other conditions in Assumption 2.1. Indeed, it is easy to verify (i) and (ii). Put $x_0 := x_1$ with x_1 given in (2.5). Then $a(x) > 0$ near x_0 , which ensures (iii). Therefore (1.1) has a sequence of solutions whose C^1 norms diverge to infinity.

Observe the following condition

$$\inf_{x \in B(x_0, \varepsilon)} \frac{f(x, u)}{|u|^{p-2}u} \rightarrow \infty \quad \text{as } |u| \rightarrow \infty. \quad (2.7)$$

It is easy to verify that (2.7) is a sufficient condition for (iii) in Assumption 2.1. Consequently, we have the following corollary.

Corollary 2.4. *Under Assumption 2.1 with (iii) replaced by (2.7), there exists a sequence of solutions for (1.1) whose $C^1(\bar{\Omega})$ -norms diverge to infinity.*

Next, we consider a sequence of solutions converging to zero. To this end, we introduce the assumption below.

Assumption 2.5. Suppose that $1 < q \leq p$ and $f(x, u)$ is a continuous function defined on $\bar{\Omega} \times [-\varepsilon_0, \varepsilon_0]$ with an $\varepsilon_0 > 0$ and it is odd with respect to u . Assume furthermore that there exist $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset \Omega$ and

$$\inf_{x \in B(x_0, \varepsilon)} \frac{F(x, u)}{|u|^q} \rightarrow \infty \quad \text{as } u \rightarrow 0. \quad (2.8)$$

We call $f(x, u)$ *locally q -sublinear* if it satisfies (2.8). This local condition produces a sequence of solutions converging to zero in $C^1(\bar{\Omega})$. Indeed, we have the second main theorem below.

Theorem 2.6. *Under Assumption 2.5, (1.1) has a sequence of non-trivial solutions converging to zero in $C^1(\bar{\Omega})$.*

It is easy to show that (2.8) follows from the condition

$$\inf_{x \in B(x_0, \varepsilon)} \frac{f(x, u)}{|u|^{q-2}u} \rightarrow \infty \quad \text{as } u \rightarrow 0. \quad (2.9)$$

Corollary 2.7. *Under Assumption 2.5 with (2.8) replaced by (2.9), there exists a sequence of non-trivial solutions of (1.1) converging to zero in $C^1(\bar{\Omega})$.*

Remark 2.8. We emphasize that Assumption 2.5 does not need any conditions of $f(x, u)$ for large $|u|$ and hence $f(x, u)$ may include a supercritical exponent. For example, Theorem 2.6 is applicable to the nonlinear function

$$f(x, u) = a(x)|u|^{r-2}u + b(x)|u|^{s-2}u \exp(|u|^m),$$

where we assume that $m > 0$ is any number, $1 < r < \min(q, s)$, $a(x)$ and $b(x)$ are continuous functions such that $a(x_0) > 0$ at some x_0 . When m is large, $f(x, u)$ is supercritical as $u \rightarrow \pm\infty$. However it is easy to verify that $f(x, u)$ satisfies (2.9). Therefore (1.1) with the above $f(x, u)$ has a sequence of solutions converging to zero.

We state the third theorem, which gives a priori lower estimates of solutions.

Theorem 2.9. *Suppose that $f(x, u)/|u|^{q-2}u$ uniformly converges to a continuous function $f_0(x)$ on $\bar{\Omega}$ as $u \rightarrow 0$ and moreover assume that there exist no non-trivial solutions of the problem*

$$-\Delta_q v = f_0(x)|v|^{q-2}v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \quad (2.10)$$

Then there exists a constant $c > 0$ such that $\|u\|_{C^1} \geq c$ for all non-trivial solutions u of (1.1).

Remark 2.10. We shall explain why the non-existence of non-trivial solutions for (2.10) yields an a priori lower estimate of solutions for (1.1). To do so, we consider the bifurcation problem of the ordinary differential equation

$$u'' + f(x, u) + \lambda u = 0 \quad \text{in } (0, 1), \quad u(0) = u(1) = 0, \quad (2.11)$$

where $u'' = d^2u/dx^2$ and $\lambda \in \mathbb{R}$ is a bifurcation parameter. Here we assume that $f(x, 0) \equiv 0$. Then $(\lambda, u) = (\lambda, 0)$ is a solution of (2.11) for all $\lambda \in \mathbb{R}$. We call $(\lambda, u) = (\lambda_0, 0)$ a bifurcation point of (2.11) if (2.11) has a sequence (λ_n, u_n) of solutions satisfying $u_n \not\equiv 0$ which converges to $(\lambda_0, 0)$. Assume that $f(x, u)$ is continuously differentiable with respect to u . Then it is known that if $(\lambda_0, 0)$ is a bifurcation point, λ_0 is an eigenvalue of the linearized operator $-d^2/dx^2 - f_u(x, 0)$, i.e., the problem below has a non-trivial solution,

$$-v'' - f_u(x, 0)v = \lambda_0 v \quad \text{in } (0, 1), \quad v(0) = v(1) = 0. \quad (2.12)$$

In other words, if the linearized equation (2.12) does not have a non-trivial solution, then there exist no sequences of solutions for (2.11) which converge to $(\lambda_0, 0)$, and therefore there exists an a priori lower estimate for norms of solutions. Theorem 2.9 likes this result with $\lambda_0 = 0$. Roughly speaking, (2.10) is a linearized equation of (1.1) at zero. Hence if (2.10) has no non-trivial solutions, there exists an a priori lower estimate of C^1 -norms of non-trivial solutions.

In what follows, $\|u\|_p$ denotes the $L^p(\Omega)$ -norm of u . We consider a sufficient condition for the non-existence of non-trivial solutions of (2.10). To this end, we define the best Sobolev constant C_i with $i = 1, 2, 3$ which are the smallest positive constants satisfying

$$\|u\|_{Nq/(N-q)} \leq C_1(N, q)\|\nabla u\|_q \quad \text{if } q < N, \quad (2.13)$$

$$\|u\|_t \leq C_2(N, t)\|\nabla u\|_q \quad \text{for } t \in [1, \infty) \text{ if } q = N, \quad (2.14)$$

$$\|u\|_\infty \leq C_3(N, q)\|\nabla u\|_q \quad \text{if } q > N, \quad (2.15)$$

for all $u \in W_0^{1,p}(\Omega)$. Recall that $\Lambda(p)$ denotes the first eigenvalue of the p -Laplacian.

Assumption 2.11. Assume that $1 < q < \infty$ and $f_0(x)$ satisfies either $f_0(x) < \Lambda(q)$ in Ω or

$$C_1(N, q)^q \|f_0^+\|_{N/q} < 1 \quad \text{if } q < N, \quad (2.16)$$

$$C_2(N, t)^q \|f_0^+\|_{t/(t-q)} < 1 \quad \text{with some } t \in (q, \infty) \text{ if } q = N, \quad (2.17)$$

$$C_3(N, q)^q \|f_0^+\|_1 < 1 \quad \text{if } q > N, \quad (2.18)$$

where $f_0^+(x) := \max\{f_0(x), 0\}$.

Proposition 2.12. *If $f_0(x)$ satisfies Assumption 2.11, then (2.10) has no non-trivial solutions.*

Combining Theorem 2.9 with the proposition above, we obtain the next corollary.

Corollary 2.13. *Suppose that $f(x, u)/|u|^{q-2}u$ uniformly converges to a continuous function $f_0(x)$ on $\bar{\Omega}$ as $u \rightarrow 0$ which satisfies Assumption 2.11. Then the same conclusion as in Theorem 2.9 remains valid.*

We consider (1.5). The left hand side of (1.5) is nearly equal to $-\Delta_p u$ for $|u|$ sufficiently large and nearly equal to $-\Delta_q u$ for $|u|$ small enough. Therefore we expect that when $s < q$, the equation is locally q -sublinear at zero and it has a sequence of solutions converging to zero; when $p < r < p^*$, the equation is locally p -superlinear at infinity and it has a sequence of solutions diverging to infinity. Indeed, applying all the theorems above to (1.5), we obtain the theorem below.

Theorem 2.14. *Let $1 < q < p$, $1 < s < r$ and $a(x)$ and $b(x)$ be continuous functions on $\bar{\Omega}$. Then the following assertions hold.*

- (i) *If $s < q$ and $b(x_0) > 0$ at some $x_0 \in \Omega$, then there exists a sequence u_n of non-trivial solutions of (1.5) such that $\|u_n\|_{C^1} \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *Let $q \leq s$. When $q = s$, we suppose that Assumption 2.11 holds with $f_0(x)$ replaced by $b(x)$. Then there exist a constant $c > 0$ such that $\|u\|_{C^1} \geq c$ for all non-trivial solutions u of (1.5).*
- (iii) *If $r < p$, then there exists a constant $C > 0$ such that $\|u\|_{C^1} \leq C$ for all solutions u of (1.5).*
- (iv) *If $s < p < r < p^*$ and $a(x_0) > 0$ at some $x_0 \in \Omega$, then there exists a sequence u_n of non-trivial solutions of (1.5) such that $\|u_n\|_{C^1} \rightarrow \infty$ as $n \rightarrow \infty$.*
- (v) *Let $q \leq s < r < p$. When $q = s$, we suppose that Assumption 2.11 holds with $f_0(x)$ replaced by $b(x)$. Then there exist constants $0 < c < C$ such that $c \leq \|u\|_{C^1} \leq C$ for all non-trivial solutions u of (1.5).*
- (iv) *Let $s < q < p < r < p^*$ and suppose that $a(x_0), b(x_1) > 0$ at some $x_0, x_1 \in \Omega$, where x_0 and x_1 may or may not equal. Then there exist at least two sequences u_n, v_n of non-trivial solutions for (1.5) such that $\|u_n\|_{C^1} \rightarrow 0$ and $\|v_n\|_{C^1} \rightarrow \infty$ as $n \rightarrow \infty$.*

In the assertion (vi), the nonlinear term is both locally q -sublinear at zero and locally p -superlinear at infinity. This behavior likes the convex-concave nonlinearity studied by Ambrosetti, Brezis and Cerami [2], who obtained at least two sequences of solutions.

3. Infinitely many large solutions

The purpose of this section is to prove Theorem 2.2. We call u a *weak solution* (or *solution* for simplicity) of (1.1) if $u \in W_0^{1,p}(\Omega)$ and it satisfies (1.1) in the distribution sense, i.e.,

$$\int_{\Omega} (|\nabla u|^{p-2} \nabla u \nabla \phi + |\nabla u|^{q-2} \nabla u \nabla \phi) dx = \int_{\Omega} f(x, u) \phi dx, \quad (3.1)$$

for any $\phi \in C_0^\infty(\Omega)$. As defined in Sect. 1, $C^{1,\theta}(\overline{\Omega})$ denotes the set of all $C^1(\overline{\Omega})$ functions whose derivatives are Hölder continuous with exponent θ . We investigate the regularity of weak solutions in the following proposition.

Proposition 3.1. *Suppose that (ii) of Assumption 2.1 holds. Then any weak solution u of (1.1) belongs to $C^{1,\theta}(\overline{\Omega})$ with some $\theta \in (0, 1)$ and the $C^{1,\theta}(\overline{\Omega})$ -norm of u is bounded above by a constant $C > 0$ depending only on the $W_0^{1,p}(\Omega)$ -norm of u . Here θ is independent of u .*

The proposition above is obtained by combining the two lemmas below. The next lemma was proved by Lieberman [25].

Lemma 3.2. ([25, p. 320]) *Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of (1.1). Then u belongs to $C^{1,\theta}(\overline{\Omega})$ with some $\theta \in (0, 1)$ and the $C^{1,\theta}(\overline{\Omega})$ -norm of u is bounded above by a constant $C > 0$ depending only on the $L^\infty(\Omega)$ -norm of u . Here θ is independent of u .*

The lemma above shows Proposition 3.1 if weak solutions belong to $L^\infty(\Omega)$. The L^∞ regularity of solutions is obtained in the following lemma.

Lemma 3.3. *Suppose that (ii) of Assumption 2.1 holds. Then any weak solution u in $W_0^{1,p}(\Omega)$ of (1.1) belongs to $L^\infty(\Omega)$. Moreover, the $L^\infty(\Omega)$ -norm of u is bounded above by a constant $C > 0$ depending only on the $W_0^{1,p}(\Omega)$ -norm of u .*

Proof. If $N < p$, then $W_0^{1,p}(\Omega)$ is embedded in $L^\infty(\Omega)$ and the lemma is obvious. Assume that $p \leq N$. We use a bootstrap argument. Let $u \in W_0^{1,p}(\Omega)$ be any weak solution. Choose a sequence $u_n \in C_0^\infty(\Omega)$ which converges to u in $W_0^{1,p}(\Omega)$. Let $\alpha > 0$ be a parameter to be chosen later on. Since (3.1) remains valid for any $\phi \in C_0^1(\Omega)$, we substitute $\phi = |u_n|^\alpha u_n \in C_0^1(\Omega)$ in (3.1). Using (2.2) and letting $n \rightarrow \infty$, we have

$$(\alpha + 1) \int_{\Omega} (|\nabla u|^p |u|^\alpha + |\nabla u|^q |u|^\alpha) dx \leq C \int_{\Omega} (|u|^{\alpha+r} + |u|^{\alpha+1}) dx. \quad (3.2)$$

Hereafter we denote a positive constant independent of u and α by C , which may vary from line to line. Since $|u|^{\alpha+1} \leq |u|^{\alpha+r} + 1$ for $u \in \mathbb{R}$ (because $r > 1$), (3.2) is reduced to

$$(\alpha + 1) \int_{\Omega} |\nabla u|^p |u|^\alpha dx \leq C \int_{\Omega} |u|^{\alpha+r} dx + C. \quad (3.3)$$

The inequality above means that if the right hand side is finite, so is the left hand side and the inequality is valid. Putting $v := |u|^{(\alpha+p)/p}$, we have the identity

$$|\nabla u|^p |u|^\alpha = (p/(\alpha + p))^p |\nabla v|^p,$$

which with (3.3) yields

$$(\alpha + 1)(p/(\alpha + p))^p \|\nabla v\|_p^p \leq C (\|u\|_{\alpha+r}^{\alpha+r} + 1). \quad (3.4)$$

First, we assume that $p < N$. Using the Sobolev embedding $\|v\|_{Np/(N-p)} \leq C \|\nabla v\|_p$ with $v = |u|^{(\alpha+p)/p}$, we rewrite (3.4) as

$$\begin{aligned} \|u\|_{N(\alpha+p)/(N-p)}^{\alpha+p} &\leq C(\alpha + 1)^{-1} ((\alpha + p)/p)^p (\|u\|_{\alpha+r}^{\alpha+r} + 1) \\ &\leq (C(\alpha + p)/p)^p \max\{\|u\|_{\alpha+r}^{\alpha+r}, 1\}, \end{aligned}$$

which leads to

$$\|u\|_{N(\alpha+p)/(N-p)} \leq (C(\alpha + p)/p)^{p/(\alpha+p)} [\max\{\|u\|_{\alpha+r}, 1\}]^{(\alpha+r)/(\alpha+p)}. \quad (3.5)$$

We define sequences α_k and β_k inductively by

$$\alpha_1 := p^* - r, \quad \beta_1 := N(\alpha_1 + p)/(N - p),$$

$$\alpha_k := \beta_{k-1} - r, \quad \beta_k := \frac{N(\alpha_k + p)}{N - p}.$$

Then

$$\alpha_{k+1} + r = \frac{N(\alpha_k + p)}{N - p}. \quad (3.6)$$

We shall show that

$$\frac{N}{N - p} \alpha_k < \alpha_{k+1}, \quad \alpha_{k+1} + p < \frac{N}{N - p} (\alpha_k + p). \quad (3.7)$$

Since $r < p^* = Np/(N - p)$, (3.6) leads to the first inequality of (3.7). Since $r > p$, (3.6) implies the second inequality of (3.7). It follows from (3.7) that

$$R^{k-1} \alpha_1 \leq \alpha_k \leq R^{k-1} (\alpha_1 + p) \quad \text{with } R := N/(N - p). \quad (3.8)$$

Substitute $\alpha = \alpha_1$ in (3.5). Since $\|u\|_{\alpha_1+r} = \|u\|_{p^*} \leq C \|\nabla u\|_p < \infty$, the both sides of (3.5) with $\alpha = \alpha_1$ are finite, and hence $\|u\|_{\beta_1} < \infty$. Substituting $\alpha = \alpha_2$ in (3.5), we find that the both sides of (3.5) are finite. Repeating this argument, we ensure that $u \in L^{\beta_k}(\Omega)$ for all k . Define $M_k := \max\{\|u\|_{\beta_k}, 1\}$. Substituting $\alpha = \alpha_k$ in (3.5), we find that

$$M_k \leq (C(\alpha_k + p)/p)^{p/(\alpha_k+p)} M_{k-1}^{(\alpha_k+r)/(\alpha_k+p)}. \quad (3.9)$$

Put

$$m_k := \log M_k, \quad \delta_k := \frac{p}{\alpha_k + p} \log(C(\alpha_k + p)/p), \quad \gamma_k := \frac{\alpha_k + r}{\alpha_k + p}.$$

Then (3.9) is rewritten as

$$m_k \leq \delta_k + \gamma_k m_{k-1} \quad \text{for } k \geq 2.$$

Using this inequality repeatedly, we get

$$\begin{aligned}
 m_k &\leq \delta_k + \gamma_k m_{k-1} \\
 &\leq \delta_k + \gamma_k \delta_{k-1} + \gamma_k \gamma_{k-1} m_{k-2} \\
 &\leq \dots \\
 &\leq \delta_k + \sum_{i=2}^{k-1} \Gamma_{k,i} \delta_i + \left(\prod_{i=2}^k \gamma_i \right) m_1,
 \end{aligned} \tag{3.10}$$

where we have put $\Gamma_{k,i} := \gamma_k \gamma_{k-1} \dots \gamma_{i+1}$. Note that

$$\gamma_i = \frac{\alpha_i + r}{\alpha_i + p} = 1 + \frac{r - p}{\alpha_i + p}.$$

Since $\sum_{i=2}^{\infty} 1/(\alpha_i + p)$ converges because of (3.8), the infinite product $\prod_{i=2}^{\infty} \gamma_i$ also converges. Denote this value by Γ . Then $\Gamma_{k,i} < \Gamma$ for all k and i . Therefore $\sum_{i=2}^{k-1} \Gamma_{k,i} \delta_i \leq \Gamma \sum_{i=2}^{\infty} \delta_i < \infty$ because of (3.8). Since $\Gamma_{k,i}$ is increasing with respect to k , so is $\sum_{i=2}^{k-1} \Gamma_{k,i} \delta_i$, and hence it converges as $k \rightarrow \infty$. Letting $k \rightarrow \infty$ in (3.10), we have $\limsup_{k \rightarrow \infty} m_k < \infty$, and therefore

$$\max\{\|u\|_{\infty}, 1\} = \limsup_{k \rightarrow \infty} M_k < \infty.$$

Consequently, $u \in L^{\infty}(\Omega)$.

In the case where $p = N$ also, the above argument works well with the Sobolev embedding $\|v\|_{Np/(N-p)} \leq C\|\nabla v\|_p$ replaced by $\|v\|_s \leq C\|\nabla v\|_p$ for any $s \in [1, \infty)$. Moreover, observing the proof above, we find that the $L^{\infty}(\Omega)$ -norm of u is bounded above by a constant $C > 0$ depending only on the $W_0^{1,p}(\Omega)$ -norm of u . The proof is complete. $\square$

We define the Lagrangian functional $I(u)$ by

$$I(u) := \int_{\Omega} \left(\frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q - F(x, u) \right) dx, \tag{3.11}$$

for $u \in W_0^{1,p}(\Omega)$, where $F(x, u)$ is defined by (2.3). Then I is a C^1 functional, whose derivative is computed as

$$I'(u)v = \int_{\Omega} (|\nabla u|^{p-2} \nabla u \nabla v + |\nabla u|^{q-2} \nabla u \nabla v - f(x, u)v) dx, \tag{3.12}$$

for $u, v \in W_0^{1,p}(\Omega)$. We verify the Palais–Smale condition in the next lemma.

Lemma 3.4. *Suppose that (ii) and (iv) of Assumption 2.1 hold. Then $I(u)$ satisfies the Palais–Smale condition.*

Proof. By (iv) of Assumption 2.1, we choose a $\lambda > 0$ satisfying

$$\limsup_{|u| \rightarrow \infty} \left(\sup_{x \in \Omega} \frac{\alpha F(x, u) - u f(x, u)}{|u|^p} \right) < \lambda < \frac{\Lambda(p)(\alpha - p)}{p}.$$

Then there exists an $M > 0$ such that

$$\alpha F(x, u) - u f(x, u) \leq \lambda |u|^p \quad \text{for } |u| \geq M, \ x \in \Omega,$$

which gives a constant $C > 0$ such that

$$\alpha F(x, u) - u f(x, u) \leq \lambda |u|^p + C \quad \text{for } u \in \mathbb{R}, \quad x \in \Omega. \quad (3.13)$$

Let (u_n) be a sequence in $W_0^{1,p}(\Omega)$ such that $\sup_n |I(u_n)| < \infty$ and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Put $\varepsilon_n := \|I'(u_n)\|$. Using (3.11), (3.12) and the boundedness of $I(u_n)$, we have a constant C_0 such that

$$\begin{aligned} \varepsilon_n \|\nabla u_n\|_p + C_0 &\geq \alpha I(u_n) - I'(u_n)u_n \\ &= \int_{\Omega} \left(\frac{\alpha - p}{p} |\nabla u_n|^p + \frac{\alpha - q}{q} |\nabla u_n|^q - \alpha F(x, u_n) + u_n f(x, u_n) \right) dx, \end{aligned}$$

where we have used $\|\nabla u\|_p$ as the $W_0^{1,p}(\Omega)$ -norm of u . The expression above with (3.13) leads to

$$\varepsilon_n \|\nabla u_n\|_p + C_0 \geq \frac{\alpha - p}{p} \|\nabla u_n\|_p^p + \frac{\alpha - q}{q} \|\nabla u_n\|_q^q - \lambda \|u_n\|_p^p - C|\Omega|,$$

where $|\Omega|$ denotes the volume of Ω . Since the first eigenvalue $\Lambda(p)$ of the p -Laplacian is the minimum of the Rayleigh quotient, we have

$$\Lambda(p) \|u\|_p^p \leq \|\nabla u\|_p^p \quad \text{for } u \in W_0^{1,p}(\Omega). \quad (3.14)$$

Using this inequality, we get

$$\begin{aligned} \varepsilon_n \|\nabla u_n\|_p + C_0 &\geq \frac{\alpha - p}{p} \|\nabla u_n\|_p^p + \frac{\alpha - q}{q} \|\nabla u_n\|_q^q - (\lambda/\Lambda(p)) \|\nabla u_n\|_p^p - C|\Omega| \\ &= \left(\frac{\alpha - p}{p} - \frac{\lambda}{\Lambda(p)} \right) \|\nabla u_n\|_p^p + \frac{\alpha - q}{q} \|\nabla u_n\|_q^q - C|\Omega|. \end{aligned}$$

Since $(\alpha - p)/p > \lambda/\Lambda(p)$ and $\varepsilon_n \rightarrow 0$, $\|\nabla u_n\|_p$ is bounded. Therefore u_n has a subsequence (denoted by u_n again) converging to a certain limit u weakly in $W_0^{1,p}(\Omega)$. By the compact embedding, u_n converges to u strongly in $L^r(\Omega)$ with r given in (ii) of Assumption 2.1. Using (2.2) with the Hölder inequality, we get

$$\int_{\Omega} |f(x, u_n)(u_n - u)| dx \leq C(\|u_n\|_r^{r-1} \|u_n - u\|_r + \|u_n - u\|_1) \rightarrow 0,$$

as $n \rightarrow \infty$. Substituting $u = u_n$ and $v = u_n - u$ in (3.12) and using the fact above, we have

$$\begin{aligned} &\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u) dx + \int_{\Omega} |\nabla u_n|^{q-2} \nabla u_n (\nabla u_n - \nabla u) dx \\ &= I'(u_n)(u_n - u) + \int_{\Omega} f(x, u_n)(u_n - u) dx \rightarrow 0, \end{aligned} \quad (3.15)$$

as $n \rightarrow \infty$. Since u_n converges to u weakly in $W_0^{1,p}(\Omega)$, it follows that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u) dx + \int_{\Omega} |\nabla u|^{q-2} \nabla u (\nabla u_n - \nabla u) dx \rightarrow 0, \quad (3.16)$$

as $n \rightarrow \infty$. We define

$$\begin{aligned} A_n := & \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx \\ & + \int_{\Omega} (|\nabla u_n|^{q-2} \nabla u_n - |\nabla u|^{q-2} \nabla u) (\nabla u_n - \nabla u) dx. \end{aligned}$$

Subtracting (3.16) from the left hand side of (3.15), we find that $A_n \rightarrow 0$ as $n \rightarrow \infty$. By the Hölder inequality, we obtain

$$\begin{aligned} A_n \geq & (\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p) \\ & + (\|\nabla u_n\|_q^{q-1} - \|\nabla u\|_q^{q-1})(\|\nabla u_n\|_q - \|\nabla u\|_q) \geq 0. \end{aligned}$$

Since $A_n \rightarrow 0$, $\|\nabla u_n\|_p$ converges to $\|\nabla u\|_p$. Since u_n converges weakly to u and $W_0^{1,p}(\Omega)$ is uniformly convex, u_n converges strongly to u . The proof is complete. $\square$

We shall show Theorem 2.2. Let $B(x_0, \varepsilon)$ be as in (iii) of Assumption 2.1. Let μ_k and $\psi_k(x)$ be the k -th eigenvalue and the eigenfunction of the Dirichlet Laplacian in $B(x_0, \varepsilon)$,

$$-\Delta \psi = \mu \psi \quad \text{in } B(x_0, \varepsilon), \quad \psi = 0 \quad \text{on } \partial B(x_0, \varepsilon).$$

We extend $\psi_k(x)$ by putting $\psi_k(x) = 0$ in $\Omega \setminus B(x_0, \varepsilon)$. Then $\psi_k(x)$ belongs to $W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$. For $k \in \mathbb{N}$, we define

$$X_k := \left\{ \sum_{i=1}^k t_i \psi_i : t_i \in \mathbb{R} \quad (1 \leq i \leq k) \right\}. \quad (3.17)$$

Then X_k has dimension k . Hereafter we define the $W_0^{1,p}(\Omega)$ -norm of u by

$$\|u\|_{1,p} := \|\nabla u\|_p = \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}.$$

From here to the end of this section, we suppose that Assumption 2.1 holds.

Lemma 3.5. *For any $k \in \mathbb{N}$, there exists an $R_k > 0$ such that*

$$I(u) < 0 \quad \text{when } u \in X_k, \quad \|u\|_{1,p} \geq R_k. \quad (3.18)$$

Proof. Since all norms are equivalent in the finite dimensional space X_k , there exist constants $a_k, b_k > 0$ depending only on k such that

$$\|\nabla u\|_q \leq a_k \|\nabla u\|_p, \quad b_k \|\nabla u\|_p \leq \|u\|_p \quad \text{for } u \in X_k. \quad (3.19)$$

Choose a constant $M_k > 0$ such that $M_k b_k^p > 1/p$ when $q < p$ and $M_k b_k^p > 1/p + a_k^q/q$ when $q = p$. By (iii) in Assumption 2.1, we can choose a $u_k > 0$ so large that

$$F(x, u) \geq M_k |u|^p \quad \text{when } |u| \geq u_k, \quad x \in B(x_0, \varepsilon).$$

From this inequality, there exists a constant $C_k > 0$ such that

$$F(x, u) \geq M_k |u|^p - C_k \quad \text{for } u \in \mathbb{R}, \quad x \in B(x_0, \varepsilon).$$

Using the inequality above with (3.19) and noting that the support of $u \in X_k$ is in $\bar{B}(x_0, \varepsilon)$, we obtain for $u \in X_k$,

$$\begin{aligned} I(u) &= \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - \int_{B(x_0, \varepsilon)} F(x, u) dx \\ &\leq \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - M_k \|u\|_p^p + C_k |B(x_0, \varepsilon)| \\ &\leq - \left(M_k b_k^p - \frac{1}{p} \right) \|\nabla u\|_p^p + \frac{a_k^q}{q} \|\nabla u\|_p^q + C_k |B(x_0, \varepsilon)| < 0 \end{aligned}$$

provided that $\|\nabla u\|_p$ is large enough, where $|B(x_0, \varepsilon)|$ denotes the volume of $B(x_0, \varepsilon)$. The proof is complete. $\square$

Determine R_k by Lemma 3.5. We define

$$D_k := \{u \in X_k : \|u\|_{1,p} \leq R_k\},$$

$$\partial D_k := \{u \in X_k : \|u\|_{1,p} = R_k\},$$

$$G_k := \{g \in C(D_k, W_0^{1,p}(\Omega)) : g \text{ is odd, } g(u) = u \text{ on } \partial D_k\}, \quad (3.20)$$

$$c_k := \inf_{g \in G_k} \max_{u \in D_k} I(g(u)). \quad (3.21)$$

We state the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz [3] (see also [32, 34]).

Lemma 3.6. (Symmetric mountain pass lemma) *Each c_k is a critical value of I .*

To prove that c_k diverges to infinity as $k \rightarrow \infty$, we need the Krasnoselskii genus, which is defined as follows.

Definition 3.7. Let W be an infinite dimensional Banach space. We call a subset A of W *symmetric* if $u \in A$ implies $-u \in A$. For a closed symmetric subset A not containing the origin, we define a *genus* $\gamma(A)$ of A by the smallest integer n such that there exists an odd continuous mapping from A to $\mathbb{R}^n \setminus \{0\}$. If there does not exist such a finite n , we define $\gamma(A) = \infty$. Furthermore, we put $\gamma(\emptyset) = 0$.

We summarize the properties of the genus in the next lemma. For the proof, we refer the readers to [32, Proposition 7.5] or [34, Proposition 5.4].

Lemma 3.8. *Let A, B be closed symmetric subsets of W which do not contain the origin.*

- (i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
- (ii) *If there exists an odd continuous mapping $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
- (iii) *If V is a symmetric bounded open neighborhood of the origin in $\mathbb{R}^N$, then $\gamma(\partial V) = N$.*
- (iv) *Let Y be a closed linear subspace of W whose codimension is finite. If $\gamma(A)$ is greater than the codimension of Y , then $A \cap Y \neq \emptyset$.*

Let $B(r, W)$ denote a ball in $W_0^{1,p}(\Omega)$ centered at origin with radius r . We denote the boundary of $B(r, W)$ by $\partial B(r, W)$. Therefore

$$B(r, W) := \{u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} < r\},$$

$$\partial B(r, W) := \{u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} = r\}.$$

Lemma 3.9. *Let Y be a closed linear subspace of $W_0^{1,p}(\Omega)$ whose codimension is less than k . Let G_k be given by (3.20). Then it holds that*

$$g(D_k) \cap \partial B(r, W) \cap Y \neq \emptyset \quad \text{for } g \in G_k, \quad 0 < r < R_k.$$

Proof. The lemma can be proved in the same method as in [32, Proposition 9.23], however we prove it for the sake of completeness. Let Y satisfy the assumption of the lemma and let $g \in G_k$ and $0 < r < R_k$. Put

$$V := \{u \in X_k : \|u\|_{1,p} < R_k, \quad g(u) \in B(r, W)\}.$$

Since g is an odd continuous mapping, V is symmetric, bounded, open in X_k and $0 \in V$. Recall that $\gamma(\cdot)$ denotes the genus, which is defined in Definition 3.7. Then $\gamma(g(\partial V)) \geq \gamma(\partial V) = k$ by Lemma 3.8. It follows from definition that $\|u\|_{1,p} \leq R_k$ for $u \in \bar{V}$. We claim that $\|u\|_{1,p} < R_k$ for $u \in \partial V$. Suppose on the contrary that $\|v\|_{1,p} = R_k$ at a certain $v \in \partial V$. Since $g \in G_k$, it holds that $g(v) = v$, which shows that $R_k = \|v\|_{1,p} = \|g(v)\|_{1,p} \leq r$. A contradiction occurs. Consequently, $\|u\|_{1,p} < R_k$ for $u \in \partial V$. By the definition of V , we have

$$g(u) \in \partial B(r, W) \quad \text{for } u \in \partial V. \quad (3.22)$$

Since $\text{codim} Y < k \leq \gamma(g(\partial V))$, it follows from (iv) of Lemma 3.8 that $Y \cap g(\partial V) \neq \emptyset$. We choose a $v \in \partial V$ satisfying $g(v) \in Y$. From (3.22), it follows that $g(v) \in Y \cap \partial B(r, W)$. The proof is complete. $\square$

To prove that $c_k \rightarrow \infty$, we need a Schauder basis. A sequence $\{e_n\}$ of the Sobolev space $W_0^{1,p}(\Omega)$ is called a *Schauder basis* if for any $u \in W_0^{1,p}(\Omega)$, there exists a unique sequence $\{\alpha_n\}$ of real numbers such that $u = \sum_{n=1}^{\infty} \alpha_n e_n$, that is, $\|u - \sum_{n=1}^k \alpha_n e_n\|_{1,p} \rightarrow 0$ as $k \rightarrow \infty$. It was proved by Fučík, John and Nečas [16] (see also [5]) that $W_0^{1,p}(\Omega)$ with $1 < p < \infty$ has a Schauder basis $\{e_n\}$. By the unique expression $u = \sum_{n=1}^{\infty} \alpha_n e_n$, we can define $e_n^*(u) := \alpha_n$. Then e_n^* is a bounded linear functional on $W_0^{1,p}(\Omega)$ (see [18, Theorem 4.13] or [26] for the proof). We define

$$Y_k := \{u \in W_0^{1,p}(\Omega) : e_i^*(u) = 0 \text{ for } i \leq k-1\},$$

$$Z_k := \{u \in W_0^{1,p}(\Omega) : e_i^*(u) = 0 \text{ for } i \geq k\}.$$

Then it is clear that $W_0^{1,p}(\Omega) = Y_k \oplus Z_k$ and Y_k has codimension $k-1$. The next lemma was proved by Silva and Xavier [33].

Lemma 3.10. ([33, Lemma 4.1]) *Let $1 < p \leq r < p^*$, where p^* is defined by (1.3). Then there exists a sequence δ_k such that $\delta_k > 0$, $\lim_{k \rightarrow \infty} \delta_k = 0$ and $\|u\|_r \leq \delta_k \|u\|_{1,p}$ for all $u \in Y_k$ with $k \in \mathbb{N}$.*

Since the codimension of Y_k is $k - 1$, Lemma 3.9 shows that

$$g(D_k) \cap \partial B(t, W) \cap Y_k \neq \emptyset \quad \text{for } g \in G_k, \quad 0 < t < R_k. \quad (3.23)$$

The next lemma is known (for example, see [22, Lemma 4.9]).

Lemma 3.11. ([22, Lemma 4.9]) *c_k is independent of the choice of R_k as far as R_k satisfies (3.18).*

Using Lemmas 3.10 and 3.11, we prove that c_k diverges to infinity in the lemma below.

Lemma 3.12. *c_k defined by (3.21) diverges to infinity as $k \rightarrow \infty$.*

Proof. It follows from (3.23) that for $0 < t < R_k$,

$$\max_{u \in D_k} I(g(u)) \geq \inf\{I(u) : u \in \partial B(t, W) \cap Y_k\},$$

which leads to

$$c_k \geq \inf\{I(u) : u \in \partial B(t, W) \cap Y_k\} \quad \text{for } 0 < t < R_k. \quad (3.24)$$

We shall compute the supremum of the right hand side for $t \in (0, R_k)$. Hereafter we denote by C various positive constants independent of u and k . Since $|f(x, u)| \leq C(|u|^{r-1} + 1)$ with $p < r < p^*$ by (ii) in Assumption 2.1, it holds that $|F(x, u)| \leq C(|u|^r + 1)$. Hence we have

$$I(u) \geq \frac{1}{p} \|\nabla u\|_p^p - C\|u\|_r^r - C.$$

Since $\|u\|_r \leq \delta_k \|\nabla u\|_p$ for $u \in Y_k$ by Lemma 3.10, we have

$$I(u) \geq \frac{1}{p} \|\nabla u\|_p^p - C\delta_k^r \|\nabla u\|_p^r - C \quad \text{for } u \in Y_k.$$

Using the inequality above with (3.24), we obtain

$$c_k \geq p^{-1}t^p - C\delta_k^r t^r - C,$$

for $0 \leq t \leq R_k$. We define $g(t) := p^{-1}t^p - C\delta_k^r t^r$. Then $c_k \geq g(t) - C$ for $0 \leq t \leq R_k$. Since $g(t)$ has a unique zero T_k in $(0, \infty)$, we replace R_k by a larger constant so that $R_k \geq T_k$. By Lemma 3.11, such a replacement leaves c_k invariant. The maximum of $g(t)$ is computed as

$$\max_{0 \leq t \leq R_k} g(t) = \frac{r-p}{rp} (Cr\delta_k^r)^{-p/(r-p)}.$$

Therefore

$$c_k \geq \frac{r-p}{rp} (Cr\delta_k^r)^{-p/(r-p)} - C,$$

which diverges to infinity as $k \rightarrow \infty$. The proof is complete. $\square$

Proof of Theorem 2.2. By Lemmas 3.6 and 3.12, c_k is a sequence of critical values diverging to infinity. A critical point u_k corresponding to c_k is a weak solution of (1.1) and its $W_0^{1,p}(\Omega)$ -norm diverges to infinity as $k \rightarrow \infty$. By Proposition 3.1, u_k belongs to $C^{1,\theta}(\overline{\Omega})$. Since $\|u_k\|_{1,p} \leq C\|u_k\|_{C^1}$, the C^1 -norm of u_k diverges to infinity. The proof is complete. $\square$

4. Infinitely many small solutions

To get a sequence of solutions converging to zero, we use another type of the symmetric mountain pass lemma. Recall that $\gamma(A)$ denotes the genus of A , which is given by Definition 3.7. To use the symmetric mountain pass lemma, we introduce the assumption below.

Assumption 4.1. Let W be an infinite dimensional Banach space and let Γ_k denote the family of closed symmetric subsets A of W such that $0 \notin A$ and $\gamma(A) \geq k$. Suppose that $I \in C^1(W, \mathbb{R})$ is a functional satisfying (I1) and (I2) below.

- (I1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS).
- (PS) If u_k is a sequence in W such that $I(u_k)$ is bounded and $I'(u_k) \rightarrow 0$ as $k \rightarrow \infty$, then it has a convergent subsequence.
- (I2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Under the assumption above, we define d_k by

$$d_k := \inf_{A \in \Gamma_k} \sup_{u \in A} I(u).$$

We state the symmetric mountain pass lemma due to Clark [12] (see also [3], [32, Theorem 9.1] and [34]).

Lemma 4.2. (Symmetric mountain pass lemma) *Suppose that Assumption 4.1 holds. Then each d_k is a critical value of I , $d_k \leq d_{k+1} < 0$ for $k \in \mathbb{N}$ and d_k converges to zero. Moreover, if $d_k = d_{k+1} = \dots = d_{k+p} \equiv d$, then $\gamma(K_d) \geq p + 1$. Here K_d is defined by*

$$K_d := \{u \in W : I'(u) = 0, I(u) = d\}.$$

For the proof of the lemma above, we refer the readers to [3, 12] or [32, Theorem 9.1]. The convergence of d_k to zero is proved in [20]. By Lemma 4.2, we have infinitely many critical values. Here we state another symmetric mountain pass lemma, which was proved in our paper [20].

Lemma 4.3. ([20] symmetric mountain pass lemma) *Under Assumption 4.1, either (i) or (ii) below holds.*

- (i) *There exists a sequence u_k such that $I'(u_k) = 0$, $I(u_k) < 0$ and u_k converges to zero.*
- (ii) *There exist two sequences u_k and v_k such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k \rightarrow \infty} I(v_k) = 0$, and v_k converges to a non-zero limit.*

In any case (i) or (ii), there exists a sequence u_k of critical points such that $I'(u_k) = 0$, $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.

Remark 4.4. We shall clarify the difference between Lemmas 4.2 and 4.3. Lemma 4.2 guarantees the existence of a sequence of critical values converging to zero in $\mathbb{R}$. On the other hand, Lemma 4.3 assures the existence of a sequence of critical points converging to zero in W . Since our purpose is to get a sequence of solutions for (1.1) converging to zero in $C^1(\bar{\Omega})$, Lemma 4.3

is appropriate for our purpose. However we have a question whether a critical point corresponding to d_k converges to zero in W . This is true if the equation

$$I(u) = 0 \quad \text{and} \quad I'(u) = 0 \quad (4.1)$$

has a unique solution $u = 0$. Clearly, under this condition, the Palais-Smale condition (PS) assures that any critical point corresponding to d_k converges to zero in W . However, if (4.1) has a non-trivial solution, a critical point corresponding to d_k does not necessarily converge to zero. In fact, we gave an example of a functional I and a Hilbert space W in [20] such that I is defined on W and it satisfies Assumption 4.1, but $\|u\| \geq c > 0$ whenever $I(u) = d_k$ and $I'(u) = 0$, where c is independent of k . Therefore no critical point corresponding to d_k converges to zero. For the Lagrangian functional I defined by (3.11), Eq. (4.1) is rewritten as

$$\int_{\Omega} \left(\frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q - F(x, u) \right) dx = 0,$$

and

$$-\Delta_p u - \Delta_q u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

However, it is uncertain whether the system of the two equations above has the trivial solution only. Therefore we use Lemma 4.3 to get a sequence of solutions converging to zero in W .

Recall that $f(x, u)$ is defined on $\overline{\Omega} \times [-\varepsilon_0, \varepsilon_0]$ in Assumption 2.5. To prove Theorem 2.6, we need to extend $f(x, u)$ to $\overline{\Omega} \times \mathbb{R}$. Choose a function $g(u)$ such that $g(u)$ is even and continuous on $\mathbb{R}$, $0 \leq g(u) \leq 1$ in $\mathbb{R}$, $g(u) = 1$ for $|u| \leq \varepsilon_0/2$ and $g(u) = 0$ for $|u| \geq \varepsilon_0$. Define $h(x, u) := f(x, u)g(u)$. Then it satisfies Assumption 2.5. Instead of (1.1), we solve the equation

$$-\Delta_p u - \Delta_q u = h(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (4.2)$$

If (4.2) would have a sequence u_k of solutions converging to zero in $C^1(\overline{\Omega})$, then $\|u_k\|_{\infty} < \varepsilon_0/2$ for all large k . Then $h(x, u_k) = f(x, u_k)$, and so u_k becomes a solution of (1.1). Thus we get a sequence of solutions for (1.1) converging to zero. Hereafter we rewrite $h(x, u)$ as $f(x, u)$. Then f and F are bounded, i.e.,

$$\sup_{\overline{\Omega} \times \mathbb{R}} |f(x, u)| < \infty, \quad \sup_{\overline{\Omega} \times \mathbb{R}} |F(x, u)| < \infty.$$

Proof of Theorem 2.6. To prove Theorem 2.6, it is enough to verify that I defined by (3.11) satisfies Assumption 4.1. Since $F(x, u)$ is bounded, $I(u)$ is bounded from below in $W_0^{1,p}(\Omega)$. Obviously, I is even and $I(0) = 0$. Since $f(x, u)$ and $F(x, u)$ are bounded, they satisfy (ii) and (iv) of Assumption 2.1. By Lemma 3.4, I satisfies the Palais-Smale condition.

Let us show that I satisfies (I2). Since X_k given by (3.17) is a finite dimensional space, any norm in X_k is equivalent to each other. Hence there exist constants $a_k, b_k > 0$ such that

$$\|\nabla u\|_p \leq a_k \|\nabla u\|_q, \quad b_k \|\nabla u\|_q \leq \|u\|_q, \quad \|u\|_{\infty} \leq a_k \|\nabla u\|_q, \quad (4.3)$$

for $u \in X_k$. We choose $M_k > 0$ so large that $1/q < M_k b_k^q$ when $q < p$ and $1/q + a_k^p/p < M_k b_k^q$ when $q = p$. By (2.8), we can choose a $\delta_k > 0$ such that

$$F(x, u) \geq M_k |u|^q \quad \text{for } |u| \leq \delta_k, \quad x \in B(x_0, \varepsilon). \quad (4.4)$$

We define

$$A := \{u \in X_k : \|\nabla u\|_q = \nu\},$$

where $\nu > 0$ will be determined later on. Since A is homeomorphic to the $(k-1)$ -dimensional sphere S^{k-1} by an odd mapping, it has genus k , i.e., $\gamma(A) = k$. We shall prove that I satisfies (I2). By the third inequality in (4.3), if $a_k \nu < \delta_k$, then $\|u\|_\infty < \delta_k$ for $u \in A$. Recall that the support of $u \in X_k$ is in $\overline{B}(x_0, \varepsilon)$. Therefore (4.4) implies that $F(x, u(x)) \geq M_k |u(x)|^q$ for $u \in A$ with $\nu \in (0, \delta_k/a_k)$. Using this inequality and (4.3), for $u \in A$ with $\nu \in (0, \delta_k/a_k)$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{q} \|\nabla u\|_q^q - \int_{B(x_0, \varepsilon)} F(x, u) dx \\ &\leq \frac{a_k^p}{p} \|\nabla u\|_q^p + \frac{1}{q} \|\nabla u\|_q^q - M_k \|u\|_q^q \\ &\leq -(M_k b_k^q - 1/q) \nu^q + \frac{a_k^p}{p} \nu^p < 0, \end{aligned}$$

provided that $\nu > 0$ is small enough. Therefore $\sup_{u \in A} I(u) < 0$ and hence I satisfies Assumption 4.1. By Lemma 4.3, (1.1) has a sequence of non-trivial weak solutions u_k in $W_0^{1,p}(\Omega)$ such that $\|u_k\|_{1,p} \rightarrow 0$ as $k \rightarrow \infty$. By Proposition 3.1, u_k belongs to $C^{1,\theta}(\overline{\Omega})$ and the $C^{1,\theta}(\overline{\Omega})$ -norm of u_k is bounded, where $\theta \in (0, 1)$ is independent of k . Since $C^{1,\theta}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$ because of the Ascoli-Arzelà theorem, the $C^1(\overline{\Omega})$ -norm of u_k converges to zero. The proof is complete. $\square$

5. A priori estimates of solutions

In this section, we shall prove all the theorems remained.

Proof of Theorem 2.9. Suppose that the theorem is false. Then there exists a sequence u_n of non-trivial solutions for (1.1) such that $\|u_n\|_{C^1} \rightarrow 0$ as $n \rightarrow \infty$. We put $\alpha_n := \|u_n\|_\infty$ and $v_n(x) := u_n(x)/\alpha_n$. Then $\|v_n\|_\infty = 1$ and v_n satisfies

$$-\alpha_n^{p-1} \Delta_p v_n - \alpha_n^{q-1} \Delta_q v_n = f(x, u_n).$$

Dividing both sides by α_n^{q-1} and putting $g_n(x) := f(x, u_n)/|u_n|^{q-2} u_n$, we obtain

$$-\alpha_n^{p-q} \Delta_p v_n - \Delta_q v_n = g_n(x) |v_n|^{q-2} v_n. \quad (5.1)$$

Since $\|v_n\|_\infty = 1$ and $g_n(x)$ uniformly converges, the right hand side is bounded in $L^\infty(\Omega)$. Although α_n converges to zero, Lieberman's theorem (Theorem 1.7 in [25]) and the sentence after Theorem 1.7) ensures that v_n is bounded in

$C^{1,\theta}(\overline{\Omega})$ with some $\theta \in (0, 1)$, where θ is independent of n . Let $\phi \in C_0^\infty(\Omega)$ be any test function. Multiplying (5.1) by ϕ and integrating it over Ω , we have

$$\int_{\Omega} (\alpha_n^{p-q} |\nabla v_n|^{p-2} \nabla v_n \nabla \phi + |\nabla v_n|^{q-2} \nabla v_n \nabla \phi - g_n(x) |v_n|^{q-2} v_n \phi) dx = 0.$$

Since v_n is bounded in $C^{1,\theta}(\overline{\Omega})$ and this space is compactly embedded in $C^1(\overline{\Omega})$, along a subsequence, v_n converges to a limit v in $C^1(\overline{\Omega})$. Since $g_n(x)$ converges to $f_0(x)$ uniformly on x , we let $n \rightarrow \infty$ to get

$$\int_{\Omega} (|\nabla v|^{q-2} \nabla v \nabla \phi - f_0(x) |v|^{q-2} v \phi) dx = 0.$$

Since $\|v_n\|_\infty = 1$, $\|v\|_\infty = 1$ also holds. Thus v is a non-trivial solution of (2.10). This contradicts the assumption of the theorem. Therefore we have an a priori lower estimate $\|u\|_{C^1} \geq c > 0$. The proof is complete. $\square$

Proof of Proposition 2.12. Let $f_0(x)$ satisfy Assumption 2.11. We use a contradiction. Suppose that (2.10) has a non-trivial solution u . Multiplying (2.10) by u and integrating it over Ω , we have

$$\int_{\Omega} |\nabla u|^q dx = \int_{\Omega} f_0(x) |u|^q dx. \quad (5.2)$$

First, we assume that $f_0(x) < \Lambda(q)$ in Ω . Then it holds that

$$\int_{\Omega} |\nabla u|^q dx < \Lambda(q) \int_{\Omega} |u|^q dx,$$

which contradicts (3.14) with p replaced by q . Therefore (2.10) has no non-trivial solution.

Next, we assume that $f_0(x)$ satisfies (2.16). Using (2.13), (2.16) and the Hölder inequality, we estimate the right hand side of (5.2) as

$$\begin{aligned} \int_{\Omega} f_0(x) |u|^q dx &\leq \int_{\Omega} f_0^+(x) |u|^q dx \leq \|f_0^+\|_{N/q} \|u\|_{Nq/(N-q)}^q \\ &\leq C_1(N, q)^q \|f_0^+\|_{N/q} \|\nabla u\|_q^q < \|\nabla u\|_q^q, \end{aligned}$$

which contradicts (5.2). When $f_0(x)$ satisfies (2.17), we compute

$$\int_{\Omega} f_0^+(x) |u|^q dx \leq \|f_0^+\|_{t/(t-q)} \|u\|_t^q \leq C_2(N, t)^q \|f_0^+\|_{t/(t-q)} \|\nabla u\|_q^q < \|\nabla u\|_q^q.$$

When $f_0(x)$ satisfies (2.18), we have

$$\int_{\Omega} f_0^+(x) |u|^q dx \leq \|f_0^+\|_1 \|u\|_\infty^q \leq C_3(N, q)^q \|f_0^+\|_1 \|\nabla u\|_q^q < \|\nabla u\|_q^q.$$

All the cases lead to a contradiction. The proof is complete. $\square$

We conclude this paper by proving Theorem 2.14.

Proof of Theorem 2.14. We shall prove (i). Let $s < q$ and $b(x_0) > 0$ at some $x_0 \in \Omega$. We put

$$f(x, u) := a(x) |u|^{r-2} u + b(x) |u|^{s-2} u.$$

Since $b(x) \geq b_0 > 0$ in $B(x_0, \varepsilon)$ with small ε , $b_0 > 0$, (2.9) holds. Corollary 2.7 shows the existence of a sequence of solutions converging to zero.

Impose the assumption of (ii). If $q < s$, then we have

$$f_0(x) := \lim_{u \rightarrow 0} f(x, u)/|u|^{q-2}u = 0.$$

Therefore (2.10) has no non-trivial solutions. If $q = s$, then $f_0(x) = b(x)$. If Assumption 2.11 holds, then (2.10) has no non-trivial solutions. Consequently, Theorem 2.9 gives an a priori lower estimate $\|u\|_{C^1} \geq c > 0$. Thus (ii) holds.

Let $r < p$. Multiplying (1.1) by u and integrating it over Ω , we have

$$\|\nabla u\|_p^p + \|\nabla u\|_q^q = \int_{\Omega} (a(x)|u|^r + b(x)|u|^s) dx \leq C_0(\|\nabla u\|_p^r + \|\nabla u\|_p^s),$$

with a certain constant $C_0 > 0$. Since $s < r < p$, the above inequality implies an a priori upper estimate $\|u\|_{C^1} \leq C$. Hence (iii) holds.

Impose the assumption of (iv). Then Assumption 2.1 is fulfilled with $\alpha = r$. Consequently, Theorem 2.2 ensures the existence of a sequence of solutions diverging to infinity.

The assertion (v) follows directly from (ii) and (iii). The assertion (vi) is obtained by combining (i) and (iv). The proof is complete. $\square$

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Received: 15 February 2016.

Accepted: 21 July 2016.