Nonlinear Differ. Equ. Appl. (2016) 23:10

1021-9722/16/020001-26
published online March 24, 2016
DOI 10.1007/s00030-016-0376-z

Nonlinear Differential Equations and Applications NoDEA

Picone's identity for biharmonic operators on Heisenberg group and its applications

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Abstract. In this paper, we establish a nonlinear analogue of Picone's identity for biharmonic operators on Heisenberg group. As an applications of Picone's identity, we obtain Hardy-Rellich type inequality, Morse index, Caccioppoli inequality, Picone inequality for biharmonic operators on Heisenberg group.

Mathematics Subject Classification. Primary 35J91 · Secondary 35B35.

Keywords. Bi-Laplacian, Variational methods, Hardy-Rellich type inequality, Morse index, Caccioppoli inequality, Picone inequality, Picone's identity on Heisenberg group.

1. Introduction

It is a well-known fact that in the qualitative theory of elliptic PDEs, Picone's identity plays an important role. The classical Picone's identity says that if u and v are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0, \tag{1.1}$$

see [30]. (1.1) has an enormous applications to second-order elliptic equations and systems, see for instance [2–4,27] and the references therein. Let us write briefly the recent developments on Picone's identity. Berestycki et al. [7] proved a generalized Picone's identity. Their identity is as follows:

Theorem 1.1. *Let* $\psi \in C^1(\mathbb{R})$ *and* $u, \phi \in C^2, u > 0,$ *then*

G. Dwivedi thanks IIT Gandhinagar for the financial support.

$$div\left[\psi\left(\frac{\phi}{u}\right)(u\nabla\phi-\phi\nabla u)\right]=\psi\left(\frac{\phi}{u}\right)(u\Delta\phi-\phi\Delta u)$$
$$+\psi'\left(\frac{\phi}{u}\right)u^2\left|\nabla\left(\frac{\phi}{u}\right)\right|^2. \qquad (1.2)$$

Equation (1.2) reduces to classical Picone's identity (1.1) in the case $\psi(t)=t$. In order to apply (1.1) to p-Laplace equations, (1.1) is extended by Allegretto and Huang [5]. The extension to (1.1) is as follows:

Theorem 1.2. [5] *Let $v>0$ and $u\geq 0$ be differentiable functions. Denote*

$$L(u,v)=|\nabla u|^p+(p-1)\frac{u^p}{v^p}|\nabla v|^p-p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v.$$
$$R(u,v)=|\nabla u|^p-\nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v)=R(u,v)$. Moreover, $L(u,v)\geq 0$ and $L(u,v)=0$ a.e. in Ω if and only if $\nabla(\frac{u}{v})=0$ a.e. in Ω.

Recently, the second author obtain a nonlinear analogue of (1.1) in [32]. The nonlinear analogue of (1.1) reads as follows:

Theorem 1.3. [32] *Let v be a differentiable function in Ω such that $v\neq 0$ in Ω and u be a non-constant differentiable function in Ω. Let $f(y)\neq 0,\ \forall\, 0\neq y\in\mathbb{R}$ and suppose that there exists $\alpha>0$ such that $f'(y)\geq\frac{1}{\alpha},\ \forall\, 0\neq y\in\mathbb{R}$. Denote*

$$L(u,\,v)=\alpha|\nabla u|^2-\frac{|\nabla u|^2}{f'(v)}+\left(\frac{u\sqrt{f'(v)}\nabla v}{f(v)}-\frac{\nabla u}{\sqrt{f'(v)}}\right)^2. \qquad (1.3)$$

$$R(u,\,v)=\alpha|\nabla u|^2-\nabla\left(\frac{u^2}{f(v)}\right)\nabla v. \qquad (1.4)$$

Then $L(u,\,v)=R(u,\,v)$. Moreover, $L(u,\,v)\geq 0$ and $L(u,\,v)=0$ in Ω if and only if $u=c_1v+c_2$ for some arbitrary constants c_1, c_2.

Bal [6] extended the nonlinear Picone's identity of [32] to deal with p-Laplace equations. The extension reads as follows:

Theorem 1.4. [6] *Let $v>0$ and $u\geq 0$ be two non-constant differentiable functions in Ω. Also assume that $f'(y)\geq(p-1)[f(y)^{\frac{p-2}{p-1}}]$ for all y. Define*

$$L(u,v)=|\nabla u|^p-\frac{pu^{p-1}\nabla u|\nabla v|^{p-2}\nabla v}{f(v)}+\frac{u^pf'(v)|\nabla v|^p}{[f(v)]^2}.$$
$$R(u,v)=|\nabla u|^p-\nabla\left(\frac{u^p}{f(v)}\right)|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v)=R(u,v)\geq 0$. Moreover $L(u,v)=0$ a.e. in Ω if and only if $\nabla(\frac{u}{v})=0$ a.e. in Ω.

There are also several interesting articles dealing with Picone's identity in different contexts. We just name a few articles, for instance, for Picone's identities to half-linear elliptic operators with $p(x)$-Laplacians, we refer to [35] and for Picone-type identity to pseudo p-Laplacian with variable power, we refer to [9].

In the past and recently there has been a good amount of interest on the existence and qualitative questions such as Hardy inequality [12,13], Picone's identity [5,30,32,35], Morse index [15], Caccioppoli inequality [17,24], Sturm comparison theorem for p-Laplace operators [4] as well as biharmonic operators, see [11] for the existence and uniqueness of a solution to the variational inequality to biharmonic operators. We remark that very little research work is known for these operators on Heisenberg group. Birindelli and Capuzzo Dolcetta obtained the Morse index and Liouville property for superlinear elliptic equations on the Heisenberg group, see [8] and the references therein. Niu et al. obtained Picone's identity for p-Laplace equations on Heisenberg group, see [29] and the references therein. Their identity reads as follows:

Theorem 1.5. [29] *For differentiable functions $u \geq 0$, $v > 0$ on $\Omega \subset \mathbb{H}^n$, where Ω is a bounded or unbounded domain in $\mathbb{H}^n$, we have*

$$L(u,v) = R(u,v) \geq 0, \quad \textit{where}$$

$$L(u,v) = |\nabla_H u|^p + (p-1)\frac{u^p}{v^p}|\nabla_H v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla_H v|^{p-2}\nabla v.$$

$$R(u,v) = |\nabla_H u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla_H v|^{p-2}\nabla v.$$

As an application of Theorem 1.5, Dou [16] obtained Picone type inequality which reads as follows:

Theorem 1.6. [16] *Let $p > 1$ and $\Omega \subset \mathbb{H}^n$ or $\Omega = \mathbb{H}^n$. If $u \in D_0^{1,p}(\Omega)$, $u \geq 0$, $v \in D_0^{1,p}(\Omega)$, $-\Delta_{H,p} v \geq 0$ is a bounded radon measure, $v|_{\partial\Omega} = 0$, $v \geq 0$ and not identically zero. Then*

$$\int_\Omega |\nabla_H u|^p \geq \int_\Omega \frac{u^p}{v^{p-1}}(-\Delta_{H,p} v),$$

where $\Delta_{H,p} u = \nabla_H(|\nabla_H u|^{p-2}\nabla_H u)$, $p > 1$ is called the p-sub Laplacian.

There are several research papers dealing with the applications of Picone's identity but we mention only those articles which are closely related to this paper. Han and Niu [22] proved Hardy type inequality for p-Laplacian on Heisenberg group using the Picone's identity, Theorem 1.5. Han et al. [23] and Liu and Luan [26] established Hardy type inequalities on half spaces of Heisenberg group. Lian et al. [25] established weighted Hardy type inequality for Heisenberg Laplacian in anisotropic Heisenberg group. Xiao [33] established Hardy type inequality with Aharanov-Bohm type magnetic field on Heisenberg group.

The aim of this paper is to obtain Picone's identity for biharmonic operators on Heisenberg group. In fact, we also establish a nonlinear analogue of Picone's identity for biharmonic operators on Heisenberg group and discuss related qualitative questions such as Morse index, monotonicity of the principle eigenvalue associated with biharmonic operator on Heisenberg group, Caccioppoli inequality etc.

The organization of this paper is as follows. Section 2 deals with the preliminaries on the Heisenberg group. In Sect. 3, we establish Picone's identity for biharmonic operators on Heisenberg group. Section 4 deals with a nonlinear analogue of Picone's identity for biharmonic operators. Section 5 deals with the auxiliary results and applications of Picone's identity.

2. Preliminaries

Let us recall the briefs on the Heisenberg group $\mathbb{H}^n$. The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \boldsymbol{.})$, is the space $\mathbb{R}^{2n+1}$ with the non-commutative law of product

$$(x, y, t)\boldsymbol{.}(x', y', t') = (x + x',\, y + y', t + t' + 2(< y, x' > - < x,\, y' >)),$$

where $x,\, y,\, x',\, y' \in \mathbb{R}^n$, $t,\, t' \in \mathbb{R}$ and $< .,. >$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t},\;\; X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t},\;\; Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},\;\; i = 1,\, 2, .3,\, \ldots,\, n.$$

These generators satisfy the non-commutative formula

$$[X_i, Y_j] = -4\delta_{ij} T,\;\; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x,\, y) \in \mathbb{R}^{2n},\;\; \xi = (z,\, t) \in \mathbb{H}^n$. The parabolic dilation

$$\delta_\lambda \xi = (\lambda x,\, \lambda y,\, \lambda^2 t)$$

satisfies

$$\delta_\lambda(\xi_0 \boldsymbol{.} \xi) = \delta_\lambda \xi \boldsymbol{.} \delta_\lambda \xi_0$$

and

$$||\xi||_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$$

is a norm with respect to the parabolic dilation which is known as Korányi gauge norm $N(z,\, t)$. In other words, $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ denotes the Heisenberg distance between ξ and the origin. Similarly, one can define the distance between $(z,\, t)$ and $(z',\, t')$ on $\mathbb{H}^n$ as follows:

$$\rho(z,\, t; z', t') = \rho((z',\, t')^{-1} \,.\, (z,\, t)).$$

It is clear that the vector fields $X_i,\, Y_i,\, i = 1,\, 2,\, \ldots,\, n$ are homogeneous of degree 1 under the norm $||.||_{\mathbb{H}^n}$ and T is homogeneous of degree 2. The Lie algebra of Heisenberg group has the stratification $\mathbb{H}^n = V_1 \oplus V_2$, where the $2n$-dimensional horizontal space V_1 is spanned by $\{X_i,\, Y_i\},\; i = 1,\, 2,\, \ldots,\, n$, while V_2 is spanned by T. The Korányi ball of center ξ_0 and radius r is defined by

$$B_{\mathbb{H}^n}(\xi_0,\, r) = \{\xi : ||\xi^{-1} \boldsymbol{.} \xi_0|| \leq r\}$$

and it satisfies

$$|B_{\mathbb{H}^n}(\xi_0,\, r)| = |B_{\mathbb{H}^n}(0,\, r)| = r^d |B_{\mathbb{H}^n}(\mathbf{0},\, 1)|,$$

where $|.|$ is the $(2n + 1)$-dimensional Lebesgue measure on $\mathbb{H}^n$ and $d = 2n + 2$ is the so called the homogeneous dimension of Heisenberg group $\mathbb{H}^n$. The

Heisenberg gradient and Heisenberg Laplacian or the Laplacian-Kohn operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)$$

and

$$\begin{aligned}\Delta_{\mathbb{H}^n} &= \sum_{i=1}^{n} X_i^2 + Y_i^2 \\ &= \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2} \right).\end{aligned}$$

By [19], the fundamental solution on $\mathbb{H}^n$ of $-\Delta_{\mathbb{H}^n}$ with pole at the origin is

$$\Gamma(\xi) = \frac{c_d}{\rho(\xi)^{d-2}},$$

where c_d is a positive constant and $d = 2n + 2$ is the homogeneous dimension of the group. The fundamental solution on $\mathbb{H}^n$ of $-\Delta_{\mathbb{H}^n}$ with pole at ξ_0 is

$$\Gamma(\xi, \xi_0) = \frac{c_d}{\rho(\xi, \xi_0)^{d-2}}.$$

Definition 2.1. ($D^{1,p}(\Omega)$ *and* $D_0^{1,p}(\Omega)$) Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ such that } u, |\nabla_{\mathbb{H}^n} u| \in L^p(\Omega)\}.$$

$D^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{D^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

$D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$

Definition 2.2. ($D^{2,p}(\Omega)$ *and* $D_0^{2,p}(\Omega)$) Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{2,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ such that } u, |\nabla_{\mathbb{H}^n} u|, |\Delta_{\mathbb{H}^n} u| \in L^p(\Omega)\}.$$

$D^{2,p}(\Omega)$ is equipped with the norm

$$\|u\|_{D^{2,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} + \|\Delta_{\mathbb{H}^n} u\|^p \right)^{\frac{1}{p}}.$$

$D_0^{2,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{2,p}(\Omega)} = \left(\int_\Omega |\Delta_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$

3. Picone's identity on Heisenberg group

Proposition 3.1. (Picone's identity) *Let u and v be twice continuously differentiable functions in Ω such that $v > 0$, $-\Delta_{\mathbb{H}^n} v > 0$ in Ω. Denote*

$$L(u,\,v) = \left(\Delta_{\mathbb{H}^n} u - \frac{u}{v}\Delta_{\mathbb{H}^n} v\right)^2 - \frac{2\Delta_{\mathbb{H}^n} v}{v}\left(\nabla_{\mathbb{H}^n} u - \frac{u}{v}\nabla_{\mathbb{H}^n} v\right)^2.$$
$$R(u,\,v) = |\Delta_{\mathbb{H}^n} u|^2 - \Delta_{\mathbb{H}^n}\left(\frac{u^2}{v}\right)\Delta_{\mathbb{H}^n} v.$$

Then (i) $L(u,\,v) = R(u,\,v)$ (ii) $L(u,\,v) \geq 0$ *and* (iii) $L(u,\,v) = 0$ *in Ω if and only if $u = \alpha v$ for some $\alpha \in \mathbb{R}$.*

In order to prove Proposition 3.1, we shall first prove following lemmas.

Lemma 3.2. *Let u and v be continuously differentiable functions in Ω such that $v > 0$ in Ω, then*

$$\begin{aligned}
\left|\nabla_{\mathbb{H}^n} u - \frac{u}{v}\nabla_{\mathbb{H}^n} v\right|^2 &= \sum_{i=1}^n {u_{x_i}}^2 + 4{u_t}^2\sum_{i=1}^n {y_i}^2 + \frac{u^2}{v^2}\sum_{i=1}^n {v_{x_i}}^2 + \frac{4u^2}{v^2}{v_t}^2\sum_{i=1}^n {y_i}^2\\
&\quad + 4u_t\sum_{i=1}^n u_{x_i}y_i - \frac{2u}{v}\sum_{i=1}^n u_{x_i}v_{x_i} - \frac{4u}{v}v_t\sum_{i=1}^n u_{x_i}y_i\\
&\quad - \frac{4u}{v}u_t\sum_{i=1}^n v_{x_i}y_i - \frac{8u}{v}u_tv_t\sum_{i=1}^n {y_i}^2 + \frac{4u^2}{v^2}v_t\sum_{i=1}^n v_{x_i}y_i\\
&\quad + \sum_{i=1}^n {u_{y_i}}^2 + 4{u_t}^2\sum_{i=1}^n {x_i}^2 + \frac{u^2}{v^2}\sum_{i=1}^n {v_{y_i}}^2 + \frac{4u^2}{v^2}{v_t}^2\sum_{i=1}^n {x_i}^2\\
&\quad - 4u_t\sum_{i=1}^n u_{y_i}x_i - \frac{2u}{v}\sum_{i=1}^n u_{y_i}v_{y_i} + \frac{4u}{v}v_t\sum_{i=1}^n u_{y_i}x_i\\
&\quad + \frac{4u}{v}u_t\sum_{i=1}^n v_{y_i}x_i - \frac{8u}{v}u_tv_t\sum_{i=1}^n {x_i}^2 - \frac{4u^2}{v^2}v_t\sum_{i=1}^n v_{y_i}x_i\\
&= \sum_{i=1}^n \left({u_{x_i}}^2 + {u_{y_i}}^2\right) + 4{u_t}^2\sum_{i=1}^n ({x_i}^2 + {y_i}^2)\\
&\quad + \frac{u^2}{v^2}\sum_{i=1}^n ({v_{x_i}}^2 + {v_{y_i}}^2) + \frac{4u^2}{v^2}{v_t}^2\sum_{i=1}^n ({x_i}^2 + {y_i}^2)\\
&\quad + 4u_t\sum_{i=1}^n (u_{x_i}y_i - u_{y_i}x_i) - \frac{2u}{v}\sum_{i=1}^n (u_{x_i}v_{x_i} + u_{y_i}v_{y_i})\\
&\quad + \frac{4u}{v}v_t\sum_{i=1}^n (u_{y_i}x_i - u_{x_i}y_i) + \frac{4u}{v}u_t\sum_{i=1}^n (v_{y_i}x_i - u_{x_i}y_i)\\
&\quad - \frac{8u}{v}u_tv_t\sum_{i=1}^n ({x_i}^2 + {y_i}^2) + \frac{4u^2}{v^2}v_t\sum_{i=1}^n (v_{x_i}y_i - v_{y_i}x_i).
\end{aligned}$$

Proof. Note that

$$\nabla_{\mathbb{H}^n} u = (u_{x_1} + 2y_1 u_t, u_{x_2} + 2y_2 u_t, \ldots, u_{x_n} \\ +2y_n u_t, u_{y_1} - 2x_1 u_t, u_{y_2} - 2x_2 u_t, \ldots, u_{y_n} - 2x_n u_t) \tag{3.1}$$

$$\frac{u}{v}\nabla_{\mathbb{H}^n} v = \frac{u}{v}\,(v_{x_1} + 2y_1 v_t, v_{x_2} + 2y_2 v_t, \ldots, v_{x_n} \\ +2y_n v_t, v_{y_1} - 2x_1 v_t, v_{y_2} - 2x_2 v_t, \ldots, v_{y_n} - 2x_n v_t) \tag{3.2}$$

By (3.1) and (3.2), we get

$$\begin{aligned}\left(\nabla_{\mathbb{H}^n} u - \frac{u}{v}\nabla_{\mathbb{H}^n} v\right) = \Bigg(& u_{x_1} + 2y_1 u_t - \frac{u}{v} v_{x_1} - \frac{2u}{v} y_1 v_t, \\ & u_{x_2} + 2y_2 u_t - \frac{u}{v} v_{x_2} - \frac{2u}{v} y_2 v_t, \ldots, \\ & u_{x_n} + 2y_n u_t - \frac{u}{v} v_{x_n} - \frac{2u}{v} y_n v_t, \\ & u_{y_1} - 2x_1 u_t - \frac{u}{v} v_{y_1} + \frac{2u}{v} x_1 v_t, \\ & u_{y_2} - 2x_2 u_t - \frac{u}{v} v_{y_2} + \frac{2u}{v} x_2 v_t, \ldots, \\ & u_{y_n} - 2x_n u_t - \frac{u}{v} v_{y_n} + \frac{2u}{v} x_n v_t\Bigg)\end{aligned} \tag{3.3}$$

and therefore

$$\begin{aligned}\left|\nabla_{\mathbb{H}^n} u - \frac{u}{v}\nabla_{\mathbb{H}^n} v\right|^2 = \sum_{i=1}^{n}\Bigg\{&\left(u_{x_i} + 2y_i u_t - \frac{u}{v} v_{x_i} - \frac{2u}{v} y_i v_t\right)^2 \\ &+ \left(u_{y_i} - 2x_i u_t - \frac{u}{v} v_{y_i} + \frac{2u}{v} x_i v_t\right)^2\Bigg\}.\end{aligned} \tag{3.4}$$

On expanding the right hand side of (3.4), we get the desired result. □

Lemma 3.3. *Let u and v be twice continuously differentiable functions in Ω such that $v > 0$ in Ω, then*

$$\begin{aligned}\frac{2u}{v}\Delta_{\mathbb{H}^n} u \Delta_{\mathbb{H}^n} v = {} & \frac{2u}{v}\Delta_x u \Delta_x v + \frac{2u}{v}\Delta_x u \Delta_y v + \frac{8u}{v}\Delta_x u \sum_{i=1}^{n}(y_i v_{x_i t} - x_i v_{y_i t}) \\ & + \frac{8u}{v} v_{tt} \Delta_x u \sum_{i=1}^{n}({x_i}^2 + {y_i}^2) + \frac{2u}{v}\Delta_y u \Delta_x v + \frac{2u}{v}\Delta_y u \Delta_y v \\ & + \frac{8u}{v}\Delta_y u \sum_{i=1}^{n}(y_i v_{x_i t} - x_i v_{y_i t}) + \frac{8u}{v}\Delta_y u v_{tt} \sum_{i=1}^{n}({x_i}^2 + {y_i}^2) \\ & + \frac{8u}{v}\Delta_x v \sum_{i=1}^{n}(y_i u_{x_i t} - x_i u_{y_i t}) + \frac{8u}{v}\Delta_y v \sum_{i=1}^{n}(y_i u_{x_i t} - x_i u_{y_i t}) \\ & + \frac{32u}{v}\left(\sum_{i=1}^{n}(y_i u_{x_i t} - x_i u_{y_i t})\right)\left(\sum_{i=1}^{n}(y_i v_{x_i t} - x_i v_{y_i t})\right)\end{aligned}$$

$$+\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\left\{\frac{32u}{v}v_{tt}\sum_{i=1}^{n}(y_iu_{x_it}-x_iu_{y_it})+\frac{8u}{v}u_{tt}\Delta_x v\right.$$
$$+\frac{8u}{v}u_{tt}\Delta_y v+\frac{32u}{v}u_{tt}\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})$$
$$\left.+\frac{32u}{v}u_{tt}v_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\right\}.$$

Proof. By the definition of Heisenberg Laplacian, we have

$$\Delta_{\mathbb{H}^n}u=\Delta_x u+\Delta_y u+4\sum_{i=1}^{n}(y_iu_{x_it}-x_iu_{y_it})+4u_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2) \quad (3.5)$$

and

$$\Delta_{\mathbb{H}^n}v=\Delta_x v+\Delta_y v+4\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})+4v_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2). \quad (3.6)$$

Using (3.5) and (3.6), we obtain the required result. □

Lemma 3.4. *Let u and v be twice continuously differentiable functions in Ω such that $v>0$ in Ω, then*

$$\frac{u^2}{v^2}(\Delta_{\mathbb{H}^n}v)^2=\frac{u^2}{v^2}(\Delta_x v)^2+\frac{u^2}{v^2}(\Delta_y v)^2+\frac{16u^2}{v^2}\left(\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})\right)^2$$
$$+\frac{16u^2}{v^2}(v_{tt})^2\left(\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\right)^2+\frac{2u^2}{v^2}\Delta_x v\Delta_y v$$
$$+\frac{8u^2}{v^2}\Delta_x v\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})+\frac{8u^2}{v^2}\Delta_x v v_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2)$$
$$+\frac{8u^2}{v^2}\Delta_y v\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})+\frac{8u^2}{v^2}\Delta_y v u_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2)$$
$$+\frac{32u^2}{v^2}v_{tt}\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})\sum_{i=1}^{n}({x_i}^2+{y_i}^2).$$

Proof. On squaring both the sides of (3.6) and then multiplying by $\frac{u^2}{v^2}$, we get the desired result. □

Lemma 3.5. *Let u and v be twice continuously differentiable functions in Ω such that $v>0$ in Ω,*

$$\Delta_{\mathbb{H}^n}\left(\frac{u^2}{v}\right)=\frac{2}{v}\sum_{i=1}^{n}({u_{x_i}}^2+{u_{y_i}}^2)+\frac{2u}{v}(\Delta_x u+\Delta_y u)-\frac{u^2}{v^2}(\Delta_x v+\Delta_y v)$$
$$-\frac{4u}{v^2}\sum_{i=1}^{n}(u_{x_i}v_{x_i}+u_{y_i}v_{y_i})+\frac{2u^2}{v^3}\sum_{i=1}^{n}({v_{x_i}}^2+{v_{y_i}}^2)$$

$$
\begin{aligned}
&+\frac{8}{v}u_t\sum_{i=1}^{n}(y_iu_{x_i}-x_iu_{y_i})+\frac{8u}{v^2}v_t\sum_{i=1}^{n}(x_iu_{y_i}-y_iu_{x_i})\\
&+\frac{8u}{v}\sum_{i=1}^{n}(y_iu_{x_it}-x_iu_{y_it})+\frac{8u}{v^2}u_t\sum_{i=1}^{n}(x_iv_{y_i}-y_iv_{x_i})\\
&-\frac{4u^2}{v^2}\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})+\frac{8u^2}{v^3}v_t\sum_{i=1}^{n}(y_iv_{x_i}-x_i-v_{y_i})\\
&+\frac{8}{v}{u_t}^2\sum_{i=1}^{n}({x_i}^2+{y_i}^2)+\frac{8u}{v}u_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\\
&-\frac{4u^2}{v^2}v_{tt}\sum_{i=1}^{n}({x_i}^2+{y_i}^2)-\frac{16u}{v^2}u_tv_t\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\\
&+\frac{8u^2}{v^3}{v_t}^2\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\\
=&\ \frac{2}{v}\sum_{i=1}^{n}({u_{x_i}}^2+{u_{y_i}}^2)+\frac{2u^2}{v^3}\sum_{i=1}^{n}({v_{x_i}}^2+{v_{y_i}}^2)+\frac{2u}{v}(\Delta_xu+\Delta_yu)\\
&-\frac{u^2}{v^2}(\Delta_xv+\Delta_yv)-\frac{4u}{v^2}\sum_{i=1}^{n}(u_{x_i}v_{x_i}+u_{y_i}v_{y_i})\\
&+\sum_{i=1}^{n}(y_iu_{x_i}-x_iu_{y_i})\left(\frac{8u_t}{v}-\frac{8u}{v^2}v_t\right)\\
&+\sum_{i=1}^{n}(y_iv_{x_i}-x_iv_{y_i})\left(\frac{8u^2}{v^3}v_t-\frac{8u}{v^2}u_t\right)\\
&+\frac{8u}{v}\sum_{i=1}^{n}(y_iu_{x_it}-x_iu_{y_it})-\frac{4u^2}{v^2}\sum_{i=1}^{n}(y_iv_{x_it}-x_iv_{y_it})\\
&+\frac{8}{v}{u_t}^2\sum_{i=1}^{n}({x_i}^2+{y_i}^2)+\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\\
&\times\left(\frac{8u}{v}u_{tt}-\frac{4u^2}{v^2}-\frac{16u}{v^2}u_tv_t+\frac{8u^2}{v^3}{v_t}^2\right).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
\Delta_{\mathbb{H}^n}\left(\frac{u^2}{v}\right)=&\sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2}\left(\frac{u^2}{v}\right)+\sum_{i=1}^{n}\frac{\partial^2}{\partial {y_i}^2}\left(\frac{u^2}{v}\right)\\
&+4\sum_{i=1}^{n}\left(y_i\frac{\partial^2}{\partial x_i\partial t}\left(\frac{u^2}{v}\right)-x_i\frac{\partial^2}{\partial y_i\partial t}\left(\frac{u^2}{v}\right)\right)\\
&+4\sum_{i=1}^{n}({x_i}^2+{y_i}^2)\frac{\partial^2}{\partial t^2}\left(\frac{u^2}{v}\right).
\end{aligned}
\tag{3.7}
$$

First we shall compute $\frac{\partial^2}{\partial {x_i}^2}\left(\frac{u^2}{v}\right)$. On differentiating $\frac{u^2}{v}$ with respect to x_i, we obtain

$$\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}\right) = \frac{2uvu_{x_i} - u^2 v_{x_i}}{v^2}. \tag{3.8}$$

By differentiating (3.8) with respect to x_i, we get

$$\begin{aligned}\frac{\partial^2}{\partial {x_i}^2}\left(\frac{u^2}{v}\right) &= \frac{1}{v^4}\left\{v^2(2{u_{x_i}}^2 + 2uu_{x_i}v_{x_i} + 2uvu_{x_ix_i} - 2uu_{x_i}v_{x_i} - u^2v_{x_ix_i})\right.\\ &\quad \left. - 2vv_{x_i}(2uvu_{x_i} - u^2v_{x_i})\right\}\\ &= \frac{2}{v}{u_{x_i}}^2 + \frac{2u}{v}u_{x_ix_i} - \frac{u^2}{v^2}v_{x_ix_i} - \frac{4u}{v^2}u_{x_i}v_{x_i} + \frac{2u^2}{v^3}{v_{x_i}}^2. \end{aligned} \tag{3.9}$$

Similarly

$$\begin{aligned}\frac{\partial^2}{\partial {y_i}^2}\left(\frac{u^2}{v}\right) &= \frac{1}{v^4}\left\{v^2(2{u_{y_i}}^2 + 2uu_{y_i}v_{y_i} + 2uvu_{y_iy_i} - 2uu_{y_i}v_{y_i} - u^2v_{y_iy_i})\right.\\ &\quad \left. - 2vv_{y_i}(2uvu_{y_i} - u^2v_{y_i})\right\}\\ &= \frac{2}{v}{u_{y_i}}^2 + \frac{2u}{v}u_{y_iy_i} - \frac{u^2}{v^2}v_{y_iy_i} - \frac{4u}{v^2}u_{y_i}v_{y_i} + \frac{2u^2}{v^3}{v_{y_i}}^2. \end{aligned} \tag{3.10}$$

On differentiating (3.8) with respect to t, we get

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial t}\left(\frac{u^2}{v}\right) &= \frac{1}{v^4}\left\{v^2(2u_tvu_{x_i} + 2uv_tu_{x_i} + 2uvu_{x_it} - 2uu_tv_{x_i} - u^2v_{x_it})\right.\\ &\quad \left. - 2vv_t(2uvu_{x_i} - u^2v_{x_i})\right\}\\ &= \frac{2}{v}u_tu_{x_i} + \frac{2u}{v^2}v_tu_{x_i} + \frac{2u}{v}u_{x_it} - \frac{2u}{v^2}u_tv_{x_i} - \frac{u^2}{v^2}v_{x_it}\\ &\quad - \frac{4u}{v^2}v_tu_{x_i} + \frac{2u^2}{v^3}v_{x_i}v_t. \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial y_i \partial t}\left(\frac{u^2}{v}\right) &= \frac{1}{v^4}\left\{v^2(2u_tvu_{y_i} + 2uv_tu_{y_i} + 2uvu_{y_it} - 2uu_tv_{y_i} - u^2v_{y_it})\right.\\ &\quad \left. -2vv_t(2uvu_{y_i} - u^2v_{y_i})\right\}\\ &= \frac{2}{v}u_tu_{y_i} + \frac{2u}{v^2}v_tu_{y_i} + \frac{2u}{v}u_{y_it} - \frac{2u}{v^2}u_tv_{y_i} - \frac{u^2}{v^2}v_{y_it}\\ &\quad - \frac{4u}{v^2}v_tu_{y_i} + \frac{2u^2}{v^3}v_{y_i}v_t. \end{aligned} \tag{3.12}$$

Using (3.11) and (3.12),

$$\left(y_i\frac{\partial^2}{\partial x_i\partial t}-x_i\frac{\partial^2}{\partial y_i\partial t}\right)\left(\frac{u^2}{v}\right)=\frac{2}{v}u_t(y_iu_{x_i}-x_iu_{y_i})+\frac{2u}{v^2}v_t(y_iu_{x_i}-x_iu_{y_i})$$
$$+\frac{2u}{v}(y_iu_{x_it}-x_iu_{y_it})+\frac{2u}{v^2}u_t(x_iv_{y_i}-y_iv_{x_i})$$
$$+\frac{u^2}{v^2}(x_iv_{y_it}-y_iv_{x_it})+\frac{4u}{v^2}v_t(x_iu_{y_i}-y_iu_{x_i})$$
$$+\frac{2u^2}{v^3}v_t(y_iv_{x_i}-x_iv_{y_i}). \tag{3.13}$$

Next, we shall compute $\frac{\partial^2}{\partial t^2}\left(\frac{u^2}{v}\right)$. On differentiating $\frac{u^2}{v}$ with respect to t, we obtain

$$\frac{\partial}{\partial t}\left(\frac{u^2}{v}\right)=\frac{1}{v^2}\{2uvu_t-u^2v_t\}. \tag{3.14}$$

On differentiating (3.14) with respect to t, we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{u^2}{v}\right)=\frac{2{u_t}^2}{v}+\frac{2u}{v}u_{tt}-\frac{u^2}{v^2}v_{tt}-\frac{4u}{v^2}u_tv_t+\frac{2u^2}{v^3}{v_t}^2. \tag{3.15}$$

On using (3.9), (3.10), (3.13) and (3.15) in (3.7), we get the required result. □

Proof of Proposition 3.1 Observe that in order to show $L(u,v)=R(u,v)$, it is enough to show $L_1(u,v)=R_1(u,v)$, where

$$L_1(u,v)=\frac{u^2}{v^2}(\Delta_{\mathbb{H}^n}v)^2-\frac{2u}{v}\Delta_{\mathbb{H}^n}u\Delta_{\mathbb{H}^n}v-\frac{2}{v}\Delta_{\mathbb{H}^n}v\left|\nabla_{\mathbb{H}^n}u-\frac{u}{v}\nabla_{\mathbb{H}^n}v\right|^2$$
$$R_1(u,v)=-\Delta_{\mathbb{H}^n}\left(\frac{u^2}{v}\right)\Delta_{\mathbb{H}^n}v.$$

Using Lemmas 3.2, 3.3, 3.4 and 3.5, we get that $L_1(u,v)=R_1(u,v)$. This completes the proof of (i).

Now using the fact that $v>0$, $-\Delta_{\mathbb{H}^n}v>0$ in Ω, one can see that $L(u,\,v)\geq 0$ and therefore (ii) is proved. Further, $L(u,\,v)=0$ in Ω implies that

$$0=\left(\Delta_{\mathbb{H}^n}u-\frac{u}{v}\Delta_{\mathbb{H}^n}v\right)^2-\frac{2\Delta_{\mathbb{H}^n}v}{v}\left(\nabla_{\mathbb{H}^n}u-\frac{u}{v}\nabla_{\mathbb{H}^n}v\right)^2, \text{ that is },$$

$$0\leq-\frac{2\Delta_{\mathbb{H}^n}v}{v}\left(\nabla_{\mathbb{H}^n}u-\frac{u}{v}\nabla_{\mathbb{H}^n}v\right)^2=-\left(\Delta_{\mathbb{H}^n}u-\frac{u}{v}\Delta_{\mathbb{H}^n}v\right)^2\leq 0,$$

which implies that there exists some $\alpha\in\mathbb{R}$ such that $u=\alpha v$. Conversely, when $u=\alpha v$, one can see easily that $L(u,\,v)=0$, and therefore (iii) is proved. □

Remark 3.6. *We note that the Proposition 3.1 also holds if we replace $v>0$ and $-\Delta_{\mathbb{H}^n}v>0$ in Ω by $v<0$ and $-\Delta_{\mathbb{H}^n}v<0$ in Ω, respectively.*

4. A nonlinear Picone's identity on Heisenberg group

Proposition 4.1. (Nonlinear analogue of Picone's identity) *Let u and v be twice continuously differentiable functions in Ω such that $v > 0$, $-\Delta_{\mathbb{H}^n} v > 0$ in Ω. Let $f\colon (0, \infty)\to(0,\infty)$ be a C^2 function such that $f''(y) \leq 0,\ f'(y) \geq 1, \forall\, 0 \neq y \in \mathbb{R}$. Denote*

$$L(u, v) = |\Delta_{\mathbb{H}^n} u|^2 - \frac{|\Delta_{\mathbb{H}^n} u|^2}{f'(v)} + \left(\frac{\Delta_{\mathbb{H}^n} u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta_{\mathbb{H}^n} v\right)^2$$
$$- \frac{2\Delta_{\mathbb{H}^n} v}{f(v)}\left(\nabla_{\mathbb{H}^n} u - \frac{uf'(v)}{f(v)}\nabla_{\mathbb{H}^n} v\right)^2 + \frac{u^2 f''(v)}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2\Delta_{\mathbb{H}^n} v.$$
$$R(u, v) = |\Delta_{\mathbb{H}^n} u|^2 - \Delta_{\mathbb{H}^n}\left(\frac{u^2}{f(v)}\right)\Delta_{\mathbb{H}^n} v.$$

Then (i) $L(u, v) = R(u, v)$ (ii) $L(u, v) \geq 0$ *and* (iii) $L(u, v) = 0$ *in* Ω *if and only if* $u = cv + d$ *for some* $c, d \in \mathbb{R}$.

Proof. Note that

$$\begin{aligned}
\Delta_{\mathbb{H}^n}\left(\frac{u^2}{f(v)}\right) &= \nabla_{\mathbb{H}^n}\left(\nabla_{\mathbb{H}^n}\left(\frac{u^2}{f(v)}\right)\right)\\
&= \nabla_{\mathbb{H}^n}\left\{\frac{1}{f^2(v)}(2uf(v)\nabla_{\mathbb{H}^n} u - u^2 f'(v)\nabla_{\mathbb{H}^n} v)\right\}\\
&= \frac{1}{f^2(v)}\left\{2|\nabla_{\mathbb{H}^n} u|^2 f(v) + 2u\Delta_{\mathbb{H}^n} u f(v)\right.\\
&\quad \left. - u^2 f''(v)|\nabla_{\mathbb{H}^n} v|^2 - u^2 f'(v)\Delta_{\mathbb{H}^n} v\right\}\\
&\quad - \frac{2f'(v)}{f^3(v)}\nabla_{\mathbb{H}^n} v(2uf(v)\nabla_{\mathbb{H}^n} u - u^2 f(v)\nabla_{\mathbb{H}^n} v)\\
&= \frac{1}{f^2(v)}\left\{2|\nabla_{\mathbb{H}^n} u|^2 f(v) + 2u\Delta_{\mathbb{H}^n} u f(v)\right.\\
&\quad \left. - u^2 f''(v)|\nabla_{\mathbb{H}^n} v|^2 - u^2 f'(v)\Delta_{\mathbb{H}^n} v\right\}\\
&\quad - \frac{2f'(v)}{f^3(v)}(2uf(v)\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v - u^2 f(v)|\nabla_{\mathbb{H}^n} v|^2)\\
\Delta_{\mathbb{H}^n}\left(\frac{u^2}{f(v)}\right)\Delta_{\mathbb{H}^n} v &= \frac{1}{f^2(v)}\left\{2|\nabla_{\mathbb{H}^n} u|^2 f(v)\Delta_{\mathbb{H}^n} v + 2uf(v)\Delta_{\mathbb{H}^n} u\Delta_{\mathbb{H}^n} v\right.\\
&\quad \left. - u^2 f''(v)|\nabla_{\mathbb{H}^n} v|^2\Delta_{\mathbb{H}^n} v - u^2 f'(v)(\Delta_{\mathbb{H}^n} v)^2\right\}\\
&\quad - \frac{2f'(v)}{f^3(v)}(2uf(v)\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v - u^2 f(v)|\nabla_{\mathbb{H}^n} v|^2)\\
&= \frac{2}{f(v)}|\nabla_{\mathbb{H}^n} u|^2\Delta_{\mathbb{H}^n} v + \frac{2u}{f(v)}\Delta_{\mathbb{H}^n} u\Delta_{\mathbb{H}^n} v\\
&\quad - \frac{u^2 f''}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2\Delta_{\mathbb{H}^n} v - \frac{u^2 f'(v)}{f^2(v)}(\Delta_{\mathbb{H}^n} v)^2\\
&\quad - \frac{4uf'(v)}{f^2(v)}\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v + \frac{2u^2 f'(v)}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2
\end{aligned}$$

$$
\begin{aligned}
-\Delta_{\mathbb{H}^n}\left(\frac{u^2}{f(v)}\right)\Delta_{\mathbb{H}^n} v &= \frac{u^2 f'(v)}{f^2(v)}(\Delta_{\mathbb{H}^n} v)^2 - \frac{2u}{f(v)}\Delta_{\mathbb{H}^n} u \Delta_{\mathbb{H}^n} v \\
&\quad + \frac{u^2 f''}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2 \Delta_{\mathbb{H}^n} v - \frac{2}{f(v)}|\nabla_{\mathbb{H}^n} u|^2 \Delta_{\mathbb{H}^n} v \\
&\quad + \frac{4u f'(v)}{f^2(v)}\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v - \frac{2u^2 f'(v)}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2 \\
&= -\frac{(\Delta_{\mathbb{H}^n} u)^2}{f'(v)} + \frac{(\Delta_{\mathbb{H}^n} u)^2}{f'(v)} + \frac{u^2 f'(v)}{f^2(v)}(\Delta_{\mathbb{H}^n} v)^2 \\
&\quad - \frac{2u}{f(v)}\Delta_{\mathbb{H}^n} u \Delta_{\mathbb{H}^n} v + \frac{u^2 f''}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2 \Delta_{\mathbb{H}^n} v \\
&\quad - \frac{2\Delta_{\mathbb{H}^n} v}{f(v)}\left|\nabla_{\mathbb{H}^n} u - \frac{u f'(v)}{f(v)}\nabla_{\mathbb{H}^n} v\right|^2 \\
&= \frac{(\Delta_{\mathbb{H}^n} u)^2}{f'(v)} + \left(\frac{\Delta_{\mathbb{H}^n} u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta_{\mathbb{H}^n} v\right)^2 \\
&\quad + \frac{u^2 f''}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2 \Delta_{\mathbb{H}^n} v \\
&\quad - \frac{2\Delta_{\mathbb{H}^n} v}{f(v)}\left|\nabla_{\mathbb{H}^n} u - \frac{u f'(v)}{f(v)}\nabla_{\mathbb{H}^n} v\right|^2. \qquad (4.1)
\end{aligned}
$$

Using (4.1), we obtain

$$R(u, v) = L(u, v),$$

which proves (i). Now using the fact that $-\Delta v > 0$, $f'(y) \geq 1$, and $f''(y) \leq 0, \forall\, 0 \neq y \in \mathbb{R}$, we get $L(u, v) \geq 0$ and therefore (ii) is proved. Next we prove (iii). We have

$$
\begin{aligned}
L(u,\, v) = &\underbrace{|\Delta_{\mathbb{H}^n} u|^2 - \frac{(\Delta_{\mathbb{H}^n} u)^2}{f'(v)}}_{\text{(I)}} + \underbrace{\left(\frac{\Delta_{\mathbb{H}^n} u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta_{\mathbb{H}^n} v\right)^2}_{\text{(II)}} \\
&\underbrace{-\frac{2\Delta_{\mathbb{H}^n} v}{f(v)}\left(\nabla_{\mathbb{H}^n} u - \frac{u f'(v)}{f(v)}\nabla_{\mathbb{H}^n} v\right)^2}_{\text{(III)}} + \underbrace{\frac{u^2 f''(v)}{f^2(v)}|\nabla_{\mathbb{H}^n} v|^2 \Delta_{\mathbb{H}^n} v.}_{\text{(IV)}}
\end{aligned}
$$

From our assumptions on v and f, we conclude that each of the term (I), (II), (III) and (IV) in the expression for $L(u, v)$ is nonnegative. Hence $L(u, v) = 0$ in Ω implies that each of (I), (II), (III) and (IV) is zero. In particular

$$|\Delta_{\mathbb{H}^n} u|^2 - \frac{|\Delta_{\mathbb{H}^n} u|^2}{f'(v)} = 0 \qquad (4.2)$$

and

$$\nabla_{\mathbb{H}^n} u - \frac{u f'(v)}{f(v)}\nabla_{\mathbb{H}^n} v = 0. \qquad (4.3)$$

On solving (4.2), we get

$$f'(v) = 1 \Rightarrow f(v) = v + c_1, \tag{4.4}$$

where c_1 is a constant.

On using (4.4) in (4.3), we get

$$(\nabla_{\mathbb{H}^n} u)(v + c_1) - u\nabla_{\mathbb{H}^n}(v + c_1) = 0 \Rightarrow \nabla_{\mathbb{H}^n}\left(\frac{u}{v + c_1}\right) = 0 \text{ i.e., } \; u = cv + d$$

for some constants c and d. Conversely, let us assume (4.2) holds. We need to show that $L(u, v) = 0$. From (4.2), we get that $f'(v) = 1$ and therefore $f''(v) = 0$. Now it remains to show that

$$\left(\frac{\Delta_{\mathbb{H}^n} u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta_{\mathbb{H}^n} v\right) = 0 \text{ i.e., } f(v)\Delta_{\mathbb{H}^n} u = uf'(v)\Delta_{\mathbb{H}^n} v.$$

From (4.2), we get

$$0 = f(v)\nabla_{\mathbb{H}^n} u - uf'(v)\nabla_{\mathbb{H}^n} v. \tag{4.5}$$

Differentiating (4.5) yields

$$\begin{aligned} 0 = {} & f(v)\Delta_{\mathbb{H}^n} u + f'(v)\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v - f'(v)\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v \\ & - uf''(v)|\nabla_{\mathbb{H}^n} v|^2 - uf'(v)\Delta_{\mathbb{H}^n} v. \end{aligned}$$

Now using the fact that $f''(v) = 0$, one can see easily that

$$f(v)\Delta_{\mathbb{H}^n} u = uf'(v)\Delta_{\mathbb{H}^n} v,$$

which completes the proof. □

5. Applications

This section deals with the applications of Picone's identity. For the existence of positive solution to biharmonic equations on Heisenberg group, we refer to the paper of Zhang and Xuebo [38]. Zhang [37] also established the existence of solution for biharmonic equation in Heisenberg group using mountain pass theorem.

To begin with the applications, we need the following auxiliary results.

Lemma 5.1. *Let $u \in D^2(\Omega) \cap D_0^1(\Omega)$ be a nonnegative weak solution (not identically zero) of*

$${\Delta_{\mathbb{H}^n}}^2 u = a(x)u \;\; \text{in } \Omega, \quad u = \Delta_{\mathbb{H}^n} u = 0 \;\; \text{on } \partial\Omega, \tag{5.1}$$

where $0 \le a \in L^\infty(\Omega)$, then $-\Delta_{\mathbb{H}^n} u > 0$ in Ω.

Proof. Let $-\Delta_{\mathbb{H}^n} u = v$. Then writing (5.1) into system form, we get

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = v \;\text{ in } \;\Omega, \\ -\Delta_{\mathbb{H}^n} v = a(x)u \;\text{ in } \;\Omega, \\ u = 0 = v \;\text{ on } \;\partial\Omega. \end{cases} \tag{5.2}$$

Since $a(x) \ge 0$ in Ω, so by maximum principle [10], we get $v \ge 0$. By strong maximum principle, either $v > 0$ or $v \equiv 0$ in Ω. If $v \equiv 0$, then we have

$$-\Delta_{\mathbb{H}^n} u = 0 \;\text{ in } \;\Omega; \;\; v = 0 \;\text{ on } \partial\Omega.$$

Again by maximum principle, we get $u \equiv 0$, which is a contradiction and therefore $v > 0$ in Ω and hence

$$-\Delta_{\mathbb{H}^n} u > 0 \text{ in } \Omega.$$

□

In next theorem, we obtain a Hardy-Rellich type inequality for biharmonic operators on Heisenberg group.

Theorem 5.2. *Assume that there is a C^2 function v satisfying*

$$\Delta_{\mathbb{H}^n}{}^2 v \geq \lambda g f(v),\ v > 0,\ -\Delta_{\mathbb{H}^n} v > 0 \ in\ \Omega, \tag{5.3}$$

for some $\lambda > 0$ and a nonnegative continuous function g on Ω and f satisfies the conditions of Proposition 4.1. Then for any $u \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx. \tag{5.4}$$

Proof. Take $\phi \in C_0^\infty(\Omega)$, by Proposition 4.1, we have

$$\begin{aligned}
0 &\leq \int_\Omega L(\phi, v) dx = \int_\Omega R(\phi, v) dx \\
&= \int_\Omega |\Delta_{\mathbb{H}^n} \phi|^2 dx - \int_\Omega \Delta_{\mathbb{H}^n} \left(\frac{\phi^2}{f(v)} \right) \Delta_{\mathbb{H}^n} v dx \\
&= \int_\Omega |\Delta_{\mathbb{H}^n} \phi|^2 dx - \int_\Omega (\Delta_{\mathbb{H}^n}{}^2 v). \frac{\phi^2}{f(v)} dx, \quad \text{(on integration)}, \\
&\leq \int_\Omega |\Delta_{\mathbb{H}^n} \phi|^2 dx - \lambda \int_\Omega \phi^2 g dx \quad \text{(by (5.3))}.
\end{aligned}$$

Letting $\phi = u$, yields

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx.$$

□

Remark 5.3. Inequality (5.3) in the case $f(v) = v$ was established by Niu et al. For the details, we refer to [28].

In the next theorem, we prove a Picone inequality for biharmonic operators on Heisenberg group. Abdellaoui and Peral [1] established Picone inequality in Euclidean space. Picone inequality for p-sublaplacian on Heisenberg group is obtained in [16].

In order to prove next theorem, we need the following lemma.

Lemma 5.4. *Let $v \in D^{2,2}(\Omega)$ be such that $v \geq \delta > 0$ in Ω, where $\Omega \subseteq \mathbb{H}^n$. Then for all $0 \leq u \in C_0^\infty(\Omega)$, $u \geq 0$, we have*

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^p dx \geq \int_\Omega \frac{u^2}{v} \Delta_{\mathbb{H}^n}{}^2 v dx. \tag{5.5}$$

Proof. Since $v \in D^{2,2}(\Omega)$ and $v \geq \delta > 0$ in Ω, therefore there exists a sequence $v_n \in C^2(\Omega)$ such that

$$v_n \to v \text{ in } D^{2,2}(\Omega),$$

and

$$v_n \to v \text{ a.e and } v_n > \delta/2 \text{ in } \Omega.$$

Since ${\Delta_{\mathbb{H}^n}}^2$ is a continuous operator from $D^{2,2}(\Omega)$ to $D^{-2,2}(\Omega)$, therefore

$${\Delta_{\mathbb{H}^n}}^2 v_n \to {\Delta_{\mathbb{H}^n}}^2 v \quad \text{in} \quad D^{2,2}(\Omega).$$

By Lemma 3.1, we have

$$\begin{aligned}\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx &\geq \int_\Omega \Delta_{\mathbb{H}^n}\left(\frac{u^2}{v_n}\right)\Delta_{\mathbb{H}^n} v_n \\ &= \int_\Omega \frac{u^2}{v_n}{\Delta_{\mathbb{H}^n}}^2 v_n, \quad \text{(By integration)} \\ &= \int_\Omega \frac{u^2}{v}{\Delta_{\mathbb{H}^n}}^2 v, \text{ (By Lebesgue dominated convergence theorem)}\end{aligned} \tag{5.6}$$

for all $0 \leq u \in C_0^\infty(\Omega)$. This completes the proof. □

Theorem 5.5. *Let $\Omega \subseteq \mathbb{H}^n$ be a smooth and bounded domain. Let $0 \leq u \in D_0^{2,2}(\Omega)$, and $0 \leq v \in D_0^{2,2}(\Omega)$ be such that $-\Delta_{\mathbb{H}^n} v \geq 0$ is a bounded Radon measure, v is not identically zero and $v = 0 = \Delta_{\mathbb{H}^n} v$ on $\partial\Omega$. Then*

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx \geq \int_\Omega \frac{u^2}{v}{\Delta_{\mathbb{H}^n}}^2 v dx. \tag{5.7}$$

Proof. Since $v \geq 0$ and $-\Delta_{\mathbb{H}^n} v \geq 0$ in Ω and $v = 0$ on $\partial\Omega$, therefore by strong maximum principle [10] either $v > 0$ or $v \equiv 0$ in Ω. But by the assumption $v \not\equiv 0$ in Ω, thus $v > 0$ in Ω. Let $v_m(\xi) = v(\xi) + \frac{1}{m}$, then ${\Delta_{\mathbb{H}^n}}^2 v_m = {\Delta_{\mathbb{H}^n}}^2 v$ and $v_m \to v$ in $D^{2,2}(\Omega)$ and almost everywhere. By Lemma 5.4 , we get

$$\int_\Omega |\Delta_{\mathbb{H}^n} \phi|^2 dx \geq \int_\Omega \frac{\phi^2}{v_m}{\Delta_{\mathbb{H}^n}}^2 v_m, \tag{5.8}$$

for each m and $0 \leq \phi \in C_0^\infty(\Omega)$. Now, we consider $0 \leq u \in D_0^{2,2}(\Omega)$, then there exists a sequence $\{u_n\}$ in $C_0^\infty(\Omega)$ such that $u \geq 0$ for each n and $u_n \to u$ in $D_0^{2,2}(\Omega)$. Using (5.8), we get

$$\int_\Omega |\Delta_{\mathbb{H}^n} u_n|^2 dx \geq \int_\Omega \frac{u_n^2}{v_m}{\Delta_{\mathbb{H}^n}}^2 v_m dx. \tag{5.9}$$

On passing the limits as $n, m \to \infty$, by Fatou's lemma and Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx \geq \int_\Omega \frac{u^2}{v}{\Delta_{\mathbb{H}^n}}^2 v dx. \tag{5.10}$$

This completes the proof. □

Next, we consider the following singular system of fourth order elliptic equations:

$$\begin{aligned} {\Delta_{\mathbb{H}^n}}^2 u &= f(v) \text{ in } \Omega, \\ {\Delta_{\mathbb{H}^n}}^2 v &= \frac{(f(v))^2}{u} \text{ in } \Omega, \\ u > 0, &\quad v > 0 \text{ in } \Omega, \\ u = \Delta_{\mathbb{H}^n} u = 0 &= v = \Delta_{\mathbb{H}^n} v \text{ on } \partial\Omega, \end{aligned} \tag{5.11}$$

where f is defined as in Proposition 4.1. In the next theorem, we show a linear relationship between the components u and v, where (u, v) is a solution of (5.11).

Theorem 5.6. *Let $(u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ be a weak solution of (5.11) and f satisfy the conditions of Proposition 4.1. Then $u = c_1 v + c_2$, where c_1, c_2 are constants.*

Proof. The proof is on the same lines as the proof of similar result in Euclidean setting [32], for sake of brevity, we omit the details. □

Let us consider the following weighted eigenvalue problem

$${\Delta_{\mathbb{H}^n}}^2 u = \lambda a(x) u \text{ in } \Omega, \quad u = \Delta_{\mathbb{H}^n} u = 0 \text{ on } \partial\Omega, \tag{5.12}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded subset with smooth boundary, $N > 4$, and $0 \leq a \in L^\infty(\Omega)$. We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of (5.12) if and only if there exists $u \in D^2(\Omega) \cap D_0^1(\Omega) \backslash \{0\}$ such that

$$\int_\Omega \Delta_{\mathbb{H}^n} u . \Delta_{\mathbb{H}^n} \phi dx = \lambda \int_\Omega a(x) u \phi dx, \ \ \forall \ \phi \in D^2(\Omega) \cap D_0^1(\Omega) \tag{5.13}$$

and u is called an eigenfunction associated with λ. For the results related to existence of eigenvalues of biharmonic operator on Heisenberg group, we refer to [21,36,37]. The least positive eigenvalue of (5.12) is defined as

$$\lambda_1 = \inf \left\{ \int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx : \quad u \in D^2(\Omega) \cap D_0^1(\Omega) \ \text{ and } \ \int_\Omega a(x) |u|^2 dx = 1 \right\}.$$

Remark 5.7. *Let $-\Delta_{\mathbb{H}^n} u = v$. Then writing (5.12) into system form, we get*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = v \ \ in \ \ \Omega, \\ -\Delta_{\mathbb{H}^n} v = a(x) u \ \ in \ \ \Omega, \\ u = 0 = v \quad on \ \ \partial\Omega. \end{cases} \tag{5.14}$$

Now by using Theorem 3.35 [14] for second equation in (5.14), we conclude that $v \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$. Then by using Theorem 3.9 [34] we get that $u \in C^{2,\alpha}(\Omega)$. Again applying Theorem 3.35 [14] and Theorem [34] for $u \in C^{2,\alpha}(\Omega)$, we conclude that $u \in C^{4,\alpha}(\Omega)$.

Lemma 5.8. *λ_1 is attained.*

Proof. For showing the above infimum is attained, let us introduce the functionals

$$J, \ G : D^2(\Omega) \cap D_0^1(\Omega) \longrightarrow \mathbb{R} \quad \text{defined by}$$

$$J(u) = \frac{1}{2}\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx, \quad G(u) = \frac{1}{2}\int_\Omega a(x)|u|^2 dx, \quad u \in D^2(\Omega) \cap D_0^1(\Omega).$$

It is easy to see that J and G are C^1 functionals. By definition, $\lambda \in \mathbb{R}$ is an eigenvalue of (5.12) if and only if there exists $u \in D^2(\Omega) \cap D_0^1(\Omega)\backslash\{0\}$ such that

$$J'(u) = \lambda G'(u).$$

Let us define

$$M = \left\{u \in D^2(\Omega) \cap D_0^1(\Omega) : \frac{1}{2}\int_\Omega a(x)|u|^2 dx = 1\right\}.$$

Since $a \geq 0$ so $M \neq \emptyset$ and M is a C^1 manifold in $D^2(\Omega) \cap D_0^1(\Omega)$. It is also easy to see that J is coercive and (sequentially) weakly lower semicontinuous on M and M is a weakly closed subset of $D^2(\Omega) \cap D_0^1(\Omega)$. Now by an application of Theorem 1.1 [31], J is bounded from below on M and attains its infimum in M. Also by Lagrange's multiplier rule

$$J'(u) = \lambda_1 G'(u)$$

and therefore λ_1 is attained. □

In the next lemma, we show that the first eigenfunction u corresponding to the first eigenvalue λ_1 of (5.12) is of one sign. We use the following theorem.

Theorem 5.9. ([20] Dual cone decomposition theorem) *Let H be a Hilbert space with scalar product $(\ .,)_H$. Let $K \subset H$ be a closed, convex nonempty cone. Let K^* be its dual cone, namely*

$$K^* = \{w \in H | (w,\, v)_H \leq 0, \ \ \forall\, v \in K\}.$$

Then for any $u \in H$, there exists a unique $(u_1,\, u_2) \in K \times K^*$ such that

$$u = u_1 + u_2, \quad (u_1,\, u_2)_H = 0. \tag{5.15}$$

In particular,

$$||u||_H^2 = ||u_1||_H^2 + ||u_2||_H^2.$$

Moreover, if we decompose arbitrary $u, v \in H$ according to (5.15), then this implies

$$||u - v||_H^2 \geq ||u_1 - v_1||_H^2 + ||u_2 - v_2||_H^2.$$

In particular, the projection onto K is Lipschitz continuous.

For a proof of the above theorem, we refer to Theorem 3.4 [20].

Lemma 5.10. *The eigenfunction u corresponding to the first eigenvalue λ_1 of (5.12) is of one sign.*

Proof. Using Theorem 5.9, and classical maximum principle for $-\Delta$, Ferrero et al. [18] obtain the positivity of the minimizers of the problem

$$S_q = \min_{w \in X/\{0\}} \frac{||\Delta w||_2^2}{||w||_q^2}, \ \ 1 \leq q < \frac{2n}{n-4},$$

where $X = H^2(B) \cap H_0^1(B)$, B denotes the unit ball in $\mathbb{R}^n$. The same proof works for eigenfunction u corresponding to the first eigenvalue λ_1 of (5.12) in Ω. For this, we refer to [18] and omit the details. □

Next, we show the strict monotonicity of the principle eigenvalue λ_1.

Theorem 5.11. *Suppose $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. Then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$, if both exist.*

Proof. Proof follows from the proof of similar result in Euclidean setting by Allegretto and Huang [5], for the sake of brevity, we omit the details. □

In the next theorem, using Picone's identity (Lemma 3.1), we show that λ_1 is simple, i.e., the eigenfunctions associated to it are a constant multiple of each other.

Theorem 5.12. *λ_1 is simple.*

Proof. Let u and v be two eigenfunctions associated with λ_1. In view of Remark 5.7, we may assume that $u,\, v \in C^{4,\alpha}(\Omega)$. From Lemma 5.10, without any loss of generality, we may also assume that u and v are positive in Ω. Now by Lemma 5.1, we have

$$-\Delta_{\mathbb{H}^n} u > 0, \quad -\Delta_{\mathbb{H}^n} v > 0 \ \text{ in } \Omega.$$

Let $\epsilon > 0$. From Lemma 3.1, we have

$$\begin{aligned} 0 &\leq \int_\Omega L(u,\, v+\epsilon)dx \\ &= \int_\Omega R(u,\, v+\epsilon)dx \\ &= \int_\Omega \left[|\Delta_{\mathbb{H}^n} u|^2 - \Delta_{\mathbb{H}^n}\left(\frac{u^2}{v+\epsilon}\right)\Delta_{\mathbb{H}^n} v \right] dx \\ &= \lambda_1 \int_\Omega a(x)u^2 dx - \int_\Omega \Delta_{\mathbb{H}^n}\left(\frac{u^2}{v+\epsilon}\right)\Delta_{\mathbb{H}^n} v dx. \end{aligned} \tag{5.16}$$

In view of Remark 5.7, $\dfrac{u^2}{v+\epsilon} \in D^{2,2}(\Omega) \cap D_0^{1,2}(\Omega)$ and is admissible in the weak formulation of

$${\Delta_{\mathbb{H}^n}}^2 v = \lambda_1 a(x) v, \quad \text{i.e.,}$$

$$\int_\Omega \Delta_{\mathbb{H}^n} v \Delta_{\mathbb{H}^n}\left(\frac{u^2}{v+\epsilon}\right) dx = \lambda_1 \int_\Omega a(x) v \left(\frac{u^2}{v+\epsilon}\right) dx. \tag{5.17}$$

From (5.16) and (5.17), we get

$$\begin{aligned} 0 &\leq \int_\Omega L(u,\, v+\epsilon)dx \\ &= \lambda_1 \int_\Omega a(x)\left[u^2 - v\left(\frac{u^2}{v+\epsilon}\right)\right] dx. \end{aligned}$$

Letting $\epsilon \longrightarrow 0$, in the above inequality, we get

$$L(u,\, v) = 0$$

and again by an application of Lemma 3.1, there exists $\alpha \in \mathbb{R}$ such that

$$u = \alpha\, v,$$

which proves the simplicity of λ_1. $\square$

Next, we show the sign changing nature of any eigenfunction v associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$.

Proposition 5.13. *Any eigenfunction v associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$ changes sign.*

Proof. Assume by contradiction that $v \geq 0$, the case $v \leq 0$ can be dealt similarly. By Lemma 5.1, $v > 0$ in Ω. Let $\phi > 0$ be an eigenfunction associated with $\lambda_1 > 0$. For any $\epsilon > 0$, we apply Lemma 3.1 to the pair ϕ, $v + \epsilon$ and get

$$\begin{aligned} 0 &\leq \int_\Omega L(\phi,\, v+\epsilon)dx \\ &= \int_\Omega R(\phi,\, v+\epsilon)dx \\ &= \int_\Omega \left[|\Delta_{\mathbb{H}^n}\phi|^2 - \Delta_{\mathbb{H}^n}\left(\frac{\phi^2}{v+\epsilon}\right)\Delta_{\mathbb{H}^n} v\right] dx \\ &= \int_\Omega \left[\lambda_1 a(x)\phi^2 - \Delta_{\mathbb{H}^n}\left(\frac{\phi^2}{v+\epsilon}\right)\Delta_{\mathbb{H}^n} v\right] dx. \end{aligned} \tag{5.18}$$

Again, we note that $\frac{\phi^2}{v+\epsilon} \in D^{2,2}(\Omega) \cap D_0^{1,2}(\Omega)$ and is admissible in the weak formulation of

$$\Delta_{\mathbb{H}^n}{}^2 v = \lambda a(x) v \text{ in } \Omega;\;\; v = \Delta_{\mathbb{H}^n} v = 0 \;\text{ on } \partial\Omega.$$

This implies that

$$\int_\Omega \Delta_{\mathbb{H}^n} v \Delta_{\mathbb{H}^n}\left(\frac{\phi^2}{v+\epsilon}\right) dx = \lambda \int_\Omega a(x) v \frac{\phi^2}{v+\epsilon} dx. \tag{5.19}$$

From (5.18) and (5.19), we get

$$0 \leq \int_\Omega \left[\lambda_1 a(x)\phi^2 - \lambda a(x) v \frac{\phi^2}{v+\epsilon}\right] dx.$$

Letting $\epsilon \longrightarrow 0$ in the above inequality, we get

$$0 \leq (\lambda_1 - \lambda) \textstyle\int_\Omega a(x)\phi^2 dx,$$

which is a contradiction, because $\int_\Omega a(x)\phi^2 dx > 0$ and hence v must change sign. $\square$

For the application of Lemma 3.1 on Morse index, let us consider the following boundary value problem

$$\Delta_{\mathbb{H}^n}{}^2 u = a(x) G(u) \;\text{ in } \Omega;\;\; u = \Delta_{\mathbb{H}^n} u = 0 \;\text{ on } \partial\Omega, \tag{5.20}$$

where $a \in C^\alpha(\Omega)$, $0 < \alpha < 1$ and $G \in C^1(\mathbb{R}, \mathbb{R})$. The Morse index is defined via the eigenvalue problem for the linearization at u :

Definition 5.14. (*Morse index*) The Morse index of a solution u of (5.20) is the number of negative eigenvalues of the linearized operator

$$\Delta_{\mathbb{H}^n}{}^2 - a(x)G'(u) \tag{5.21}$$

acting on $D^{2,2}(\Omega) \cap D_0^1(\Omega)$, i.e., the number of eigenvalues λ such that $\lambda < 0$, and the boundary value problem

$$\Delta_{\mathbb{H}^n}{}^2 w - a(x)G'(u)w = \lambda w \text{ in } \Omega; \ \ w = 0 = \Delta_{\mathbb{H}^n} w \text{ on } \partial\Omega \tag{5.22}$$

has a nontrivial solution w in $D^{2,2}(\Omega) \cap D_0^1(\Omega)$.

The next theorem gives an application of Lemma 3.1.

Theorem 5.15. *Let us consider* (5.20). *Let* $a \in C^{\alpha}(\Omega)$, $0 < \alpha < 1$ *and* $G \in C^1(\mathbb{R}, \mathbb{R})$ *be such that*

$$\frac{G(v)}{v} \geq G'(0) \geq 0, \ \ \forall\, 0 < v \in \mathbb{R}.$$

Then the trivial solution of (5.20) *has Morse index* 0.

Proof. Let $v \in C^2(\bar{\Omega})$ be a positive weak solution of (5.20). Then

$$\int_{\Omega} \Delta_{\mathbb{H}^n} v \Delta_{\mathbb{H}^n} \psi dx = \int_{\Omega} a(x)G(v)\psi dx, \ \forall\, \psi \in C_c^{\infty}(\Omega). \tag{5.23}$$

For any $w \in C_c^{\infty}(\Omega)$, let us take $\frac{w^2}{v}$ as a test function in (5.23) and obtain

$$\int_{\Omega} \Delta_{\mathbb{H}^n} v \Delta_{\mathbb{H}^n} \left(\frac{w^2}{v}\right) dx = \int_{\Omega} a(x) \frac{G(v)}{v} w^2 dx. \tag{5.24}$$

Since v is a positive solution of (5.20) so using the fact that $G(v) \geq 0$ and in view of Lemma 5.1 , one can see that

$$-\Delta_{\mathbb{H}^n} v > 0.$$

Now an application of Lemma 3.1 for $u = w$ proves the required result, see [32] for a similar proof in the Euclidean setting. $\square$

Let us consider

$$\Delta_{\mathbb{H}^n}{}^2 v = g(x)v, \tag{5.25}$$

where $0 \leq g \in L^{\infty}(\Omega)$. We say that $v \in D^{2,2}(\Omega) \cap C(\bar{\Omega})$ is a continuous weak subsolution to (5.25) if

$$\int_{\Omega} \Delta_{\mathbb{H}^n} v \Delta_{\mathbb{H}^n} \psi dx \leq \int_{\Omega} g(x) v \psi dx, \ \ \forall\, 0 \leq \psi \in D^{2,2}(\Omega) \cap C(\bar{\Omega}). \tag{5.26}$$

In the next theorem, we establish a Caccioppoli type inequality for the subsolutions of (5.25).

Theorem 5.16. (Caccioppoli Inequality) *Let* $0 < v \in D^{2,2}(\Omega) \cap C(\bar{\Omega})$ *be a continuous weak subsolution of* (5.25) *such that* $\Delta_{\mathbb{H}^n} v > 0$ *a.e in* Ω *and* $0 \leq \eta \in C_c^\infty(\Omega)$, *then*

$$\int_\Omega |\eta \Delta_{\mathbb{H}^n} v|^2 dx \leq \int_\Omega g(x)\eta^2 v dx + \left(\int_\Omega |\eta \Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}} \left[2\left(\int_\Omega |v\Delta_{\mathbb{H}^n}\eta|^2 dx\right)^{\frac{1}{2}} + 2\left(\int_\Omega \left|\frac{\eta}{v}\right|^2 |\nabla_{\mathbb{H}^n} v|^4\right)^{\frac{1}{2}}\right]. \tag{5.27}$$

Proof. Let us choose $u = \eta v$ in (3.1), where $v \in D^{2,2}(\Omega) \cap C(\bar{\Omega})$ is a positive solution of (5.26) and $\eta \in C_c^\infty(\Omega)$ is a nonnegative test function, we get

$$\int_\Omega \left[|\Delta_{\mathbb{H}^n}(\eta v)|^2 - \Delta_{\mathbb{H}^n}(\eta^2 v)\Delta_{\mathbb{H}^n} v\right] dx = \int_\Omega \Bigg[|\Delta_{\mathbb{H}^n}(\eta v)|^2 + \eta^2|\Delta_{\mathbb{H}^n} v|^2 - 2\eta\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}(\eta v) - \frac{2\Delta_{\mathbb{H}^n} v}{v}\left|\nabla_{\mathbb{H}^n}(\eta v) - \eta\nabla_{\mathbb{H}^n} v\right|^2\Bigg] dx \tag{5.28}$$

We can compute

$\nabla_{\mathbb{H}^n}(\eta v) = \eta\nabla_{\mathbb{H}^n} v + v\nabla_{\mathbb{H}^n}\eta$ and $\Delta_{\mathbb{H}^n}(\eta v) = \eta\Delta_{\mathbb{H}^n} v + v\Delta_{\mathbb{H}^n}\eta + 2\nabla_{\mathbb{H}^n}\eta.\nabla_{\mathbb{H}^n} v$.

Inserting these expressions in (5.28), we obtain

$$\begin{aligned}
&-\int_\Omega \Delta_{\mathbb{H}^n}(\eta^2 v)\Delta_{\mathbb{H}^n} v dx \\
&= \int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}(\eta v) - 2\int_\Omega v\Delta_{\mathbb{H}^n} v|\nabla_{\mathbb{H}^n}\eta|^2 \\
&= \int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta v\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}\eta dx \\
&\quad -4\int_\Omega \eta\Delta_{\mathbb{H}^n} v(\nabla_{\mathbb{H}^n} v.\nabla_{\mathbb{H}^n}\eta) dx - 2\int_\Omega v|\Delta_{\mathbb{H}^n} v||\nabla_{\mathbb{H}^n}\eta|^2 dx \\
&= -\int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta v\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}\eta dx \\
&\quad -4\int_\Omega \eta\Delta_{\mathbb{H}^n} v(\nabla_{\mathbb{H}^n} v.\nabla_{\mathbb{H}^n}\eta) dx - 2\int_\Omega v|\Delta_{\mathbb{H}^n} v||\nabla_{\mathbb{H}^n}\eta|^2 dx \\
&= -\int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta v\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}\eta dx + 2\int_\Omega \frac{1}{v}\eta^2\Delta_{\mathbb{H}^n} v|\nabla_{\mathbb{H}^n} v|^2 dx \\
&\quad -2\int_\Omega \Delta_{\mathbb{H}^n} v\left\{2v|\nabla_{\mathbb{H}^n}\eta|^2 + \frac{|\nabla_{\mathbb{H}^n} v|^2\eta^2}{v} + 2\eta\nabla_{\mathbb{H}^n} v.\nabla_{\mathbb{H}^n}\eta\right\} dx \\
&= -\int_\Omega \eta^2|\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta v\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}\eta dx + 2\int_\Omega \frac{1}{v}\eta^2\Delta_{\mathbb{H}^n} v|\nabla_{\mathbb{H}^n} v|^2 dx \\
&\quad -2\int_\Omega \Delta_{\mathbb{H}^n} v\left|\sqrt{v}\nabla_{\mathbb{H}^n}\eta + \frac{\eta\nabla_{\mathbb{H}^n} v}{\sqrt{v}}\right|^2 dx \\
&\leq -\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx - 2\int_\Omega \eta v\Delta_{\mathbb{H}^n} v\Delta_{\mathbb{H}^n}\eta dx + 2\int_\Omega \frac{1}{v}\eta^2\Delta_{\mathbb{H}^n} v|\nabla_{\mathbb{H}^n} v|^2 dx \\
&\qquad (\text{since } v > 0,\ \Delta_{\mathbb{H}^n} v > 0 \text{ and } \eta \geq 0)
\end{aligned}$$

$$
\begin{aligned}
&\leq -\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx + 2\int_\Omega |\eta\Delta_{\mathbb{H}^n} v||v\Delta_{\mathbb{H}^n}\eta| dx + 2\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|\frac{|\eta||\nabla_{\mathbb{H}^n} v|^2}{|v|} dx \\
&\leq -\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx + 2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |v\Delta_{\mathbb{H}^n}\eta|^2 dx\right)^{\frac{1}{2}} \\
&\quad +2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\eta}{v}\right|^2 |\nabla_{\mathbb{H}^n} v|^4\right)^{\frac{1}{2}} \\
&\leq -\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx + 2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |v\Delta_{\mathbb{H}^n}\eta|^2 dx\right)^{\frac{1}{2}} + \\
&\quad +2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\eta}{v}\right|^2 |\nabla_{\mathbb{H}^n} v|^4\right)^{\frac{1}{2}}. \qquad (5.29)
\end{aligned}
$$

From the last inequality (5.29), we have

$$
\begin{aligned}
-\int_\Omega \Delta_{\mathbb{H}^n}(\eta^2 v)\Delta_{\mathbb{H}^n} v dx \leq& -\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx \\
&+2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |v\Delta_{\mathbb{H}^n}\eta|^2 dx\right)^{\frac{1}{2}} \\
&+2\left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\eta}{v}\right|^2 |\nabla_{\mathbb{H}^n} v|^4\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.30}
$$

Let us choose $\psi = \eta^2 v$ as a test function in (5.26), where $v \in D^{2,1}(\Omega)\cap C(\bar{\Omega})$, and we obtain

$$
\int_\Omega \Delta_{\mathbb{H}^n}(\eta^2 v)\Delta_{\mathbb{H}^n} v dx \leq \int_\Omega g(x)\eta^2 v dx, \quad \forall\, 0 \leq \eta \in C_c^\infty(\Omega). \tag{5.31}
$$

Using (5.31) in (5.30), we get

$$
\begin{aligned}
\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx \leq \int_\Omega g(x)\eta^2 v dx + \left(\int_\Omega |\eta\Delta_{\mathbb{H}^n} v|^2 dx\right)^{\frac{1}{2}}&\left[2\left(\int_\Omega |v\Delta_{\mathbb{H}^n}\eta|^2 dx\right)^{\frac{1}{2}}\right. \\
&\left. +2\left(\int_\Omega \left|\frac{\eta}{v}\right|^2 |\nabla_{\mathbb{H}^n} v|^4\right)^{\frac{1}{2}}\right],
\end{aligned}
$$

which is the required Inequality (5.27). This completes the proof. □

Corollary 5.17. *Let $g(x) \equiv 0$ in (5.25). Let $0 < v \in D^{2,2}(\Omega)\cap C(\bar{\Omega})$ be a continuous weak subsolution of (5.25) such that $\Delta_{\mathbb{H}^n} v > 0$ a.e in Ω and $0 \leq \eta \in C_c^\infty(\Omega)$, then*

$$
\|\eta\Delta_{\mathbb{H}^n} v\|_{L^2(\Omega)} \leq 2\left[\|v\Delta_{\mathbb{H}^n}\eta\|_{L^2(\Omega)} + \left\|\frac{\eta}{v}|\Delta_{\mathbb{H}^n} v|^2\right\|_{L^2(\Omega)}\right]. \tag{5.32}
$$

Proof. The proof follows from (5.27) by putting $g(x) = 0$ and adjusting the terms. For the sake of brevity, we omit the details. □

Acknowledgments

Authors are highly thankful to the referee whose critical comments and suggestions improved the paper considerably. The first author thanks IIT Gandhinagar for providing him financial support.

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Received: 27 August 2015.
Accepted: 9 February 2016.