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A stochastic nonlinear Schrödinger problem in variational formulation

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Abstract. This paper investigates a Schrödinger problem with power-type nonlinearity and Lipschitz-continuous diffusion term on a bounded one-dimensional domain. Using the Galerkin method and a truncation, results from stochastic partial differential equations can be applied and uniform a priori estimates for the approximations are shown. Based on these boundedness results and the structure of the nonlinearity, it follows the unique existence of the variational solution.

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Keywords. Stochastic nonlinear Schrödinger equation, Power-type nonlinearity, Variational solution, Galerkin method.

1. Introduction

The one-dimensional complex Ginzburg–Landau equation

$$dX(t) = (a_1 + ia_2)\Delta X(t)\,dt + (b_1 + ib_2)|X(t)|^{2\sigma} X(t)\,dt + c_1 X(t)\,dt \tag{1.1}$$

with $a_1, a_2, b_1, b_2, c_1 \in \mathbb{R}$ and $\sigma > 0$ represents a nonlinear generalized Schrödinger equation with complex coefficients. It involves non-steady, diffusive and dispersive terms as well as nonlinear and linear effects that can be interpreted physically, see for example [19,20,26,32]. This equation has many applications in physics such as fluid mechanics, nonlinear optics, wave propagation, the theory of phase transitions, hydrodynamic instabilities and wave envelopes, chemical and biological dynamics. Furthermore, it describes physical phenomena like Rayleigh–Bénard convection, Taylor–Couette flow, Bose–Einstein condensation, superconductivity and superfluidity. Throughout this work, we consider stochastic degeneracies of the one-dimensional complex Ginzburg–Landau equation.

In [17], we deal with the stochastic nonlinear Schrödinger equation

$$dX(t) = i\Delta X(t)\,dt + i\lambda|X(t)|^{2\sigma}X(t)\,dt + ig(t,X(t))\,dW(t)$$

for all $t \in [0,T]$, where $\lambda > 0$, g is a special linear function in X and W is an infinite-dimensional Wiener process. We investigate the existence and uniqueness of the variational solution on a closed interval in $\mathbb{R}^1$. Due to the approach, σ is restricted to the interval $(0,2)$. Deterministic Schrödinger equations of this type are already considered for different types of solutions, from classical solutions in [14,22,28] over strong solutions in [2,15,27] and mild solutions in [3,4,12] right up to generalized (weak/variational) solutions in [10,11,15,26]. Since physical experiments are burdened with random disturbances, stochastic Schrödinger equations, based on the semigroup approach, are treated with additive noise in [6,9] and with multiplicative noise in [1,5,6,23]. So far, variational solutions are only investigated for locally Lipschitz-continuous nonlinear drift and diffusion terms in [13].

In this paper, we focus on the unique existence of the variational solution of the following stochastic nonlinear Schrödinger equation

$$dX(t) = i\Delta X(t)\,dt - \lambda|X(t)|^{2\sigma}X(t)\,dt + g(t,X(t))\,dW(t)$$

over a finite time horizon and a bounded one-dimensional domain, where $\lambda > 0$, $\sigma \geq 1$, g is a Lipschitz-continuous function of bounded growth and W is a cylindrical Wiener process. Deterministic Schrödinger equations of this type are treated in [15,18,22]. Our aim is to enlarge the ideas of [18] to the stochastic case. Note that the power-type nonlinearity does not satisfy the local Lipschitz-continuity assumption from [13]. However, in comparison with [17], the missing imaginary unit in front of the nonlinear drift term is crucial for this approach.

Since the variational solution we are concerned with is more regular than the mild solution, we obtain more general results than the papers using the semigroup approach. Notice that each variational solution is also a mild solution but not vice versa. The price we have to pay is that we only get results for the one-dimensional bounded domain. Instead of using Strichartz' estimates that are only valid for the unbounded space domain, we establish some inequalities for our special nonlinear drift term (compare the Appendix).

The paper proceeds in the following way: Sect. 2 contains some useful notations and the formulation of the stochastic nonlinear Schrödinger problem and its variational solution. The uniqueness of such a solution is shown in Sect. 3. Then the Schrödinger problem is approximated by the Galerkin method and a special truncation is introduced to state and prove a priori estimates thereafter. In Sect. 5, we deduce global existence results for the stochastic nonlinear Schrödinger equation, which is followed by a last section about possible generalization where we especially discuss the unique existence of the variational solution of the stochastic one-dimensional complex Ginzburg–Landau equation by using similar ideas. The Appendix at the end includes some auxiliary results applied within the paper.

2. Formulation of the problem

Let K be a separable real Hilbert space, $H := L^2(0,1)$ and $V := H^1(0,1)$. Then the inner product in H is given by

$$\big(u,v\big) := \int_0^1 u(x)\,\overline{v}(x)\,dx, \qquad \text{for all } u,v \in H,$$

where $\overline{v}$ is the complex conjugate of v, while the inner product in V is constituted by

$$\big(u,v\big)_V := \int_0^1 \left[u(x)\,\overline{v}(x) + \frac{d}{dx}u(x)\,\frac{d}{dx}\overline{v}(x)\right] dx, \qquad \text{for all } u,v \in V.$$

The norms in H and V are represented by $\|\cdot\|$ and $\|\cdot\|_V$, respectively. Furthermore, let V^* be the dual space of V and $\langle\cdot,\cdot\rangle$ denotes the duality pairing of V^* and V. The appropriate choice of H and V and the identification of H with its dual space H^* allow to work on a triplet of rigged Hilbert spaces (V,H,V^*) with continuous and dense embeddings, which is also known as a Gelfand triplet. Moreover, we regard the operator $A : V \to V^*$ given by the bilinear form

$$\big\langle Au, v\big\rangle := \int_0^1 \frac{d}{dx}u(x)\,\frac{d}{dx}\overline{v}(x)\,dx, \qquad \text{for all } u,v \in V. \tag{2.1}$$

Let $(\mu_k)_{k\in\mathbb{N}}$ be the increasing sequence of real-valued eigenvalues and let $(h_k)_{k\in\mathbb{N}}$ be the corresponding eigenfunctions of A with respect to homogeneous Neumann boundary conditions. The eigenfunctions $(h_k)_{k\in\mathbb{N}}$ form an orthonormal system in H and they are orthogonal in V. Obviously, for all $u \in H$ and all $v \in V$, it holds that

$$u = \sum_{k=1}^{\infty} \big(u, h_k\big)\,h_k, \qquad Av = \sum_{k=1}^{\infty} \mu_k \big(v, h_k\big)\,h_k$$

and

$$\big\langle Av, v\big\rangle = \sum_{k=1}^{\infty} \mu_k \left|\big(v, h_k\big)\right|^2 \ge 0.$$

Now, we consider the stochastic nonlinear Schrödinger equation

$$dX(t,x) = -\,iAX(t,x)\,dt - \lambda f(X(t,x))\,dt + g(t,X(t,x))\,dW(t) \tag{2.2}$$

with initial condition $X(0,\cdot) = \varphi(\cdot) \in V$ and homogeneous Neumann boundary conditions. Here, X is the complex-valued wave function depending on $t \in [0,T]$ and $x \in [0,1]$, i is the imaginary unit, A represents the one-dimensional negative Laplacian defined by (2.1), $\lambda > 0$, $T > 0$ and the function $f : V \to H$ has the form

$$f(v) := |v|^{2\sigma} v, \qquad \text{for all } v \in V,$$

where $\sigma \ge 1$ is fixed. Furthermore, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered complete probability space and $L_2(K,H)$ the space of all Hilbert-Schmidt operators from K into H. Then $(W(t))_{t\in[0,T]}$ in (2.2) is a K-valued cylindrical Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and the diffusion function

$g : \Omega \times [0,T] \times H \to L_2(K,H)$ is measurable, $\mathcal{F}_t$-adapted and assumed to fulfill the following assumptions:

- there exists a constant $c_g > 0$ such that
$$\|g(t,u) - g(t,v)\|^2_{L_2(K,H)} \le c_g \|u - v\|^2 \tag{2.3}$$
for all $t \in [0,T]$, all $u, v \in H$ and a.e. $\omega \in \Omega$;
- there exists a constant $k_g > 0$ such that
$$\|g(t,v)\|^2_{L_2(K,V)} \le k_g \left(1 + \|v\|^2_V\right) \tag{2.4}$$
for all $t \in [0,T]$, all $v \in V$ and a.e. $\omega \in \Omega$.

Note that the noise term appearing in (2.2) includes additive as well as multiplicative Gaussian noise.

Definition 2.1. An $\mathcal{F}_t$-adapted process
$$X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$$
is called a variational solution of the stochastic nonlinear Schrödinger equation (2.2) with initial condition $\varphi \in V$ if it fulfills
$$\begin{aligned}\big(X(t), v\big) = \big(\varphi, v\big) - i \int_0^t \big\langle AX(s), v\big\rangle \, ds - \lambda \int_0^t \big(f(X(s)), v\big) \, ds \\ + \left(\int_0^t g(s, X(s)) \, dW(s), v \right)\end{aligned} \tag{2.5}$$
for all $t \in [0,T]$, all $v \in V$ and a.e. $\omega \in \Omega$.

In this paper, we will investigate the existence and uniqueness of the variational solution of (2.5) by using the Galerkin method and a truncation. At first, we will point out some important properties of the nonlinear functions f and g. Due to Lemmas 7.1 and 7.2 in the Appendix, we have for all $u, v \in V$ that
$$\|f(v)\|^2 \le 2^{2\sigma+1} \|v\|^{4\sigma+2}_V, \tag{2.6}$$
$$\|f(u) - f(v)\|^2 \le 2^{2\sigma+1}(4\sigma - 1)^2 \left(\|u\|^{4\sigma}_V + \|v\|^{4\sigma}_V\right) \|u - v\|^2. \tag{2.7}$$
Hence, $f : V \to H$ is correctly defined and Lemma 7.3 implies for all $u, v \in V$ that
$$\operatorname{Re}\big(f(v), v\big) \ge 0, \qquad \operatorname{Re}\big(f(u) - f(v), u - v\big) \ge 0. \tag{2.8}$$
A more general form of the nonlinear function f is discussed in Sect. 6. Because of the representation of the Hilbert-Schmidt norm and the properties (2.3) and (2.4) of g, it follows for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ that
$$\begin{aligned}&\|g(t,0)\|^2_{L_2(K,H)} \le \|g(t,0)\|^2_{L_2(K,V)} \le k_g,\\ &\|g(t,u)\|^2_{L_2(K,H)} \le 2c_g\|u\|^2 + 2k_g, \qquad \text{for all } u \in H.\end{aligned} \tag{2.9}$$
Finally, let $(u(t))_{t\in[0,T]}$ be an H-valued process with
$$\sup_{t\in[0,T]} \|u(t)\|^2 < \infty, \qquad \text{for a.e. } \omega \in \Omega.$$

Then we introduce for $R \in \mathbb{N}$ the stopping time

$$\tau_R^u := \begin{cases} T, & \text{if } \sup_{t\in[0,T]} \|u(t)\|^2 < R^2, \\ \inf\left\{t \in [0,T] \,\middle|\, \|u(t)\|^2 \geq R^2\right\}, & \text{if } \sup_{t\in[0,T]} \|u(t)\|^2 \geq R^2. \end{cases} \tag{2.10}$$

Note that $(\tau_R^u)_R$ is an increasing sequence with

$$\lim_{R\to\infty} \tau_R^u = T, \qquad \text{for a.e. } \omega \in \Omega. \tag{2.11}$$

Below, $C(p_1, p_2, \ldots, p_m)$ represents a generic positive constant depending on certain parameters $p_1, p_2, \ldots, p_m$. The value of this constant may vary from line to line.

3. Uniqueness of the variational solution

While the existence of a variational solution of the stochastic nonlinear Schrödinger problem (2.5) will be shown in Sect. 5, we first treat its uniqueness.

Theorem 3.1. *If $X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ is a variational solution of the Schrödinger problem (2.5), then it is unique.*

Proof. Assume that there are two variational solutions

$$X, \hat{X} \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$$

of problem (2.5). By denoting $\delta X := X - \hat{X}$ and applying the stochastic energy equality, we obtain

$$\begin{aligned} \|\delta X(t)\|^2 =& 2\,\mathrm{Im} \int_0^t \left\langle A\delta X(s), \delta X(s)\right\rangle ds \\ & - 2\lambda\,\mathrm{Re} \int_0^t \left(f(X(s)) - f(\hat{X}(s)), \delta X(s)\right) ds \\ & + 2\,\mathrm{Re} \sum_{j=1}^{\infty} \int_0^t \left(\left[g(s, X(s)) - g(s, \hat{X}(s))\right] e_j, \delta X(s)\right) d\beta_j(s) \\ & + \int_0^t \left\|g(s, X(s)) - g(s, \hat{X}(s))\right\|_{L_2(K,H)}^2 ds \end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. The first addend on the right-hand side vanishes since $\mathrm{Im}\left\langle Av, v\right\rangle = 0$ for all $v \in V$, and the second one is less or equal to zero because of the second property in (2.8). With the help of the Burkholder–Davis–Gundy inequality (see [25, p. 44, Theorem 7]) and the

Lipschitz-continuity (2.3) of g, we estimate the Itô integral by

$$2E \sup_{t\in[0,T]} \left| \sum_{j=1}^{\infty} \int_0^t \left(\left[g(s,X(s)) - g(s,\hat{X}(s)) \right] e_j, \delta X(s) \right) d\beta_j(s) \right|$$
$$\le 6E \left[\int_0^T \left\| g(s,X(s)) - g(s,\hat{X}(s)) \right\|^2_{L_2(K,H)} \|\delta X(s)\|^2 \, ds \right]^{\frac{1}{2}}$$
$$\le \frac{1}{2} E \sup_{t\in[0,T]} \|\delta X(t)\|^2 + 18 c_g \int_0^T E \sup_{s\in[0,t]} \|\delta X(s)\|^2 \, dt.$$

Finally, the Hilbert–Schmidt norm will be treated analogously referring to (2.3) such that it follows

$$E \sup_{t\in[0,T]} \|\delta X(t)\|^2 \le 38 c_g \int_0^T E \sup_{s\in[0,t]} \|\delta X(s)\|^2 \, dt.$$

Gronwall's lemma yields $E \sup_{t\in[0,T]} \|\delta X(t)\|^2 = 0$, which implies

$$X(t) = \hat{X}(t) \qquad \text{for all } t \in [0,T] \text{ and a.e. } \omega \in \Omega.$$

□

4. Galerkin method and a priori estimates

We need some preliminaries for the finite-dimensional approximations. For each $n \in \mathbb{N}$, we regard the finite-dimensional space $H_n := \mathrm{sp}\{h_1, h_2, \ldots, h_n\}$ and the orthogonal projection $\pi_n : H \to H_n$ given by

$$\pi_n u := \sum_{k=1}^{n} (u, h_k)\, h_k, \qquad \text{for all } u \in H. \tag{4.1}$$

It especially holds for all $u \in H$ and all $h \in H_n$ that

$$(\pi_n u, h) = (u, h), \qquad \|\pi_n u\|^2 \le \|u\|^2. \tag{4.2}$$

Next, observe that the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent on H_n, which means that

$$\|u\|^2 \le \|u\|_V^2 = \|u\|^2 + \langle Au, u \rangle \le (1+\mu_n)\|u\|^2, \quad \text{for all } u \in H_n, \tag{4.3}$$

since $\mu_n := \max\{\mu_k \mid k \in \{1,2,\ldots,n\}\}$ and the operator $A : H_n \to H_n$ is linear, continuous and satisfies

$$\mathrm{Im}\,\langle Au, u \rangle = 0, \quad Au = \sum_{k=1}^{n} \mu_k (u, h_k)\, h_k, \quad \|Au\|^2 \le \mu_n^2 \|u\|^2,$$
$$\langle Au, u \rangle = \sum_{k=1}^{n} \mu_k \left| (u, h_k) \right|^2 = \left\| \frac{d}{dx} u \right\|^2, \qquad \text{for all } u \in H_n, \tag{4.4}$$

and

$$(v, Au) = \overline{\langle Au, v \rangle} \le \left\| \frac{d}{dx} u \right\| \left\| \frac{d}{dx} v \right\|, \qquad \text{for all } u, v \in H_n. \tag{4.5}$$

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis in K and $K_n := \mathrm{sp}\,\{e_1, e_2, \ldots, e_n\}$. Then we use the notations $\varphi_n := \pi_n\varphi$, $f_n(u) := \pi_n f(u)$ and $g_n(\cdot,u)w := \pi_n g(\cdot,u)w$ for all $u \in H_n$ and all $w \in K_n$ hereafter. The finite-dimensional Wiener process in K_n is represented by

$$W_n(s) := \sum_{j=1}^{n} e_j\,\beta_j(s).$$

Thus, adapting the Galerkin method for deterministic nonlinear Schrödinger equations for the case of problem (2.5), we regard for each $n \in \mathbb{N}$ the finite-dimensional Galerkin equations

$$\begin{aligned}\big(X_n(t), h_k\big) = \big(\varphi_n, h_k\big) - i\int_0^t \big\langle AX_n(s), h_k\big\rangle\,ds - \lambda\int_0^t \big(f_n(X_n(s)), h_k\big)\,ds \\ + \left(\int_0^t g_n(s, X_n(s))\,dW_n(s), h_k\right) \qquad (4.6)\end{aligned}$$

for all $t \in [0,T]$, all $k \in \{1, 2, \ldots, n\}$ and a.e. $\omega \in \Omega$.

For fixed $M \in \mathbb{N}$, we further introduce the Lipschitz-continuous real-valued truncation function $\psi^M : [0,\infty) \to [0,\infty)$ by

$$\psi^M(r) := \begin{cases} 1, & \text{if } 0 \le r \le M,\\ M+1-r, & \text{if } M < r < M+1,\\ 0, & \text{if } r \ge M+1\end{cases}$$

and choose $f_n^M : H_n \to H_n$ defined by $f_n^M(u) := \psi^M(\|u\|)f_n(u)$ for each $u \in H_n$. We consider the following finite-dimensional equations

$$\begin{aligned}\big(X_n^M(t), h_k\big) = \big(\varphi_n, h_k\big) - i\int_0^t \big\langle AX_n^M(s), h_k\big\rangle\,ds - \lambda\int_0^t \big(f_n^M(X_n^M(s)), h_k\big)\,ds \\ + \left(\int_0^t g_n(s, X_n^M(s))\,dW_n(s), h_k\right) \qquad (4.7)\end{aligned}$$

for all $t \in [0,T]$, all $k \in \{1, 2, \ldots, n\}$ and a.e. $\omega \in \Omega$. Referring to the equivalence of norms (4.3) and the property (2.7) of f, one can show that the nonlinear truncated function $f_n^M : H_n \to H_n$ is Lipschitz-continuous and growth-bounded on H_n for fixed $M, n \in \mathbb{N}$. Since the noise term g is also Lipschitz-continuous by (2.3) and satisfies (2.9), g_n fulfills similar properties. Hence, we know from the theory of finite-dimensional stochastic differential equations (with Lipschitz-continuous mappings) that there exists a unique solution $X_n^M \in L^2(\Omega; C([0,T]; H_n))$. Due to the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_V$ on H_n, we further get that $X_n^M \in L^2(\Omega\times[0,T]; V)$. In the subsequent theorems, we state uniform a priori estimates for the Galerkin approximation X_n^M in $L^{2p}(\Omega; C([0,T]; H))$ and $L^{2p}(\Omega\times[0,T]; V)$ for $p \ge 1$.

Theorem 4.1. *Let $M, n \in \mathbb{N}$ be arbitrarily fixed and $p \ge 1$. Then there exists a positive constant C depending on p, T, c_g and k_g such that*

$$E\sup_{t\in[0,T]}\big\|X_n^M(t)\big\|^{2p} \le C(p,T,c_g,k_g)\Big[1 + \|\varphi\|^{2p}\Big].$$

Proof. For simplicity, we use the notation $Y(t) := X_n^M(t)$, apply the stochastic energy equality to (4.7), use (4.2) and get

$$\begin{aligned}\|Y(t)\|^2 = \|\varphi_n\|^2 &+ 2\,\mathrm{Im}\int_0^t \big\langle AY(s), Y(s)\big\rangle\, ds \\ &- 2\lambda\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\,\big(f(Y(s)), Y(s)\big)\, ds \\ &+ 2\,\mathrm{Re}\sum_{j=1}^n \int_0^t \big(g_n(s, Y(s))e_j, Y(s)\big)\, d\beta_j(s) \\ &+ \int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n,H_n)}\, ds\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Note that $\mathrm{Im}\big\langle AY(s), Y(s)\big\rangle = 0$ because of the first property in (4.4). Let $p > 1$ and apply the Itô formula such that

$$\begin{aligned}\|Y(t)\|^{2p} = \|\varphi_n\|^{2p} &- 2\lambda p\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\,\big(f(Y(s)), Y(s)\big)\|Y(s)\|^{2(p-1)}\, ds \\ &+ 2p\,\mathrm{Re}\sum_{j=1}^n \int_0^t \big(g_n(s, Y(s))e_j, Y(s)\big)\,\|Y(s)\|^{2(p-1)}\, d\beta_j(s) \\ &+ 2p(p-1)\int_0^t \sum_{j=1}^n \big|\mathrm{Re}\,\big(g_n(s, Y(s))e_j, Y(s)\big)\big|^2\,\|Y(s)\|^{2(p-2)}\, ds \\ &+ p\int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n,H_n)}\,\|Y(s)\|^{2(p-1)}\, ds\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Observe that the term with the nonlinearity f is less or equal to zero by the first property in (2.8) and we also have

$$\begin{aligned}2p(p-1)&\int_0^t \sum_{j=1}^n \big|\mathrm{Re}\,\big(g_n(s, Y(s))e_j, Y(s)\big)\big|^2\,\|Y(s)\|^{2(p-2)}\, ds \\ &+ p\int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n,H_n)}\,\|Y(s)\|^{2(p-1)}\, ds \\ &\le p(2p-1)\int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n,H_n)}\,\|Y(s)\|^{2(p-1)}\, ds.\end{aligned}$$

Consequently, one obtains

$$\begin{aligned}\|Y(t)\|^{2p} \le \|\varphi_n\|^{2p} &+ 2p\,\mathrm{Re}\sum_{j=1}^n \int_0^t \big(g_n(s, Y(s))e_j, Y(s)\big)\,\|Y(s)\|^{2(p-1)}\, d\beta_j(s) \\ &+ p(2p-1)\int_0^t \|g_n(s, Y(s))\|^2_{L_2(K_n,H_n)}\,\|Y(s)\|^{2(p-1)}\, ds \qquad (4.8)\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. We use the notation $\tau := \tau_R^Y$ with $R \in \mathbb{N}$ and observe that for each $\gamma \in [0,\infty)$ it holds

$$\gamma E \left(\int_0^{t\wedge\tau} \|Y(s)\|^{4p}\, ds \right)^{\frac{1}{2}} \leq \gamma E \left[\sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^p \left(\int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds \right)^{\frac{1}{2}} \right] \leq \frac{1}{2} E \sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^{2p} + \frac{\gamma^2}{2} E \int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds. \tag{4.9}$$

The Burkholder–Davis–Gundy inequality, Young's inequality, estimate (2.9) and relation (4.9) lead to

$$\begin{aligned}
&2pE \sup_{s\in[0,t\wedge\tau]} \operatorname{Re} \sum_{j=1}^n \int_0^s \big(g_n(r,Y(r))e_j, Y(r)\big) \|Y(r)\|^{2(p-1)}\, d\beta_j(r) \\
&\leq 6pE \left[\int_0^{t\wedge\tau} \|g(s,Y(s))\|^2_{L_2(K,H)} \|Y(s)\|^{4p-2}\, ds \right]^{\frac{1}{2}} \\
&\leq 6pE \left[\frac{1}{2p} \int_0^{t\wedge\tau} \|g(s,Y(s))\|^{4p}_{L_2(K,H)}\, ds + \frac{2p-1}{2p} \int_0^{t\wedge\tau} \|Y(s)\|^{4p}\, ds \right]^{\frac{1}{2}} \\
&\leq C(p,T,k_g) + \frac{1}{2} E \sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^{2p} + C(p,c_g) E \int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds.
\end{aligned}$$

Moreover, the inequality of Young with $p > 1$ and (2.9) yield

$$\begin{aligned}
&p(2p-1)E \sup_{s\in[0,t\wedge\tau]} \int_0^s \|g_n(r,Y(r))\|^2_{L_2(K_n,H_n)} \|Y(r)\|^{2(p-1)}\, dr \\
&\leq (2p-1) \left[E \int_0^{t\wedge\tau} \|g(s,Y(s))\|^{2p}_{L_2(K,H)}\, ds + (p-1)E \int_0^{t\wedge\tau} \|Y(s)\|^{2p}\, ds \right] \\
&\leq C(p,c_g) E \int_0^{t\wedge\tau} \|Y(s))\|^{2p}\, ds + C(p,T,k_g).
\end{aligned}$$

Based on (4.8), we obtain

$$E \sup_{s\in[0,t\wedge\tau]} \|Y(s)\|^{2p} \leq 2\|\varphi_n\|^{2p} + C(p,T,k_g) + C(p,c_g) \int_0^t E \sup_{s\in[0,r\wedge\tau]} \|Y(s)\|^{2p}\, dr.$$

By applying Gronwall's lemma we receive

$$E \sup_{s\in[0,T\wedge\tau]} \|Y(s)\|^{2p} \leq C(p,T,c_g,k_g) \Big[1 + \|\varphi_n\|^{2p} \Big].$$

Let $R \to \infty$, use (2.11), the notation $Y(t) = X_n^M(t)$ and $\|\varphi_n\| \leq \|\varphi\|$ to get

$$E \sup_{t\in[0,T]} \left\| X_n^M(t) \right\|^{2p} \leq C(p,T,c_g,k_g) \Big[1 + \|\varphi\|^{2p} \Big].$$

The proof for the case $p = 1$ follows analogous ideas as above. □

Theorem 4.2. *Let $M, n \in \mathbb{N}$ be arbitrarily fixed and $p \geq 1$. Then there exists a positive constant C depending on p, T, c_g and k_g such that the solution X_n^M of* (4.7) *satisfies the estimate*

$$E \sup_{t\in[0,T]} \left\|X_n^M(t)\right\|_V^{2p} \leq C(p,T,c_g,k_g)\left[1+\|\varphi\|_V^{2p}\right].$$

Proof. We denote again $Y(t) := X_n^M(t)$, consider (4.7) and use the energy equality and (4.2) to write

$$\begin{aligned}\left|\big(Y(t),h_k\big)\right|^2 = \left|\big(\varphi_n,h_k\big)\right|^2 &+ 2\,\mathrm{Im}\int_0^t \big\langle AY(s),\big(Y(s),h_k\big)\,h_k\big\rangle\,ds \\ &- 2\lambda\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\,\big(f(Y(s)),\big(Y(s),h_k\big)\,h_k\big)\,ds \\ &+ 2\,\mathrm{Re}\sum_{j=1}^n\int_0^t \big(g_n(s,Y(s))e_j,\big(Y(s),h_k\big)\,h_k\big)\,d\beta_j(s) \\ &+ \int_0^t\sum_{j=1}^n\left|\big(g_n(s,Y(s))e_j,h_k\big)\right|^2 ds\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where the second term on the right-hand side vanishes. Multiplication with the real-valued eigenvalues μ_k of A, summing up over all $k \in \{1,2,\ldots,n\}$ and using the relations (4.4) result in

$$\begin{aligned}\left\|\frac{\partial}{\partial x}Y(t)\right\|^2 = \left\|\frac{d}{dx}\varphi_n\right\|^2 &- 2\lambda\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\,\big(f(Y(s)),AY(s)\big)\,ds \\ &+ 2\,\mathrm{Re}\sum_{j=1}^n\int_0^t \big(g_n(s,Y(s))e_j,AY(s)\big)\,d\beta_j(s) \\ &+ \int_0^t\sum_{j=1}^n\left\|\frac{\partial}{\partial x}\left[g_n(s,Y(s))e_j\right]\right\|^2 ds\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Let again $p > 1$, then the application of the Itô formula yields

$$\begin{aligned}\left\|\frac{\partial}{\partial x}Y(t)\right\|^{2p} &= \left\|\frac{d}{dx}\varphi_n\right\|^{2p} \\ &- 2\lambda p\,\mathrm{Re}\int_0^t \psi^M(\|Y(s)\|)\,\big(f(Y(s)),AY(s)\big)\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)} ds \\ &+ 2p\,\mathrm{Re}\sum_{j=1}^n\int_0^t \big(g_n(s,Y(s))e_j,AY(s)\big)\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)} d\beta_j(s) \\ &+ 2p(p-1)\int_0^t\sum_{j=1}^n\left|\mathrm{Re}\,\big(g_n(s,Y(s))e_j,AY(s)\big)\right|^2\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-2)} ds \\ &+ p\int_0^t\sum_{j=1}^n\left\|\frac{\partial}{\partial x}\left[g_n(s,Y(s))e_j\right]\right\|^2\left\|\frac{\partial}{\partial x}Y(s)\right\|^{2(p-1)} ds\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Due to Lemma 7.4 (see Appendix), the second term on the right-hand side is less or equal to zero. For the Itô integral, we apply the Burkholder-Davis-Gundy inequality, property (4.5), the growth-boundedness (2.4), Young's inequality and relation (4.9) to receive for $\tau := \tau_R^Y$ that

$$
\begin{aligned}
&2pE \sup_{s\in[0,t\wedge\tau]} \mathrm{Re} \sum_{j=1}^{n} \int_0^s \left(g_n(r,Y(r))e_j, AY(r)\right) \left\| \frac{\partial}{\partial x} Y(r) \right\|^{2(p-1)} d\beta_j(r) \\
&\quad \le 6pE \left[\int_0^{t\wedge\tau} \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x} \left[g_n(s,Y(s))e_j\right] \right\|^2 \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p-2} ds \right]^{\frac{1}{2}} \\
&\quad \le 6p\sqrt{k_g}\, E \left[\int_0^{t\wedge\tau} \left(1 + \|Y(s)\|_V^2\right) \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p-2} ds \right]^{\frac{1}{2}} \\
&\quad \le 6p\sqrt{k_g}\, E \left[\frac{1}{2p} T + \frac{1}{2p} \int_0^T \|Y(s)\|^{4p} \, ds + \frac{6p-2}{2p} \int_0^{t\wedge\tau} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{4p} ds \right]^{\frac{1}{2}} \\
&\quad \le C(p,T,k_g) + \frac{1}{2} E \sup_{s\in[0,T]} \|Y(s)\|^{2p} + C(p,k_g) E \int_0^T \|Y(s)\|^{2p} \, ds \\
&\qquad + \frac{1}{2} E \sup_{s\in[0,t\wedge\tau]} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} + C(p,k_g) E \int_0^{t\wedge\tau} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} ds.
\end{aligned}
$$

Next, we write

$$
\begin{aligned}
&E \sup_{s\in[0,t\wedge\tau]} 2p(p-1) \int_0^s \sum_{j=1}^{n} \left|\mathrm{Re} \left(g_n(r,Y(r))e_j, AY(r)\right)\right|^2 \left\| \frac{\partial}{\partial x} Y(r) \right\|^{2(p-2)} dr \\
&\qquad + E \sup_{s\in[0,t\wedge\tau]} p \int_0^s \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x} \left[g_n(r,Y(r))e_j\right] \right\|^2 \left\| \frac{\partial}{\partial x} Y(r) \right\|^{2(p-1)} dr \\
&\quad \le E \sup_{s\in[0,t\wedge\tau]} p(2p-1) \int_0^s \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x} \left[g_n(r,Y(r))e_j\right] \right\|^2 \left\| \frac{\partial}{\partial x} Y(r) \right\|^{2(p-1)} dr \\
&\quad \le C(p,T,k_g) + C(p,k_g) E \int_0^T \|Y(s)\|^{2p} \, ds + C(p,k_g) E \int_0^{t\wedge\tau} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} ds
\end{aligned}
$$

because of the same calculations like in the case of the stochastic integral. Combining these estimates, we obtain

$$E \sup_{s\in[0,t\wedge\tau]} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} \leq 2 \left\| \frac{\partial}{\partial x} \varphi_n \right\|^{2p} + C(p,T,k_g) + E \sup_{s\in[0,T]} \|Y(s)\|^{2p} + C(p,k_g) E \int_0^T \|Y(s)\|^{2p}\, ds + C(p,k_g) \int_0^t E \sup_{s\in[0,r\wedge\tau]} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} dr.$$

Applying Gronwall's lemma, it results that

$$E \sup_{s\in[0,T\wedge\tau]} \left\| \frac{\partial}{\partial x} Y(s) \right\|^{2p} \leq C(p,T,k_g) \left[1 + \left\| \frac{\partial}{\partial x} \varphi_n \right\|^{2p} + E \sup_{s\in[0,T]} \|Y(s)\|^{2p} + E \int_0^T \|Y(s)\|^{2p}\, ds \right].$$

Letting $R \to \infty$ and taking into account (2.11) and Theorem 4.1, we get

$$E \sup_{t\in[0,T]} \left\| \frac{\partial}{\partial x} Y(t) \right\|^{2p} \leq C(p,T,c_g,k_g) \left[1 + \|\varphi_n\|_V^{2p} \right].$$

Hence, with $Y(t) = X_n^M(t)$, $\|\varphi_n\|_V \leq \|\varphi\|_V$ and Theorem 4.1 it follows that

$$E \sup_{t\in[0,T]} \left\| X_n^M(t) \right\|_V^{2p} \leq C(p,T,c_g,k_g) \left[1 + \|\varphi\|_V^{2p} \right].$$

The proof for the case $p = 1$ can be proved with analogous ideas as above. □

5. Existence of the variational solution

Based on the a priori estimates from the last section, we are now able to show the unique existence of the variational solution of the finite-dimensional problem (4.6) and of the infinite-dimensional problem (2.5) thereafter.

Theorem 5.1. *For each fixed $n \in \mathbb{N}$ and $p \geq 1$ there exists a unique variational solution $X_n \in L^{2p}(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V)$ of the finite-dimensional stochastic nonlinear Schrödinger problem (4.6). Besides, we have for each fixed $n \in \mathbb{N}$ and $p \geq 1$ the estimates*

$$E \sup_{t\in[0,T]} \|X_n(t)\|^{2p} \leq C(p,T,c_g,k_g) \Big[1 + \|\varphi\|^{2p} \Big],$$

$$E \int_0^T \|X_n(t)\|_V^{2p}\, dt \leq C(p,T,c_g,k_g) \left[1 + \|\varphi\|_V^{2p} \right].$$

Proof. For the whole proof, we fix $n \in \mathbb{N}$. Then the uniqueness of the variational solution follows similarly to the proof of Theorem 3.1. Consider the stopping time $\tau_M := \tau_M^u$ for $u := X_n^M$, which is equal to the stopping time in

(2.10) for $R = M$. From the definition of τ_M, Markov's inequality and Theorem 4.1 (for $p = 1$) we obtain

$$P(\tau_M < T) \leq P\left(\sup_{t\in[0,T]} \left\|X_n^M(t)\right\|^2 \geq M^2\right) \leq \frac{C(T, c_g, k_g)}{M^2}\Big[1 + \|\varphi\|^2\Big]. \tag{5.1}$$

Thus, the increasing sequence of stopping times $(\tau_M)_M$ converges a.s. to T. Let Ω^M be the set of all $\omega \in \Omega$ such that $X_n^M(\omega, \cdot)$ satisfies (4.7) for all $t \in [0, T]$ and all $k \in \{1, 2, \ldots, n\}$ and $X_n^M(\omega, \cdot)$ has continuous trajectories in H. We introduce $\Omega' := \bigcap_{M=1}^{\infty} \Omega^M$ with $P(\Omega') = 1$. Furthermore, we define

$$S := \bigcup_{M=1}^{\infty} \bigcup_{1\leq K\leq M} \left\{\omega \in \Omega' \,\middle|\, \tau_K = T \text{ and } \exists\, t \in [0, T] : X_n^K(\omega, t) \neq X_n^M(\omega, t)\right\}.$$

It holds that $P(S) = 0$ because otherwise there exist two natural numbers K_0, M_0 with $K_0 < M_0$ such that

$$S_{M_0,K_0} := \left\{\omega \in \Omega' \,\middle|\, \tau_{K_0} = T \text{ and } \exists\, t \in [0, T] : X_n^{K_0}(\omega, t) \neq X_n^{M_0}(\omega, t)\right\}$$

has the probability $P(S_{M_0,K_0}) > 0$. Denoting

$$X^*(\omega, t) := \begin{cases} X_n^{K_0}(\omega, t), & \text{if } \omega \in S_{M_0,K_0}, \\ X_n^{M_0}(\omega, t), & \text{if } \omega \in \Omega' \backslash S_{M_0,K_0} \end{cases}$$

for each $t \in [0, T]$, we see that for all $\omega \in S_{M_0,K_0}$ there exists a $t \in [0, T]$ such that $X^*(\omega, t) \neq X_n^{M_0}(\omega, t)$. This contradicts the almost sure uniqueness of the variational solution of (4.7) (for $M = M_0$), and it follows that $P(S) = 0$. Letting

$$\Omega'' := \Omega' \cap \left(\bigcup_{M=1}^{\infty} \{\tau_M = T\}\backslash S\right),$$

using (5.1) and the definition of S, we get

$$P(\Omega'') = \lim_{M\to\infty} P\left(\{\tau_M = T\}\backslash S\right) = 1 - \lim_{M\to\infty} P(\tau_M < T) = 1.$$

Now, we choose $\omega \in \Omega''$. For this ω there exists an $M_0 \in \mathbb{N}$ such that $\tau_M = T$ for all $M \geq M_0$. Therefore, $\psi^M\left(\left\|X_n^M(s)\right\|\right) = 1$ for all $s \in [0, T]$ and all $M \geq M_0$, and consequently

$$\begin{aligned}\left(X_n^M(t), h_k\right) = \left(\varphi_n, h_k\right) - i\int_0^t \left\langle AX_n^M(s), h_k\right\rangle ds - \lambda\int_0^t \left(f_n(X_n^M(s)), h_k\right) ds \\ + \left(\int_0^t g_n(s, X_n^M(s))\, dW_n(s), h_k\right)\end{aligned}$$

for all $t \in [0, T]$, all $M \geq M_0$ and all $k \in \{1, 2, \ldots, n\}$. For this fixed $\omega \in \Omega''$ we define

$$X_n(\omega, \cdot) := X_n^M(\omega, \cdot), \qquad \text{for each } t \in [0, T] \text{ and all } M \geq M_0. \tag{5.2}$$

Hence, we have

$$\begin{aligned}\big(X_n(t),h_k\big)=\big(\varphi_n,h_k\big)-i\int_0^t\big\langle AX_n(s),h_k\big\rangle\,ds-\lambda\int_0^t\big(f_n(X_n(s)),h_k\big)\,ds\\+\left(\int_0^t g_n(s,X_n(s))\,dW_n(s),h_k\right)\end{aligned}$$

for all $\omega\in\Omega''$, all $t\in[0,T]$ and all $k\in\{1,2,\ldots,n\}$. By the way, this equals Eq. (4.6). Due to the properties of X_n^M, the process $(X_n(t))_{t\in[0,T]}$ is H-valued, $\mathcal{F}\times\mathcal{B}_{[0,T]}$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has almost surely continuous trajectories in H. Because of (5.2), it results for each $p\geq 1$ that

$$\lim_{M\to\infty}\sup_{t\in[0,T]}\big\|X_n^M(t)-X_n(t)\big\|^{2p}=0,\qquad\text{for a.e. }\omega\in\Omega,$$

$$\lim_{M\to\infty}\int_0^T\big\|X_n^M(t)-X_n(t)\big\|_V^{2p}\,dt=0,\qquad\text{for a.e. }\omega\in\Omega.$$

The Lemma of Fatou, Theorems 4.1 and 4.2 finally yield

$$\begin{aligned}E\sup_{t\in[0,T]}\|X_n(t)\|^{2p}&\leq\liminf_{M\to\infty}E\sup_{t\in[0,T]}\big\|X_n^M(t)\big\|^{2p}\\&\leq C(p,T,c_g,k_g)\Big[1+\|\varphi\|^{2p}\Big],\\E\int_0^T\|X_n(t)\|_V^{2p}\,dt&\leq\liminf_{M\to\infty}E\int_0^T\big\|X_n^M(t)\big\|_V^{2p}\,dt\\&\leq C(p,T,c_g,k_g)\left[1+\|\varphi\|_V^{2p}\right].\end{aligned}$$

Thus, $X_n\in L^{2p}(\Omega;C([0,T];H))\cap L^{2p}(\Omega\times[0,T];V)$ is the unique variational solution of (4.6) for all $p\geq 1$. □

Now, we state our main result concerning the unique existence of the variational solution of the stochastic nonlinear Schrödinger problem (2.5).

Theorem 5.2. *The stochastic nonlinear Schrödinger problem (2.5) possesses a unique variational solution $X\in L^{2p}(\Omega;C([0,T];H))\cap L^{2p}(\Omega\times[0,T];V)$ for each $p\geq 1$, which satisfies*

$$E\sup_{t\in[0,T]}\|X(t)\|^{2p}\leq C(p,T,c_g,k_g)\Big[1+\|\varphi\|^{2p}\Big],$$

$$E\int_0^T\|X(t)\|_V^{2p}\,dt\leq C(p,T,c_g,k_g)\left[1+\|\varphi\|_V^{2p}\right].$$

Moreover, the sequence of Galerkin approximations $(X_n)_n$ converges to X strongly in $L^2(\Omega;C([0,T];H))$ and weakly in $L^{2p}(\Omega\times[0,T];V)$.

Proof. It suffices to focus on the existence of a solution because the uniqueness can be found in Theorem 3.1. We know from Theorem 5.1 that there exists a unique variational solution $X_n\in L^{2p}(\Omega;C([0,T];H))\cap L^{2p}(\Omega\times[0,T];V)$

of (4.6) and its corresponding uniform a priori estimates. Due to (2.6) and Theorem 5.1, the nonlinear drift term obeys

$$E\int_0^T \|f(X_n(t))\|^2\,dt \le C(\sigma,T,c_g,k_g)\left[1+\|\varphi\|_V^{2(2\sigma+1)}\right], \tag{5.3}$$

which implies the uniform boundedness of $(f(X_n))_n$ in $L^2(\Omega\times[0,T];H)$. The property (2.9) and Theorem 5.1 lead to

$$E\int_0^T \|g(t,X_n(t))\|^2_{L_2(K,H)}\,dt \le C(T,c_g,k_g)\Big[1+\|\varphi\|^2\Big],$$

hence, $(g(\cdot,X_n))_n$ is uniformly bounded in $L^2(\Omega\times[0,T];L_2(K,H))$.

First, we fix $p\ge 4\sigma$. By the above uniform boundedness properties and Lemma 7.5, it follows that there exist a subsequence, which we denote by $(X_n)_n$ as well, and functions $Z\in L^{2p}(\Omega\times[0,T];V)$, $f^*\in L^2(\Omega\times[0,T];H)$ and $g^*\in L^2(\Omega\times[0,T];L_2(K,H))$ such that we receive for $n\to\infty$ that

$$\begin{aligned} X_n \rightharpoonup Z, \qquad & \text{in } L^2(\Omega\times[0,T];H), L^2(\Omega\times[0,T];V) \\ & \text{and } L^{2p}(\Omega\times[0,T];V), \end{aligned}\tag{5.4}$$

$$f(X_n)\rightharpoonup f^*, \qquad \text{in } L^2(\Omega\times[0,T];H), \tag{5.5}$$

$$g(\cdot,X_n)\rightharpoonup g^*, \qquad \text{in } L^2(\Omega\times[0,T];L_2(K,H)). \tag{5.6}$$

Taking $n\to\infty$ in (4.6) and using these weak convergence results, we get for a.e. $(\omega,t)\in\Omega\times[0,T]$ and all $k\in\mathbb{N}$ that

$$\begin{aligned}\big(Z(t),h_k\big) =\ &\big(\varphi,h_k\big) - i\int_0^t \big\langle AZ(s),h_k\big\rangle\,ds - \lambda\int_0^t \big(f^*(s),h_k\big)\,ds \\ &+\bigg(\int_0^t g^*(s)\,dW(s),h_k\bigg).\end{aligned}\tag{5.7}$$

There exists an $\mathcal{F}_t$-measurable H-valued process which is equal to $Z(t)$ for a.e. $(\omega,t)\in\Omega\times[0,T]$ and equal to the right-hand side of (5.7) for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. We also denote this process by $(Z(t))_{t\in[0,T]}$. Therefore,

$$\begin{aligned}\big(Z(t),h_k\big) =\ &\big(\varphi,h_k\big) - i\int_0^t \big\langle AZ(s),h_k\big\rangle\,ds - \lambda\int_0^t \big(f^*(s),h_k\big)\,ds \\ &+\bigg(\int_0^t g^*(s)\,dW(s),h_k\bigg)\end{aligned}\tag{5.8}$$

for all $t\in[0,T]$, all $k\in\mathbb{N}$ and a.e. $\omega\in\Omega$. The process $(Z(t))_{t\in[0,T]}$ has in H almost surely continuous trajectories (see [25, p. 73, Theorem 2]).

Next, we denote by $Z_n:=\pi_n Z$ and $g_n^*(\cdot)w:=\pi_n g^*(\cdot)w$ for all $w\in K_n$ the finite-dimensional approximations of Z and $g^*(\cdot)w$, respectively, and we choose $\xi(t):=\exp\{-2(1+c_g)t\}$ for all $t\in[0,T]$. Using (4.6), (5.8), the stochastic energy equality and the properties (4.4) of A, we obtain

$$\begin{aligned} & E\,\xi(T)\,\|X_n(T)-Z_n(T)\|^2 \\ & \quad = -2\lambda E\,\mathrm{Re}\int_0^T \xi(t)\,\big(f(X_n(t))-f^*(t),X_n(t)-Z_n(t)\big)\,dt \\ & \qquad -2(1+c_g)E\int_0^T \xi(t)\,\|X_n(t)-Z_n(t)\|^2\,dt \\ & \qquad +E\int_0^T \xi(t)\,\|g_n(t,X_n(t))-g_n^*(t)\|^2_{L_2(K_n,H_n)}\,dt \\ & \qquad +E\int_0^T \xi(t)\sum_{j>n}\sum_{k=1}^n \big|\big(g^*(t)e_j,h_k\big)\big|^2\,dt. \end{aligned} \tag{5.9}$$

Due to monotone convergence and since $g^*\in L^2(\Omega\times[0,T];L_2(K,H))$, we have for $n\to\infty$ that

$$E\int_0^T \xi(t)\sum_{j>n}\sum_{k=1}^n \big|\big(g^*(t)e_j,h_k\big)\big|^2\,dt \le E\int_0^T \xi(t)\sum_{j>n}\|g^*(t)e_j\|^2\,dt\to 0.$$

For simplicity, we omit to write the dependence on $t\in[0,T]$ in the following two auxiliary results. The second property in (2.8) entails

$$\begin{aligned} & -2\lambda\,\mathrm{Re}\big(f(X_n)-f^*,X_n-Z_n\big) \\ & \quad = -2\lambda\,\mathrm{Re}\big(f(X_n)-f(Z_n),X_n-Z_n\big)-2\lambda\,\mathrm{Re}\big(f(Z_n)-f(Z),X_n-Z_n\big) \\ & \qquad -2\lambda\,\mathrm{Re}\big(f(Z)-f^*,X_n-Z_n\big) \\ & \quad \le \lambda^2\|f(Z_n)-f(Z)\|^2+\|X_n-Z_n\|^2-2\lambda\,\mathrm{Re}\big(f(Z)-f^*,X_n-Z_n\big), \end{aligned}$$

and, regarding (2.3), it results that

$$\begin{aligned} & \|g_n(\cdot,X_n)-g_n^*\|^2_{L_2(K_n,H_n)} \le \|g(\cdot,X_n)-g^*\|^2_{L_2(K,H)} \\ & \quad = \|g(\cdot,X_n)-g(\cdot,Z)\|^2_{L_2(K,H)}+\big(g(\cdot,X_n)-g^*,g(\cdot,Z)-g^*\big)_{L_2(K,H)} \\ & \qquad +\big(g(\cdot,Z)-g^*,g(\cdot,X_n)-g^*\big)_{L_2(K,H)}-\|g(\cdot,Z)-g^*\|^2_{L_2(K,H)} \\ & \quad \le 2c_g\|X_n-Z_n\|^2+2c_g\|Z_n-Z\|^2+\big(g(\cdot,X_n)-g^*,g(\cdot,Z)-g^*\big)_{L_2(K,H)} \\ & \qquad +\big(g(\cdot,Z)-g^*,g(\cdot,X_n)-g^*\big)_{L_2(K,H)}-\|g(\cdot,Z)-g^*\|^2_{L_2(K,H)}. \end{aligned}$$

We use that $Z_n=\pi_n Z$ and $2p\ge 8\sigma\ge 4$ to state

$$Z_n,Z\in L^{2p}(\Omega\times[0,T];V)\hookrightarrow L^4(\Omega\times[0,T];V)\hookrightarrow L^4(\Omega\times[0,T];H).$$

Observe that $\|Z_n(t)-Z(t)\|\to 0$ for all $t\in[0,T]$ and a.e. $\omega\in\Omega$ as $n\to\infty$ such that

$$\lim_{n\to\infty} E\int_0^T \|Z_n(t)-Z(t)\|^4\,dt = E\int_0^T \lim_{n\to\infty}\|Z_n(t)-Z(t)\|^4\,dt=0. \tag{5.10}$$

Based on (2.7), the Cauchy–Schwarz inequality and $\xi(t) \le 1$ for all $t \in [0, T]$, we have

$$E \int_0^T \xi(t) \left\| f(Z_n(t)) - f(Z(t)) \right\|^2 dt$$
$$\le C(\sigma) \left(E \int_0^T \left(\|Z_n(t)\|_V^{8\sigma} + \|Z(t)\|_V^{8\sigma} \right) dt \right)^{\frac{1}{2}} \left(E \int_0^T \|Z_n(t) - Z(t)\|^4 \, dt \right)^{\frac{1}{2}}, \tag{5.11}$$

which yields, due to $Z_n, Z \in L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^{8\sigma}(\Omega \times [0,T]; V)$ and (5.10), that

$$f(Z_n) \to f(Z) \qquad \text{in } L^2(\Omega \times [0,T]; H) \text{ as } n \to \infty. \tag{5.12}$$

Because $Z_n \to Z$ in $L^2(\Omega \times [0,T]; H)$ (since $Z_n = \pi_n Z$), $X_n - Z_n \rightharpoonup 0$ in $L^2(\Omega \times [0,T]; H)$ (by (5.4)) and $g(\cdot, X_n) \rightharpoonup g^*$ in $L^2(\Omega \times [0,T]; L_2(K, H))$ (see (5.6)), it follows by (5.9) for $n \to \infty$ that

$$E\, \xi(T) \left\| X_n(T) - Z_n(T) \right\|^2 \to 0, \qquad E \int_0^T \xi(t) \left\| X_n(t) - Z_n(t) \right\|^2 dt \to 0,$$

and, therefore,

$$E \int_0^T \left\| X_n(t) - Z_n(t) \right\|^2 dt \to 0. \tag{5.13}$$

Furthermore, we obtain

$$E \int_0^T \xi(t) \left\| g(t, Z(t)) - g^*(t) \right\|_{L_2(K,H)}^2 dt = 0,$$

which implies

$$g(t, Z(t)) = g^*(t), \qquad \text{for a.e. } (\omega, t) \in \Omega \times [0, T].$$

Consider $\eta \in L^2(\Omega \times [0,T]; H)$ to be a simple function. Hence, it is uniformly bounded with respect to the variables ω and t. We use Lemmas 7.1 and 7.2 in the Appendix and the Cauchy–Schwarz inequality to compute

$$\left| E \int_0^T \left(f(X_n(t)) - f(Z_n(t)), \eta(t) \right) dt \right|$$
$$\le C(\sigma) \left(E \int_0^T \|\eta(t)\|^2 \left(\|X_n(t)\|_V^{4\sigma} + \|Z_n(t)\|_V^{4\sigma} \right) dt \right)^{\frac{1}{2}}$$
$$\times \left(E \int_0^T \|X_n(t) - Z_n(t)\|^2 \, dt \right)^{\frac{1}{2}}.$$

As a result of (5.13) and the fact that $(X_n)_n$ is bounded in $L^{4\sigma}(\Omega \times [0,T]; V)$ (see Theorem 5.1) and $(Z_n)_n$ is bounded in $L^{4\sigma}(\Omega \times [0,T]; V)$ (due to the embedding $L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^{4\sigma}(\Omega \times [0,T]; V)$), we conclude that

$$E \int_0^T \left(f(X_n(t)) - f(Z_n(t)), \eta(t) \right) dt \to 0.$$

Based on $f(X_n) \rightharpoonup f^*$ (by (5.5)) and $f(Z_n) - f(Z) \to 0$ in $L^2(\Omega \times [0,T]; H)$ (by (5.12)), it follows

$$E \int_0^T \big(f^*(t) - f(Z(t)), \eta(t)\big)\, dt = 0.$$

However, the set of simple functions in $L^2(\Omega \times [0,T]; H)$ is dense in the space $L^2(\Omega \times [0,T]; H)$, so we deduce that

$$E \int_0^T \big(f^*(t) - f(Z(t)), \eta(t)\big)\, dt = 0, \qquad \text{for all } \eta \in L^2(\Omega \times [0,T]; H).$$

Thus,

$$f(Z(t)) = f^*(t), \qquad \text{for a.e. } (\omega, t) \in \Omega \times [0,T].$$

Then (5.8) yields for all $t \in [0,T]$, all $k \in \mathbb{N}$ and a.e. $\omega \in \Omega$ that

$$\begin{aligned}\big(Z(t), h_k\big) = \big(\varphi, h_k\big) - i \int_0^t \big\langle AZ(s), h_k \big\rangle\, ds - \lambda \int_0^t \big(f(Z(s)), h_k\big)\, ds \\ + \bigg(\int_0^t g(s, Z(s))\, dW(s), h_k \bigg).\end{aligned}$$

Since $\mathrm{sp}\{h_1, h_2, \ldots, h_n, \ldots\}$ is dense in V, the above equation also holds for all $v \in V$. Hence, $X := Z$ is the variational solution of the stochastic nonlinear Schrödinger problem (2.5), and $X \in L^2(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V)$ for fixed $p \geq 4\sigma$.

For $p \in [1, 4\sigma)$ we use the continuous embedding result

$$L^{8\sigma}(\Omega \times [0,T]; V) \hookrightarrow L^{2p}(\Omega \times [0,T]; V).$$

Consequently, weak convergence in $L^{8\sigma}(\Omega \times [0,T]; V)$ implies weak convergence in $L^{2p}(\Omega \times [0,T]; V)$ (see [30, p. 265, Proposition 21.35(c)]). By now, we know that a subsequence of $(X_n)_n$ converges to X strongly in $L^2(\Omega \times [0,T]; H)$ and weakly in $L^{2p}(\Omega \times [0,T]; V)$. In fact, the whole sequence has these properties because of [29, p. 480, Proposition 10.13(1) and (2)] and since (2.5) possesses a unique solution. Using the weak convergence of $(X_n)_n$ towards X in $L^{2p}(\Omega \times [0,T]; V)$ for all $p \geq 1$ and the result of Theorem 5.1, we get

$$E \int_0^T \|X(t)\|_V^{2p}\, dt \leq \liminf_{n \to \infty} E \int_0^T \|X_n(t)\|_V^{2p}\, dt \leq C(p, T, c_g, k_g) \Big[1 + \|\varphi\|_V^{2p}\Big]. \tag{5.14}$$

The estimate

$$E \sup_{t \in [0,T]} \|X(t)\|^{2p} \leq C(p, T, c_g, k_g) \Big[1 + \|\varphi\|^{2p}\Big]$$

can be shown similarly to Theorem 4.1. Therefore, it holds that

$$X \in L^{2p}(\Omega; C([0,T]; H)) \cap L^{2p}(\Omega \times [0,T]; V).$$

To verify the strong convergence of $(X_n)_n$ to X in $L^2(\Omega; C([0,T]; H))$, we take (2.5) and (4.6), apply the stochastic energy equality to their difference and obtain

$$
\begin{aligned}
&\|X(t)-X_n(t)\|^2 \\
&\quad = \|\varphi-\varphi_n\|^2 + 2\operatorname{Im}\int_0^t \big\langle A\left[X(s)-X_n(s)\right], X(s)-X_n(s)\big\rangle\, ds \\
&\qquad - 2\lambda \operatorname{Re}\int_0^t \big(f(X(s))-f_n(X_n(s)), X(s)-X_n(s)\big)\, ds \\
&\qquad + 2\operatorname{Re}\sum_{j=1}^n \int_0^t \big(\left[g(s,X(s))-g_n(s,X_n(s))\right] e_j, X(s)-X_n(s)\big)\, d\beta_j(s) \\
&\qquad + 2\operatorname{Re}\sum_{j>n} \int_0^t \big(g(s,X(s))e_j, X(s)-X_n(s)\big)\, d\beta_j(s) \\
&\qquad + \int_0^t \sum_{j=1}^n \|\left[g(s,X(s))-g_n(s,X_n(s))\right] e_j\|^2\, ds + \int_0^t \sum_{j>n} \|g(s,X(s))e_j\|^2\, ds
\end{aligned}
$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Based on the series representation, the first term on the right-hand side converges to zero as $n \to \infty$ and the second one vanishes. Due to (4.1) and (5.13) (with $Z_n = \pi_n Z = \pi_n X$), observe that

$$
\begin{aligned}
&E\int_0^T \|X(s)-X_n(s)\|^2\, ds \\
&\quad \le 2E\int_0^T \|X(s)-\pi_n X(s)\|^2\, ds + 2E\int_0^T \|\pi_n X(s)-X_n(s)\|^2\, ds \to 0
\end{aligned} \tag{5.15}
$$

as $n \to \infty$. The Cauchy–Schwarz inequality yields

$$
\begin{aligned}
&E\sup_{t\in[0,T]} -2\lambda \operatorname{Re}\int_0^t \big(f(X(s))-f_n(X_n(s)), X(s)-X_n(s)\big)\, ds \\
&\quad \le 2\lambda \left(E\int_0^T \|f(X(s))-f_n(X_n(s))\|^2\, ds\right)^{\frac{1}{2}} \left(E\int_0^T \|X(s)-X_n(s)\|^2\, ds\right)^{\frac{1}{2}},
\end{aligned}
$$

which converges to zero as $n \to \infty$ because the first expression in parentheses is bounded (compare (5.3) and (5.14)), while the second one goes to zero (by (5.15)). Using the Burkholder–Davis–Gundy inequality, we get

$$
\begin{aligned}
&2E\sup_{t\in[0,T]} \operatorname{Re}\sum_{j=1}^n \int_0^t \big(\left[g(s,X(s))-g_n(s,X_n(s))\right] e_j, X(s)-X_n(s)\big)\, d\beta_j(s) \\
&\quad \le 6E\left[\int_0^T \sum_{j=1}^n \|\left[g(s,X(s))-g_n(s,X_n(s))\right] e_j\|^2 \|X(s)-X_n(s)\|^2\, ds\right]^{\frac{1}{2}} \\
&\quad \le 36E\int_0^T \sum_{j=1}^n \|\left[g(s,X(s))-g_n(s,X_n(s))\right] e_j\|^2\, ds \\
&\qquad + \frac{1}{4}E\sup_{t\in[0,T]} \|X(t)-X_n(t)\|^2 .
\end{aligned}
$$

Relations (4.1) and (4.2), the definition of $g_n(\cdot,u)w$ for all $u \in H_n$ and all $w \in K_n$, Cauchy's Double Series Theorem (see [8, p. 22]) and the Lipschitz-continuity (2.3) of g entail

$$\begin{aligned} &E\int_0^T \sum_{j=1}^n \|[g(s,X(s)) - g_n(s,X_n(s))]\, e_j\|^2\, ds \\ &\quad\le 2E\int_0^T \sum_{j=1}^n \|g(s,X(s))e_j - \pi_n g(s,X(s))e_j\|^2\, ds \\ &\qquad + 2E\int_0^T \sum_{j=1}^n \|\pi_n g(s,X(s))e_j - g_n(s,X_n(s))e_j\|^2\, ds \\ &\quad\le 2E\int_0^T \sum_{k>n}\sum_{j=1}^\infty \big|\big(g(s,X(s))e_j, h_k\big)\big|^2\, ds + 2c_g E\int_0^T \|X(s) - X_n(s)\|^2\, ds. \end{aligned}$$

This expression also converges to zero as $n \to \infty$ because of relation (5.15), the fact that $g \in L^2(\Omega \times [0,T]; L_2(K,H))$ and the rest of a convergent series goes to zero. Furthermore, the Burkholder-Davis-Gundy inequality leads to

$$\begin{aligned} &2E \sup_{t\in[0,T]} \operatorname{Re} \sum_{j>n} \int_0^t \big(g(s,X(s))e_j, X(s) - X_n(s)\big)\, d\beta_j(s) \\ &\quad\le 6E\left[\int_0^T \sum_{j>n} \|g(s,X(s))e_j\|^2\, \|X(s) - X_n(s)\|^2\, ds\right]^{\frac{1}{2}} \\ &\quad\le 36E\int_0^T \sum_{j>n} \|g(s,X(s))e_j\|^2\, ds + \frac{1}{4}E \sup_{t\in[0,T]} \|X(t) - X_n(t)\|^2 \end{aligned}$$

where the same reasoning of a convergent series is valid such that

$$E\int_0^T \sum_{j>n} \|g(s,X(s))e_j\|^2\, ds \to 0 \qquad \text{as } n \to \infty.$$

Finally, the equation given by the stochastic energy equality results in

$$E \sup_{t\in[0,T]} \|X(t) - X_n(t)\|^2 \to 0 \qquad \text{as } n \to \infty.$$

Thus, the sequence of Galerkin approximations $(X_n)_n$ converges to X strongly in $L^2(\Omega; C([0,T]; H))$ and weakly in $L^{2p}(\Omega \times [0,T]; V)$. □

6. Generalizations

1. Instead of homogeneous Neumann boundary conditions, we can also think of homogeneous Dirichlet or periodic boundary conditions. Then all results of this paper for the stochastic nonlinear Schrödinger problem stay the same. Furthermore, during this work, we used Lipschitz-continuity and

growth-boundedness conditions of the diffusion function g. These assumptions can be weakened to local Lipschitz-continuity in $L_2(K, H)$ and growth-boundedness in $L_2(K, H)$ and $L_2(K, V)$:

- for each $L \in \mathbb{N}$ there exists a constant $c_{g,L} > 0$ such that

$$\|g(t, u) - g(t, v)\|^2_{L_2(K,H)} \leq c_{g,L}\|u - v\|^2$$

for all $t \in [0, T]$, all $u, v \in H$ with $\|u\| \leq L$, $\|v\| \leq L$ and a.e. $\omega \in \Omega$;
- there exist constants $c_g, k_g > 0$ such that

$$\|g(t, u)\|^2_{L_2(K,H)} \leq c_g \left(1 + \|u\|^2\right),$$
$$\|g(t, v)\|^2_{L_2(K,V)} \leq k_g \left(1 + \|v\|^2_V\right)$$

for all $t \in [0, T]$, all $u \in H$, all $v \in V$ and a.e. $\omega \in \Omega$.

Similar existence and uniqueness results as in the case of a globally Lipschitz-continuous function g hold: we use the Galerkin method and the truncation to prove the existence of the finite-dimensional equation and then we shift the results to the investigated equation (the steps are similar as in [13, Section 3.2]).

2. More general types of nonlinearities may be considered. In place of the power-term $f(v) = |v|^{2\sigma}v$ for all $v \in V$, we can take $f : V \to H$ defined by $f(v) := F(|v|^2)v$, where $F : [0, \infty) \to [0, \infty)$ is a C^1-function with $F'(x) \geq 0$ for each $x \geq 0$, and there exist $C > 0$ and $\sigma > 1$ such that for each $x_1, x_2 \geq 0$

$$|F(x_1) - F(x_2)| \leq C \left(1 + |x_1|^{\sigma-1} + |x_2|^{\sigma-1}\right) |x_1 - x_2|. \tag{6.1}$$

The case $\sigma = 1$ may also be included by assuming that F is globally Lipschitz-continuous. Assumption (6.1) substitutes the inequality from Lemma 7.2. With the help of Lemma 7.1 and Young's inequality, one can verify the analogues of (2.6) and (2.7)

$$\|f(v)\| \leq C(\sigma) \left(1 + \|v\|^{2\sigma+1}_V\right),$$
$$\|f(u) - f(v)\| \leq C(\sigma) \left(1 + \|u\|^{2\sigma}_V + \|v\|^{2\sigma}_V\right) \|u - v\|$$

for each $u, v \in V$. These inequalities permit to derive similar estimates as (5.3) and (5.11) needed in Theorem 5.2. The result from Lemma 7.3 is replaced by

$$\mathrm{Re}\left\{\left(F(|z_1|^2)z_1 - F(|z_2|^2)z_2\right)(\overline{z}_1 - \overline{z}_2)\right\} \geq 0, \qquad \text{for all } z_1, z_2 \in \mathbb{C},$$

which is proved analogously to Lemma 7.3 while using the fact that F is an increasing and positive function. Furthermore, the inequality from Lemma 7.4 is exchanged by

$$\mathrm{Re}\left\{\left(F(|v|^2)v, Av\right)\right\} \geq 0, \qquad \text{for each } v \in V \text{ such that } Av \in H,$$

which is shown similarly to Lemma 7.4 since F and F' are positive functions.

The case $F(x) = x^\sigma$ with $\sigma \geq 1$ corresponds to $f(v) = |v|^{2\sigma}v$ and (6.1) is replaced by the inequality from Lemma 7.2. Such nonlinearities appear for example in the deterministic papers [16, 21, 22]. We can also take a polynomial of the form $F(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2$ with $\lambda_i \geq 0$ for $i = 0, 1, 2$, which represents a cubic-quintic nonlinearity. In that case our method also works and yields the same results. Without loss of generality, this idea can be transferred to polynomials of finite degree with positive coefficients and also to linear combinations of power-type nonlinearities of the form $F(x) = \lambda_1 x^{\sigma_1} + \lambda_2 x^{\sigma_2}$ with $\lambda_i > 0$, $\sigma_i \geq 1$ for $i = 1, 2$.

3. Due to our approach, we have considered the stochastic nonlinear Schrödinger problem over a bounded one-dimensional domain. Taking $\mathbb{R}$ instead of the interval $(0, 1)$ with $H := L^2(\mathbb{R})$ and $V := W^{1,2}(\mathbb{R})$, we can further ensure the same results as in this paper if we
   - regard the continuity of the embedding $V \hookrightarrow H$ (see [31, p. 1027, (45e)]);
   - use an analogue of Lemma 7.1 in form of
   $$\sup_{x\in\mathbb{R}} |v(x)|^2 \leq c\|v\|_V^2, \qquad \text{for all } v \in V,$$
   with the embedding constant c of $V \hookrightarrow C(\mathbb{R})$ (see [31, p. 1027, (45d)]);
   - replace the operator A in our Schrödinger equation by $\mathcal{A} := A + Q$, where $Q \in L^1_{loc}(\mathbb{R})$ is bounded from below and satisfies $Q(x) \to \infty$ as $|x| \to \infty$, such that $\mathcal{A}$ has a purely discrete spectrum of eigenvalues and a complete set of eigenfunctions (see [24, p. 249, Theorem XIII.67]).
4. Referring to the deterministic equation (1.1), we investigate the unique existence of the variational solution of the stochastic one-dimensional complex Ginzburg–Landau equation
$$\begin{aligned}\big(X(t), v\big) = \big(\varphi, v\big) - (a_1 + ia_2)\int_0^t \big\langle AX(s), v\big\rangle\, ds \\ + (b_1 + ib_2)\int_0^t \big(f(X(s)), v\big)\, ds + c_1\int_0^t \big(X(s), v\big)\, ds \\ + \left(\int_0^t g(s, X(s))\, dW(s), v\right) \qquad (6.2)\end{aligned}$$
for all $t \in [0, T]$, all $v \in V$ and a.e.$\omega \in \Omega$, where we choose $a_1, a_2, b_1, b_2, c_1 \in \mathbb{R}$ and $\sigma > 0$ additionally to the assumptions in Sect. 2. Note that the results of the previous sections correspond to Eq. (6.2) with the coefficients $a_1 = 0, a_2 = 1, b_1 < 0, b_2 = 0, c_1 = 0$ and $\sigma \geq 1$.

   Now, we discuss the case $a_1 > 0, a_2 \in \mathbb{R}, b_1 < 0, b_2 = 0, c_1 \in \mathbb{R}$ and $\sigma \geq 1$ by using the same ideas as in the study of Eq. (2.5). The main result is the analogue of Theorem 5.2 stating that the stochastic one-dimensional complex Ginzburg–Landau equation (6.2) possesses a unique variational solution $X \in L^{2p}(\Omega; C([0, T]; H)) \cap L^{2p}(\Omega \times [0, T]; V)$ for each $p \geq 1$. Moreover, similar results are true by choosing $a_1 > 0, a_2 \in \mathbb{R}, b_1 = 0, b_2 > 0, c_1 \in \mathbb{R}$, $\sigma \in (0, 2)$ and linear multiplicative noise, compare the ansatz in [17]. Then we get a unique variational solution $X \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V)$ which also satisfies $X \in L^2(\Omega; L^\infty([0, T]; V))$. Note that we apply two different methods (in [17] and here) that cannot be mixed.

Appendix

Here, we provide some results used throughout the paper.

Lemma 7.1. [10, Lemma 1.1] *For $v \in V$ it holds that*

$$\sup_{x\in[0,1]} |v(x)|^2 \leq \|v\| \left(\|v\| + 2 \left\| \frac{dv}{dx} \right\| \right) \leq 2\|v\|_V^2.$$

Lemma 7.2. *Let z_1 and z_2 be two complex-valued numbers and $\sigma \geq \frac{1}{2}$. Then the following inequality is fulfilled*

$$\left| |z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right| \leq (4\sigma - 1) \left(|z_1|^{2\sigma} + |z_2|^{2\sigma} \right) |z_1 - z_2|.$$

Proof. Initially, we prove the auxiliary inequality

$$x^s - 1 \leq s(x-1)x^{s-1}, \qquad \text{for all } x \geq 1 \text{ and all } s \geq 1. \tag{7.1}$$

We regard $F : [1, \infty) \to \mathbb{R}$ defined by $F(x) := (s-1)x^s - sx^{s-1} + 1$. So, we have $F(1) = 0$ and $F'(x) = s(s-1)x^{s-2}(x-1) \geq 0$ for all $x \geq 1$ and all $s \geq 1$. Therefore, F is a monotonically increasing real-valued function on $[1, \infty)$ and $F(x) \geq F(1)$ for all $x \geq 1$. Thus, inequality (7.1) is true.

Now, we face the assertion of our lemma and assume that $|z_1| > |z_2|$ and $z_2 \neq 0$ (in the case $|z_1| = |z_2|$ or $|z_2| = 0$ the inequality is obvious). Since

$$\left| |z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right| \leq |z_1|^{2\sigma} |z_1 - z_2| + \left(|z_1|^{2\sigma} - |z_2|^{2\sigma} \right) |z_2|$$

and inequality (7.1) applied for $x = \left| \frac{z_1}{z_2} \right|$ and $s = 2\sigma$ yields

$$\left(|z_1|^{2\sigma} - |z_2|^{2\sigma} \right) \leq 2\sigma \left(|z_1| - |z_2| \right) |z_1|^{2\sigma-1} \leq 2\sigma |z_1 - z_2| |z_1|^{2\sigma-1}$$

for all $\sigma \geq \frac{1}{2}$, we receive with Young's inequality that

$$\begin{aligned} \left| |z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right| &\leq 2\sigma \left(|z_1|^{2\sigma} + |z_1|^{2\sigma-1} |z_2| \right) |z_1 - z_2| \\ &\leq (4\sigma - 1) \left(|z_1|^{2\sigma} + |z_2|^{2\sigma} \right) |z_1 - z_2|. \end{aligned}$$

□

Lemma 7.3. *Let z_1 and z_2 be two complex-valued numbers and $\sigma > 0$, then*

$$\operatorname{Re} \left\{ \left(|z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right) \left(\overline{z_1} - \overline{z_2} \right) \right\} \geq 0.$$

Proof. Let $z_1 = r_1(\cos\alpha_1 + i \sin\alpha_1)$ and $z_2 = r_2(\cos\alpha_2 + i \sin\alpha_2)$, where $r_1, r_2 \geq 0$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$. By using trigonometric formulas and taking into account that the codomain of the cosine function is $[-1, 1]$, we compute

$$\begin{aligned} &\operatorname{Re} \left\{ \left(|z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right) \left(\overline{z_1} - \overline{z_2} \right) \right\} \\ &\quad = r_1^{2\sigma+2} + r_2^{2\sigma+2} - r_1 r_2^{2\sigma+1} \cos(\alpha_1 - \alpha_2) - r_2 r_1^{2\sigma+1} \cos(\alpha_1 - \alpha_2) \\ &\quad \geq r_1^{2\sigma+2} + r_2^{2\sigma+2} - r_1 r_2^{2\sigma+1} - r_2 r_1^{2\sigma+1} = \left(r_1^{2\sigma+1} - r_2^{2\sigma+1} \right) (r_1 - r_2) \geq 0. \end{aligned}$$

□

Lemma 7.4. *Let $v \in V$ such that $Av \in H$ and let $\sigma \geq 1$, then*

$$\operatorname{Re}\left\{\left(|v|^{2\sigma} v, Av\right)\right\} \geq 0.$$

Proof. Observe that

$$\operatorname{Re}\left\{\left(|v|^{2\sigma} v, Av\right)\right\} = \operatorname{Re}\left\{\overline{\left(Av, |v|^{2\sigma} v\right)}\right\} = \operatorname{Re}\left\{\left(Av, |v|^{2\sigma} v\right)\right\}.$$

Moreover, $|v|^{2\sigma} v \in V$ for each $v \in V$ such that the definition of A and the relation $\frac{d}{dx}|v|^2 = \left(\frac{d}{dx}v\right)\overline{v} + v\left(\frac{d}{dx}\overline{v}\right)$ entail

$$\begin{aligned}\left(Av, |v|^{2\sigma} v\right) &= \int_0^1 \left(\frac{d}{dx}v\right)\left(\frac{d}{dx}(|v|^{2\sigma}\overline{v})\right) dx \\ &= \int_0^1 |v|^{2\sigma} \left|\frac{d}{dx}v\right|^2 dx + \sigma \int_0^1 |v|^{2(\sigma-1)}\, \overline{v} \left(\frac{d}{dx}v\right)\left(\frac{d}{dx}|v|^2\right) dx.\end{aligned}$$

Taking the real part, one obtains

$$\operatorname{Re}\left\{\left(Av, |v|^{2\sigma} v\right)\right\} = \int_0^1 |v|^{2\sigma} \left|\frac{d}{dx}v\right|^2 dx + \frac{1}{2}\sigma \int_0^1 |v|^{2(\sigma-1)} \left(\frac{d}{dx}|v|^2\right)^2 dx,$$

which is non-negative. □

Lemma 7.5. *Let $(U_n)_n$ be a bounded sequence in $L^{2p}(\Omega \times [0,T]; V)$ with $p \geq 1$. Then there exist a subsequence $(U_{n'})_{n'}$ and a function $U \in L^{2p}(\Omega \times [0,T]; V)$ such that $(U_{n'})_{n'}$ converges weakly to U in $L^2(\Omega \times [0,T]; H)$, $L^2(\Omega \times [0,T]; V)$ and $L^{2p}(\Omega \times [0,T]; V)$.*

Proof. Note that $L^{2p}(\Omega \times [0,T]; V)$ is a reflexive Banach space (see [7, p. 100, Corollary 2]). Hence, (see [30, p. 258, Proposition 21.23(i)]) there exist a subsequence $(U_{n'})_{n'}$ and a function $U \in L^{2p}(\Omega \times [0,T]; V)$ such that $(U_{n'})_{n'}$ converges weakly to U in $L^{2p}(\Omega \times [0,T]; V)$. Using the continuity of the embeddings

$$L^{2p}(\Omega \times [0,T]; V) \hookrightarrow L^2(\Omega \times [0,T]; V) \hookrightarrow L^2(\Omega \times [0,T]; H),$$

we receive the weak convergences of $(U_{n'})_{n'}$ to U in $L^2(\Omega \times [0,T]; V)$ and $L^2(\Omega \times [0,T]; H)$ as well (see [30, p. 265, Proposition 21.35(c)]). □

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