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Three solutions for a class of higher dimensional singular problems

Francesca Faraci and George Smyrlis

Abstract. In the present paper we establish the existence of three positive weak solutions for a quasilinear elliptic problem involving a singular term of the type $u^{-\gamma}$. As far as we know this is the first contribution in the higher dimensional case for arbitrary $\gamma > 0$.

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1. Introduction

In the present paper we consider the following singular quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu a(x) u^{-\gamma}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P}_{\lambda,\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(1 < p \le N)$, $f : \Omega \times [0,+\infty[\to [0,+\infty[$ is a Carathéodory function not identically zero, $a : \Omega \to \mathbb{R}$ is a function in $L^{(p^*)'}(\Omega)$ which is positive almost everywhere (being p^* the critical Sobolev exponent and $(p^*)'$ its conjugate), λ, μ are positive parameters and γ is a positive real number.

There is a wide literature dealing with existence and multiplicity results for quasilinear elliptic problems depending on a singular term of the type $u^{-\gamma}$ when $0 < \gamma < 1$. In such a case it is natural to associate to problem $(\mathcal{P}_{\lambda,\mu})$ the following energy functional

$$\mathscr{E}_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \lambda \int_\Omega \int_0^{u(x)} f(x,t)dt\, dx + \frac{\mu}{\gamma - 1}\int_\Omega a(x) u(x)^{-\gamma+1} dx$$

which is well defined on the Sobolev space $W_0^{1,p}(\Omega)$. Although not continuously Gâteaux differentiable, under standard assumptions $\mathscr{E}_{\lambda,\mu}$ is continuous and variational methods are still applicable (see for example [4,5,13] and the references therein).

When $\gamma \geq 1$, such kind of problems have been less investigated. Notice in fact that the above functional (when $\gamma > 1$) is not defined on the whole space $W_0^{1,p}(\Omega)$. However, the existence of one or two solutions can be still obtained in the framework of variational setting by using suitable truncation methods (see [7,8]) or techniques from non smooth analysis (see [1,6]).

As far as we know, the existence of three solutions for arbitrary $\gamma > 0$ has been investigated only in [12]. More precisely, the authors employ an abstract three critical points theorem in the lower dimensional case, i.e. when $p > N$, being crucial the continuous embedding of $W_0^{1,p}(\Omega)$ into $C(\overline{\Omega})$.

In the present paper we derive three positive solutions to the problem $(\mathcal{P}_{\lambda,\mu})$ for all $\gamma > 0,\ 1 < p \leq N$, for λ large enough and μ sufficiently small. In our approach we combine an abstract multiplicity result by Ricceri [9] with techniques from non smooth analysis. With respect to [1] and [6] where a suitable variational approach is provided, we employ some topological arguments to guarantee the existence of two local minimizers. The third solution is given by a suitable version of the Mountain Pass Theorem for Szulkin functionals. To the best of our knowledge, this is the first contribution establishing three solutions in the higher dimensional case and for any $\gamma > 0$.

Under further restrictions on the function a we also show that the solutions belong to $C^1(\Omega)$. This work extends a previous result of the authors [4] in which three solutions are obtained for $0 < \gamma < 1$ by means of a different approach which exploits truncation techniques and classical critical point theory. It is to be mentioned that in [4], extra properties of the solutions are derived, namely, the solutions lie in $\text{int}(C_0^1(\overline{\Omega}))_+$ and are bounded in norms uniformly with respect to the parameters.

First of all we clarify the meaning of solution we adopt in the sequel.

Definition 1.1. A weak solution of $(\mathcal{P}_{\lambda,\mu})$ is a function $u \in W_0^{1,p}(\Omega)$ such that

(i) $u > 0$ almost everywhere in Ω,

(ii) $au^{-\gamma}\varphi \in L^1(\Omega)$, for all $\varphi \in W_0^{1,p}(\Omega)$,

(iii) $\displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \ dx = \int_\Omega [\lambda f(x,u) + \mu a u^{-\gamma}]\varphi \ dx$, for all $\varphi \in W_0^{1,p}(\Omega)$.

We assume that for almost all $x \in \Omega$, $f(x,0) = 0$ and for almost all $x \in \Omega$ and all $t \geq 0$

(H) $\quad f(x,t) \leq c(1 + t^{r-1})$

where $c > 0,\ 1 < r < p^*$.

Let also $F : \Omega \times [0,+\infty[\to \mathbb{R}$ be the primitive of f, i.e.

$$F(x,t) = \int_0^t f(x,\tau)d\tau.$$

We require that:

(H_1) $\quad \displaystyle\lim_{t\to 0^+} \frac{\sup_{x\in\Omega} F(x,t)}{t^p} = 0;$

(H_2) $$\lim_{t\to+\infty} \frac{\sup_{x\in\Omega} F(x,t)}{t^p} = 0;$$

(H_3) there exists $\bar{u} \in C_0^1(\overline{\Omega})$ such that $\bar{u} > 0$ on Ω and $a\bar{u}^{-\gamma} \in L^{(p^*)'}(\Omega)$.

Assumption (H_3) is a standard hypothesis in the setting of singular problems with arbitrary $\gamma > 0$ (see for example [8]).

As usual, if $u \in W_0^{1,p}(\Omega)$, we denote by u^+ its positive part, that is, $u^+ = \max\{u, 0\}$.

Our main multiplicity result is the following

Theorem 1.1. *In addition to* (H), *assume* $(H_1), (H_2), (H_3)$.

Set

$$\lambda^* = \frac{1}{p} \inf \left\{ \frac{\int_\Omega |\nabla u(x)|^p\, dx}{\int_\Omega F(x, u^+(x))\, dx} \ : \ \int_\Omega F(x, u^+(x))\, dx > 0 \right\}. \tag{1}$$

Then, for each $\lambda > \lambda^*$ *there exists* $\mu^* > 0$ *such that for each* $\mu \in]0, \mu^*]$, *the problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least three weak solutions.*

In order to derive weak solutions in $C^1(\Omega)$, we replace hypothesis (H_3) by the following stronger one:

(H_3') there exist $\bar{u} \in C_0^1(\overline{\Omega})$, $q > p'N$ such that $\bar{u} > 0$ on Ω and $a\bar{u}^{-\gamma} \in L^q(\Omega)$ (where $p' = p/(p-1)$).

We have the following

Theorem 1.2. *In addition to* (H), *assume* $(H_1), (H_2), (H_3')$ *and let* λ^* *be as in* (1).

Then, for each $\lambda > \lambda^*$ *there exists* $\mu^* > 0$ *such that for each* $\mu \in]0, \mu^*]$, *the problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least three weak solutions in* $C^1(\Omega)$.

2. Preliminaries

Let us recall first some well known facts from non-smooth analysis (see [11]).

Let X be a Banach space and Φ, Ψ be two functionals with $\Phi \in C^1(X)$ and $\Psi : X \to]-\infty, +\infty]$ proper, convex and lower semicontinuous. A point $u \in X$ is said to be *critical for the Szulkin functional* $I = \Phi + \Psi$ if $u \in \mathrm{dom}\Psi(u) = \{u \in X : \ \Psi(u) < +\infty\}$ and

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0 \quad \text{for all } v \in X.$$

It is well known that a local minimum of I is a critical point of I.

We also say that I satisfies the *Palais Smale condition* if the following holds:

(PS) If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\| \ \text{ for all } v \in X, \ \ n \in \mathbb{N}$$

(where $\{\varepsilon_n\}$ is a sequence of positive numbers converging to zero) then, it has a strongly convergent subsequence.

Theorem 2.1. ([11], Corollary 3.3) *Suppose that* I *satisfies* (PS). *If* I *has two local minima, then it has at least three critical points.*

If X is a reflexive Banach space, a map $A : X \to X^*$ satisfies the $(S)_+$ condition if the following holds:

$(S)_+$ If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $u_n \rightharpoonup u$ (weakly) and

$$\limsup_n \langle A(u_n), u_n - u\rangle \leq 0,$$

then, it strongly converges to u.

Remark 2.1. Let X be a reflexive Banach space, $\Phi \in C^1(X)$, $\Psi : X \to]-\infty, +\infty]$ such that Φ is sequentially weakly lower semicontinuous with Φ' of type $(S)_+$ and Ψ is proper, convex and lower semicontinuous. If $I = \Phi + \Psi$ is coercive, then I satisfies (PS).

Proof. Indeed, choose $\{u_n\}_{n\in\mathbb{N}} \subseteq X$, $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq]0, +\infty[$ such that $I(u_n) \to c \in \mathbb{R}$, $\varepsilon_n \to 0$ and

$$\langle \Phi'(u_n),\ v - u_n\rangle + \Psi(v) - \Psi(u_n) \ \geq\ -\varepsilon_n\|u_n - v\|, \quad \text{for all } v \in X,\ n \geq 1.$$

By coercivity of I and by passing to subsequences, we may assume that $u_n \rightharpoonup u$. Since I is sequentially weakly lower semicontinuous, we have $I(u) \leq \liminf_n I(u_n) = c < +\infty$, thus $\Psi(u) < +\infty$. Putting $v = u$ in the above inequality we obtain

$$\langle \Phi'(u_n),\ u_n - u\rangle \ \leq\ \Psi(u) - \Psi(u_n) + \varepsilon_n\|u_n - u\|, \quad \text{for all } n \geq 1$$

and hence, $\limsup_n \langle \Phi'(u_n),\ u_n - u\rangle \ \leq\ 0$ (recall that Ψ is weakly lower semicontinuous). Exploiting the fact that Φ' is of type $(S)_+$, we infer that $u_n \to u$ strongly. □

We will need the following theorem which we state here in a convenient form for our purposes:

Theorem A([10], Theorem C) *Let X be a reflexive and separable real Banach space, $I : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous functional and $p > 0$. Denote by $\mathcal{I} : X \to \mathbb{R}$ the functional*

$$\mathcal{I}(u) = \frac{1}{p}\|u\|^p + I(u)$$

and assume that $\mathcal{I}$ is coercive. Then, any strict local minimizer of $\mathcal{I}$ in the strong topology is so in the weak topology.

The following is a convenient form of the *Weak Comparison Principle* whose proof is standard and we omit:

Proposition 2.1. Let $g : \Omega\times]0, +\infty[\to \mathbb{R}$ be a Carathéodory function, nonincreasing with respect to the real variable and $u, w \in W^{1,p}(\Omega)$ are such that $u > 0$, $w > 0$ almost everywhere in Ω and $w\,|_{\partial\Omega} \leq u\,|_{\partial\Omega}$ in the sense of trace. Suppose that

$$-\Delta_p u(x) - g(x, u(x)) \ \geq\ -\Delta_p w(x) - g(x, w(x)),$$

that is, for each non negative $\varphi \in W_0^{1,p}(\Omega)$ we have $g(x,u)\varphi,\ g(x,w)\varphi \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_\Omega g(x,u)\varphi dx \geq \int_\Omega |\nabla w|^{p-2}\nabla w \nabla\varphi dx - \int_\Omega g(x,w)\varphi dx.$$

Then, $u \geq w$ almost everywhere in Ω.

We conclude this section with a topological remark which will be useful in the sequel.

Proposition 2.2. Let X be a Hausdorff topological space and $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty compact subsets of X s.t. $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$ and

$$\bigcap_{n=1}^{\infty} K_n = D_1 \cup D_2, \quad D_1 \cap D_2 = \emptyset,$$

where $D_1,\ D_2$ are nonempty and compact. Then, there exist $\bar{n} \in \mathbb{N}$ and $C_1,\ C_2$ nonempty compact sets such that

$$K_{\bar{n}} = C_1 \cup C_2, \quad C_1 \cap C_2 = \emptyset, \quad D_1 \subseteq C_1, \quad D_2 \subseteq C_2.$$

Proof. Since X is a Hausdorff topological space, the compact sets $D_1,\ D_2$ can be separated by two disjoint open sets, i.e. there exist disjoint, nonempty, open sets $A,\ B$ such that $D_1 \subseteq A,\ D_2 \subseteq B$. So, $K_1 \subseteq \bigcup_{n=2}^{\infty}(X \setminus K_n) \cup A \cup B$. By compactness and since $\{K_n\}_{n\in\mathbb{N}}$ is nonincreasing, we may find $\bar{n} \geq 2$ such that $K_1 \subseteq (X\setminus K_{\bar{n}}) \cup A \cup B$, thus $K_{\bar{n}} \subseteq A \cup B$. Then we set $C_1 = K_{\bar{n}} \setminus B,\ C_2 = K_{\bar{n}} \setminus A$. □

3. Proofs of Theorems 1.1, 1.2.

In order to prove our main results, let us introduce the functional $\mathscr{E}_{\lambda,\mu}$ associated to problem $(\mathcal{P}_{\lambda,\mu})$ and some of its properties.

Without loss of generality we can assume that $f(x,t) = 0$ for almost all $x \in \Omega$ and all $t \leq 0$. Set, for all $\lambda > 0$, $\Phi_\lambda = H - \lambda J$, being $H, J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$H(u) = \frac{1}{p}\|u\|^p = \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx,$$

$$J(u) = \int_\Omega F(x, u(x))dx,$$

respectively, where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is the function

$$F(x,t) = \int_0^t f(x,\tau)d\tau.$$

Because of assumption (H), the functional Φ_λ turns out to be of class C^1 on the Sobolev space $W_0^{1,p}(\Omega)$.

Define $G : \Omega \times \mathbb{R} \to]-\infty, +\infty]$ as it follows:

$$\text{if} \quad 0<\gamma<1, \quad G(x,t) = \begin{cases} -\dfrac{a(x)}{1-\gamma} t^{-\gamma+1}, & \text{if } x \in \Omega \ \text{ and } \ t \geq 0 \\ +\infty, & \text{if } x \in \Omega \ \text{ and } \ t < 0 \end{cases}$$

$$\text{if} \quad \gamma = 1, \quad G(x,t) = \begin{cases} -a(x) \ln t, & \text{if } x \in \Omega \ \text{ and } \ t > 0 \\ +\infty, & \text{if } x \in \Omega \ \text{ and } \ t \leq 0 \end{cases}$$

$$\text{if} \quad \gamma > 1, \quad G(x,t) = \begin{cases} \dfrac{a(x)}{\gamma-1} t^{-\gamma+1}, & \text{if } x \in \Omega \ \text{ and } \ t > 0 \\ +\infty, & \text{if } x \in \Omega \ \text{ and } \ t \leq 0. \end{cases}$$

Remark 3.1. For almost every $x \in \Omega$

1. $G(x, \cdot)$ is convex,
2. $G(x, \cdot)$ is lower semicontinuous,
3. $G(x, \cdot)$ belongs to $C^1(]0, +\infty[)$ and $G'(x,t) = -a(x)t^{-\gamma}$ for all $t > 0$.

Remark 3.2. 1. From assumption (H_3), it follows at once that $a \in L^{(p^*)'}(\Omega)$.

2. For all $u \in W_0^{1,p}(\Omega)$, the function $a \cdot u \in L^1(\Omega)$ and from Hölder's inequality, there exists a constant c (the embedding constant of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$)such that

$$\int_\Omega a(x)|u(x)|dx \ \leq c\|a\|_{(p^*)'} \ \|u\|.$$

Moreover, if $u \in W_0^{1,p}(\Omega)$ with $u \geq 0$ a.e. in Ω, then for $0 < \gamma < 1$ we have $au^{1-\gamma} \in L^1(\Omega)$. [This follows from the above inequality combined with the inequality $t^{1-\gamma} \leq 1 + t$ which holds for $t \geq 0$.]

3. For all $u \in W_0^{1,p}(\Omega)$, the superposition operator associated to the function G satisfies the following inequalities:

$$\begin{array}{lll} \text{if} \quad 0<\gamma<1, & G(\cdot, u(\cdot)) \geq -\dfrac{a(\cdot)}{1-\gamma}(u(\cdot)+1) \in L^1(\Omega) \\ \text{if} \quad \gamma = 1, & G(\cdot, u(\cdot)) \geq -a(\cdot)u(\cdot) \in L^1(\Omega) \\ \text{if} \quad \gamma > 1, & G(\cdot, u(\cdot)) \geq 0. \end{array}$$

Define $\Psi : W_0^{1,p}(\Omega) \to]-\infty, +\infty]$ by

$$\Psi(u) = \begin{cases} \displaystyle\int_\Omega G(x,u)dx, & \text{if } G(x,u) \in L^1(\Omega) \\ +\infty, & \text{if } G(x,u) \notin L^1(\Omega). \end{cases}$$

Using Remark 3.1, Ψ turns out to be convex and lower semicontinuous (thus, sequentially weakly lower semicontinuous). Moreover, Ψ is proper. In fact, we have the following

Lemma 3.1. Assume (H_3). Then,

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) \subset \operatorname{dom}(\Psi)$$

where $\mathrm{int}(C_0^1(\overline{\Omega})_+)$ denotes the interior in the ordered Banach space $C_0^1(\overline{\Omega})$ of the positive cone

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \ \ \forall\, x \in \Omega\}.$$

Proof. Let $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Choose a C^1- open ball $B(u,\delta)$ $(\delta > 0)$ such that $B(u,\delta) \subset \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Therefore, for $\varepsilon > 0$ small enough, $u - \varepsilon\bar{u} \in B(u,\delta) \subset \mathrm{int}(C_0^1(\overline{\Omega})_+)$, that implies $u(x) > \varepsilon\bar{u}(x)$ for every $x \in \Omega$. Then, $0 \leq au^{-\gamma} \leq \varepsilon^{-\gamma} a\bar{u}^{-\gamma} \in L^{(p^*)'}(\Omega)$ (see (H_3)).

If $\gamma = 1$, then

$$|G(\cdot,u)| = a|\ln u| \leq \frac{a}{u} \max_{t\in]0,\|u\|_\infty]} |t\ln t| \in L^{(p^*)'}(\Omega) \subset L^1(\Omega).$$

If $\gamma \neq 1$, then,

$$|G(\cdot,u)| = a\frac{u^{-\gamma+1}}{|-\gamma+1|}\,.$$

By Hölder inequality, $au^{-\gamma+1} \in L^1(\Omega)$ as the product of $au^{-\gamma} \in L^{(p^*)'}(\Omega)$ and of $u \in L^{p^*}(\Omega)$, so the latter is a function of $L^1(\Omega)$. □

Define $\mathscr{E}_{\lambda,\mu} : W_0^{1,p}(\Omega) \to]-\infty, +\infty]$ by

$$\mathscr{E}_{\lambda,\mu}(u) = \Phi_\lambda(u) + \mu\Psi(u), \quad \lambda,\ \mu \geq 0.$$

$\mathscr{E}_{\lambda,\mu}$ is a Szulkin functional according to the definition given at the beginning of Sect. 2.

Lemma 3.2. If u is a critical point of $\mathscr{E}_{\lambda,\mu}$, then u is a weak solution of $(\mathcal{P}_{\lambda,\mu})$.

Proof. By definition, u is a critical point of $\mathscr{E}_{\lambda,\mu}$ if $u \in \mathrm{dom}(\Psi)$ and

$$\langle \Phi_\lambda'(u), v - u\rangle + \mu(\Psi(v) - \Psi(u)) \ \geq \ 0 \quad \text{for all} \quad v \in W_0^{1,p}(\Omega). \tag{2}$$

Let us prove first that $u > 0$ almost everywhere in Ω.

Since $\int_\Omega |G(x,u(x))|dx < +\infty$, then $G(\cdot,u(\cdot))$ is finite almost everywhere. We distinguish two cases.

When $\gamma \geq 1$, according to the definition of G, $u > 0$ almost everywhere.

When $0 < \gamma < 1$, from the definition of G, $u \geq 0$ almost everywhere.

Assume that there exists a set of positive measure A such that $u = 0$ in A. Let $\varphi : \Omega \to \mathbb{R}$ be a function in $W_0^{1,p}(\Omega)$, positive in Ω. Let $\varepsilon > 0$ small enough and define $v = u + \varepsilon\varphi$. Notice that since $G(x,\cdot)$ is a decreasing function in $[0,+\infty[$, we have $G(\cdot,v(\cdot)) \leq G(\cdot,u(\cdot)) \in L^1(\Omega)$. This fact together with Remark 3.2 implies that $v \in \mathrm{dom}(\Psi)$. Plugging v into (2), dividing by ε and using the monotonicity of $G(x,\cdot)$, that is $G(\cdot,u(\cdot)+\varepsilon\varphi(\cdot)) \leq G(\cdot,u(\cdot))$ almost everywhere in Ω, we get

$$
\begin{aligned}
0 \le & \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx - \lambda\int_\Omega f(x,u)\varphi\, dx \\
& + \frac{\mu}{\varepsilon}\int_A [G(x,\varepsilon\varphi)-G(x,0)]\, dx + \frac{\mu}{\varepsilon}\int_{\Omega\setminus A}[G(x,u+\varepsilon\varphi)-G(x,u)]\, dx \\
\le & \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx - \lambda\int_\Omega f(x,u)\varphi\, dx \\
& - \frac{\mu}{(1-\gamma)}\varepsilon^{-\gamma}\int_A a(x)\varphi(x)^{-\gamma+1}dx \to -\infty \text{ as } \varepsilon \to 0^+
\end{aligned}
$$

(notice that $\int_A a(x)\varphi(x)^{-\gamma+1}dx \in \mathbb{R}^+$, thanks to Remark 3.2). The contradiction ensures that $u > 0$ almost everywhere in Ω.

Next, let us prove that

$$
au^{-\gamma}\varphi \in L^1(\Omega) \quad \text{for all } \varphi \in W_0^{1,p}(\Omega) \tag{3}
$$

and that

$$
\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\, dx - \lambda\int_\Omega f(x,u)\varphi\, dx - \mu\int_\Omega a(x)u^{-\gamma}\varphi\, dx \ge 0 \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \quad \varphi \ge 0. \tag{4}
$$

Choose $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \ge 0$. Fix a decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq]0,1]$ with $\lim_n \varepsilon_n = 0$. The functions

$$
h_n(x) = \frac{G(x,u(x)) - G(x,u(x)+\varepsilon_n\varphi(x))}{\varepsilon_n}
$$

are measurable, non-negative and $\lim_n h_n(x) = a(x)u(x)^{-\gamma}\varphi(x)$ for almost all $x \in \Omega$. From Fatou's lemma, we deduce

$$
\int_\Omega a(x)u^{-\gamma}\varphi\, dx \le \liminf_n \int_\Omega h_n\, dx. \tag{5}
$$

From the definition of critical point, using again as test function $v = u + \varepsilon_n\varphi$, it follows that

$$
\mu\int_\Omega h_n\, dx \le \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\, dx - \lambda\int_\Omega f(x,u)\varphi\, dx.
$$

Passing to the liminf as $n \to \infty$ in the above inequality and taking into account (5), we deduce at once (3) (it is enough to prove the integrability for nonnegative test functions) and (4).

To proceed, first note that for each $c > 0$, we have that $cu \in \mathrm{dom}(\Psi)$. Indeed, by the definition of G we get that for $\gamma \ne 1$,

$$
|G(\cdot, cu(\cdot))| \le \frac{1}{|\gamma-1|}\, c^{1-\gamma}a(\cdot)u(\cdot)^{1-\gamma} \in L^1(\Omega) \quad \text{(see (3))}.
$$

For $\gamma = 1$, we have $G(\cdot, cu(\cdot)) = -\ln c\, a(\cdot) + G(\cdot, u(\cdot)) \in L^1(\Omega)$.

Now fix $\varepsilon \in]0,1[$ and plug $v=(1-\varepsilon)u$ into (2). Dividing by ε, there exists $\tau=\tau(\varepsilon)\in]0,\varepsilon[$ such that

$$\begin{aligned} 0 &\leq -\int_\Omega |\nabla u|^p\,dx + \lambda\int_\Omega f(x,u)u\,dx + \mu\int_\Omega \frac{G(x,u-\varepsilon u)-G(x,u)}{\varepsilon}\,dx \\ &= -\int_\Omega |\nabla u|^p\,dx + \lambda\int_\Omega f(x,u)u\,dx + \mu(1-\tau)^{-\gamma}\int_\Omega a(x)u^{-\gamma+1}\,dx. \end{aligned}$$

Passing to the limit as $\varepsilon\to 0$ and taking into account (4) we get

$$\int_\Omega |\nabla u|^p\,dx = \lambda\int_\Omega f(x,u)u\,dx + \mu\int_\Omega a(x)u^{-\gamma+1}\,dx. \tag{6}$$

Let $\varphi\in W_0^{1,p}(\Omega)$ and plug into (4) the test function $v=(u+\varepsilon\varphi)^+$ where ε is a positive number. Hence, by using (6) we have

$$\begin{aligned} 0 \leq &\int_{\{u+\varepsilon\varphi\geq 0\}} |\nabla u|^{p-2}\nabla u\nabla(u+\varepsilon\varphi)\,dx - \lambda\int_{\{u+\varepsilon\varphi\geq 0\}} f(x,u)(u+\varepsilon\varphi)\,dx \\ &-\mu\int_{\{u+\varepsilon\varphi\geq 0\}} a(x)u^{-\gamma}(u+\varepsilon\varphi)\,dx \\ = &\int_\Omega |\nabla u|^p\,dx + \varepsilon\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx \\ &-\lambda\int_\Omega f(x,u)u\,dx - \varepsilon\lambda\int_\Omega f(x,u)\varphi\,dx - \mu\int_\Omega a(x)u^{-\gamma+1}\,dx \\ &-\varepsilon\mu\int_\Omega a(x)u^{-\gamma}\varphi\,dx - \int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^p\,dx - \varepsilon\int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx \\ &+\lambda\int_{\{u+\varepsilon\varphi<0\}} f(x,u)(u+\varepsilon\varphi)\,dx + \mu\int_{\{u+\varepsilon\varphi<0\}} a(x)u^{-\gamma}(u+\varepsilon\varphi)\,dx \leq \\ \leq &\,\varepsilon\left[\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx - \lambda\int_\Omega f(x,u)\varphi\,dx - \mu\int_\Omega a(x)u^{-\gamma}\varphi\,dx\right] \\ &-\varepsilon\int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx. \end{aligned}$$

Notice that as $\varepsilon\to 0$, the measure of the set $\{u+\varepsilon\varphi<0\}$ tends to zero, so

$$\int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx \to 0.$$

Hence, dividing by ε and passing to the limit as $\varepsilon\to 0$, we get that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx - \lambda\int_\Omega f(x,u)\varphi\,dx - \mu\int_\Omega a(x)u^{-\gamma}\varphi\,dx \geq 0.$$

From the arbitrariness of φ, we get at once that u is a weak solution of $(\mathcal{P}_{\lambda,\mu})$. □

Lemma 3.3. Under hypotheses (H), (H_2) and (H_3), the functional $\mathscr{E}_{\lambda,\mu}$ satisfies (PS).

Proof. By using hypothesis (H) together with the Sobolev embedding theorem and the strong monotonicity of $-\Delta_p$, one may show that Φ'_λ is of type of $(S)_+$. Then due to Remark 2.1, it suffices to prove that $\mathscr{E}_{\lambda,\mu}$ is coercive.

By virtue of Remark 3.2, we may find positive constants c_1, c_2 s.t.

$$\Psi(u) \geq -c_1\|u\| - c_2, \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{7}$$

Next, fix $\varepsilon > 0$. From (H) and (H_2), we may choose $c_\varepsilon > 0$ such that for almost all $x \in \Omega$ and all $t \geq 0$

$$F(x,t) < \varepsilon t^p + c_\varepsilon.$$

Then for all $u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \lambda\int_\Omega F(x,u)dx \geq \frac{1}{p}\|u\|^p - \lambda\varepsilon\|u\|_p^p - \lambda c_\varepsilon|\Omega| \\ &\geq \left(\frac{1}{p} - \lambda c_3\varepsilon\right)\|u\|^p \; - \; \lambda c_\varepsilon|\Omega|,\end{aligned}$$

for some positive constant c_3.

Hence,

$$\mathscr{E}_{\lambda,\mu}(u) = \Phi_\lambda(u) + \mu\Psi(u) \geq \left(\frac{1}{p} - \lambda c_3\varepsilon\right)\|u\|^p \; - \; \lambda c_\varepsilon|\Omega| - \mu c_1\|u\| - \mu c_2 \,.$$

Choosing $\varepsilon > 0$ sufficiently small, we obtain the coercivity of $\mathscr{E}_{\lambda,\mu}$. □

We can prove now our multiplicity result. The existence of two local minimizers is obtained by following the ideas of Theorem 1 in [9], while the third solution of $(P_{\lambda,\mu})$ derives from Theorem 2.1.

Proof of Theorem 1.1.

From hypotheses $(H), (H_1)$ and (H_2) one has

$$\lim_{u\to 0}\frac{J(u)}{H(u)} = 0 \tag{8}$$

and

$$\lim_{\|u\|\to+\infty}\frac{J(u)}{H(u)} = 0. \tag{9}$$

Therefore, for each $\lambda > 0$, the functional Φ_λ is sequentially weakly lower semicontinuous and coercive [as it follows from (9)]. Let u_0 be a global minimizer of Φ_λ.

For every $s \in \mathbb{R}$, the set $\Phi_\lambda^{-1}(]-\infty, s])$ is weakly compact and metrizable (thus, weakly sequentially compact) with respect to the weak topology.

Choose $\lambda > \lambda^*$ [see (1)]. One has that the null function 0 turns out to be a strict local minimizer of Φ_λ [see (8)]. Applying Theorem A we get also that 0 is a strict local minimizer of Φ_λ in the weak topology. Thus, we may choose a weak neighborhood of zero U_w such that

$$0 = \Phi_\lambda(0) < \Phi_\lambda(u) \quad \text{for every } u \in U_w \setminus \{0\}.$$

By the choice of λ it follows that 0 is not a global minimum, that is

$$\Phi_\lambda(u_0) = \min_{W_0^{1,p}(\Omega)} \Phi_\lambda < 0 = \Phi_\lambda(0).$$

We may write

$$\Phi_\lambda^{-1}(]-\infty, 0]) = D_1 \cup D_2, \quad D_1 = \{0\}, \quad D_2 = \Phi_\lambda^{-1}(]-\infty, 0]) \setminus U_w \,.$$

Clearly $D_1,\ D_2$ are disjoint and weakly compact in X.

On the other hand, one has

$$\Phi_\lambda^{-1}(]-\infty,0]) = \bigcap_{n=1}^{\infty} \Phi_\lambda^{-1}\left(\left]-\infty, \frac{1}{n}\right]\right).$$

Exploiting Proposition 2.2 we may find $\bar{n} \geq 1$ and weakly compact sets $C_1,\ C_2$ s.t.

$$\Phi_\lambda^{-1}\left(\left]-\infty, \frac{1}{\bar{n}}\right]\right) = C_1 \cup C_2, \quad C_1 \cap C_2 = \emptyset, \quad D_1 \subseteq C_1, \quad D_2 \subseteq C_2.$$

Note that $C_1,\ C_2$ are also weakly sequentially compact and that $0 \in C_1,\ u_0 \in C_2$.

Put $\sigma = \frac{1}{\bar{n}}$. We separate $C_1,\ C_2$ by two disjoint weakly open sets A_1 and A_2 and we consider the sets

$$G_i = \{u \in A_i :\ \Phi_\lambda(u) < \sigma\}, \quad i = 1, 2.$$

Obviously, $G_i \subseteq C_i,\ \ i = 1, 2$ and $0 \in G_1,\ u_0 \in G_2$.

The sets $A_1,\ A_2$ are also open with respect to the strong topology. Moreover, the functional Φ_λ is continuous with respect to the strong topology, as the sum of two continuous functionals. It follows that the sets $G_1\ , G_2$ are open with respect to the strong topology.

We wish to prove that $\mathrm{dom}(\Psi) \cap G_i \neq \emptyset$, for $i = 1, 2$. By virtue of Lemma 3.1, it suffices to check that

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) \cap G_i \neq \emptyset, \quad i = 1, 2.$$

Since $0 \in G_1$, there exists a positive number ε such that $B(0, \varepsilon) \subset G_1$. Choose $\tilde{u} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Then $\frac{\varepsilon}{2\|\tilde{u}\|}\tilde{u} \in G_1$ and this proves that $\mathrm{dom}(\Psi) \cap G_1 \neq \emptyset$.

On the other hand, $u_0 \in G_2$ and u_0 is a critical point of Φ_λ, thus it is a weak solution of

$$-\Delta_p u = \lambda f(x, u), \quad u\ |_{\partial\Omega} = 0.$$

By classical regularity theory, $u_0 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and thus, $\mathrm{dom}(\Psi) \cap G_2 \neq \emptyset$.

Since Ψ is sequentially weakly lower semicontinuous and C_i are sequentially weakly compact sets, the infimum of Ψ on each C_i is attained.

Set

$$m_i = \inf\left\{\frac{\Psi(u) - \min_{C_i}\Psi}{\sigma - \Phi_\lambda(u)} : u \in \mathrm{dom}(\Psi) \cap G_i\right\} \quad i = 1, 2$$

and choose $\mu^* > 0$ s.t.

$$1/\mu^* \ > \ \max\{m_1,\ m_2\}.$$

Let $\mu \in]0, \mu^*]$. For each $i \in \{1, 2\}$, we have $1/\mu > m_i$, so there exists $y_i \in \mathrm{dom}(\Psi) \cap G_i$ s.t.

$$\sigma > \Phi_\lambda(y_i) + \mu\Psi(y_i) \ - \ \mu \min_{C_i} \Psi. \tag{10}$$

Since $\Phi_\lambda + \mu\Psi$ is sequentially weakly lower semicontinuous we may find $u_1 \in \mathrm{dom}(\Psi) \cap C_1$, $u_2 \in \mathrm{dom}(\Psi) \cap C_2$ s.t.

$$\min_{C_i}(\Phi_\lambda + \mu\Psi) = \Phi_\lambda(u_i) + \mu\Psi(u_i), \quad i = 1, 2.$$

We claim that $u_i \in G_i$, $i = 1, 2$. Suppose on the contrary that $\Phi_\lambda(u_i) \geq \sigma$, for some $i \in \{1, 2\}$. Then, by (10)

$$\Phi_\lambda(u_i) + \mu\Psi(u_i) \geq \sigma + \mu \min_{C_i} \Psi > \Phi_\lambda(y_i) + \mu\Psi(y_i),$$

which is a contradiction.

Since G_1, G_2 are open in the strong topology, we infer that u_1, u_2 are two local minima of $\Phi_\lambda + \mu\Psi$ (recall that $G_i \subseteq C_i$). Thus, from Theorem 2.1, we deduce the existence of a third critical point for $\mathscr{E}_{\lambda,\mu}$. Bearing in mind that local minimizers of $\mathscr{E}_{\lambda,\mu}$ are critical points of $\mathscr{E}_{\lambda,\mu}$, Lemma 3.2 ensures that problem $(\mathcal{P}_{\lambda,\mu})$ has at least three weak solutions.

Remark 3.3. It is clear that for $\mu = 0$ the thesis still holds. Indeed, 0 and the function u_0 are respectively a local and a global minimizer of Φ_λ. The existence of a third critical point for Φ_λ follows in a standard way through the classical Mountain Pass Theorem.

Proof of Theorem 1.2. It suffices to prove that for all λ, $\mu > 0$, each positive weak solution of the problem $(\mathcal{P}_{\lambda,\mu})$ lies in $C^1(\Omega)$. Fix $\lambda, \mu > 0$. Since $q > p'N > N$, from Proposition 2.1 of [8], there exists a unique weak solution $\nu \in \mathrm{int}(C_0^1(\overline{\Omega}))$ of the problem

$$-\Delta_p \nu = \mu a, \quad \nu\,|_{\partial\Omega} = 0.$$

Put $\underline{u} = \varepsilon^{\frac{1}{p-1}} \nu \in \mathrm{int}(C_0^1(\overline{\Omega}))$ with $\varepsilon \in]0, 1]$ and such that $\|\underline{u}\|_\infty \leq 1$. One has that

$$-\Delta_p \underline{u} - \mu a \underline{u}^{-\gamma} = \mu a(\varepsilon - \underline{u}^{-\gamma}) \leq 0.$$

Arguing as in Lemma 3.1, $a\underline{u}^{-\gamma} \in L^q(\Omega) \subseteq L^{(p^*)'}(\Omega)$ (recall that $q > N > (p^*)'$). Thus, $a\underline{u}^{-\gamma}\varphi \in L^1(\Omega)$, for all $\varphi \in W_0^{1,p}(\Omega)$. To proceed, let u be a positive weak solution of the problem $(\mathcal{P}_{\lambda,\mu})$. We have

$$-\Delta_p u \ - \ \mu a u^{-\gamma} = \lambda f(\cdot, u(\cdot)) \ \geq \ 0 \ \geq \ -\Delta_p \underline{u} \ - \ \mu a \underline{u}^{-\gamma}$$

which implies that $u \geq \underline{u}$ (see Proposition 2.1). Hence, $au^{-\gamma} \leq a\underline{u}^{-\gamma}$, so $au^{-\gamma} \in L^q(\Omega)$ and $q > N > N/p = (p^*/p)'$. Now by virtue of hypothesis (H), we have

$$-\Delta_p u \ \leq \ \lambda c u^{r-1} \ + \ \mu a u^{-\gamma} + \lambda c,$$

where $1 < r < p^*$. Exploiting Theorem 2.3 of [2], we infer that u is locally bounded. It follows that

$$-\Delta_p u = \lambda f(\cdot, u(\cdot)) \ + \ \mu a u^{-\gamma} \in L^q_{loc}(\Omega)$$

with $q > p'N$. Consequently, $u \in C^1(\Omega)$, due to [3, Corollary,p.830].

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Francesca Faraci
Dipartimento di Matematica e Informatica
Università degli Studi di Catania
Viale A. Doria 6, 95125, Catania, Italy
e-mail: ffaraci@dmi.unict.it

George Smyrlis
Department of Mathematics
National Technical University of Athens
Zografou Campus, 15780 Athens, Greece
e-mail: gsmyrlis@math.ntua.gr

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