Fully nonlinear operators with Hamiltonian: Hölder regularity of the gradient

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Abstract. The aim of this work is to prove $C^{1,\gamma}$ regularity up to the boundary for solutions of some fully nonlinear degenerate elliptic equations with a “sublinear” Hamiltonian term.

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1. Introduction

In this paper we shall establish some regularity results for the gradient of the solutions of the following family of equations

$$|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta b(x) = f(x).$$

Here and in the rest of the paper $\alpha > -1$ and, unless otherwise stated, $\beta \in [0, 1 + \alpha]$. The hypothesis on the data are that f and b are continuous and F is uniformly elliptic, i.e. there exist $\lambda < \Lambda$ such that for any M in the space S of symmetric matrices in $\mathbb{R}^N$, and for any $N \geq 0$ in S :

$$\lambda \operatorname{tr}(N) \leq F(M + N) - F(M) \leq \Lambda \operatorname{tr}(N),$$

with $F(0) = 0$.

This paper is a natural continuation of [4, 10]. We now state the main result under the above hypothesis, and then we will contextualize it.

Theorem 1.1. *Suppose that Ω is a bounded C^2 domain of $\mathbb{R}^N$. Let $f \in C(\overline{\Omega})$, $b \in C(\overline{\Omega})$, and $\varphi \in C^{1,\gamma_o}(\partial\Omega)$. Then there exist $\gamma \in (0, \gamma_o)$ depending on ellipticity constants of F , β and γ_o , and $C = C(\gamma)$ such that any bounded viscosity solution u of*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta b(x) = f(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

is $\mathcal{C}^{1,\gamma}$ and for $\beta < \alpha + 1$

$$\|u\|_{\mathcal{C}^{1,\gamma}(\bar{\Omega})} \leq C \left(\|u\|_\infty + \|b\|_\infty^{\frac{1}{1+\alpha-\beta}} + \|f\|_\infty^{\frac{1}{1+\alpha}} + \|\varphi\|_{\mathcal{C}^{1,\gamma_0}(\partial\Omega)} \right).$$

For $\beta = \alpha + 1$, γ also depends on the $\|b(x)\|_\infty$.

In [10], the authors proved an interior Hölder regularity of the gradient for solutions of

$$|\nabla u|^\alpha F(D^2 u) = f(x)$$

when $\alpha \geq 0$. After this important breakthrough, for this same class of solutions, through a geometric analysis of the behaviour of solutions along the set of critical points, an optimal $\mathcal{C}^{1,\gamma}$ estimate was obtained in [1]. While in [4] we proved $\mathcal{C}^{1,\gamma}$ regularity up to the boundary for solutions of

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(x) \cdot \nabla u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Finally, for Eq. (1.1), with $\alpha = 0$, $\mathcal{C}^{1,\gamma}$ regularity was proved in [3].

Remark 1.2. When $\beta = \alpha + 1$ in Theorem 1.1, the Hölder regularity of the gradient of the solution can easily be obtained following the lines of the proofs in [4] replacing $h(x) \cdot \nabla u |\nabla u|^\alpha$ with $b(x) |\nabla u|^{1+\alpha}$. This is the reason why, in the proofs of this paper, we only consider the case $\beta < 1 + \alpha$.

Concerning the solutions of Eq. (1.1) the Lipschitz interior regularity holds when $\beta \leq \alpha + 2$ (see Remark 2.3), instead we don't expect higher regularity for $\beta > \alpha + 1$. A more complete study of the case $\beta > \alpha + 1$ will be the subject of a subsequent paper.

Observe that we will prove Theorem 1.1 and its local counterpart only when $\alpha \geq 0$. Indeed, for $\alpha \in (-1, 0)$, in [5], we proved:

Proposition 1.3. *Suppose that for $\alpha \in (-1, 0)$, u is a viscosity solution of (1.1), then u is a viscosity solution of*

$$F(D^2 u) + b(x) |\nabla u|^{\beta-\alpha} = f(x) |\nabla u|^{-\alpha}.$$

Hence, the case $\alpha \in (-1, 0)$ in Theorem 1.1, could be obtained from the case $\alpha = 0$, as long as one can choose the Hamiltonian as the sum of terms with different homogeneity:

$$b(x, \nabla u) = \sum b_i(x) |\nabla u|^{\beta_i}.$$

We shall give in Sect. 3 of this paper the means to prove regularity in this case.

We also wish to point out that, in general, uniqueness of the Dirichlet problem does not hold when the Hamiltonian is not Lipschitz continuous. For

example, when Ω is the ball of radius 1, then $u \equiv 0$ and $u(x) = C(1 - |x|^\gamma)$ with $\gamma = \frac{2-\beta}{1-\beta}$ and $C = \gamma^{-1}(\gamma + N - 2)^{\frac{1}{\beta-1}}$ are both solutions of equation

$$\begin{cases} \Delta u + |\nabla u|^\beta = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We conclude this introduction by recalling that the results here and in the inspiring work [10], use the $\mathcal{C}^{1,\gamma}$ regularity of viscosity solutions of fully nonlinear equations, that are uniformly elliptic. Precisely, we shall use the following result

Proposition 1.4. *If Ω is a bounded C^2 domain, f is bounded, $\varphi \in \mathcal{C}^{1,\gamma_0}$ and u is a bounded viscosity solution of*

$$\begin{cases} F(D^2u) = f & \text{in } \Omega \cap B_1 \\ u = \varphi & \text{on } \partial\Omega \cap B_1, \end{cases}$$

then u is in $\mathcal{C}^{1,\gamma}(B_{\frac{1}{2}} \cap \Omega)$ with estimates depending on the L^∞ bounds of f and u , and the $\mathcal{C}^{1,\gamma_0}$ norm of φ .

This result is known and can be obtained gathering results of several experts, starting with the acclaimed work of Caffarelli [6, 7] and Evans [9] for the interior regularity; while boundary regularity is obtained through the results of Milakis and Silvestre [14] and Winter [16]. See also the first part of Theorem 1.4 of [15] where other important regularity issues are treated.

Finally let us recall that, equations of the form

$$F(D^2u) - |\nabla u|^p = f(x),$$

have been extensively studied for $F(D^2u) = \Delta u$. Interestingly most of the literature is concerned with the case $p \geq 1$, especially the so called natural growth i.e. $p = 2$. The behaviours of the solutions are quite different if $p \geq 2$ or if $1 < p < 2$. We will just mention the fundamental paper of Lasry and Lions [13], and more recently, in the fully nonlinear case, the papers of Capuzzo Dolcetta et al. [8] and Barles et al. [2]. In the latter the Lipschitz regularity of the solution is proved for non local uniformly elliptic operators, and with lower order terms that may be subquadratic. Note that in that case, to our knowledge, no further regularity is known.

2. Regularity results up to the boundary

In the rest of the paper we suppose that $\alpha \geq 0$. Let B be any ball in $\mathbb{R}^N$, such that $B \cap \Omega \subset \{y_N > a(y')\}$, $\partial\Omega \cap B = \{y_N = a(y')\} \cap B$ where a is supposed to be in $\mathcal{C}^2(\mathbb{R}^{N-1})$, and B_r will indicate a ball with the same center than B and r such that $\overline{B_r} \subset B$. On the other hand $B_R(x_o)$ will indicate a ball of radius R and center x_o .

It is clear that it is sufficient to prove a local result, i.e. for any $x \in B \cap \{y_N > a(y')\}$ with $B_1(x) \subset B$ and for any $r < 1$ there exists $\gamma \in]0, 1[$ such that the solutions are in $\mathcal{C}^{1,\gamma}(B_r(x) \cap \{y_N \geq a(y')\})$.

2.1. Lipschitz estimates

In order to prove the Hölder regularity of the gradient, we will need some Lipschitz regularity results up to the boundary. The lemmata which follow will be proved in the appendix.

Lemma 2.1. *Let $\varphi \in C^{1,\gamma_0}(B \cap \{y_N = a(y')\})$. Let d be the distance to $\{y_N = a(y')\} \cap B$. Suppose that H is continuous on $\bar{B} \times \mathbb{R}^N$ and satisfies for some constant C and for all $q \in \mathbb{R}^N$ and for any $y \in \bar{B}$*

$$|H(y, q) - H(y, 0)| \leq C(|q|^\beta + |q|). \quad (2.1)$$

Then, for all $r < 1$ there exists C_{a,B_r} depending on $(\|f\|_\infty, \gamma_0, \beta, \lambda, \Lambda, C, a, r, \text{Lip}(\varphi))$, such that if u is a solution of

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + H(y, \nabla u) = f(y) & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

and $\text{osc } u \leq 1$, then it satisfies

$$|u(y', y_N) - \varphi(y')| \leq C_{a,B_r} d(y) \quad \text{in } B_r \cap \{y_N > a(y')\}.$$

Using this estimate together with arguments as in [12], and [2], one finally gets the Hölder regularity of the solution, which can be stated as follows with the same hypothesis on a , while f will be taken continuous and bounded.

Lemma 2.2. *Suppose that H , continuous on $\mathbb{R}^N \times \mathbb{R}^N$, satisfies (2.1). Any bounded viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + H(y, \nabla u) = f(y) & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\}, \end{cases}$$

is Hölder continuous on $B_r \cap \{y_N \geq a(y')\}$ for any $r < 1$.

Remark 2.3. In the setting of the previous lemma, a more involved proof would give the Lipschitz regularity of the solution, but we don't include it here since we only need the compactness results of the corollary below in order to prove the regularity of the gradient of the solution for $\beta < \alpha + 1$. On the other hand, when $B_r \subset B$, note that the techniques used to prove Lemma 2.2 would give the Hölder regularity of the solution even for $\beta \leq \alpha + 2$ and hence, it would give interior Lipschitz regularity. For the boundary regularity, the bound $\beta \leq \alpha + 1$ is required in order to construct a super solution for the estimate of Lemma 2.1.

In the proof of the improvement of flatness lemma, we will also need the following estimate for solutions of some slightly different equations.

Lemma 2.4. *Let e be a unit vector. Suppose that H is continuous on $\bar{B} \times \mathbb{R}^N$, and satisfies for any $y \in \bar{B}$*

$$|H(y, q + e) - H(y, e)| \leq C|q|, \quad \text{for any } q \text{ such that } |q| \leq \frac{1}{2}. \quad (2.2)$$

For any $r > 0$, there exists c_0 such that for all $c < c_0$, any solution of

$$\begin{cases} |e + c\nabla u|^\alpha F(D^2 u) + H(y, e + c\nabla u) = f(y) & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases} \quad (2.3)$$

is Lipschitz continuous in $B_r \cap \{y_N \geq a(y')\}$.

Of course these two lemmata will imply the corollaries

Corollary 2.5. *Suppose that H_n is a sequence of continuous functions on $\bar{B} \times \mathbb{R}^N$, which satisfy*

$$|H_n(y, q) - H_n(y, 0)| \leq C_n(|q|^\beta + |q|), \quad (2.4)$$

with C_n a sequence going to zero.

Suppose that (u_n) is a bounded sequence of solutions of

$$\begin{cases} |\nabla u_n|^\alpha F(D^2 u_n) + H_n(y, \nabla u_n) = f_n(y) & \text{in } B \cap \{y_N > a(y')\} \\ u_n = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

and suppose that (f_n) goes uniformly to zero and $H_n(\cdot, 0)$ is uniformly convergent to H_∞ , then, up to subsequence, (u_n) converges uniformly in any compact set of $B \cap \{y_N \geq a(y')\}$ to u_∞ which satisfies

$$|\nabla u_\infty|^\alpha F(D^2 u_\infty) + H_\infty(y) = 0.$$

Corollary 2.6. *Suppose that e_n is a sequence of vectors with norm 1, that H_n is a sequence of functions continuous on $\bar{B} \times \mathbb{R}^N$ which satisfy, for any $y \in \bar{B}$ and for some constant C independent of n :*

$$|H_n(y, q + e_n) - H_n(y, e_n)| \leq C|q|, \quad \text{for any } q \text{ such that } |q| \leq \frac{1}{2}. \quad (2.5)$$

Suppose that (u_n) is a bounded sequence of solutions of

$$\begin{cases} |e_n + c_n \nabla u_n|^\alpha F(D^2 u_n) + H_n(y, e_n + c_n \nabla u_n) = f_n & \text{in } B \cap \{y_N > a(y')\} \\ u_n = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

Suppose that f_n goes uniformly to zero, $H_n(\cdot, e_n)$ goes uniformly to some function H_∞ , and c_n tends to zero. Then u_n converges uniformly in any compact set of $B \cap \{y_N \geq a(y')\}$, up to subsequence to u_∞ which satisfies

$$\begin{cases} F(D^2 u_\infty) + H_\infty(y) = 0 & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\}. \end{cases}$$

All these technical lemmata will be used in the proof of the improvement of flatness lemma up to the boundary.

2.2. Local regularity

Following the ideas of Imbert and Silvestre, we start by proving the following flatness lemma. The modulus of continuity of a function g is defined by $\omega_g(\delta) = \sup_{|x-y| \leq \delta} |g(x) - g(y)|$. While ω will denote some continuous increasing function on a neighborhood of 0 on the right, such that $\omega(0) = 0$.

Lemma 2.7. *For any ω as above, there exist $\epsilon_o > 0$ and $\rho \in (0, 1)$ which depend on (λ, Λ) , $\|a\|_{C^2}$, ω , and on $\|\varphi\|_{C^{1, \gamma_o}}$ such that for any F with ellipticity constants (λ, Λ) and for any $p \in \mathbb{R}^N$, if u a viscosity solution of*

$$\begin{cases} |p + \nabla u|^\alpha F(D^2 u) + b(y)|p + \nabla u|^\beta = f & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

then the following holds:

For any b and f such that $\omega_b(\delta) \leq \|b\|_\infty \omega(\delta)$, $\|b\|_\infty \leq \epsilon_o$, $\|f\|_\infty \leq \epsilon_o$, for all $x \in B$ such that $B_1(x) \subset B$, and $\text{osc}_{B_1(x) \cap \{y_N > a(y')\}} u \leq 1$,

there exists $q_{x,\rho} \in \mathbb{R}^N$ such that
$$\text{osc}_{B_\rho(x) \cap \{y_N > a(y')\}} (u(y) - q_{x,\rho} \cdot y) \leq \frac{\rho}{2}.$$

Proof. We argue by contradiction and suppose that for all n , there exist $x_n \in \overline{B}$ and $p_n \in \mathbb{R}^N$, $\|f_n\|_{L^\infty(\Omega)} \leq \frac{1}{n}$, $\|b_n\|_\infty \leq \frac{1}{n}$, $\omega_{b_n}(\delta) \leq \omega(\delta)\|b_n\|_\infty$, a sequence of operators F_n with ellipticity constants (λ, Λ) and there exists u_n , solution of

$$\begin{cases} |p_n + \nabla u_n|^\alpha F_n(D^2 u_n) + b_n(x) |\nabla u_n + p_n|^\beta = f_n & \text{in } B \cap \{y_N > a(y')\} \\ u_n(y) = \varphi(y') & \text{on } B \cap \{y_N = a(y')\} \end{cases} \quad (2.6)$$

with e.g. $\text{osc}(u_n) \leq 1$, such that, for any $q \in \mathbb{R}^N$ and any $\rho \in]0, 1[$,

$$\text{osc}_{B_\rho(x_n) \cap \{y_N > a(y')\}} (u_n(y) - q \cdot y) > \frac{\rho}{2}. \quad (2.7)$$

Extract from $(x_n)_n$ a subsequence which converges to $x_\infty \in \overline{B} \cap \{y_N \geq a(y')\}$. Observe furthermore that F_n converges locally uniformly to another (λ, Λ) -elliptic operator F_∞ . We denote in the sequel $B_\infty = \cap_{n \geq N_1} B_1(x_n)$, which contains $B_\rho(x_\infty)$ as soon as N_1 is large enough.

If $B_\rho(x_\infty) \cap \{y_N = a(y')\} \neq \emptyset$, then the boundedness of φ implies the boundedness of u_n . On the other hand, if $B_\rho(x_\infty) \cap \{y_N = a(y')\} = \emptyset$, we will replace u_n by $u_n - u_n(x_n)$ which is bounded and has the same oscillation than u_n .

Suppose first that $(p_n)_n$ is bounded so, up to subsequences, it converges to p_∞ . We want to apply Corollary 2.5, with

$$H_n(\cdot, q) = b_n(\cdot) |q + p_n|^\beta,$$

in order to say that u_n converges to some u_∞ , uniformly on $\overline{B_r} \cap \{y_N \geq a(y')\}$ for every $B_r \subset B_\infty$. Note that this is possible because (2.4) is satisfied with $C = \sup_n \|b_n\|_\infty (1 + 2^\beta + \beta 2^{\beta-1} |p_n|^{(\beta-1)^+})$. Indeed, if $\beta \leq 1$

$$||p_n + q|^\beta - |p_n|^\beta| \leq |q|^\beta.$$

While, if $\beta > 1$ there are two cases, either $|q| \leq |p_n|$ and by the mean value's theorem

$$||p_n + q|^\beta - |p_n|^\beta| = \beta |p_n + \theta q|^{\beta-1} |q| \leq \beta 2^{\beta-1} |p_n|^{\beta-1} |q|$$

or, $|q| \geq |p_n|$, and then

$$||p_n + q|^\beta - |p_n|^\beta| \leq (2^\beta + 1) |q|^\beta.$$

Note also that $H_n(\cdot, 0) = b_n(\cdot) |p_n|^\beta$ tends uniformly to 0.

Furthermore, (u_∞, p_∞) solves

$$\begin{cases} |p_\infty + \nabla u_\infty|^\alpha F_\infty(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty = \varphi & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases}$$

We can conclude the case p_n bounded, using Lemma 6 in [10], which says that u_∞ is also a solution of

$$\begin{cases} F_\infty(D^2u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty = \varphi & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases} \quad (2.8)$$

At the same time, by passing to the limit in (2.7), one obtains that for all ρ and $q \in \mathbb{R}^N$,

$$\operatorname{osc}_{B_\rho(x_\infty) \cap \{y_N > a(y')\}}(u_\infty(y) - q \cdot y) \geq \frac{\rho}{2}.$$

This contradicts the fact that, as a classical consequence of Proposition 1.4, there exists ρ_1 and $q_1 = q_{x, \rho_1}$ so that any u , solution of (2.8), satisfies

$$\operatorname{osc}_{B_{\rho_1}(x) \cap \{y_N > a(y')\}}(u(y) - q_1 \cdot y) \leq \frac{\rho_1}{2}.$$

We treat now the case when $(p_n)_n$ is unbounded. Dividing (2.6) by $|p_n|^\alpha$ it becomes

$$\begin{cases} |e_n + \frac{1}{|p_n|} \nabla u_n|^\alpha F_n(D^2u_n) \\ \quad + b_n(x) |p_n|^{\beta-\alpha} |e_n + \frac{1}{|p_n|} \nabla u_n|^\beta = \frac{f_n}{|p_n|^\alpha} & \text{in } B_\infty^+ \\ u_n = \varphi & \text{on } B_\infty \cap \{y_N = a(y')\} \end{cases}$$

where $e_n = \frac{p_n}{|p_n|}$ and $B_\infty^+ = B_\infty \cap \{y_N > a(y')\}$.

We begin by supposing that $\|b_n\|_\infty |p_n|^{\beta-\alpha}$ is bounded. By Lemma 2.4, u_n is uniformly Lipschitz and then, for n large enough, $\frac{1}{|p_n|} |\nabla u_n| \leq \frac{1}{2}$. Note that $H_n(\cdot, q) = b_n(\cdot) |p_n|^{\beta-\alpha} |q|^\beta$ satisfies (2.5) with $C = \sup_n \|b_n\|_\infty |p_n|^{\beta-\alpha} \beta (\sup((\frac{3}{2})^{\beta-1}, 2^{1-\beta}))$. Then, by Ascoli's theorem, up to subsequences, on any compact set of B_∞^+ , $H_n(y, e_n)$ and u_n converge respectively to H_∞ , and u_∞ which satisfy

$$\begin{cases} F_\infty(D^2u_\infty) + H_\infty(y) = 0 & \text{in } B_\infty^+ \\ u_\infty = \varphi & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases}$$

Passing to the limit in (2.7) one gets that $\operatorname{osc}_{B_\rho(x_\infty)}(u_\infty - q \cdot y) > \frac{\rho}{2}$, we obtain the same contradiction as in the first case.

We now assume that $\|b_n\|_\infty |p_n|^{\beta-\alpha}$ is unbounded. For $c_n = |p_n|^{\beta-\alpha} \|b_n\|_\infty$, the hypothesis $\beta < 1 + \alpha$ implies $\frac{c_n}{|p_n|} \rightarrow 0$. We can define $v_n(y) = \frac{u_n(y)}{c_n}$ and $e_n = \frac{p_n}{|p_n|}$, then v_n satisfies

$$\begin{cases} |e_n + \frac{c_n}{|p_n|} \nabla v_n|^\alpha F_n(D^2v_n) + H_n(y, \frac{c_n}{|p_n|} \nabla v_n + e_n) = \frac{f_n(y)}{c_n |p_n|^\alpha} & \text{in } B_\infty^+ \\ v_n(y') = \frac{\phi(y')}{c_n} & \text{on } B_\infty \cap \{y_N = a(y')\} \end{cases}$$

with $\tilde{F}_n(X) = \frac{1}{c_n} F_n(c_n X)$, $H_n(y, q) = \frac{b_n(y)}{\|b_n\|_\infty} |e_n + q|^\beta$ which satisfies (2.5) with $C = \beta (\sup((\frac{3}{2})^{\beta-1}, 2^{1-\beta}))$. Note that, by Lemma 2.4, v_n is uniformly Lipschitz and then, for n large enough, $\frac{c_n}{|p_n|} |\nabla v_n| \leq \frac{1}{2}$. In particular we obtain that v_n

converges uniformly to zero and, by passing to the limit in the equation, this gives

$$\lim_{n \rightarrow +\infty} H_n(y, e_n) = 0.$$

This is a contradiction, since $H_n(\cdot, e_n)$ has L^∞ norm 1 and it is uniformly convergent. $\square$

The Hölder regularity for the gradient can be derived from the iteration process below and Lemma 2.6 of [5].

Lemma 2.8. *Suppose that $\rho, \epsilon_o \in]0, 1[$ are as in Lemma 2.7 and suppose that u is a viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + b(y)|\nabla u|^\beta = f & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases} \quad (2.9)$$

with $\|b\|_\infty \leq \epsilon_o$ and $\|f\|_\infty \leq \epsilon_o$, then, there exists $\gamma \in]0, 1[$, such that for all k and for all $x \in B$ such that $B_1(x) \subset B$, there exists $p_k \in \mathbb{R}^N$ for which, if $\text{osc}_{B_1(x) \cap \{y_N > a(y')\}} u \leq 1$, then

$$\text{osc}_{B_{r_k}(x) \cap \{y_N > a(y')\}} (u(y) - p_k \cdot y) \leq r_k^{1+\gamma} \quad (2.10)$$

where $r_k = \rho^k$.

Proof. We use a recursive argument as in Lemma 3.4 of [4] and in Lemma 2.5 of [5].

Let ρ be given by Lemma 2.7. Choose γ small enough in order that $\rho^\gamma > \frac{1}{2}$, and define $r_k = \rho^k$. For $k = 0$, (2.10) holds with $p_o = 0$.

If x is on the boundary, say $x_N = a(x')$, in the computation below we replace u by $v(x) = u(x) - u(x', a(x'))$ which satisfies

$$\begin{cases} |\nabla v|^\alpha F(D^2 v) + b(y)|\nabla v|^\beta = f & \text{in } B \cap \{y_N > a(y')\} \\ v(y', a(y')) = \varphi(y') - \varphi(x') & \text{for } (y', a(y')) \in B \end{cases}$$

so we can suppose in the iteration below that $\varphi(x') = 0$. This will be used when we will later evaluate the $\mathcal{C}^{1,\gamma}$ norm of φ_k .

If x is not on the boundary there exists k_o such that, for $k \geq k_o$, $B(x, r_k) \cap \{y_N = a(y')\} = \emptyset$, and then φ_k does not intervene for $k \geq k_o$.

Suppose by induction that, for a value k , (2.10) holds true with some p_k . We want to construct p_{k+1} . Let

$$u_k(y) = r_k^{-1-\gamma} (u(r_k(y-x) + x) - p_k \cdot (r_k(y-x) + x)).$$

u_k is well defined on $B_1(x) \cap \{y_N > a_k(y')\}$, where $a_k(y') = x_N(1 - \frac{1}{r_k}) + r_k^{-1}a(r_k(y' - x') + x')$, and it satisfies

$$\text{osc}_{B_1(x)} u_k = r_k^{-1-\gamma} \text{osc}_{B_{r_k}(x)} (u - p_k \cdot y) \leq r_k^{-1-\gamma} r_k^{1+\gamma} = 1.$$

Furthermore u_k is a solution of

$$\begin{cases} |p_k r_k^{-\gamma} + \nabla u_k|^\alpha F_k(D^2 u_k) + b_k |p_k r_k^{-\gamma} + \nabla u_k|^\beta = f_k & \text{in } B_1(x) \cap \{y_N > a_k(y')\} \\ u_k(y') = \varphi_k(y') & \text{on } B_1(x) \cap \{y_N = a_k(y')\} \end{cases}$$

with $F_k(M) = r_k^{1-\gamma} F(r_k^{\gamma-1} M)$ which has the same constant of ellipticity of F , $f_k(y) = r_k^{1-\gamma(1+\alpha)} f(r_k(y-x) + x)$, $b_k(x) = r_k^{\gamma(\beta-(1+\alpha))+1} b(r_k(y-x) + x)$ and

$$\begin{aligned} \varphi_k(y') &= r_k^{-1-\gamma} (\varphi(r_k(y-x)' + x') - p'_k \\ &\quad \cdot (r_k(y-x) + x)' - p_k^N a(r_k(y-x)' + x')). \end{aligned}$$

Note that, as long as $\gamma < \frac{1}{1+\alpha}$,

$$\|f_k\|_\infty \leq \epsilon_o, \quad \omega_{b_k}(\delta) \leq \omega(\delta) \|b_k\|_\infty \quad \text{and} \quad \|b_k\|_\infty \leq \|b\|_\infty.$$

Furthermore, $|\nabla a_k(y')| = |\nabla a(r_k(y-x)' + x')|$, while $D^2 a_k(y') = r_k D^2 a(r_k(y-x)' + x')$, and hence $\|D^2 a_k\|_\infty = r_k \|D^2 a\|_\infty \leq \|D^2 a\|_\infty$.

Finally we need to prove that φ_k is uniformly bounded in $\mathcal{C}^{1,\gamma}$, only when x is on the boundary, as remarked at the beginning of the proof. Let us recall that, for all y' and z' ,

$$|\nabla \varphi(y') - \nabla \varphi(z')| \leq C |y' - z'|^{\gamma_o}. \quad (2.11)$$

Remark as well, that, since

$$B_{r_k}(x) \overset{\text{osc}}{\cap \{y_N > a(y')\}} (u(y) - p_k \cdot y) \leq r_k^{1+\gamma}$$

then

$$B_{r_k}(x) \overset{\text{osc}}{\cap \{y_N = a(y')\}} (\varphi(y') - p'_k \cdot y' - p_k^N a(y')) \leq r_k^{1+\gamma}. \quad (2.12)$$

Using $\varphi(x') = u(x) = 0$, since x is on the boundary, using (2.11) together with the fact that $D^2 a$ is bounded, one easily obtains the required uniform bound on the $\mathcal{C}^{1,\gamma}$ norm of φ_k .

Hence, using Lemma 2.7 for u_k and $p = p_k$, there exists $q_{k+1} \in \mathbb{R}^N$ such that

$$B_\rho(x) \overset{\text{osc}}{\cap \{y_N > a_k(y')\}} (u_k(y) - q_{k+1} \cdot y) \leq \frac{\rho}{2} B_1(x) \overset{\text{osc}}{\cap \{y_N > a_k(y')\}} u_k.$$

Defining $p_{k+1} = p_k + q_{k+1} r_k^\gamma$, with the assumptions on γ and ρ , one gets:

$$B_{r_{k+1}}(x) \overset{\text{osc}}{\cap \{y_N > a(y')\}} (u(y) - p_{k+1} \cdot y) \leq \frac{\rho}{2} r_k^{1+\gamma} \leq r_{k+1}^{1+\gamma}.$$

This is the desired conclusion. $\square$

Proof of Theorem 1.1. Suppose now that u is a solution with bounded oscillation of (1.1), for general f bounded in L^∞ , and b continuous. The function $v(x) = \epsilon u(x)$ with $\epsilon^{-1} = \text{osc } u + \frac{1}{\epsilon_o} (\|f\|_\infty^{\frac{1}{1+\alpha}} + \|b\|_\infty^{\frac{1}{1+\alpha-\beta}})$ satisfies the equation

$$|\nabla v|^\alpha \epsilon F(\epsilon^{-1} D^2 v) + b(x) \epsilon^{\alpha+1-\beta} |\nabla v|^\beta = \epsilon^{\alpha+1} f(x).$$

Our choice of ϵ implies that we are under the conditions of Lemma 2.8, in any ball that intersects $\bar{\Omega}$. So for any $x \in \bar{\Omega}$, (2.10) is satisfied. A standard procedure, illustrated in Lemma 2.6 of [5], implies that v is in $C^{1,\gamma}(B_\rho(x) \cap \bar{\Omega})$. This ends the proof of Theorem 1.1. $\square$

3. A generalization

We consider the equation

$$|\nabla u|^\alpha F(D^2 u) + \sum_{i=1}^k b_i(x) |\nabla u|^{\beta_i} = f(x) \quad (3.1)$$

where the functions b_i and f are continuous, $\alpha \geq 0$ and $\beta_i \in [0, 1 + \alpha)$ for all $i \in [1, k]$. In this section we want to show how to generalize to this equation, the results obtained for the case of only one non linearity in the gradient. For simplicity sake we shall just discuss the case of the interior regularity, the boundary case can be treated in analogy.

Theorem 3.1. *Suppose that $\alpha \geq 0$, that F is uniformly elliptic, that $\beta_i \in [0, 1 + \alpha)$ for all $i \in [1, k]$, $f \in C(B(0, 1))$ and that $b_i \in C(B(0, 1))$. For any u , bounded viscosity solution of (3.1) in $B(0, 1)$ and for any $r < 1$, there exist $\gamma \in (0, 1)$ depending on λ, Λ , on the modulus of continuity of b_i , on β_i and $C = C(\gamma)$ such that*

$$\|u\|_{C^{1,\gamma}(B_r(0))} \leq C \left(\|u\|_\infty + \sup_i \|b_i\|_\infty^{\frac{1}{1+\alpha-\beta_i}} + \|f\|_{C^0}^{\frac{1}{1+\alpha}} \right).$$

Without loss of generality we will suppose that $\beta_i \neq \beta_j$ for all $i \neq j$, that $\beta_1 = \sup \beta_i$ and $b_i \not\equiv 0$ for any i .

Under these hypothesis we establish an improvement of flatness lemma. Again let ω be a modulus of continuity.

Lemma 3.2. *There exist ϵ_o and $\rho < 1$ depending on the ellipticity constants λ, Λ , on b_1 and ω , such that for any F with ellipticity constants λ, Λ , and for any (u, p) solution of*

$$|\nabla u + p|^\alpha F(D^2 u) + \sum_{i=1}^k b_i(x) |\nabla u + p|^{\beta_i} = f \quad \text{in } B_1 \quad (3.2)$$

such that $\text{osc}_{B_1} u \leq 1$ and for all $f, b_i, i \in [2, k]$ such that $\|f\|_\infty, \|b_i\|_\infty \leq \epsilon_o$ and $\omega_{b_i}(\delta) \leq \|b_i\|_\infty \omega(\delta)$, there exists $q \in \mathbb{R}^N$ such that

$$\text{osc}_{B_\rho}(u(x) - q \cdot x) \leq \frac{\rho}{2}.$$

Proof. As before, this lemma is proved by contradiction. Suppose that for all n , there exist u_n, p_n, f_n, b_{i_n} such that $\|f_n\|_\infty, \|b_{i_n}\|_\infty \leq \frac{1}{n}$, $\omega_{b_{i_n}}(\delta) \leq \|b_{i_n}\|_\infty \omega(\delta)$ for $i \in [2, k]$ and u_n is a solution of (3.2), with $p = p_n$, $f = f_n$ and $b = b_{i_n}$ such that for all $q \in \mathbb{R}^N$ and all $\rho < 1$:

$$\text{osc}_{B_\rho}(u_n(x) - q \cdot x) > \frac{\rho}{2}.$$

If the sequence of (p_n) is bounded, up to subsequences, it converges to $\bar{p}$, then by an easy generalization of the compactness results in Sect. 2, we obtain, by passing to the limit in extracted subsequences, that $u_n(x) + p_n \cdot x$ converges to u solution of

$$|\nabla u|^\alpha F(D^2 u) + b_1(x) |\nabla u|^{\beta_1} = 0$$

and

$$\operatorname{osc}_{B_\rho}(u(x) - q \cdot x) > \frac{\rho}{2}.$$

This contradicts the regularity result obtained in the Sect. 2.

If (p_n) is unbounded, there are two cases

First case, $\beta_1 < \alpha$: Dividing by $|p_n|^\alpha$ the equation satisfied by u_n one gets

$$\begin{aligned} & \left| e_n + \frac{1}{|p_n|} \nabla u_n \right|^\alpha F(D^2 u_n) + b_1(x) |p_n|^{\beta_1 - \alpha} \left| e_n + \frac{1}{|p_n|} \nabla u_n \right|^{\beta_1} \\ & + \sum_2^k b_{i_n}(x) |p_n|^{\beta_i - \alpha} \left| e_n + \frac{1}{|p_n|} \nabla u_n \right|^{\beta_i} = \frac{f_n}{|p_n|^\alpha} \end{aligned}$$

By an obvious extension of Corollary 2.6 in the interior case, one obtains by passing to the limit that u_n converges to u_∞ which satisfies

$$F(D^2 u_\infty) = 0$$

and

$$\operatorname{osc}_{B_\rho}(u_\infty(x) - q \cdot x) > \frac{\rho}{2}$$

for any q . This contradicts the classical regularity results.

Second case, $\beta_1 \geq \alpha$: We divide the equation by $|p_n|^{\beta_1}$ and define $v_n = \frac{u_n}{|p_n|^{\beta_1 - \alpha}}$ then the equation becomes

$$\begin{aligned} & \left| e_n + |p_n|^{\beta_1 - 1 - \alpha} \nabla v_n \right|^\alpha F_n(D^2 v_n) + b_1(x) \left| e_n + |p_n|^{\beta_1 - 1 - \alpha} \nabla v_n \right|^{\beta_1} \\ & + \sum_2^k b_{i_n}(x) |p_n|^{\beta_i - \beta_1} \left| e_n + |p_n|^{\beta_1 - 1 - \alpha} \nabla v_n \right|^{\beta_i} = \frac{f_n}{|p_n|^{\beta_1}} \end{aligned}$$

where $F_n(X) = |p_n|^{\beta_1 - \alpha} F(\frac{X}{|p_n|^{\beta_1 - \alpha}})$, and once more using some compactness result for p large, one obtains

$$b_1(x) = 0.$$

This is a contradiction and it ends the proof of improvement of flatness lemma. $\square$

To conclude the proof of Theorem 3.1, it is enough to proceed with the equivalent of Lemma 2.8.

4. Proofs of Lemmata 2.1, 2.2 and 2.4

In the sequel we will need a comparison principle.

Proposition 4.1. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$. Suppose that f and g are continuous in $\overline{\Omega}$, that H is continuous in $\overline{\Omega} \times \mathbb{R}^N$, that v is a lower semi-continuous bounded super-solution of*

$$|\nabla v|^\alpha F(D^2 v) + H(x, \nabla v) = f \quad \text{in } \Omega$$

and u is a upper semi-continuous bounded sub-solution of

$$|\nabla u|^\alpha F(D^2 u) + H(x, \nabla u) = g \quad \text{in } \Omega.$$

Suppose that either u or v is Lipschitz continuous and that $f < g$ in $\overline{\Omega}$.

If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω .

This result may be known to specialists but it requires some subtleties so we include the proof for completeness sake.

Proof. If M is some Lipschitz constant for u or v , then for all $(x, y) \in \Omega^2$

$$u(x) - v(y) \leq \sup(u - v) + M|x - y|. \quad (4.1)$$

We now consider

$$\psi_j(x, y) = u(x) - v(y) - \frac{j}{2}|x - y|^2.$$

Suppose by contradiction that $u > v$ somewhere, then the supremum of $u - v$ is achieved in $\bar{x} \in \Omega$. The fact that $\|u\|_\infty + \|v\|_\infty \geq \sup \psi_j(x, y) \geq \sup(u - v)$ implies that the positive maximum of ψ_j is reached in $(x_j, y_j) \in \overline{\Omega}^2$. And the boundedness of $j|x_j - y_j|^2$ implies that up to a subsequence (x_j, y_j) tends to $(\bar{x}, \bar{x})$, $\bar{x}$ being a maximum point for $u - v$. Finally, since $u \leq v$ on $\partial\Omega$, $\bar{x}$ is an interior point and so are x_j and y_j .

Furthermore by (4.1), $j|x_j - y_j|^2 \leq 2M|x_j - y_j|$ and then, on bounded sets, the uniform continuity with respect to x of H , gives

$$|H(x_j, j(x_j - y_j)) - H(y_j, j(x_j - y_j))| = o(1).$$

By Theorem 3 in [11] there exist X_j and Y_j such that $(j(x_j - y_j), X_j) \in \bar{J}^{2,+}u(x_j)$, $(j(x_j - y_j), -Y_j) \in \bar{J}^{2,-}v(y_j)$ and

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq 3j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

In particular $X_j + Y_j \leq 0$, and by the equation

$$\begin{aligned} f(x_j) &\leq |j(x_j - y_j)|^\alpha F(X_j) + H(x_j, j(x_j - y_j)) \\ &\leq |j(x_j - y_j)|^\alpha F(-Y_j) + H(y_j, j(x_j - y_j)) + o(1) \\ &\leq g(y_j) + o(1). \end{aligned}$$

Using $f < g$, we have reached a contradiction. $\square$

Proof of Lemma 2.1. We can assume that B has radius 1 and, without loss of generality, that $\|u\|_\infty \leq 1$. Let $r < 1$ and Ω be some bounded $\mathcal{C}^2$ domain which contains $\{y_N > a(y')\} \cap B_{\frac{1+r}{2}}$ and is contained in $B \cap \{y_N > a(y')\}$. Note that it is sufficient to prove that there exists C_{a,B_r} such that, for $y \in \Omega$,

$$|u(y', y_N) - \varphi(y')| \leq C_{a,B_r} d(y, \partial\Omega).$$

Recall that the distance is $\mathcal{C}^2$ near the boundary and it is semi-concave elsewhere, hence there exists some constant C_1 such that $D^2d \leq C_1 I$. Let us denote by D the diameter of Ω . We begin by proving an upper bound for $u - \varphi$. Let ψ be a solution of

$$\begin{cases} \mathcal{M}_{\lambda,\Lambda}^+(D^2\psi) = 0 & \text{in } B \cap \{y_N > a(y')\} \\ \psi = \varphi & \text{on } B \cap \{y_N = a(y')\}. \end{cases}$$

Using the results in Proposition 1.4, there exists some constant c which depends on $\lambda, \Lambda, N, \Omega$, such that $\|\nabla\psi\|_\infty + \|\psi\|_\infty \leq c\|\varphi\|_{\mathcal{C}^{1,\beta_0}}$. Furthermore by classical regularity results [6, 7, 9], $\psi \in \mathcal{C}^2(B \cap \{y_N > a(y')\})$.

Let k be so that $N\Lambda C_1(1+D) < \frac{(k+1)\lambda}{2}$. We also define

$$c_1 := \frac{k^{1+\alpha}(k+1)}{(2(1+D))^{(k+2)\alpha+k+1}}, \quad c_2 := \frac{2k}{(1+D)^{k+1}}$$

and choose M large enough in order that both

$$Mk > 2 \left(\|\nabla\psi\|_\infty + \frac{2^7}{1-r} \right) (1+D)^{k+1}$$

and

$$M^{1+\alpha}c_1 - C_H(Mc_2 + (Mc_2)^\beta) > \|f\|_\infty + \|H(\cdot, 0)\|_\infty$$

hold, this is possible since $\beta < 1 + \alpha$. In Ω , our barrier function will be

$$w(y) = \begin{cases} M \left(1 - \frac{1}{(1+d(y))^k} \right) + \psi(y) & \text{for } |y| \leq r \\ M \left(1 - \frac{1}{(1+d(y))^k} \right) + \frac{2^5}{(1-r)^3} (|y| - r)^3 + \psi(y) & \text{for } |y| > r. \end{cases}$$

The function $w_o(y) := \left(1 - \frac{1}{(1+d(y))^k} \right)$ satisfies, in the viscosity sense

$$\nabla w_o = \frac{k \nabla d}{(1+d)^{k+1}}, \quad D^2 w_o = \frac{k D^2 d}{(1+d)^{k+1}} - k(k+1) \frac{\nabla d \otimes \nabla d}{(1+d)^{k+2}};$$

this uses the semi-concavity of d . Then, by the assumption on k ,

$$\mathcal{M}^+(D^2 w_o) \leq -\frac{k(k+1)\lambda}{2(1+D)^{k+2}}.$$

Finally, this gives,

$$\begin{aligned} |\nabla w|^\alpha \mathcal{M}_{\lambda,\Lambda}^+(D^2 w) + H(\cdot, \nabla w) &\leq - \left(\frac{Mk}{2(1+D)^{k+1}} \right)^\alpha \left(\frac{k(k+1)\lambda}{2(1+D)^{k+2}} \right) \\ &\quad + C_H \left(\left(\frac{2Mk}{(1+D)^{k+1}} \right)^\beta + \frac{2Mk}{(1+D)^{k+1}} \right) \\ &\quad + \|H(\cdot, 0)\|_\infty \leq -\|f\|_\infty. \end{aligned}$$

We now check that

$$w \geq u \quad \text{on } \partial\Omega.$$

Note that on $\partial\Omega \cap \{y_N = a(y')\}$, $w \geq \varphi = u$. While on $\partial\Omega \cap B_1$, one has

$$\frac{1+r}{2} \leq |y| < 1 \quad \text{and then } w \geq 2 + \psi(y) \geq 1.$$

Proposition 4.1 implies that $w \geq u$ in Ω . This ends the proof since

$$u(y) - \varphi(y') \leq Mw_o(y) + \psi(y) - \varphi(y') \leq C_o d(y, \Omega).$$

For the lower bound, replace w by $2\psi - w$. This ends the proof of Lemma 2.1. $\square$

We are now ready to give the

Proof of Lemma 2.2. The proof proceeds in a standard way (see the proof of Proposition 2.3 in [4]). Precisely for $r < r' < 1$ and for $x_o \in B_r \cap \{y_N > a(y')\}$, choosing $M > \text{Lip } \varphi + C_{a, B_{r'}}$ we define

$$\phi(x, y) = u(x) - u(y) - M|x - y|^\gamma - L|x - x_o|^2 - L|y - x_o|^2.$$

We assume that $L > \frac{8 \sup u}{(r' - r)^2}$. One needs to prove that $\phi \leq 0$ in $(B_{r'} \cap \{y_N > a(y')\})^2$, this will imply, by taking first $x = x_o$ and then exchanging x and y , that u is Hölder continuous of exponent γ .

We shall not give the details of this computation since it does not require any new tool, we just remark that the hypothesis (2.1) on H , implies

$$H(\bar{x}, q_x) \leq H(\bar{y}, q_y) + 2\|H(\cdot, 0)\|_\infty + C_H(|q_x|^\beta + |q_y|^\beta + |q_x| + |q_y|),$$

for any $\bar{x}, \bar{y}$ and for any q_x, q_y . $\square$

Proof of Lemma 2.4. In order to prove Lipschitz regularity up to the boundary, we need first a boundary estimates i.e. that any solution of (2.3) satisfies, in $B_{r'} \cap \{y_N > a(y')\}$,

$$|u(y', y_N) - \varphi(y')| \leq C_{a, B_{r'}} d(y).$$

To prove this, note that the comparison principle in Proposition 4.1 with one sub- or super-solution Lipschitz for the equation

$$|e + c\nabla u|^\alpha F(D^2 u) + H(y, e + c\nabla u) = f$$

is still true (with obvious changes in the proof).

Then it is sufficient to prove that, as soon as c is small enough, one can choose the constants in the definition of w in Lemma 2.1 in order to have in $B_{r'} \cap \{y_N > a(y')\}$

$$|e + c\nabla w|^\alpha F(D^2 w) + H(y, e + c\nabla w) \leq -\|f\|_\infty. \quad (4.2)$$

This will be achieved by choosing k so that

$$(k+1)\lambda - N\Lambda C_1(1 + \text{diam } \Omega) > \frac{(k+1)\lambda}{2}$$

and M large enough that $M2^{-\alpha} > \|f\|_\infty + \frac{3}{2}C_H + \|H(y, e)\|_\infty$. Finally we pick c so that $c(Mk + \frac{3 \times 2^5}{1-r} + \|\nabla \psi\|_\infty) < \frac{1}{2}$. Then w satisfies (4.2).

We can now proceed with the standard procedure. Let $\omega(s) = s - \omega_o s^{\frac{3}{2}}$ if $s \leq s_o = \left(\frac{2}{3\omega_o}\right)^2$, continued constantly for $s \geq s_o$, also ω_o is chosen so that $s_o > 1$. Note that $\omega(s) \geq \frac{s}{3}$ for $s < s_o$. Let $r < r' < 1$, choose $L > 3(\text{Lip } \varphi + C_{a,r'})$, $L > 1 + \frac{4}{(r'-r)^2}$, let $x_o \in B_r \cap \{y_N > a(y')\}$, and define on $(B_{r'} \cap \{y_N > a(y')\})^2$,

$$\psi(z, y) = u(z) - u(y) - L^2\omega(|z - y|) - L|z - x_o|^2 - L|y - x_o|^2.$$

If we prove that $\psi(z, y) \leq 0$ in $B_{r'} \times B_{r'}$, since L is independent of x_o , by choosing $z = x_o$ one gets for any $y \in B_{r'} \cap \{y_N > a(y')\}$,

$$u(x_o) - u(y) \leq L^2|x_o - y| + L|x_o - y|^2.$$

Next choosing $y = x_o$ for all $z \in B_{r'}, z_N > a(z')$,

$$u(z) - u(x_o) \leq L^2|x_o - z| + L|x_o - z|^2.$$

Finally we get, for $(y, z) \in B_r \times B_r$, $|u(y) - u(z)| \leq L^2|z - y| + L|z - y|^2 \leq (L^2 + 4L)|z - y|$, which implies the desired result. $\square$

We begin to observe that if the supremum is achieved on $(\bar{x}, \bar{y}) \in (\overline{B_{r'}})^2$ then, with our choice of L , neither $\bar{x}$ nor $\bar{y}$ can belong to the part $\{z_N = a(z')\}$ according to the remark at the beginning of the proof. Furthermore the assumption on L implies that $(\bar{x}, \bar{y}) \in B_{\frac{r+r'}{2}}^2$.

By Lemma 9 in [11], for all $\iota > 0$ there exist $(X_\iota, Y_\iota) \in S$ such that

$$(q_x, X_\iota) \in \bar{J}^{2,+}(u(\bar{x})), \quad (q_y, -Y_\iota) \in \bar{J}^{2,-}(u(\bar{y}))$$

with $q_x = L^2\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + 2L(\bar{x} - x_o)$ and $q_y = L^2\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - 2L(\bar{y} - x_o)$.

Furthermore

$$\begin{pmatrix} X_\iota & 0 \\ 0 & Y_\iota \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + 2(L + \iota) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and $B = L^2 D^2\omega(|\bar{x} - \bar{y}|)$. In the sequel we drop the index ι up to change L . In particular, $\text{tr}(X + Y)^- \geq \kappa L^2 \frac{1}{|\bar{x} - \bar{y}|^{\frac{3}{2}}}$ and $\text{tr}(X + Y)^+ \leq 4LN$, for some κ depending on ω_o and N (see [12], proof of Proposition III.1). We suppose that $\lambda L^2 > \frac{8N\Lambda}{\kappa}$, and we choose L also large enough in order that

$$\left(\frac{1}{2}\right)^\alpha \frac{\kappa \lambda L^2}{2} > \|f\|_\infty + \|H(\cdot, e)\|_\infty + C_H \quad (4.3)$$

and next c small enough in order that $2cL^2 < \frac{1}{6}$. Note that $\frac{1}{2} \leq \inf(|e + cq_x|, |e + cq_y|)$ by the choice of c above, hence

$$|H(\bar{x}, e + cq_x)|, |H(\bar{y}, e + cq_y)| \leq \|H(\cdot, e)\|_\infty + \frac{C_H}{2}.$$

Finally, one gets that

$$\begin{aligned} \frac{f(\bar{x}) + H(\bar{x}, e + cq_x)}{|e + cq_x|^\alpha} &\leq F(X) \leq F(-Y) - \lambda \operatorname{tr}(X + Y)^- + \Lambda(\operatorname{tr}(X + Y)^+) \\ &\leq \frac{f(\bar{y}) + H(\bar{y}, e + cq_y)}{|e + cq_y|^\alpha} - \lambda \kappa L^2 + 4LN\Lambda \\ &\leq \frac{f(\bar{y}) + H(\bar{y}, e + cq_y)}{|e + cq_y|^\alpha} - \frac{\lambda \kappa L^2}{2}. \end{aligned}$$

This clearly contradicts (4.3).

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