Infinitely many solutions for the spinorial Yamabe problem on the round sphere

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Abstract. In this paper we prove the existence of a sequence of solutions to the spinorial Yamabe problem using the action of a subgroup of the isometries of the sphere.

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1. Introduction

We consider the standard round sphere (S^n, g_0) for $n \geq 3$, with its standard spin structure ΣS^n . We want to solve the spinorial Yamabe problem on the sphere, that is the equation

$$D_{S^n} u = |u|^{p-1} u, \quad (1)$$

for $p = \frac{n+1}{n-1}$ and D_{S^n} the Dirac operator. This equation has an important link with conformal geometry. We refer the reader to papers [1–3, 16] and the references within for more details on the geometric aspect of the problem. From now on, we will write D instead of D_{S^n} . This problem is variational on any spin manifold M and its energy functional E can be written as

$$E(u) = \frac{1}{2} \int_M \langle u, Du \rangle dx - \frac{1}{p+1} \int_M |u|^{p+1} dx.$$

The main difficulty here is that the energy functional is strongly indefinite since the operator D has a sequence of eigenvalues unbounded from below and above. Moreover, the non-linearity on the right-hand-side is critical. This problem was heavily investigated in the literature in order to find a solution, we cite here the work of Isobe [10], where the author proved a result similar to the Brezis–Nirenberg one for the classical Yamabe problem [4]. We also cite the works [2, 16] that consider the problem in full generality. The subcritical case was also investigated by the author in [12] using the Rabinowitz–Floer

homology and by Isobe [9], using a linking argument.

In this work, we adapt a similar approach of [9] when dealing with the subcritical case, but here we use a different space of functions, that is functions invariant under a certain isometry group of the sphere to capture compactness. A similar approach was used for the Yamabe problem in [6, 17] and the CR-Yamabe problem in [13, 14] to find infinitely many changing-sign solutions to the problem. We present first the general setting of the problem.

2. Preliminaries

Let (M, g) be an n -dimensional compact closed Riemannian manifold and let us consider the $SO(n)$ principal bundle $P_{SO}(M)$ consisting of positively oriented frames on (M, g) . A spin structure on (M, g) is a pair $\sigma = (P_{spin}(M), \mathcal{V})$, where $P_{spin}(M)$ is a $Spin(n)$ -principal bundle over M and $\mathcal{V} : P_{spin}(M) \rightarrow P_{SO}(M)$ is such that the following diagram commutes:

$$\begin{array}{ccccc} P_{spin}(M) \times Spin(n) & \longrightarrow & P_{spin} & \longrightarrow & M \\ \downarrow \mathcal{V} \times \Phi & & \downarrow \mathcal{V} & \nearrow & \\ P_{SO}(M) \times SO(n) & \longrightarrow & P_{SO}(M) & & \end{array}$$

With $\Phi : Spin(n) \rightarrow SO(n)$ the non-trivial double covering of $SO(n)$. We consider now an n -dimensional complex representation $T : Spin(n) \rightarrow End(\Sigma_n)$ of $Spin(n)$. Here the vector space Σ_n is of dimension $2^{\lfloor \frac{n}{2} \rfloor}$. The spinor bundle is then defined as the associate vector bundle $\Sigma(M, g, \sigma) = P_{spin}(M) \times_T \Sigma_n$. This bundle carries a natural Clifford multiplication, a Hermitian metric and a natural metric connection. So we define the Dirac operator D on this spinor bundle as the composition of the following:

$$D : \Gamma(\Sigma M) \xrightarrow{\nabla^\Sigma} \Gamma(T^*M \otimes \Sigma M) \longrightarrow \Gamma(TM \otimes \Sigma M) \xrightarrow{Cliff} \Gamma(\Sigma M).$$

Now we recall a very important identity satisfied by this operator:

Proposition 2.1. (Schrödinger–Lichnerowicz formula) *Let (M, g, σ) be a compact Riemannian spin manifold and R its scalar curvature, then we have*

$$D^2 = \Delta + \frac{R}{4}.$$

For further information on Spin manifolds one can consult [7, 8]. Now we give some analytical properties of the Dirac operator that we will use in what follow.

Let D be the Dirac Operator of a compact spin manifold. Then we know that D is essentially self adjoint as an operator in $L^2(\Sigma M)$ and admits a basis of smooth eigenspinors, that is

$$L^2(\Sigma M) = \overline{\bigoplus_{\lambda \in \mathbb{R}} \ker(D - \lambda Id)}.$$

Moreover it has the space $H^{\frac{1}{2}}(\Sigma M)$ as its domain. Let $\{\psi_i\}_{i \in \mathbb{Z}}$ be an L^2 -orthonormal basis of eigenspinors with associated eigenvalues $(\lambda_i)_{i \in \mathbb{Z}}$, then if $u \in L^2(\Sigma M)$ it has a representation in this basis as follows:

$$u = \sum_{i \in \mathbb{Z}} a_i \psi_i.$$

Define the unbounded operator $|D|^s$ by

$$|D|^s(u) = \sum_{i \in \mathbb{Z}} a_i |\lambda_i|^{2s} \psi_i.$$

Using this operator we can define the inner product

$$\langle u, v \rangle_s = \langle |D|^s u, |D|^s v \rangle_{L^2} + \langle u, v \rangle_{L^2}.$$

This inner product induces a norm that is equivalent to the one in $H^s(\Sigma M)$: so we will take $\|u\|^2 = \langle u, u \rangle_{\frac{1}{2}}$ as our standard norm. For $1 \leq q < \infty$ we will use the notation

$$\|u\|_q = \left(\int_M |u|^q dx \right)^{\frac{1}{q}}.$$

We will denote the eigenfunctions corresponding to positive eigenvalues by $\{\varphi_j^+\}_{j \geq 1}$ and the ones corresponding to negative eigenvalues by $\{\varphi_j^-\}_{j \geq 1}$. Since the kernel is finite dimensional, we will write the spanning eigenfunctions by $\{\varphi_j^0\}_{1 \leq j \leq k_0}$ where k_0 is the dimension of the kernel. Hence we have the following decomposition

$$H^{\frac{1}{2}}(\Sigma S^n) = H^- \oplus H_0 \oplus H^+$$

where $H^- = \overline{\text{span}\{\varphi_j^-\}_{j \geq 1}}$, $H^+ = \overline{\text{span}\{\varphi_j^+\}_{j \geq 1}}$ and $H_0 = \ker D$. Our main result can be stated as follows:

Theorem 2.2. *There exists an infinite sequence $(u_j)_{j \geq 1}$ of solutions to problem (1), with $E(u_j) \rightarrow \infty$ as $j \rightarrow \infty$.*

Moreover, using our previous result in [12], if we consider the problem

$$Du = |u|^{p-1}u + h(x, u) \quad (2)$$

with $h(x, u) = \frac{\partial}{\partial u} H(x, u)$ where H is a smooth equivariant function, then

Theorem 2.3. *For a generic C^3 , G_k -equivariant function H , problem (2) has an infinite sequence of solutions with energy tending to infinity.*

As a corollary of this theorem we have the result of Theorem 2.2. This follows by taking a sequence of perturbations h_ε bounded in L^∞ such that $h_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ hence by compactness one gets infinitely many solutions.

3. The action group

We consider S^n as a submanifold of $\mathbb{R}^{n+1}$ and for $k \geq 2$, we consider the following subgroup $G_k = O(n+1-k) \times O(k)$ acting on $\mathbb{R}^{n+1}$ as follows:

For $x = (x_1, x_2) \in \mathbb{R}^{n+1-k} \times \mathbb{R}^k$, and $(A_1, A_2) \in G_k$

$$(A_1, A_2) \cdot x = (A_1 x_1, A_2 x_2).$$

This action leaves the sphere S^n globally invariant: therefore G_k is a subgroup of $Iso(S^n)$ the isometry group of S^n . Notice that this group acts on S^n with no fixed points and by stereographic projection, any function on the sphere invariant under the action of G_k is not radially symmetric. This group was used in [6, 17] to prove the existence of infinitely many solutions for the changing sign Yamabe problem.

This action lifts to an action on the spinor bundle ΣS^n : see for instance of Chichilnisky [5, Section 5], where the author discusses the lifting of isometric actions on a spin manifold to the spinor bundle. In fact, in this specific case the action can be easily seen on spinors of $\mathbb{R}^{n+1}$ and then induced on S^n as a submanifold (see [8, Section 1.4] for details).

We consider now the Sobolev space $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ of Spinors in $H^{\frac{1}{2}}(\Sigma S^n)$ invariant under G_k , that is

$$H_{G_k}^{\frac{1}{2}}(\Sigma S^n) = \{u \in H^{\frac{1}{2}}(\Sigma S^n); \tilde{g} \cdot u(x) = u(g \cdot x), \forall g \in G_k, x \in S^n\},$$

where $\tilde{g}$ in the above definition is the lift of the action of G_k to the spinor bundle. For the sake of notation we will just write $g \cdot u$ instead of $\tilde{g} \cdot u$.

In a similar way, one can define the space $L_{G_k}^2(\Sigma S^n)$ as the space of Spinors in L^2 invariant under the action of G_k . Since the Dirac operator D is invariant under isometry, we have that

$$D: H_{G_k}^{\frac{1}{2}}(\Sigma S^n) \rightarrow L_{G_k}^2(\Sigma S^n),$$

and since the operator is auto-adjoint, it has a basis of eigenspinors $(\varphi_j)_{j \in \mathbb{Z}}$ with corresponding eigenvalues $(\lambda_j)_{j \in \mathbb{Z}}$. We will adopt the same splitting notations as in Sect. 2.

4. Compactness and the Palais–Smale condition

Theorem 4.1. *The embedding $H_{G_k}^{\frac{1}{2}}(\Sigma S^n) \hookrightarrow L_{G_k}^{p+1}(\Sigma S^n)$ is compact.*

Proof. In the proof we need the following lemma of concentration compactness [11].

Lemma *Let $(u_j)_j$ be a bounded sequence in $H^{\frac{1}{2}}(\Sigma S^n)$. We may assume that (u_j) converges weakly to u and that $|u|^{\frac{2n}{n-1}}$ converges weakly to a measure ν , then we have*

$$\nu = |u|^{\frac{2n}{n-1}} + \sum_{i \in J} \nu_i \delta_{x_i}$$

where J is at most countable and $\nu_i \geq 0$.

We consider now a bounded sequence (u_j) in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n) \subset H^{\frac{1}{2}}(\Sigma S^n)$. Then from the Sobolev embedding (u_j) is bounded in L^p , thus it converges weakly to u in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$. From the previous lemma $|u_j|^{p+1} \rightharpoonup \nu$. Thus

$$\nu = |u|^{p+1} + \sum_{i \in J} \nu_i \delta_{x_i}.$$

So that

$$\int_{S^n} |u_j|^{p+1} dx \rightarrow \int_{S^n} |u|^{p+1} dx + \sum_{i \in J} \nu_i.$$

But for $g \in G_k$,

$$|g \cdot u_j|^{p+1} \rightarrow |g \cdot u|^{p+1} + \sum_{i \in J} \nu_i \delta_{gx_i}$$

and since u_j and u are equivariant under the action of G_k , then we have

$$\sum_{i \in J} \nu_i \delta_{gx_i} = \sum_{i \in J} \nu_i \delta_{x_i}.$$

Thus if $\nu_i > 0$ then the set $\{x_i\}_{i \in J}$ is invariant under the action of G_k and this leads to a contradiction since G_k has a continuous action and the set J is at most countable. Thus $\nu_i = 0, \forall i \in J$ and we have convergence in norm and hence strong convergence of (u_j) in $L_{G_k}^{p+1}(\Sigma S^n)$. $\square$

Let us recall here a version of the concentration-compactness proved for the Dirac operator in the critical case on any compact spin manifold M .

Consider the following critical problem

$$Du = \eta u + |u|^{\frac{2}{n-1}} u \quad (P_\eta)$$

with $\eta \in \mathbb{R}$. Its relative energy functional is then

$$E^\eta(u) = \frac{1}{2} \left(\int_M \langle Du, u \rangle dx - \eta \int_M u^2 dx \right) - \frac{n-1}{2n} \int_M |u|^{\frac{2n}{n-1}} dx.$$

Hence the following holds:

Lemma 4.2. (Isobe [10]) *Let u_i be a (PS) sequence for the functional E^η then there exist $k \geq 0$ and sequences $a_i^j \rightarrow a^j \in M$, for $1 \leq j \leq k$ and a sequence of numbers R_i^j converging to zero, a solution $u_\infty \in H^{\frac{1}{2}}(M)$ of problem (P_η) and solutions $w_i^j \in H^{\frac{1}{2}}(\mathbb{R}^n)$ of $Du = |u|^{\frac{2}{n-1}} u$ on $\mathbb{R}^n$, such that up to subsequence, we have*

$$u_i = u_\infty + \sum_{j=1}^k \omega_i^j + o(1) \quad \text{in } H^{\frac{1}{2}}(M),$$

where

$$\omega_i^j = (R_i^j)^{-\frac{n-1}{2}} \beta_j(x) (\rho_{i,j}^{-1})^*(u^j)$$

and

$$\rho_{i,j}(x) = \exp_{a^j}(R_i^j x),$$

here β_k is a non negative function equals to 1 in $B_1(a^j)$ and zero outside $B_2(a^j)$. Moreover, we have

$$E^\eta(u_i) = E^\eta(u_\infty) + \sum_{j=0}^k E(u^j) + o(1).$$

Notice that in particular this makes the functional E satisfies the $(P-S)$ condition, Indeed, if $(u_j)_{j \geq 0}$ is a (PS) sequence in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ then as the previous lemma states, we have the existence of $k \geq 0$ and sequences $a_i^j \rightarrow a^j \in M$, for $1 \leq j \leq k$ and a sequence of numbers R_i^j converging to zero, a solution $u_\infty \in H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ of problem (P_η) and solutions $u^j \in H^{\frac{1}{2}}(\mathbb{R}^n)$ of $Du = |u|^{\frac{2}{n-1}}u$ on $\mathbb{R}^n$, such that up to subsequence, we have

$$u_i = u_\infty + \sum_{j=1}^k \omega_i^j + o(1) \quad \text{in } H^{\frac{1}{2}}(M).$$

But again using the invariance under the action of G_k the set $I = \{a_i, 0 \leq i \leq k\}$ is invariant under the action of G_k but since it is discrete, this is impossible and hence it is empty. Therefore we have strong convergence in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$.

5. Proof of Theorem 2.2

The proof will go through two steps. The first step corresponds to proving the existence of a sequence of critical points for an approximated problem. Then we show the convergence of each critical point of the approximated problem to a critical point of the original functional. In fact one has two ways of proving this result, either via the Fadell-Rabinowitz equivariant index as in the original paper of Rabinowitz [15] for Hamiltonian systems, or via a fountain theorem as in the work of Isobe [9]. We will follow the second approach even though they are similar in nature and they are based on a linking argument.

Step 1

We first start by recalling an abstract critical point theorem.

Let H be a Hilbert space such that $H = \overline{\text{span}\{\varphi_j\}_{j \geq 1}}$ and for $d \geq 1$ we set $H_1 = \text{span}\{\varphi_j\}_{1 \leq j \leq d}$ and $H_2 = \overline{\text{span}\{\varphi_j\}_{j > d}}$ so that $H = H_1 \oplus H_2$.

Theorem 5.1. (The fountain theorem [18]) *Let $L \in C^1(H, \mathbb{R})$ be an even functional. For $B = \{u \in H_1; \|u\| \leq \rho\}$, where $\rho > 0$ and*

$$\Gamma = \{\gamma \in C^1(B, H); \gamma(u) = \gamma(-u), \gamma|_{\partial B} = \text{id}_{\partial B}\},$$

we define

$$c = \inf_{\gamma \in \Gamma} \max L(\gamma(B)).$$

For $0 < r < \rho$, let

$$b = \inf\{L(u); u \in H_2; \|u\| = r\}.$$

And

$$a = \inf\{L(u); u \in H_1; \|u\| = \rho\}$$

If L satisfies $(PS)_c$, and there exists $0 < r < \rho$ such that $b > a$, then c is a critical value for L with $c \geq b$.

We will use this theorem by approximating the subspaces and showing that the critical values are independent of the approximation. So first we recall the splitting

$$H_{G_k}^{\frac{1}{2}} = H^- \oplus H_0 \oplus H^+.$$

We define then

$$H(d) = \text{span}\{(\varphi_j^-)_{1 \leq j \leq d}\} \oplus H_0 \oplus H^+$$

and

$$H(d, m) = \text{span}\{(\varphi_j^-)_{1 \leq j \leq d}\} \oplus H_0 \oplus \text{span}\{(\varphi_j^+)_{1 \leq j \leq m}\}$$

and

$$H^+(d, m) = \overline{\text{span}\{(\varphi_j^+)_{j > m}\}}.$$

So that

$$H(d) = H(d, m) \oplus H^+(d, m).$$

Now we will apply the fountain theorem for this splitting as in the theorem. So one needs to exhibit r and ρ satisfying the assumptions of Theorem 5.1.

Lemma 5.2. *There exists $\rho(m) > 0$ such that*

$$E(u) \leq 0$$

for $u \in H(d, m)$ and $\|u\| \geq \rho(m)$.

Proof. Indeed,

$$E(u) = \frac{1}{2} \int_{S^n} \langle Du, u \rangle dx - \frac{1}{p+1} \int_{S^n} |u|^{p+1} dx.$$

From Hölder's inequality, we have

$$\begin{aligned} E(u) &= \frac{1}{2} \int_{S^n} \langle Du^+, u^+ \rangle dx + \frac{1}{2} \int_{S^n} \langle Du^-, u^- \rangle dx - \frac{1}{p+1} \int_{S^n} |u|^{p+1} dx \\ &\leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - C(p) \|u\|_2^{p+1} \end{aligned}$$

where $C(p) = |S^n|^{-\frac{p-1}{2}}$, therefore

$$E(u) \leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - C(p) \|u^+\|_2^{p+1} - C(p) \|u^-\|_2^{p+1} - C(p) \|u^0\|_2^{p+1}.$$

Since H_0 is finite dimensional and $H(d, m) \cap H^+$ is also finite dimensional, we have that the L^2 norm and the $H^{\frac{1}{2}}$ norm are equivalent, hence there exists $C(m)$ such that

$$E(u) \leq \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - C(m) \|u^+\|^{p+1} - C(p) \|u^0\|^{p+1}.$$

Hence the existence of $\rho(m)$ follows. But it is important to have a bound from below on $\rho(m)$.

Notice first that the term $\frac{1}{2}\|u^+\|^2 - C(m)\|u^+\|^{p+1}$ is negative for

$$\|u^+\|^2 > \frac{1}{(2C(m))^{\frac{2}{p-1}}} =: A_m.$$

We take $R > A_m$ and assume that $\|u\|^2 > (\lambda + 1)R$ for $\lambda > 1$. The first case that can happen is when $\|u^+\|^2 > A_m$ then we have that $E(u) \leq 0$. The other case is when $\|u^+\|^2 \leq A_m$, in that case we have that

$$\|u^-\|^2 + \|u^0\|^2 \geq \lambda R.$$

We assume without loss of generality that $\|u_0\| \geq 1$, then we have

$$\begin{aligned} E(u) &\leq \frac{1}{2}R - C(m)R^{\frac{p+1}{2}} - c(\|u^-\|^2 + \|u^0\|^2) \\ &\leq \frac{1}{2}R - C(m)R^{\frac{p+1}{2}} - \lambda cR. \end{aligned}$$

Notice that c depends only on p hence we can choose $\lambda > \frac{1}{2c}$ to have $E(u) \leq 0$. Therefore one can take $\rho(m) > (\frac{1}{2c} + 1)A_m$. It is important to notice that since $C(m) \rightarrow 0$ as $m \rightarrow \infty$, we do have $\rho(m) \rightarrow \infty$ as $m \rightarrow \infty$. $\square$

Lemma 5.3. *For $m \geq 1$, we define*

$$\beta_m = \sup\{\|u\|_{p+1}; u \in H^+(d, m); \|u\| = 1\}.$$

Then $\beta_m \rightarrow 0$ as $m \rightarrow \infty$. Moreover the result holds for every $\beta_{m,q}$ with the L^{p+1} norm replaced by L^q norm for all $1 < q \leq p + 1$.

Proof. By construction of β_m , we have the existence of a sequence

$$(u_m)_{m \geq 1} \subset \{u \in H^+(d, m); \|u\| = 1\}$$

such that

$$\frac{\beta_m}{2} \leq \|u_m\|_{L^{p+1}}. \quad (3)$$

Since the sequence is bounded in $H_G^{\frac{1}{2}}(\Sigma S^n)$ we may assume it is weakly convergent to $u \in H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ and by the compactness Theorem 4.1, we have $u_m \rightarrow u$ in $L_{G_k}^{p+1}$. But since $u_m \in H^+(d, m)$ then $u = 0$. Indeed, this follows from the fact that for any $f \in H^{\frac{1}{2}}$,

$$\langle u_m, f \rangle_{H^{\frac{1}{2}}} \rightarrow \langle u, f \rangle_{H^{\frac{1}{2}}}$$

and if we decompose $u = \sum_{j \in \mathbb{N}} a_j \varphi_j$, then by taking $f = \varphi_j$ we see that

$$\langle u_m, f \rangle_{H^{\frac{1}{2}}} = 0$$

for m big enough, therefore $a_j = 0$ and hence $u = 0$. Thus, passing to the limit in (3), we see that

$$\beta_m \rightarrow 0.$$

$\square$

It is important to notice that since $H^+(d, m)$ is independent of d , β_m is also independent of d .

Lemma 5.4. *There exists $r(m) > 0$ such that*

$$b_m = \inf\{E(u); u \in H^+(d, m), \|u\| = r(m)\} \rightarrow \infty$$

as $m \rightarrow \infty$.

Proof. let $u \in H^+(d, m)$ such that $\|u\| = r$, then

$$\begin{aligned} E(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\geq \frac{r^2}{2} - r^2 \frac{\beta_{m,2}}{2} - \frac{1}{p+1} r^{p+1} \beta_m^{p+1}. \end{aligned}$$

Since $\beta_{m,2} \rightarrow 0$ as $m \rightarrow \infty$ we can assume that for m large enough, we have $\beta_{m,2} \leq \frac{1}{2}$. Thus

$$E(u) \geq \frac{r^2}{4} - \frac{1}{p+1} r^{p+1} \beta_m^{p+1}.$$

But the function $r \rightarrow \frac{r^2}{4} - \frac{1}{p+1} r^{p+1} \beta_m^{p+1}$ has a maximum equals to

$$\overline{C}(m) = \left(\frac{1}{4} - \frac{1}{2(p+1)} \right) 2^{-\frac{2}{p-1}} \beta_m^{-\frac{2(p+1)}{p-1}}$$

and this is achieved at

$$r(m) = 2^{-\frac{1}{p-1}} \beta_m^{\frac{-(p+1)}{p-1}}.$$

Now since $\beta_m \rightarrow 0$ as $m \rightarrow \infty$ we have that $\overline{C}(m) \rightarrow \infty$ and hence

$$b(m) \rightarrow \infty.$$

□

By taking $0 < r(m) < \rho(m)$, (this choice is possible since we can take $\rho(m)$ large and independent of m), we define as in the setting of Theorem 5.1,

$$B_{d,m} = \{u \in H(d, m); \|u\| \leq \rho(m)\},$$

$$\Gamma_{d,m} = \{\gamma \in C^0(B_{d,m}, H(d)); \gamma(u) = \gamma(-u); \gamma|_{\partial B_{d,m}} = id_{\partial B_{d,m}}\}.$$

And

$$c(d, m) = \inf_{\gamma \in \Gamma_{d,m}} \max E(\gamma(B_{d,m})).$$

Since $b(m) \rightarrow \infty$ as $m \rightarrow \infty$, and $\max\{E(u); u \in H(d, m); \|u\| = \rho(j)\} \leq 0$, $c(d, m)$ is then a critical point of $E|_{H(d)}$ if E satisfies the (PS) condition on $H(d)$. We prove now that E , indeed, satisfies the (PS) condition on $H(d)$. So we consider a (PS) sequence $(u_j)_{j \geq 1} \in H(d)$ so that

$$dE|_{H(d)}(u_j) = o(1)$$

and

$$E(u_j) = c + o(1).$$

Since $u_j \in H(d)$ we have that

$$\int_{S^n} \langle Du_j, u_j \rangle dx - \int_{S^n} |u_j|^{p+1} dx = o(\|u_j\|).$$

Hence,

$$E(u_j) - \frac{1}{2} dE|_{H(d)}(u_j)u_j = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{S^n} |u_j|^{p+1} dx = c + o(\|u_j\|). \quad (4)$$

And since u_j^+, u_j^- and u_j^0 are all elements of $H(d)$, we have

$$\|u_j^+\|^2 - \|u_j^+\|_2^2 - \int_{S^n} |u_j|^{p-1} \langle u_j, u_j^+ \rangle dx = o(\|u_j^+\|) \quad (5)$$

$$\|u_j^-\|^2 - \|u_j^-\|_2^2 + \int_{S^n} |u_j|^{p-1} \langle u_j, u_j^- \rangle dx = o(\|u_j^-\|) \quad (6)$$

and

$$\int_{S^n} |u_j|^{p-1} \langle u_j, u_j^0 \rangle dx = o(\|u_j^0\|). \quad (7)$$

By adding (5) and (6), and using Holder's inequality one has

$$\|u_j\|^2 \leq \|u_j\|_2^2 + C\|u_j\|_{p+1}^p \|u_j\|_{p+1} + o(\|u_j\|).$$

By the continuous Sobolev embedding and (4) we have

$$\|u_j\|^2 \leq C_1 + o(\|u_j\|^{\frac{p}{p+1}+1}).$$

Therefore, we have boundedness of u_j in $H(d)$, hence we can extract a weakly convergent subsequence that we denote again by $u_j \rightharpoonup u$ and the convergence is strong in L^{p+1} by Theorem 4.1. Now if we go back and add (5) and (6) we have that

$$\|u_j\|^2 \rightarrow \|u\|_2^2 + \int_{S^n} |u|^{p-1} \langle u, u^+ - u^- \rangle dx = \|u\|^2.$$

Hence we do have convergence in norm and therefore (PS) holds.

Step 2

Notice that, since E is bounded on bounded sets of $H^{\frac{1}{2}}$, we have

$$b(m) \leq c(d, m) \leq a(m) = \max E(B_m)$$

where

$$B_m = \{u \in H^- \oplus H_0 \oplus \text{span}\{\varphi_j^+\}_{1 \leq j \leq m}; \|u\| \leq \rho(m)\}.$$

Then up to subsequence, we may assume that $c(d, m) \rightarrow c(m)$ as $d \rightarrow \infty$. Now we need to show that $c(m)$ is a critical value for E in $H_{G_k}^{\frac{1}{2}}$.

Notice first that so far there exists $u_{m,d} \in H(d)$ such that $E(u_{m,d}) = c(d, m)$ and $dE|_{H(d)}(u_{m,d}) = 0$. The last step is then to show that $u_{m,d}$ converges in $H_{G_k}^{\frac{1}{2}}$.

Recall that the equation $dE|_{H(d)}(u_{m,d}) = 0$, reads as

$$\int_{S^n} \langle Du_{m,d} - |u_{m,d}|^{p-1} u_{m,d}, v \rangle dx = 0; \quad (8)$$

for all $v \in H(d)$. In particular since $u_{m,n} \in H(d)$, we have that

$$\int_{S^n} \langle Du_{m,d}, u_{m,d} \rangle dx - \int_{S^n} |u_{m,d}|^p dx = 0.$$

And since $E(u_{m,d}) \rightarrow c(m)$ as $d \rightarrow \infty$ we have:

$$E(u_{m,d}) - \frac{1}{2} dE(u_{m,d}) u_{m,d} = \left(\frac{1}{2} - \frac{1}{p} \right) \|u_{m,d}\|_p^p = c(m) + o(1).$$

Hence $\|u_{m,d}\|_p$ is bounded as $d \rightarrow \infty$ and from Holder's inequality, we have that $\|u_{m,d}\|_2$ is also bounded. Therefore $\|u_{m,d}^0\|$ is bounded. Now by going back to (8) and by taking $v = u_{m,d}^+$, we have

$$\|u_{m,d}^+\|^2 - \|u_{m,d}^+\|_2^2 - \int_{S^n} |u_{m,d}|^{p-1} \langle u_{m,d}, u_{m,d}^+ \rangle dx = 0 \quad (9)$$

and

$$\|u_{m,d}^-\|^2 - \|u_{m,d}^+\|_2^2 + \int_{S^n} |u_{m,d}|^{p-1} \langle u_{m,d}, u_{m,d}^+ \rangle dx = 0. \quad (10)$$

By Holder's inequality, we have again

$$\|u_{m,d}^+\|^2 \leq \|u_{m,d}^+\|_2^2 + C \|u_{m,d}\|_p^{p-1} \|u_{m,d}^+\|^p.$$

From the Sobolev embedding we have

$$\|u_{m,d}^+\|^2 \leq C_1 + C_2 \|u_{m,d}^+\|.$$

Thus, $\|u_{m,d}^+\|$ is bounded, and similarly $\|u_{m,d}^-\|$, thus $u_{m,d}$ is bounded in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$. Therefore we can extract a subsequence converging weakly to u_m in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ and from Theorem 4.1, strongly in L^p . Hence, by adding (9) and (10), we have that

$$\|u_{m,d}\|^2 \rightarrow \int_{S^n} |u_m|^{p-2} \langle u_m, u_m^+ - u_m^- \rangle dx + \|u_{m,d}\|_2^2 = \|u_m\|^2$$

which leads to a convergence in norm and therefore the sequence converges strongly to $u_m \in H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$.

Now, if we let P_d be the orthogonal projector on $H(d)$, we have for $v \in H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$,

$$\int_{S^n} \langle Du_{m,d} - |u_{m,d}|^{p-2} u_{m,d}, v \rangle dx = \int_{S^n} \langle Du_{m,d} - |u_{m,d}|^{p-2} u_{m,d}, v - P_d v \rangle dx.$$

But notice now that $v - P_d v$ converges weakly to zero in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$, thus by strong convergence of $u_{m,d}$,

$$\int_{S^n} \langle Du_{m,d} - |u_{m,d}|^{p-2} u_{m,d}, v \rangle dx \rightarrow 0.$$

Thus

$$\int_{S^n} \langle Du_m - |u_m|^{p-2} u_m, v \rangle dx = 0.$$

Thus u_m is indeed a critical point for E in $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$ with energy $E(u_m) = c(m) \geq b(m) \rightarrow \infty$.

6. Generic existence and proof of Theorem 2.3

Similar problem was treated in [12] using the Rabinowitz–Floer homology. What is needed for the construction is mainly the compactness via the (PS) condition which holds as it is proved in Section 4, and the geometry of the non-linearity. We recall here the result for functionals of the form

$$E(u, \lambda) = \frac{1}{2} \int_M \langle Du, u \rangle dx - \lambda \int_M (H(x, u) - 1) dx \quad (11)$$

where M is a compact Spin manifold.

Theorem 6.1. [12] *Assume that:*

(H1) *There exist $1 < q < \frac{n+1}{n-1}$, a constant C and $r_0 > 0$ so that for $|s| > r_0 > 0$.*

$$\left| \frac{\partial^2 h(x, s)}{\partial s^2} \right| < C (1 + |s|^{q-1}).$$

(H2) *The surface $S = \{u \in L^2(M); \int_M H(x, u) - 1 dx = 0\}$ bounds a strictly starshaped domain.*

If Moreover H is S^1 -equivariant in u , then the equivariant Rabinowitz–Floer homology is well defined and

$$H_*^{S^1}(H) = \mathbb{Z}_2$$

if $$ is even.*

Notice that in our case, the nonlinearity satisfies the assumption (H1)–(H2). Hence, it remains to show transversality in the space of equivariant functions $H_{G_k}^{\frac{1}{2}}(\Sigma S^n)$.

We will consider the equation

$$Du = |u|^{p-1}u + \frac{\partial}{\partial u} H(x, u)$$

and we want to show transversality for a generic H . So we will denote by E_H the corresponding functional as in (11).

Lemma 6.2. *For a generic perturbation K of H in $C^3(\Gamma(\Sigma S^n))$, the energy functional E_{H+K} is Morse.*

Proof. We consider the functional

$$\psi: H_{G_k}^{\frac{1}{2}}(\Sigma S^n) \times \mathbb{R} \times C_{G_k}^3(\Gamma(\Sigma S^n)) \longrightarrow H_{G_k}^{\frac{1}{2}}(\Sigma S^n) \times \mathbb{R}$$

defined by

$$\psi(z, K) = \nabla E_{H+K}(z).$$

Notice first that the inverse image of zero corresponds to critical points of the functional related to $H + K$. Also, for $(z, K) \in \psi^{-1}(0)$,

$$\partial_z \psi(z, h)v = \text{Hess}(E_{H+K}(z))v,$$

which is a perturbation of a compact operator, and hence it is a zero index Fredholm operator. Now it remains to show that $\nabla \psi(z, K)$ is surjective. So let us compute the differential with respect to K :

$$\partial_K \psi(z, K)G = \begin{pmatrix} -\lambda|D|^{-1}G_z(x, u(x)) \\ -\int_{S^n} G(x, u(x))dx \end{pmatrix}.$$

So first by taking G to be constant, we see that we can span the $\mathbb{R}$ component. For the other component we see that by taking $G(x, s(x)) = \langle f(x), s(x) \rangle$ where f is a G_k -equivariant section, then we have that the range of the first component is dense since G_z can be any section of the spinor bundle and the operator $|D|^{-1}$ maps C^3 to a dense subspace. Thus we have the surjectivity. Therefore by the transversality theorem, 0 is a regular point of $\psi(\cdot, K)$ for a generic K and this is equivalent to saying that E_{H+K} is Morse. $\square$

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