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Nonlinear Differential Equations and Applications NoDEA

The one dimensional parabolic $p(x)$-Laplace equation

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Abstract. The Dirichlet problem for the degenerate and singular parabolic $p(x)$-Laplace equation with one spatial variable is considered. We prove the existence of the unique weak solution such that the derivatives u_t and u_x of a solution u belong to L_∞. Moreover for the singular case we prove the existence of the strong solution i.e. such that u_t, u_x and u_{xx} belong to L_∞.

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1. Introduction and formulation of the results

Consider the following quasilinear parabolic equation

$$u_t = (|u_x|^{p(x)}u_x)_x + f(x,u,u_x) \quad \text{in } Q_T = (0,T)\times(-l,l), \tag{1.1}$$

coupled with the initial and homogeneous Dirichlet boundary conditions

$$u(0,x) = u_0(x) \ \text{ for } \ |x|\le l \quad \text{and} \quad u(t,\pm l) = 0 \quad \text{for} \quad t\in[0,T], \tag{1.2}$$

where

$$\max_{|x|\le l}\left|(|u_{0x}|^{p(x)}u_{0x})_x\right| < \infty \quad \text{and} \quad u_0(\pm l) = 0. \tag{1.3}$$

Here T, l are arbitrary positive constants and the function $p(x) > -1$ for $x\in[-l,l]$

The case when p is constant was studied by a lot of authors and optimal results concerning this equation was obtained (see, for example, [5]). There recently appeared a large number of publications with non constant p, see [1–4,6,8,9,12] and the references therein. In [8] the multidimensional case was considered and it was proved that if $p(x) > 0$ is a measurable function, $f = f(t,x,u)$ is C^1 function and $u_0 \in L^\infty \cap W_0^{1,p(x)}$, then there exists a global bounded weak solution of the problem such that

$$u \in L^\infty(0,T;W_0^{1,p(x)}), \quad u_t \in L^2.$$

For more details see [8]. Our goal in the present paper is to obtain the global solution with essentially better differential properties in the one dimensional case. To this end we need $p(x)$ to be C^1 function, but in contrast with [8] we consider the singular case as well (i.e. $-1 < p(x) \le 0$). Concerning the function f, first we assume that $f = f(x, u, u_x)$ and second we do not need f to be C^1 function but C^γ, with $\gamma \in (0, 1)$. Concerning the assumption on u_0 see (1.3). We show that the derivatives u_t and u_x are L^∞ functions, moreover if $-1 < p(x) \le 0$ then u_{xx} belongs to L^∞ as well.

Assume that

$$uf(x, u, 0) \le \alpha u^2 + \beta, \tag{1.4}$$

here α and β are some nonnegative constants,

$$|f(x, u, q)| \le |q|^{p(x)} \psi(|q|) \quad \text{for} \quad |x| \le l, \quad |u| \le M, \tag{1.5}$$

the constant M will be defined below (see (2.3)), the function ψ is smooth, nonnegative, nondecreasing function such that

$$\int^\infty \frac{\rho d\rho}{\psi(\rho)} = +\infty,$$

$$f(x, u_2, q) - f(x, u_1, q) \le 0 \quad \text{for} \quad u_2 > u_1, \tag{1.6}$$

$$p(x) \in C^1([-l, l]), \quad f(x, u, q) \in C^\gamma([-l, l] \times \mathbf{R^2}), \quad \gamma \in (0, 1). \tag{1.7}$$

Definition 1. We say that a Lipschitz continuous function $u(t, x) : Q_T \to \mathbf{R}$ is a strong solution of problem (1.1), (1.2) if $u_{xx} \in L_\infty(Q_T)$ and the equation

$$u_t = (1 + p(x))|u_x|^{p(x)} u_{xx} + p'(x) u_x |u_x|^{p(x)} ln|u_x| + f(x, u, u_x)$$

is satisfied almost everywhere in Q_T. Initial and boundary conditions are satisfied in the classical sense.

We put

$$b(x, 0) = 0 \quad \text{for} \quad b(x, z) \equiv p'(x) z |z|^{p(x)} ln|z|.$$

Theorem 1. *Assume that conditions* (1.3)–(1.7) *are fulfilled. If* $p(x) \in (-1, 0]$ *for* $x \in [-l, l]$*, then for an arbitrary* $T > 0$*, there exists a strong solution of problem* (1.1), (1.2).

If, in addition, f *is Lipschitz continuous function, then the solution is unique.*

Definition 2. We say that a Lipschitz continuous function $u(t, x) : Q_T \to \mathbf{R}$ is a weak solution of problem (1.1), (1.2) if it satisfies the following integral identity

$$\int_{Q_T} \left(u_t\, \phi + |u_x|^{p(x)} u_x \phi_x \right) dt dx = \int_{Q_T} f(x, u, u_x) \phi dt dx$$

for an arbitrary smooth function $\phi(t, x)$ such that $\phi(t, \pm l) = 0$ on $(0, T)$. Initial and boundary condition are satisfied in the classical sense.

In order to formulate the next theorem we need to substitute condition (1.5) by more restrictive one, namely, we suppose that f is linear with respect to u_x i.e.

$$f(x,u,u_x) = g_1(x,u)u_x + g_2(x,u). \tag{1.8}$$

Theorem 2. *Assume that conditions* (1.3), (1.4), (1.6)–(1.8) *are fulfilled. Then for an arbitrary* $T > 0$, *there exists a weak solution* u *of problem* (1.1), (1.2).

If, in addition, f *is Lipschitz continuous function, then the solution is unique.*

2. A priori estimates for the regularized problem

2.1. Regularization

Consider the regularized equation

$$u_{\varepsilon t} = \left((u_{\varepsilon x}^{\alpha} + \varepsilon)^{\frac{p(x)}{\alpha}} u_{\varepsilon x} \right)_x + f(x, u_\varepsilon, u_{\varepsilon x}). \tag{2.1}$$

Rewrite this equation in the equivalent form

$$u_{\varepsilon t} = a(\varepsilon, x, u_{\varepsilon x})u_{\varepsilon xx} + b(\varepsilon, x, u_{\varepsilon x}) + f(x, u_\varepsilon, u_{\varepsilon x}). \tag{2.2}$$

where

$$a(\varepsilon, x, u_{\varepsilon x}) = (u_{\varepsilon x}^{\alpha} + \varepsilon)^{\frac{p(x)}{\alpha}} \left(1 + p(x) \frac{u_{\varepsilon x}^{\alpha}}{u_{\varepsilon x}^{\alpha} + \varepsilon} \right)$$

and

$$b(\varepsilon, x, u_{\varepsilon x}) = p'(x) u_{\varepsilon x} \left(u_{\varepsilon x}^{\alpha} + \varepsilon \right)^{\frac{p(x)}{\alpha}} ln \left(u_{\varepsilon x}^{\alpha} + \varepsilon \right)^{\frac{1}{\alpha}}.$$

Here constants ε and α belong to $(0, 1)$.

We additionally suppose that $\alpha = r/m$ with positive integers r and m such that $r < m$ and m is even. For such α

$$z^{\alpha} = |z|^{\alpha} \quad \text{and} \quad (z^{\alpha})^{p/\alpha} = |z|^p.$$

The existence of a classical solution u_ε of problem (2.1), (1.2), follows from [11].

2.2. A priori estimates

Our goal in this section is to obtain uniform with respect to ε estimates of this solution which would enable us to pass to the limit as $\varepsilon \to 0$.

First we mention that (1.4) implies the estimate

$$|u_\varepsilon(t,x)| \leq M = \inf_{\lambda > \alpha} e^{\lambda T} \left[\max_{\Gamma_T} |u|, \left(\frac{\beta}{\lambda - \alpha} \right)^{1/2} \right], \tag{2.3}$$

for every $\varepsilon \in (0, 1)$ (see [7, relation (2.31)]). By Γ_T we denote the parabolic boundary of Q_T i.e.

$$\Gamma_T = \partial \overline{Q}_T \backslash \{t = T, \, |x| < l\}.$$

Denote by K the following quantity:

$$K=\max_{x,\varepsilon}\left|\left((u_{0x}^{\alpha}+\varepsilon)^{\frac{p(x)}{\alpha}}u_{0x}\right)_x\right|+\max_{x,u}|f(x,u,u_{0x})|<\infty,$$

here maximum is taking over the set $x\in[-l,l]$, $\varepsilon\in[0,1]$, $u\in[-M,M]$.

We start with the estimate of $u_{\varepsilon t}$ at $t=0$.

Lemma 2.1. *For every $\varepsilon\in(0,1)$ the following inequality*

$$|u_\varepsilon(t,x)-u_0(x)|\leq K\,t,\quad \forall(t,x)\in Q_T$$

takes place.

Proof. For simplicity we will omit in the proof the subindex ε.

Introduce the function

$$h(t)=(K+\delta)t\quad\text{in}\quad[0,T],$$

where $\delta>0$. Let us prove the following inequality

$$u(t,x)-u_0(x)\leq h(t)\quad\text{for}\quad(t,x)\in\overline{Q}_T. \tag{2.4}$$

Consider the linear operator

$$L\equiv\frac{\partial}{\partial x}\left((|u_{0x}|^{\alpha}+\varepsilon)^{p(x)/\alpha}\frac{\partial}{\partial x}\right)-\frac{\partial}{\partial t}.$$

Define the function $\phi^+(t,x)\equiv u(t,x)-[u_0(x)+h(t)]$, obviously

$$\begin{aligned}L\phi^+&=\frac{\partial}{\partial x}\left((|u_{0x}|^{\alpha}+\varepsilon)^{p(x)/\alpha}\frac{\partial u}{\partial x}\right)-\frac{\partial u}{\partial t}\\&\quad-\frac{\partial}{\partial x}\left((|u_{0x}|^{\alpha}+\varepsilon)^{p(x)/\alpha}\frac{\partial u_0}{\partial x}\right)+K+\delta\\&>\left((|u_{0x}|^{\alpha}+\varepsilon)^{p(x)/\alpha}u_x\right)_x-u_t+|f(x,u,u_{0x})|.\end{aligned} \tag{2.5}$$

Suppose that at some point $N\in\overline{Q}_T\setminus\Gamma_T$ the function ϕ^+ attains its maximum, then at this point

$$\phi^+_x=0\quad\Leftrightarrow\quad u_x=u_{0x}$$

and hence

$$\begin{aligned}&\left((|u_{0x}|^{\alpha}+\varepsilon)^{p(x)/\alpha}u_x\right)_x-u_t+|f(x,u,u_{0x})|\Big|_N\\&=\left((|u_x|^{\alpha}+\varepsilon)^{p(x)/\alpha}u_x\right)_x-u_t+|f(x,u,u_x)|\Big|_N\\&=-f(x,u,u_x)+|f(x,u,u_x)|\Big|_N\geq 0.\end{aligned}$$

Thus, from (2.5)

$$L\phi^+\Big|_N>0$$

which contradicts the assumption that ϕ^+ attains its maximum at N.

Consider ϕ^+ on Γ_T:

for $x=\pm l$, $t\in[0,T]$ we have $\phi^+=-h(t)\leq 0$;

for $t=0$, $|x|\leq l$ we have $\phi^+=-h(0)=0$.

Thus $\phi^+ \leq 0$ on Γ_T and consequently

$$\phi^+ \leq 0 \quad \text{in} \quad \overline{Q}_T.$$

Inequality (2.4) is proved.

Let us show now that

$$u(t,x) - u_0(x) \geq -h(t) \quad \text{for} \quad (t,x) \in \overline{Q}_T. \tag{2.6}$$

For the function $\phi^-(t,x) \equiv u(t,x) - [u_0(x) - h(t)]$ we have

$$\begin{aligned} L\phi^- &= \Big((|u_{0x}|^\alpha + \varepsilon)^{p(x)/\alpha} u_x\Big)_x - u_t - \Big((|u_{0x}|^\alpha + \varepsilon)^{p(x)/\alpha} u_{0x}\Big)_x - K - \delta \\ &< \Big((|u_{0x}|^\alpha + \varepsilon)^{p(x)/\alpha} u_x\Big)_x - u_t - |f(x,u,u_{0x})|. \end{aligned} \tag{2.7}$$

Suppose that at some point $N_1 \in \overline{Q}_T \backslash \Gamma_T$ the function ϕ^- attains its minimum, then at this point

$$\phi^-_x = 0 \quad \Leftrightarrow \quad u_x = u_{0x}$$

and

$$\begin{aligned} &\Big((|u_{0x}|^\alpha + \varepsilon)^{p(x)/\alpha} u_x\Big)_x - u_t - |f(x,u,u_{0x})|\Big|_{N_1} \\ &\quad = \Big((|u_x|^\alpha + \varepsilon)^{p/\alpha} u_x\Big)_x - u_t - |f(x,u,u_x)|\Big|_{N_1} \\ &\quad = -f(x,u,u_x) - |f(x,u,u_x)|\Big|_{N_1} \leq 0, \end{aligned}$$

hence, from (2.7)

$$L\phi^-\Big|_{N_1} < 0$$

which contradicts the assumption that ϕ^- attains its minimum at N_1. On Γ_T we have $\phi^- \geq 0$, hence

$$\phi^- \geq 0 \quad \text{in} \quad \overline{Q}_T$$

and (2.6) is proved. From (2.4) and (2.6) we obtain that

$$|u(t,x) - u_0(x)| \leq h(t) \quad \text{in} \quad \overline{Q}_T.$$

Passing to the limit when $\delta \to 0$ we finish the prove of Lemma 2.1. □

We turn now to the global estimate of the time derivative.

Lemma 2.2. *For every $\varepsilon \in (0,1)$ the following estimate*

$$|u_{\varepsilon t}| \leq K, \quad \forall (t,x) \in Q_T,$$

takes place.

Proof. As in the prove of the previous lemmas, here we also omit the subindex ε.

Consider Eq. (2.2) in two different points:

$$u_t(t,x) = a(\varepsilon, x, u_x(t,x)) u_{xx}(t,x) + b(\varepsilon, x, u_x(t,x)) + f(x, u(t,x), u_x(t,x)) \tag{2.8}$$

and

$$u_\tau(\tau,x) = a(\varepsilon,x,u_x(\tau,x))u_{xx}(\tau,x) + b(\varepsilon,x,u_x(\tau,x)) + f(x,u(\tau,x),u_x(\tau,x)) \tag{2.9}$$

where $t \neq \tau$.

Subtracting (2.9) from (2.8) for the function $v(t,\tau,x) \equiv u(t,x) - u(\tau,x)$ we have

$$\begin{aligned} &v_t + v_\tau - a(\varepsilon,x,u_x(t,x))v_{xx} \\ &\quad = [a(\varepsilon,x,u_x(t,x)) - a(\varepsilon,x,u_x(\tau,x))]u_{xx}(\tau,x) \\ &\qquad + [b(\varepsilon,x,u_x(t,x)) - b(\varepsilon,x,u_x(\tau,x))] \\ &\qquad + [f(x,u(t,x),u_x(t,x)) - f(x,u(\tau,x),u_x(\tau,x))]. \end{aligned}$$

To obtain the above equation we use the following obvious relations

$$\begin{aligned} v_t(t,\tau,x) &= u_t(t,x), \quad v_\tau(t,\tau,x) = -u_\tau(\tau,x), \\ v_{xx}(t,\tau,x) &= u_{xx}(t,x) - u_{xx}(\tau,x), \end{aligned}$$

Define the function

$$\mathrm{w} \equiv v - K(t-\tau) = u(t,x) - u(\tau,x) - K(t-\tau) \tag{2.10}$$

in the domain

$$P = \{(t,\tau,x) : t \in (0,T), \tau \in (0,T), |x| < l, t > \tau\}.$$

The function w satisfies the following relation:

$$\begin{aligned} &\mathrm{w}_t + \mathrm{w}_\tau - a(\varepsilon,x,u_x(t,x))\mathrm{w}_{xx} \\ &\quad = [a(\varepsilon,x,u_x(t,x)) - a(\varepsilon,x,u_x(\tau,x))]u_{xx}(\tau,x) \\ &\qquad + [b(\varepsilon,x,u_x(t,x)) - b(\varepsilon,x,u_x(\tau,x))] \\ &\qquad + [f(x,u(t,x),u_x(t,x)) - f(x,u(\tau,x),u_x(\tau,x))]. \end{aligned}$$

Introduce the function

$$\omega \equiv \mathrm{w}\, e^{-\tau}$$

which satisfies in P the following linear ultraparabolic equation

$$\begin{aligned} L\,\omega &\equiv \omega_t + \omega_\tau + \omega - a(\varepsilon,x,u_x(t,x))\omega_{xx} \\ &= e^{-\tau}\left([a(\varepsilon,x,u_x(t,x)) - a(\varepsilon,x,u_x(\tau,x))]u_{xx}(\tau,x)\right. \\ &\quad \left. + [b(\varepsilon,x,u_x(t,x)) - b(\varepsilon,x,u_x(\tau,x))]\right) \\ &\quad + e^{-\tau}\left([f(x,u(t,x),u_x(t,x)) - f(x,u(\tau,x),u_x(\tau,x))]\right). \end{aligned} \tag{2.11}$$

Let

$$\Gamma_\tau = \partial P \backslash \{(t,\tau,x) : t = T,\ 0 < \tau < T,\ |x| < l\}.$$

Suppose that the function ω attains its positive maximum at some point $N(t_0,\tau_0,x_0) \in \overline{P}\backslash\Gamma_\tau$. At this point it should be

$$L\,\omega\Big|_N > 0,$$

since

$$\omega_t(N) \geq 0, \quad \omega_\tau(N) \geq 0, \quad \omega(N) > 0 \quad \text{and} \quad -\omega_{xx}(N) \geq 0.$$

On the other hand at this point $\omega_x = 0$ i.e.

$$u_x(t_0, x_0) = u_x(\tau_0, x_0),$$

hence

$$a(\varepsilon, x_0, u_x(t_0, x_0)) = a(\varepsilon, x_0, u_x(\tau_0, x_0)),$$
$$b(\varepsilon, x_0, u_x(t_0, x_0)) = b(\varepsilon, x_0, u_x(\tau_0, x_0))$$

and, since $u(t_0, x_0) > u(\tau_0, x_0)$,

$$f(x_0, u(t_0, x_0), u_x(t_0, x_0)) \leq f(x_0, u(\tau_0, x_0), u_x(\tau_0, x_0)),$$

the last is due to condition (1.6). Thus (2.11) implies that

$$L\,\omega\Big|_N \leq 0.$$

From this contradiction we conclude that ω can not attain its positive maximum in $\overline{P}\backslash\Gamma_\tau$.

Consider ω on Γ_τ:
for $|x| = l$, $t \in [0, T]$, $\tau \in [0, T]$ we have $\omega = -K(t - \tau)e^{-\tau} \leq 0$;
for $t = \tau$, $|x| < l$, $t \in [0, T]$ we have $\omega = 0$;
for $\tau = 0$, $t \in [0, T]$, $|x| < l$ we have $\omega = u(t, x) - u_0(x) - Kt \leq 0$ due to Lemma 2.1.

Consequently $\omega \leq 0$ in $\overline{P}$ i.e.

$$u(t, x) - u(\tau, x) \leq K(t - \tau).$$

Now subtracting (2.8) from (2.9) for the function $\tilde{v}(t, \tau, x) \equiv u(\tau, x) - u(t, x)$ we obtain

$$\begin{aligned}
&\tilde{v}_t + \tilde{v}_\tau - a(\varepsilon, x, u_x(\tau, x))\tilde{v}_{xx} \\
&\quad = [a(\varepsilon, x, u_x(\tau, x)) - a(\varepsilon, x, u_x(t, x))]u_{xx}(t, x) \\
&\qquad + [b(\varepsilon, x, u_x(\tau, x)) - b(\varepsilon, x, u_x(t, x))] \\
&\qquad + [f(x, u(\tau, x), u_x(\tau, x)) - f(x, u(t, x), u_x(t, x))].
\end{aligned}$$

Define the function

$$\tilde{\mathrm{w}} \equiv \tilde{v} - K(t - \tau) = u(\tau, x) - u(t, x) - K(t - \tau),$$

which satisfies in P the following relation:

$$\begin{aligned}
&\tilde{\mathrm{w}}_t + \tilde{\mathrm{w}}_\tau - a(\varepsilon, x, u_x(\tau, x))\tilde{\mathrm{w}}_{xx} \\
&\quad = [a(\varepsilon, x, u_x(\tau, x)) - a(\varepsilon, x, u_x(t, x))]u_{xx}(t, x) \\
&\qquad + [b(\varepsilon, x, u_x(\tau, x)) - b(\varepsilon, x, u_x(t, x))] \\
&\qquad + [f(x, u(\tau, x), u_x(\tau, x)) - f(x, u(t, x), u_x(t, x))].
\end{aligned}$$

Introduce the function

$$\tilde{\omega} \equiv \tilde{\mathrm{w}}\, e^{-\tau}$$

which satisfies in P the following linear ultraparabolic equation

$$\begin{aligned} L\,\tilde{\omega} &\equiv \tilde{\omega}_t + \tilde{\omega}_\tau + \tilde{\omega} - a(\varepsilon, x, u_x(\tau, x))\tilde{\omega}_{xx} \\ &= e^{-\tau}\left([a(\varepsilon, x, u_x(\tau, x)) - a(\varepsilon, x, u_x(t, x))]u_{xx}(t, x)\right] \\ &\qquad + [b(\varepsilon, x, u_x(\tau, x)) - b(\varepsilon, x, u_x(t, x))]\big) \\ &\qquad + e^{-\tau}\left([f(x, u(\tau, x), u_x(\tau, x)) - f(x, u(t, x), u_x(t, x))]\right). \end{aligned}$$

Similarly to the previous case we obtain that $\tilde{\omega}$ can not attain its positive maximum in $\overline{P}\backslash\Gamma_\tau$ and that $\tilde{\omega} \le 0$ on Γ_τ.

Consequently $\tilde{\omega} \le 0$ in $\overline{P}$ i.e.

$$u(\tau, x) - u(t, x) \le K(t - \tau). \tag{2.12}$$

From (2.12) and (2.12) we conclude that in $\overline{P}$

$$|u(t, x) - u(\tau, x)| \le K(t - \tau).$$

Taking into account the symmetry of the variables t and τ we similarly consider the case $t < \tau$ to obtain that in

$$\{(t, \tau, x) : t \in [0, T],\ \tau \in [0, T], x\ \in [-l, l]$$

the inequality

$$|u(t, x) - u(\tau, x)| \le K|t - \tau|$$

holds. The last implies the required estimate. □

We also need the estimates of the spatial derivative of the solution.

Lemma 2.3. *There exists a constant C_0 independent of ε such that*

$$|u_{\varepsilon x}| \le C_0,$$

for every $\varepsilon \in (0, 1)$.

Proof. This lemma follows from [11]. In fact, taking into account (1.5) we see that there exists a smooth, nonnegative, nondecreasing function $\overline{\psi}$ such that

$$\int^{\infty} \frac{\rho d\rho}{\overline{\psi}(\rho)} = +\infty$$

and

$$|f(x, u_\varepsilon, q) + b(\varepsilon, x, q)| \le a(\varepsilon, x, q)\overline{\psi}(|q|) \quad \forall\varepsilon \ge 0.$$

Thus (see Lemma 3 from [11]) the estimate $|u_{\varepsilon x}| \le C_0$ is true with C_0 depending only on M and $\overline{\psi}$. □

Let us obtain the estimate of the second derivative for the singular case.

Lemma 2.4. *Suppose that $-1 < p(x) \le 0$ for $x \in [-l, l]$. Then, there exists a constant C_1 independent of ε such that for every $\varepsilon \in (0, 1)$ the following estimate takes place*

$$|u_{\varepsilon xx}| \le C_1.$$

Proof. From (2.2) we have (we omit the subindex ε):

$$|u_{xx}| \le \frac{|u_t|}{a} + \frac{|b|}{a} + \frac{|f|}{a}.$$

In order to obtain the needed estimate (taking into account the estimates obtained in the previous lemmas) it is sufficient to estimate the term $a(\varepsilon, x, u_x)$ from the below uniformly with respect to ε. We have

$$(u_x^\alpha + \varepsilon)^{\frac{p(x)}{\alpha}} \ge (C_0^\alpha + 1)^{\frac{p(x)}{\alpha}}$$

taking into account that

$$0 < 1 + p^- \le 1 + p(x)\frac{u_x^\alpha}{u_x^\alpha + \varepsilon}, \quad \text{where} \quad p^- = \min_{|x|\le l} p(x) > -1,$$

we obtain

$$a(\varepsilon, x, u_x) = (u_x^\alpha + \varepsilon)^{\frac{p(x)}{\alpha}} \left(1 + p(x)\frac{u_x^\alpha}{u_x^\alpha + \varepsilon}\right) \ge (C_0^\alpha + 1)^{\frac{p(x)}{\alpha}} (1 + p^-).$$

□

Note that both in Lemmas 2.3 and 2.4 we essentially use that we have only one spatial variable.

3. Proof of Theorems 1 and 2

We will obtain a strong (and weak) solution to problem (1.1), (1.2) as a limit of the approximate solutions u_ε constructed in the previous section. The uniqueness in both theorems can be proved by standard considerations taking into account the monotonicity of the elliptic part of the operator (see, for example, [10]).

Let us start with the existence in Theorem 1.

Consider problem (2.2), (1.2).

From the estimates of Lemmas 2.2–2.4 it follows that

$$|u_{\varepsilon x}(t, x) - u_{\varepsilon x}(\tau, x)| \le C_2|t - \tau|^{1/2} \tag{3.1}$$

with constant C_2 depending only on K, C_0 and C_1 (see [7, Chapter II, Lemma 3.1]). Thus, taking into account inequality (3.1) and the estimates obtained in previous section we conclude that there exist a subsequence ε_k such that

$$u_{\varepsilon_k} \to u, \quad u_{\varepsilon_k x} \to u_x \quad \text{uniformly},$$

and

$$u_{\varepsilon_k t} \to u_t, \quad u_{\varepsilon_k xx} \to u_{xx} \quad \text{*-weakly in} \quad L_\infty(Q_T),$$

as $\varepsilon_k \to 0$. Hence

$$a(\varepsilon_k, x, u_{\varepsilon_k x})u_{\varepsilon_k xx} \to a(0, x, u_x)u_{xx} = (1 + p(x))|u_x|^{p(x)}u_{xx} \quad \text{*-weakly in} \quad L_\infty(Q_T),$$

$$b(\varepsilon_k, x, u_{\varepsilon_k x}) \to b(0, x, u_x) = p'(x)u_x|u_x|^{p(x)}ln|u_x| \quad \text{uniformly},$$

$$f(x, u_{\varepsilon_k}, u_{\varepsilon_k x}) \to f(x, u, u_x) \quad \text{uniformly}.$$

Note that $b(0, x, u_x) = b(x, u_x)$ and recall that we put $b(x, 0) = 0$. Obviously $b(\varepsilon, x, 0) = 0$.

Multiplying Eq. (2.2) by an arbitrary smooth function ϕ, integrating over Q_T and passing to the limit when $\varepsilon_k \to 0$ we obtain the strong solution according to Definition 1.

Theorem 1 is proved.

Let us turn to the proof of Theorem 2.

Consider problem (2.1), (1.2). From the estimates obtained in Lemmas 2.2 and 2.3 we have that there exists a subsequence ε_k such that

$$u_{\varepsilon_k} \to u \quad \text{uniformly},$$

and

$$u_{\varepsilon_k t} \to u_t, \quad u_{\varepsilon_k x} \to u_x \quad \text{*-weakly in} \quad L_\infty(Q_T),$$

as $\varepsilon_k \to 0$.

Multiplying Eq. (2.1) by an arbitrary smooth function ϕ which vanishes on $x = \pm l$ and integrating by parts we obtain

$$\int_{Q_T} u_{\varepsilon t}\, \phi\, dxdt + \int_{Q_T} (|u_{\varepsilon x}|^\alpha + \varepsilon)^{p(x)/\alpha}\, u_{\varepsilon x}\, \phi_x\, dxdt \\ = \int_{Q_T} (g_1(x, u_\varepsilon)u_{\varepsilon x} + g_2(x, u_\varepsilon))\, \phi\, dxdt. \tag{3.2}$$

Obviously

$$g_1(x, u_{\varepsilon_k})u_{\varepsilon_k x} + g_2(x, u_{\varepsilon_k}) \to g_1(x, u)u_x + g_2(x, u) \quad \text{*-weakly in} \quad L_\infty(Q_T).$$

Thus, in order to pass to the limit in (3.2), we only have to prove that

$$\int_{Q_T} (|u_{\varepsilon_k x}|^\alpha + \varepsilon_k)^{p(x)/\alpha}\, u_{\varepsilon_k x}\, \phi_x\, dxdt \to \int_{Q_T} |u_x|^{p(x)} u_x\, \phi_x\, dtdx \quad \text{as} \quad \varepsilon_k \to 0.$$

This can be done similarly to as it was done in [10](page 3018).

Theorem 2 is proved.

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