Existence of nonnegative viscosity solutions for a class of problems involving the ∞ -Laplacian

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Abstract. The issue of existence of nonnegative solutions for a class of problems depending on a real parameter and involving the ∞ -Laplacian is considered. The problem is treated as the limiting case of a family of perturbed eigenvalue problems for the $(p, q(p))$ -Laplacian as $p \rightarrow \infty$. It is shown that nontrivial nonnegative viscosity solutions for this class of problems exist if and only if the parameter is greater than or equal to the reciprocal of the maximum of the distance to the boundary of the domain.

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1. Introduction

The goal of this paper is to analyze the behaviour as $p \rightarrow \infty$ of the sequence of positive solutions of the following perturbed eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta_{q(p)} u = \lambda |u|^{p-2} u & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain, $p \in (N, \infty)$, $q : (1, \infty) \rightarrow (N, \infty)$ is a function satisfying $\lim_{p \rightarrow \infty} \frac{q(p)}{p} = Q \in (0, \infty) \setminus \{1\}$, and where either $q(p) < p$ if $Q \in (0, 1)$ or $q(p) > p$ if $Q \in (1, \infty)$.

We note that in the particular case where $q(p) = p$ the problem (1.1) reduces to the classical eigenvalue problem for the p -Laplacian

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$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega. \end{cases} \quad (1.2)$$

It is well-known that for each real number $p \in (1, \infty)$ the minimum of the Rayleigh quotient associated to problem (1.2), i.e.,

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}, \quad (1.3)$$

stands for the lowest eigenvalue of problem (1.2) whose corresponding eigenfunctions are minimizers of $\lambda_1(p)$ that do not change sign in Ω .

The asymptotic behavior as $p \rightarrow \infty$ of problems (1.2) has been extensively studied in the literature (see Fukagai et al. [7], Juutinen et al. [10], Juutinen and Lindqvist [9]). For instance, in the case when $\lambda = \lambda_1(p)$ and $u_p > 0$ is a solution of (1.2) Juutinen et al. showed in [10] that there exists a subsequence of $\{u_p\}$ which converges uniformly in Ω to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min \{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases} \quad (1.4)$$

where Δ_∞ is the ∞ -Laplace operator, which on sufficiently smooth functions $u : \Omega \rightarrow \mathbb{R}$ is given by $\Delta_\infty u := \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$ and

$$\Lambda_\infty := \frac{1}{\max_{x \in \Omega} \text{dist}(x, \partial\Omega)} \quad (1.5)$$

(see [10, Theorem 1.21] or [7]). As pointed out in [10], Λ_∞ can be characterized variationally as

$$\Lambda_\infty = \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}. \quad (1.6)$$

Further, by [10, Lemma 1.5], we have

$$\lim_{p \rightarrow \infty} (\lambda_1(p))^{1/p} = \Lambda_\infty. \quad (1.7)$$

On the other hand, the characterization of the limit, as $p \rightarrow \infty$, of the positive solutions to

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{q(p)-2} u & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega \end{cases} \quad (1.8)$$

with $\lambda_p > 0$ and $\lim_{p \rightarrow \infty} \frac{q(p)}{p} = Q \in (0, \infty)$ was investigated by Charro and Peral [4] (when $q(p) < p$) and by Charro and Parini [5] (when $q(p) > p$). In particular, it was shown that positive solutions of (1.8) converge, as $p \rightarrow \infty$, to the viscosity solution of

$$\begin{cases} \min \{|\nabla u| - \Lambda u^Q, -\Delta_\infty u\} = 0 & \text{if } x \in \Omega \\ u = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

provided that the limit $\lim_{p \rightarrow \infty} \lambda_p^{1/p} = \Lambda$ exists.

The problem (1.1) considered in this paper can be regarded as a perturbation of the eigenvalue problem (1.2) when the p -Laplacian on the left-hand side of this equation is perturbed by a $q(p)$ -Laplacian with $q(p) \neq p$. Such problems have already been investigated in [2, Section 2] for the case where $p < q(p)$ by using as main argument the Direct Method of the Calculus of Variations applied to the energy functional associated to the equation. Here we also consider the case where $p > q(p)$, for which the energy functional associated to (1.1) is not coercive anymore. We overcome this additional difficulty by making use of a minimization technique on a Nehari manifold. Further, we investigate the asymptotic behavior of positive solutions of the problems (1.1) as $p \rightarrow \infty$. We are able to show that for any $\Lambda \in [\Lambda_\infty, \infty)$ and each sequence $\{\lambda_p\}$, with $\lambda_p \in (\lambda_1(p), \infty)$, such that $\lim_{p \rightarrow \infty} (\lambda_p)^{1/p} = \Lambda$ the sequence of positive weak solutions of (1.1) with $\lambda = \lambda_p$ possesses a subsequence which converges to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min \{ \max\{|\nabla u|, |\nabla u|^Q\} - \Lambda u, -\Delta_\infty u \} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.9)$$

On the other hand, we can show that for all $\Lambda \in (-\infty, \Lambda_\infty)$ there are no nonnegative and nontrivial solutions of problem (1.9). Thus, in comparison to the well-known problem (1.4), the analysis of (1.9) reveals a markedly different situation: while for the original problem a single value of Λ , namely Λ_∞ , given by (1.5), is known for which the corresponding viscosity solution is nonnegative, in the case of problem (1.9) this situation extends to the entire interval $[\Lambda_\infty, \infty)$.

We now recall, for the convenience of the reader, the definition of weak solutions for the family of problems (1.1). For given $p > N$ and $\lambda \in \mathbb{R}$, we say that u_λ is a *weak solution* of (1.1) if $u_\lambda \in W_0^{\max\{p, q(p)\}}(\Omega)$ and

$$\int_{\Omega} (|\nabla u_\lambda|^{p-2} + |\nabla u_\lambda|^{q(p)-2}) \nabla u_\lambda \cdot \nabla v \, dx = \lambda \int_{\Omega} |u_\lambda|^{p-2} u_\lambda v \, dx \quad (1.10)$$

for all $v \in W_0^{\max\{p, q(p)\}}(\Omega)$.

In Sect. 2 of the paper we prove the following result regarding the family of problems (1.1), with $p > N$.

Theorem 1.1. *The set of real numbers λ for which (1.1) possesses nontrivial weak solutions is given by the open interval $(\lambda_1(p), \infty)$. Moreover, for each $\lambda \in (\lambda_1(p), \infty)$ there exists a nontrivial and nonnegative weak solution for problem (1.1).*

Before we state the main result of the paper, we recall the following definition (see e.g., [6]).

Definition 1.2. (i) An upper semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity subsolution of

$$\begin{cases} F(u, \nabla u, D^2 u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.11)$$

if $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then we have $F(\Psi(x_0), \nabla \Psi(x_0), D^2 \Psi(x_0)) \leq 0$;

(ii) A lower semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity supersolution of (1.11) if $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) > \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then $F(\Psi(x_0), \nabla \Psi(x_0), D^2 \Psi(x_0)) \geq 0$;

(iii) A continuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity solution of (1.11) if it is both a viscosity subsolution and a viscosity supersolution of (1.11).

Our main result regarding (1.9) is the following:

Theorem 1.3. *If $\Lambda \in (-\infty, \Lambda_\infty)$ then the problem (1.9) has no nontrivial and nonnegative viscosity solutions, while if $\Lambda \in [\Lambda_\infty, \infty)$ problem (1.9) possesses at least one nontrivial and nonnegative viscosity solution.*

The proof of Theorem 1.3 is presented in Sect. 3 of the paper.

2. Proof of Theorem 1.1

For each $p > N$ define

$$\mu_1(p) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{q(p)} \int_\Omega |\nabla u|^{q(p)} dx}{\frac{1}{p} \int_\Omega |u|^p dx}, \quad (2.1)$$

and

$$\nu_1(p) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^{q(p)} dx}{\int_\Omega |u|^p dx}. \quad (2.2)$$

It was shown in [2, Proposition 1] that if $q(p) > p$ then we have $\mu_1(p) = \nu_1(p) = \lambda_1(p)$, where $\lambda_1(p)$ is given by (1.3). Similar arguments can be used to show that this result still applies in the case where $q(p) < p$. We present these arguments below for the convenience of the reader.

Lemma 2.1. *For each $p > q(p)$ we have $\mu_1(p) = \nu_1(p) = \lambda_1(p)$.*

Proof. We will only prove the fact that $\mu_1(p) = \lambda_1(p)$; the identity $\nu_1(p) = \lambda_1(p)$ can be obtained similarly. To this aim, first note that for each $u \in C_0^\infty(\Omega) \setminus \{0\}$ we have

$$\lambda_1(p) \leq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \leq \frac{\frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{q(p)} \int_\Omega |\nabla u|^{q(p)} dx}{\frac{1}{p} \int_\Omega |u|^p dx}.$$

Passing to the infimum over all $u \in C_0^\infty(\Omega) \setminus \{0\}$ we obtain that $\lambda_1(p) \leq \mu_1(p)$. Next, note that for every $t > 0$ and $u \in C_0^\infty(\Omega) \setminus \{0\}$ we have

$$\begin{aligned} \mu_1(p) &\leq \frac{\frac{1}{q(p)} \int_{\Omega} |\nabla(tu)|^{q(p)} dx + \frac{1}{p} \int_{\Omega} |\nabla(tu)|^p dx}{\frac{1}{p} \int_{\Omega} |tu|^p dx} \\ &= \frac{pt^{q(p)-p} \int_{\Omega} |\nabla u|^{q(p)} dx}{q(p) \int_{\Omega} |u|^p dx} + \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}. \end{aligned}$$

Letting $t \rightarrow \infty$, we find

$$\mu_1(p) \leq \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}, \quad \forall u \in C_0^\infty(\Omega) \setminus \{0\}.$$

Thus, $\mu_1(p) \leq \lambda_1(p)$. Overall, we have shown that $\mu_1(p) = \lambda_1(p)$. $\square$

In the case $q(p) > p$, Theorem 1.1 was proved in [2]. Thus, we only need to show that the conclusion still holds when $q(p) < p$. To this aim, we introduce the energy functional $I_\lambda : W_0^{1, \max\{p, q(p)\}}(\Omega) \rightarrow \mathbb{R}$ associated to the problem (1.1), which is defined by

$$I_\lambda(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q(p)} \int_{\Omega} |\nabla u|^{q(p)} dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx. \quad (2.3)$$

It can be easily seen that $I_\lambda \in C^1(W_0^{1, \max\{p, q(p)\}}(\Omega), \mathbb{R})$, with the Gâteaux derivative given by

$$\langle I'_\lambda(u), \varphi \rangle = \int_{\Omega} (|\nabla u|^{p-2} + |\nabla u|^{q(p)-2}) \nabla u \nabla \varphi dx - \lambda \int_{\Omega} |u|^{p-2} u \varphi dx,$$

for all $u, \varphi \in W_0^{1, \max\{p, q(p)\}}(\Omega)$. The strategy of proof in [2, Theorem 1] for the case where $q(p) > p$ uses the fact that one can apply the direct method of the calculus of variations in order to show that for each $\lambda \in (\lambda_1(p), \infty)$ the functional I_λ possesses a nontrivial nonnegative minimum point (and, consequently, a critical point) $u_\lambda \in W_0^{1, \max\{p, q(p)\}}(\Omega)$ such that $I_\lambda(u_\lambda) < 0$. Unfortunately, when $q(p) < p$, which is our focus in this proof, the Direct Method cannot be used anymore since the functional I_λ is no longer coercive. To overcome this difficulty we will use a minimization technique on a Nehari manifold (see [1, Section 2.3.3] or [13] for a description of the method).

Lemma 2.2. *If $q(p) < p$, then for each $\lambda \in (-\infty, \lambda_1(p)]$ problem (1.1) has no nontrivial solution.*

Proof. If $\lambda \leq 0$ the conclusion of the lemma is obvious. Assume, by contradiction, that there exists $\lambda \in (0, \lambda_1(p))$ for which problem (1.1) possesses a

solution $u_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$. Thus, testing with $v = u_\lambda$ in (1.10), and using Poincaré's inequality, we obtain

$$\begin{aligned} 0 < \frac{\lambda_1(p) - \lambda}{p} \int_{\Omega} |u_\lambda|^p dx &\leq \frac{1}{p} \int_{\Omega} |\nabla u_\lambda|^p dx - \frac{\lambda}{p} \int_{\Omega} |u_\lambda|^p dx \\ &\leq \frac{1}{p} \int_{\Omega} |\nabla u_\lambda|^p dx + \frac{1}{p} \int_{\Omega} |\nabla u_\lambda|^{q(p)} dx \\ &\quad - \frac{\lambda}{p} \int_{\Omega} |u_\lambda|^p dx = 0, \end{aligned}$$

which is a contradiction. Hence, for all $\lambda \in (0, \lambda_1(p))$ problem (1.1) has no nonzero solution.

To complete the proof of the lemma we will show that for $\lambda = \lambda_1(p)$ problem (1.1) possesses no nontrivial solution. Indeed, if we assume that there exists $u_{\lambda_1(p)} \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that (1.10) holds with $\lambda = \lambda_1(p)$, then letting $v = u_{\lambda_1(p)}$ in (1.10) gives

$$\int_{\Omega} |\nabla u_{\lambda_1(p)}|^p dx + \int_{\Omega} |\nabla u_{\lambda_1(p)}|^{q(p)} dx = \lambda_1(p) \int_{\Omega} |u_{\lambda_1(p)}|^p dx.$$

On the other hand, (1.3) gives $\lambda_1(p) \int_{\Omega} |u_{\lambda_1(p)}|^p dx \leq \int_{\Omega} |\nabla u_{\lambda_1(p)}|^p dx$. It follows that $\int_{\Omega} |\nabla u_{\lambda_1(p)}|^{q(p)} dx = 0$. This, combined with the fact that $u_{\lambda_1(p)} \in W_0^{1,p}(\Omega) \subset W_0^{1,q(p)}(\Omega)$ implies $\int_{\Omega} |u_{\lambda_1(p)}|^{q(p)} dx = 0$, and hence $u_{\lambda_1(p)} \equiv 0$, which is a contradiction with $u_{\lambda_1(p)} \in W_0^{1,p}(\Omega) \setminus \{0\}$. This concludes the proof. $\square$

Lemma 2.3. *If $q(p) < p$ then for all $\lambda \in (\lambda_1(p), \infty)$ problem (1.1) has a nontrivial solution.*

Proof. Fix $\lambda > \lambda_1(p)$ and define $I_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by (2.3). We consider the Nehari manifold

$$\mathcal{N}_\lambda := \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla u|^{q(p)} dx = \lambda \int_{\Omega} |u|^p dx \right\}.$$

Since $\lambda > \lambda_1(p)$ we deduce, in view of (1.3), that there exists $v_\lambda \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla v_\lambda|^p dx < \lambda \int_{\Omega} |v_\lambda|^p dx.$$

If we now let

$$t := \left(\frac{\lambda \int_{\Omega} |v_\lambda|^p dx - \int_{\Omega} |\nabla v_\lambda|^p dx}{\int_{\Omega} |\nabla v_\lambda|^{q(p)} dx} \right)^{1/(q(p)-p)} > 0,$$

then

$$t^p \int_{\Omega} |\nabla v_{\lambda}|^p dx + t^{q(p)} \int_{\Omega} |\nabla v_{\lambda}|^{q(p)} dx = \lambda t^p \int_{\Omega} |v_{\lambda}|^p dx,$$

that is, $tv_{\lambda} \in \mathcal{N}_{\lambda}$. This shows that $\mathcal{N}_{\lambda} \neq \emptyset$.

In what follows we prove several properties of the functional I_{λ} which will be used in the sequel. First, we claim that I_{λ} is coercive on $\mathcal{N}_{\lambda}$, i.e., $I_{\lambda}(u_k) \rightarrow \infty$ as $k \rightarrow \infty$ whenever $\{u_k\}$ is a sequence in $\mathcal{N}_{\lambda}$ such that $\lim_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^p dx = \infty$. Indeed, let us consider such a sequence, and note that if $u \in \mathcal{N}_{\lambda}$ then

$$\begin{aligned} I_{\lambda}(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q(p)} \int_{\Omega} |\nabla u|^{q(p)} dx \\ &\quad - \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{1}{p} \int_{\Omega} |\nabla u|^{q(p)} dx \\ &= \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u|^{q(p)} dx. \end{aligned}$$

Thus, for each $k \in \mathbb{N}$, we have $I_{\lambda}(u_k) = \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_k|^{q(p)} dx$. On the other hand, since $u_k \in \mathcal{N}_{\lambda}$, we have

$$\lambda \int_{\Omega} |u_k|^p dx = \int_{\Omega} |\nabla u_k|^p dx + \int_{\Omega} |\nabla u_k|^{q(p)} dx \geq \int_{\Omega} |\nabla u_k|^p dx \quad \forall k \in \mathbb{N}.$$

Thus, since $\int_{\Omega} |\nabla u_k|^p dx \rightarrow \infty$ as $k \rightarrow \infty$, we must also have $\int_{\Omega} |u_k|^p dx \rightarrow \infty$ as $k \rightarrow \infty$. Now, in view of the fact that $p > q(p) > N$, the space $W^{1,q(p)}(\Omega)$ is continuously embedded in $L^p(\Omega)$. Consequently, there exists a constant $D_p > 0$ such that

$$\|w\|_{L^p(\Omega)} \leq D_p \|\nabla w\|_{L^{q(p)}(\Omega)} \quad \forall w \in W_0^{1,q(p)}(\Omega) \supset W_0^{1,p}(\Omega). \quad (2.4)$$

We deduce that for each $k \in \mathbb{N}$ we have

$$\left(\frac{1}{D_p} \right)^{q(p)} \left(\int_{\Omega} |u_k|^p dx \right)^{q(p)/p} \leq \int_{\Omega} |\nabla u_k|^{q(p)} dx.$$

Consequently, $\int_{\Omega} |\nabla u_k|^{q(p)} dx \rightarrow \infty$ as $k \rightarrow \infty$, and hence $I_{\lambda}(u_k) \rightarrow \infty$ as $k \rightarrow \infty$, as claimed.

Next, we will show that $m := \inf_{w \in \mathcal{N}_{\lambda}} I_{\lambda}(w) > 0$. To see why this is the case, note that in view of (2.4) we have, for each $u \in \mathcal{N}_{\lambda}$,

$$\begin{aligned} \int_{\Omega} |\nabla u|^{q(p)} dx &\leq \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla u|^{q(p)} dx \\ &= \lambda \int_{\Omega} |u|^p dx \leq \lambda D_p^p \left(\int_{\Omega} |\nabla u|^{q(p)} dx \right)^{p/q(p)}, \end{aligned}$$

which gives

$$\left(\frac{1}{\lambda D_p^p} \right)^{q(p)/(p-q(p))} \leq \int_{\Omega} |\nabla u|^{q(p)} dx \quad \forall u \in \mathcal{N}_{\lambda}. \quad (2.5)$$

Since

$$I_\lambda(u) = \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_\Omega |\nabla u|^{q(p)} dx \quad \forall u \in \mathcal{N}_\lambda,$$

we deduce that $m > 0$.

Finally, we claim that there exists $u \in \mathcal{N}_\lambda$, $u \geq 0$ a.e. in Ω , such that $I_\lambda(u) = m$. We begin by considering $\{u_k\} \subset \mathcal{N}_\lambda$ a minimizing sequence, that is, such that $I_\lambda(u_k) \rightarrow m$ as $k \rightarrow \infty$. Note that $u_k \in \mathcal{N}_\lambda$ implies that we also have $|u_k| \in \mathcal{N}_\lambda$ and, moreover, $I_\lambda(u_k) = I_\lambda(|u_k|)$. Thus, $\{|u_k|\}$ is a minimizing sequence for m as well. Consequently, we may assume that $u_k(x) \geq 0$ for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since I_λ is coercive on $\mathcal{N}_\lambda$ we deduce that the sequence $\{u_k\}$ must be bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $u \in W_0^{1,p}(\Omega)$ such that, after eventually extracting a subsequence (not relabeled), $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $W_0^{1,q(p)}(\Omega)$, $u_k \rightarrow u$ strongly in $L^p(\Omega)$, and $u_k(x) \rightarrow u(x)$ for a.e. $x \in \Omega$ as $k \rightarrow \infty$. In particular, we deduce that $u(x) \geq 0$ for a.e. $x \in \Omega$. Standard lower semicontinuity arguments now imply that

$$I_\lambda(u) \leq \liminf_{k \rightarrow \infty} I_\lambda(u_k) = m. \quad (2.6)$$

Since $u_k \in \mathcal{N}_\lambda$ for all $k \in \mathbb{N}$, we have

$$\int_\Omega |\nabla u_k|^p dx \leq \int_\Omega |\nabla u_k|^p dx + \int_\Omega |\nabla u_k|^{q(p)} dx = \lambda \int_\Omega |u_k|^p dx \quad \forall k \in \mathbb{N}. \quad (2.7)$$

We claim that $u \not\equiv 0$. Indeed, if $u \equiv 0$ in Ω then, since $u_k \rightarrow u$ in $L^p(\Omega)$, we would have $\int_\Omega |u_k|^p dx \rightarrow 0$ as $k \rightarrow \infty$, but this is impossible since by (2.5) we have

$$0 < \left(\frac{1}{\lambda D_p^p} \right)^{q(p)/(p-q(p))} \leq \int_\Omega |\nabla u_k|^{q(p)} dx \quad \forall k \in \mathbb{N}.$$

Now, letting $k \rightarrow \infty$ in (2.7) we deduce by the weak lower semicontinuity of the norms in $W_0^{1,p}(\Omega)$ and $W_0^{1,q(p)}(\Omega)$ that

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^{q(p)} dx \leq \lambda \int_\Omega |u|^p dx.$$

We claim that the above inequality is, in fact, an equality. Indeed, if we assume that

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla u|^{q(p)} dx < \lambda \int_\Omega |u|^p dx \quad (2.8)$$

then, with

$$t := \left(\frac{\lambda \int_\Omega |u|^p dx - \int_\Omega |\nabla u|^p dx}{\int_\Omega |\nabla u|^{q(p)} dx} \right)^{1/(q(p)-p)},$$

we have $tu \in \mathcal{N}_\lambda$. Also, note that $t \in (0, 1)$, since $p > q(p)$ and by (2.8),

$$\frac{\lambda \int_{\Omega} |u|^p dx - \int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |\nabla u|^{q(p)} dx} > 1.$$

Hence,

$$\begin{aligned} 0 < m \leq I_\lambda(tu) &= \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla(tu)|^{q(p)} dx \\ &= t^{q(p)} \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u|^{q(p)} dx = t^{q(p)} I_\lambda(u) \\ &\leq t^{q(p)} \liminf_{k \rightarrow \infty} I_\lambda(u_k) = t^{q(p)} m < m, \end{aligned}$$

which is a contradiction. Thus, we must have $\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla u|^{q(p)} dx = \lambda \int_{\Omega} |u|^p dx$, which means that $u \in \mathcal{N}_\lambda$. In view of (2.6), this implies that $I_\lambda(u) = m$.

We are now ready to complete the proof of Lemma 2.3. Let $u \in \mathcal{N}_\lambda$ be such that $I_\lambda(u) = m$. We have

$$\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla u|^{q(p)} dx = \lambda \int_{\Omega} |u|^p dx,$$

and the fact that $u \not\equiv 0$ implies $\int_{\Omega} |\nabla u|^p dx < \lambda \int_{\Omega} |u|^p dx$. Let $v \in W_0^{1,p}(\Omega)$ be arbitrary but fixed, and let $\delta > 0$ be sufficiently small so that for each $s \in (-\delta, \delta)$ the function $u + sv$ does not vanish everywhere in Ω and

$$\lambda \int_{\Omega} |u + sv|^p dx > \int_{\Omega} |\nabla(u + sv)|^p dx.$$

Note that the latter follows from the fact that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(s) := \lambda \int_{\Omega} |u + sv|^p dx - \int_{\Omega} |\nabla(u + sv)|^p dx$$

is continuously differentiable and $f(0) > 0$. For $s \in \mathbb{R}$, define

$$t(s) := \left(\frac{f(s)}{\int_{\Omega} |\nabla(u + sv)|^{q(p)} dx} \right)^{1/(q(p)-p)} > 0.$$

Note that the map $s \mapsto t(s)$ is differentiable, $t(s) \cdot (u + sv) \in \mathcal{N}_\lambda \quad \forall s \in \mathbb{R}$, and since $u \in \mathcal{N}_\lambda$ we have $t(0) = 1$.

Next, define $\gamma : (-\delta, \delta) \rightarrow \mathbb{R}$ by $\gamma(s) := I_\lambda(t(s)(u + sv))$. By construction, we have $\gamma \in C^1(-\delta, \delta)$ and $\gamma(0) \leq \gamma(s)$ for all $s \in (-\delta, \delta)$. Thus, we deduce that

$0 = \gamma'(0) = \langle I'_\lambda(t(0)u), t'(0)u + t(0)v \rangle = t'(0)\langle I'_\lambda(u), u \rangle + \langle I'_\lambda(u), v \rangle = \langle I'_\lambda(u), v \rangle$. Since $v \in W_0^{1,p}(\Omega)$ was arbitrary, the proof of Lemma 2.3 is complete. $\square$

3. Proof of Theorem 1.3

3.1. Existence

Let $\Lambda \in (\Lambda_\infty, \infty)$ be fixed. Since $\Lambda_\infty = \lim_{p \rightarrow \infty} (\lambda_1(p))^{1/p}$, it follows that there exists a sequence $\{\lambda_p\}$ with $\lambda_p \in (\lambda_1(p), \infty)$ for all $p \in \mathbb{N}$ and such that $\lim_{p \rightarrow \infty} (\lambda_p)^{1/p} = \Lambda$. For each $p > N$ consider the problem (1.1) with $\lambda = \lambda_p$, and either $N < p < q(p)$ if $Q \in (1, \infty)$ or $N < q(p) < p$ if $Q \in (0, 1)$. Then, by Theorem 1.1, for each $\lambda = \lambda_p$ problem (1.1) possesses a solution $u_p \in W_0^{1, \max\{p, q(p)\}}(\Omega) \setminus \{0\}$, where u_p can be chosen such that $u_p(x) \geq 0$ for a.e. $x \in \Omega$. In fact, using an argument based on Harnack's inequality à la Lieberman (see [11, Theorem 1.3]) similar with those used in [3, pp. 5–6] we can show that $u_p > 0$ in Ω .

Lemma 3.1. *There exists a subsequence of $\{u_p\}$ which converges uniformly in Ω , as $p \rightarrow \infty$, to some function $u_\Lambda \in C(\bar{\Omega}) \setminus \{0\}$, with $u_\Lambda(x) \geq 0$ for a.e. $x \in \Omega$.*

Proof. We begin with the case where $Q \in (0, 1)$. Assume $N < q(p) < p$. By Lemma 2.3 we have that

$$I_{\lambda_p}(u_p) = \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_p|^{q(p)} dx.$$

For each $p > N$, let $\Psi_p \in W_0^{1,p}(\Omega)$ be an eigenfunction for the p -Laplacian with homogeneous Dirichlet boundary conditions corresponding to the first eigenvalue, $\lambda_1(p)$. Hence, $\Psi_p > 0$ in Ω , $\int_{\Omega} |\nabla \Psi_p|^p dx = \lambda_1(p)$, and $\int_{\Omega} \Psi_p^p dx = 1$. Let

$$t_p := \left(\frac{\lambda_p \int_{\Omega} |\Psi_p|^p dx - \int_{\Omega} |\nabla \Psi_p|^p dx}{\int_{\Omega} |\nabla \Psi_p|^{q(p)} dx} \right)^{\frac{1}{q(p)-p}} = \left(\frac{\int_{\Omega} |\nabla \Psi_p|^{q(p)} dx}{\lambda_p - \lambda_1(p)} \right)^{\frac{1}{p-q(p)}}.$$

Since $t_p \Psi_p \in \mathcal{N}_\lambda$ we have $I_{\lambda_p}(u_p) \leq I_{\lambda_p}(t_p \Psi_p)$. We deduce that

$$\begin{aligned} \int_{\Omega} |\nabla u_p|^{q(p)} dx &\leq t_p^{q(p)} \int_{\Omega} |\nabla \Psi_p|^{q(p)} dx \\ &= \left(\frac{1}{\lambda_p - \lambda_1(p)} \right)^{\frac{q(p)}{p-q(p)}} \left(\int_{\Omega} |\nabla \Psi_p|^{q(p)} dx \right)^{\frac{p}{p-q(p)}} \\ &\leq \left(\frac{1}{\lambda_p - \lambda_1(p)} \right)^{\frac{q(p)}{p-q(p)}} \left[\left(\int_{\Omega} |\nabla \Psi_p|^p dx \right)^{\frac{q(p)}{p}} |\Omega|^{\frac{p-q(p)}{p}} \right]^{\frac{p}{p-q(p)}} \\ &= \left(\frac{\lambda_1(p)}{\lambda_p - \lambda_1(p)} \right)^{\frac{q(p)}{p-q(p)}} |\Omega| \end{aligned}$$

for all $p > N$, where we have used Hölder's inequality.

Next, let us fix $m > N$ a positive integer. For any real number $p > q(p) > m$, the above estimates and Hölder's inequality imply that

$$\begin{aligned}
 \int_{\Omega} |\nabla u_p|^m dx &\leq \left(\int_{\Omega} |\nabla u_p|^{q(p)} dx \right)^{m/q(p)} |\Omega|^{(q(p)-m)/q(p)} \\
 &\leq \left(\int_{\Omega} |\nabla u_p|^{q(p)} dx \right)^{m/q(p)} (1 + |\Omega|) \\
 &\leq \left(\frac{\lambda_1(p)}{\lambda_p - \lambda_1(p)} \right)^{m/(p-q(p))} |\Omega|^{m/q(p)} (1 + |\Omega|) \\
 &\leq \left[\left(\frac{1}{\frac{\lambda_p}{\lambda_1(p)} - 1} \right)^{1/p} \right]^{m/[1-q(p)/p]} (1 + |\Omega|)^2.
 \end{aligned} \tag{3.1}$$

Since $\lim_{p \rightarrow \infty} (\lambda_p)^{1/p} = \Lambda$ and $\lim_{p \rightarrow \infty} (\lambda_1(p))^{1/p} = \Lambda_{\infty} < \Lambda$, we deduce that there exists ε such that $\Lambda_{\infty} + \varepsilon < \Lambda - \varepsilon$ and, for $p \in \mathbb{N}$ sufficiently large, $\Lambda - \varepsilon < (\lambda_p)^{1/p}$ and $(\lambda_1(p))^{1/p} < \Lambda_{\infty} + \varepsilon$. Therefore, for sufficiently large $p \in \mathbb{N}$, we have

$$\frac{\lambda_p}{\lambda_1(p)} \geq \left(\frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} \right)^p > 1,$$

and since $\left(\frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} \right)^p \geq \frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} > 1$, we obtain

$$\frac{1}{\frac{\lambda_p}{\lambda_1(p)} - 1} \leq \frac{1}{\left(\frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} \right)^p - 1} \leq \frac{1}{\frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} - 1}.$$

We deduce that for all $p \in \mathbb{N}$ sufficiently large we have

$$\int_{\Omega} |\nabla u_p|^m \leq \left[\left(\frac{1}{\frac{\Lambda - \varepsilon}{\Lambda_{\infty} + \varepsilon} - 1} \right)^{1/p} \right]^{m/[1-q(p)/p]} (1 + |\Omega|)^2 \leq c(m),$$

where $c(m)$ is a constant that depends only on m . Thus, $\{|\nabla u_p|\}$ is bounded in $L^m(\Omega)$, and hence $\{u_p\}$ is bounded in $W_0^{1,m}(\Omega)$. Since we have assumed that $m > N$, the embedding of $W_0^{1,m}(\Omega)$ into $C(\overline{\Omega})$ is compact. Taking into account the reflexivity of the space $W_0^{1,m}(\Omega)$ it follows that there exists a subsequence (not relabeled) of $\{u_p\}$ and a function $u_{\Lambda} \in C(\overline{\Omega})$ such that $u_p \rightharpoonup u_{\Lambda}$ weakly in $W_0^{1,m}(\Omega)$ and $u_p \rightarrow u_{\Lambda}$ uniformly in Ω .

It remains to show that $u_{\Lambda} \not\equiv 0$. To this aim, we first recall a result that can be found in [5, Proposition 2] (see also [4, Proposition 3.1]): if $p \in (N, \infty)$, then letting

$$C_p := p|B_1(0)|^{-1/p} N^{-N(p+1)/p^2} (p-1)^{N(p-1)/p^2} (p-N)^{N/p^2-1} \lambda_1(p)^{(N-p)/p^2},$$

we have

$$\|w\|_{L^{\infty}(\Omega)} \leq C_p \|\nabla w\|_{L^p(\Omega)} \quad \forall w \in W_0^{1,p}(\Omega). \tag{3.2}$$

Furthermore,

$$\lim_{p \rightarrow \infty} C_p = \frac{1}{\Lambda_\infty}. \quad (3.3)$$

In particular, we deduce by (3.2) that for each $p > q(p) > N$ we have

$$\|u_p\|_{L^\infty(\Omega)}^{q(p)} \leq C_{q(p)}^{q(p)} \int_{\Omega} |\nabla u_p|^{q(p)} dx. \quad (3.4)$$

On the other hand,

$$\begin{aligned} \int_{\Omega} |\nabla u_p|^{q(p)} dx &\leq \int_{\Omega} |\nabla u_p|^{q(p)} dx + \int_{\Omega} |\nabla u_p|^p dx \\ &= \lambda_p \int_{\Omega} |u_p|^p dx \leq \lambda_p \|u_p\|_{L^\infty(\Omega)}^p |\Omega|. \end{aligned}$$

This inequality, combined with (3.4), implies that for large enough $p \in \mathbb{N}$ we have

$$\left(\frac{1}{C_{q(p)}^{q(p)} \lambda_p |\Omega|} \right)^{1/(p-q(p))} \leq \|u_p\|_{L^\infty(\Omega)}. \quad (3.5)$$

Since $\lim_{p \rightarrow \infty} q(p)/p = Q \in (0, 1)$, $\lim_{p \rightarrow \infty} (\lambda_p)^{1/p} = \Lambda$, and $\lim_{p \rightarrow \infty} C_{q(p)} = 1/\Lambda_\infty$ we deduce

$$\begin{aligned} \lim_{p \rightarrow \infty} \left(\frac{1}{C_{q(p)}^{q(p)} \lambda_p |\Omega|} \right)^{1/(p-q(p))} &= \lim_{p \rightarrow \infty} \left(\frac{1}{C_{q(p)}^{q(p)/p} \lambda_p^{1/p} |\Omega|^{1/p}} \right)^{1/(1-q(p)/p)} \\ &= \left(\frac{\Lambda_\infty^Q}{\Lambda} \right)^{1/(1-Q)}. \end{aligned}$$

Thus, letting $p \rightarrow \infty$ in (3.5),

$$0 < \left(\frac{\Lambda_\infty^Q}{\Lambda} \right)^{1/(1-Q)} \leq \limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\Omega)} = \|u_\Lambda\|_{L^\infty(\Omega)}.$$

It follows that $u_\Lambda \not\equiv 0$.

Next, we consider the case where $Q \in (1, \infty)$, and we assume that $N < p < q(p)$. Recall that $u_p \in W_0^{1,q(p)}(\Omega) \subset W_0^{1,p}(\Omega)$ is the global minimum of I_{λ_p} over $W_0^{1,q(p)}(\Omega)$, and that $\langle I'_{\lambda_p}(u_p), u_p \rangle = 0$. Thus, we have

$$\int_{\Omega} |\nabla u_p|^p dx + \int_{\Omega} |\nabla u_p|^{q(p)} dx = \lambda_p \int_{\Omega} u_p^p dx$$

and

$$\begin{aligned} I_{\lambda_p}(u_p) &= \frac{1}{p} \int_{\Omega} |\nabla u_p|^p dx + \frac{1}{q(p)} \int_{\Omega} |\nabla u_p|^{q(p)} dx - \frac{\lambda_p}{p} \int_{\Omega} u_p^p dx \\ &= \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_p|^{q(p)} dx < 0. \end{aligned}$$

The above estimates, combined with Hölder's and Poincaré's inequalities, imply

$$\begin{aligned} \int_{\Omega} |\nabla u_p|^{q(p)} dx &\leq \lambda_p \int_{\Omega} u_p^p dx \leq \frac{\lambda_p}{\lambda_1(p)} \int_{\Omega} |\nabla u_p|^p dx \\ &\leq \frac{\lambda_p}{\lambda_1(p)} \left(\int_{\Omega} |\nabla u_p|^{q(p)} dx \right)^{p/q(p)} |\Omega|^{(q(p)-p)/q(p)}. \end{aligned}$$

Consequently,

$$\left(\int_{\Omega} |\nabla u_p|^{q(p)} dx \right)^{1/q(p)} \leq \left[\left(\frac{\lambda_p}{\lambda_1(p)} \right)^{1/p} (1 + |\Omega|)^{1/p} \right]^{\frac{1}{\frac{q(p)}{p} - 1}}.$$

Since the right hand side in the above inequality converges to $\left(\frac{\Lambda}{\Lambda_{\infty}}\right)^{1/(Q-1)}$ as $p \rightarrow \infty$, we deduce that $\{\|u_p\|_{W_0^{1,q(p)}(\Omega)}\}$ is bounded. Fix $m > N$ a positive integer. For each $p > m$ we have $q(p) > m$, and since $W_0^{1,q(p)}(\Omega) \subset W_0^{1,m}(\Omega)$ we can use Hölder's inequality to deduce that $\{|\nabla u_p|\}$ is bounded in $L^m(\Omega)$. It follows that $\{u_p\}$ is bounded in $W_0^{1,m}(\Omega)$. Taking into account the reflexivity of the space $W_0^{1,m}(\Omega)$, and using the fact that the embedding of $W_0^{1,m}(\Omega)$ into $C(\overline{\Omega})$ is compact, we deduce that there exists a subsequence (not relabeled) of $\{u_p\}$ and a function $u_{\Lambda} \in C(\overline{\Omega})$ such that $u_p \rightharpoonup u_{\Lambda}$ weakly in $W_0^{1,m}(\Omega)$ and $u_p \rightarrow u_{\Lambda}$ uniformly in Ω .

As in the previous case, we need to show that $u_{\Lambda} \not\equiv 0$. Let δ be the distance function to the boundary of Ω . We have $\delta \in W_0^{1,q(p)}(\Omega)$ and since $I_{\lambda_p}(u_p) = \min_{w \in W_0^{1,q(p)}(\Omega)} I_{\lambda_p}(w)$ we deduce that

$$I_{\lambda_p}(u_p) = \left(\frac{1}{q(p)} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_p|^{q(p)} dx \leq I_{\lambda_p}(\delta) = \frac{|\Omega|}{p} + \frac{|\Omega|}{q(p)} - \frac{\lambda_p}{p} \int_{\Omega} \delta(x)^p dx$$

or, equivalently,

$$\lambda_p \int_{\Omega} \delta(x)^p dx \leq \left(1 - \frac{p}{q(p)} \right) \int_{\Omega} |\nabla u_p|^{q(p)} dx + |\Omega| + \frac{|\Omega|p}{q(p)}.$$

This inequality and the fact that $\int_{\Omega} |\nabla u_p|^{q(p)} dx \leq \lambda_p \int_{\Omega} u_p^p dx$ yield

$$\begin{aligned} \lambda_p \int_{\Omega} \delta(x)^p dx &\leq \left(1 - \frac{p}{q(p)} \right) \lambda_p \int_{\Omega} u_p^p dx + |\Omega| + \frac{|\Omega|p}{q(p)} \\ &\leq \left(1 - \frac{p}{q(p)} \right) \lambda_p |\Omega| \|u_p\|_{L^{\infty}(\Omega)}^p + |\Omega| + \frac{|\Omega|p}{q(p)}. \end{aligned} \quad (3.6)$$

Since $\lim_{p \rightarrow \infty} (\lambda_p)^{1/p} = \Lambda > \Lambda_{\infty} = \frac{1}{\|\delta\|_{L^{\infty}(\Omega)}}$, $\lim_{p \rightarrow \infty} \left(\int_{\Omega} \delta(x)^p dx \right)^{1/p} = \|\delta\|_{L^{\infty}(\Omega)}$, we deduce $\lim_{p \rightarrow \infty} \left(\lambda_p \int_{\Omega} \delta(x)^p dx \right)^{1/p} = \Lambda \|\delta\|_{L^{\infty}(\Omega)} > 1$. Hence,

we must have $\lim_{p \rightarrow \infty} \lambda_p \int_{\Omega} \delta(x)^p dx = \infty$. It follows that

$$|\Omega| + \frac{|\Omega|p}{q(p)} \leq \frac{\lambda_p}{2} \int_{\Omega} \delta(x)^p dx \quad (3.7)$$

for all $p \in \mathbb{N}$ sufficiently large. In view of (3.6) and (3.7) we obtain that

$$\left(\frac{1}{2} \int_{\Omega} \delta(x)^p dx \right)^{1/p} \leq \left(1 - \frac{p}{q(p)} \right)^{1/p} |\Omega|^{1/p} \|u_p\|_{L^\infty(\Omega)}.$$

Passing to the limit as $p \rightarrow \infty$ we find $0 < \|\delta\|_{L^\infty(\Omega)} \leq \|u_\Lambda\|_{L^\infty(\Omega)}$, which shows that $u_\Lambda \not\equiv 0$. $\square$

Remark 3.2. Lemma 3.1 remains true if $\Lambda = \Lambda_\infty$ and $Q \in (0, 1)$. Indeed, taking $\lambda_p := 2\lambda_1(p)$ in the first part of the proof, the estimate (3.1) now becomes $\int_{\Omega} |\nabla u_p|^m dx \leq (1 + |\Omega|)^2$. Thus, the norm of ∇u_p in $L^m(\Omega)$ is still bounded by a constant $c(m)$ that only depends on m . The rest of the proof can then be carried out using essentially the same arguments.

The proof of the following result is standard in the literature (see e.g., [10], [12]) and consequently we omit it here.

Lemma 3.3. *Let $p > N$. If $\lambda \in (\lambda_1(p), \infty)$ and u_p is a weak solution of the problem (1.1) then u_p is a viscosity solution of (1.1).*

We are now ready to continue the proof of the existence part in Theorem 1.3. We will show that the limit function u_Λ given by Lemma 3.1 is a viscosity solution for the problem (1.9). We begin by showing that u_Λ is a viscosity supersolution. To this aim, let $x_0 \in \Omega$ and $\Psi \in C^2(\bar{\Omega})$ be such that $u_\Lambda(x_0) = \Psi(x_0)$ and $u_\Lambda - \Psi$ has a strict minimum at x_0 . The uniform convergence of u_p to u_Λ implies that there exists a sequence $\{x_p\} \subset \Omega$ such that $x_p \rightarrow x_0$ and $u_p - \Psi$ has a minimum at x_p . Indeed, we have $u_\Lambda(x_0) = \Psi(x_0)$ and $u_\Lambda(x) > \Psi(x)$ for all $x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and verifying $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ we have $\inf\{u_\Lambda - \Psi \mid \text{in } B(x_0, R) \setminus B(x_0, r)\} > 0$. Since $u_p \rightarrow u_\Lambda$ uniformly in $\bar{B}(x_0, R)$, for p sufficiently large, the function $u_p - \Psi$ attains its minimum value in $B(x_0, r)$. Let us denote by $x_p \in B(x_0, r)$ the corresponding minimum point. Letting $r \rightarrow 0^+$ we get a subsequence $\{x_{p_r}\}$ such that $x_{p_r} \rightarrow x_0$ as $p_r \rightarrow \infty$. To simplify the notation we relabel the subindex p_r by p in what follows. Since u_p is a continuous viscosity solution of (1.1) (by Lemma 3.3), we have

$$\begin{aligned} & -(|\nabla \Psi(x_p)|^{p-2} + |\nabla \Psi(x_p)|^{q(p)-2}) \Delta \Psi(x_p) - [(p-2)|\nabla \Psi(x_p)|^{p-4} \\ & + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}] \Delta_\infty \Psi(x_p) \geq \lambda_p |\Psi(x_p)|^{p-2} \Psi(x_p). \end{aligned} \quad (3.8)$$

Let us first assume that $u_\Lambda(x_0) > 0$. We claim that $\Psi(x_p) \geq u_p(x_p) - \varepsilon > 0$ for all $p \in \mathbb{N}$ sufficiently large provided that $u_\Lambda(x_0) > 3\varepsilon > 0$. Indeed, let us fix $\varepsilon > 0$ such that $u_\Lambda(x_0) - 3\varepsilon > 0$. It is easy to see that for $p \in \mathbb{N}$ sufficiently large we have

$$\varepsilon > u_p(x_0) - u_\Lambda(x_0) = u_p(x_0) - \Psi(x_0) \geq u_p(x_p) - \Psi(x_p),$$

and since u_p converges uniformly to u_Λ on $B(x_0, r)$ we have $|u_p(x_p) - u_\Lambda(x_p)| < \varepsilon$ and $|u_\Lambda(x_p) - u_\Lambda(x_0)| < \varepsilon$. Thus, for all $p \in \mathbb{N}$ sufficiently large,

$$\Psi(x_p) \geq u_p(x_p) - \varepsilon \geq u_\Lambda(x_p) - 2\varepsilon \geq u_\Lambda(x_0) - 3\varepsilon > 0.$$

In view of (3.8) we deduce that $|\nabla \Psi(x_p)| > 0$ for all sufficiently large $p \in \mathbb{N}$. This means that we can divide both sides of (3.8) by $(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}$ to get

$$\begin{aligned} & - \frac{|\nabla \Psi(x_p)|^{p-2} + |\nabla \Psi(x_p)|^{q(p)-2}}{(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}} \Delta \Psi(x_p) - \Delta_\infty \Psi(x_p) \\ & \geq \frac{\lambda_p \Psi(x_p)^{p-1}}{(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}} \\ & = \left(\frac{\lambda_p^{1/p} \Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}]^{1/p}} \right)^p. \end{aligned}$$

Note that

$$\begin{aligned} & \frac{|\nabla \Psi(x_p)|^{p-2} + |\nabla \Psi(x_p)|^{q(p)-2}}{(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}} \\ & \leq \max \left\{ \frac{1}{p-2}, \frac{1}{q(p)-2} \right\} |\nabla \Psi(x_p)|^2. \end{aligned}$$

Letting $p \rightarrow \infty$ in the above estimates leads to

$$\begin{aligned} & - \Delta_\infty \Psi(x_0) \\ & \geq \limsup_{p \rightarrow \infty} \left[\frac{\lambda_p^{1/p} \Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}]^{1/p}} \right]^p. \end{aligned} \quad (3.9)$$

In particular we have

$$- \Delta_\infty \Psi(x_0) \geq 0. \quad (3.10)$$

On the other hand,

$$\begin{aligned} & \max\{(p-2)^{1/p} |\nabla \Psi(x_p)|^{1-4/p}, (q(p)-2)^{1/p} |\nabla \Psi(x_p)|^{(q(p)-4)/p}\} \\ & \leq [(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}]^{1/p} \\ & \leq 2^{1/p} \max\{(p-2)^{1/p} |\nabla \Psi(x_p)|^{1-4/p}, (q(p)-2)^{1/p} |\nabla \Psi(x_p)|^{(q(p)-4)/p}\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \lim_{p \rightarrow \infty} [(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}]^{1/p} \\ & = \max\{|\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q\}. \end{aligned} \quad (3.11)$$

We now claim that

$$\max\{|\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q\} - \Delta \Psi(x_0) \geq 0. \quad (3.12)$$

We argue by contradiction. If we assume that (3.12) is false then

$$\max\{|\nabla\Psi(x_0)|, |\nabla\Psi(x_0)|^Q\} < \Lambda\Psi(x_0).$$

By (3.11) we deduce that there exists $\theta > 1$ such that for each $p \in \mathbb{N}$ sufficiently large we have

$$\frac{\lambda_p^{1/p}\Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla\Psi(x_p)|^{p-4} + (q(p)-2)|\nabla\Psi(x_p)|^{q(p)-4}]^{1/p}} \geq \theta > 1.$$

This implies that

$$\begin{aligned} \limsup_{p \rightarrow \infty} \left[\frac{\lambda_p^{1/p}\Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla\Psi(x_p)|^{p-4} + (q(p)-2)|\nabla\Psi(x_p)|^{q(p)-4}]^{1/p}} \right]^p \\ \geq \lim_{p \rightarrow \infty} \theta^p = +\infty, \end{aligned}$$

which is a contradiction with (3.9). We conclude that, in fact, (3.12) holds.

Overall, in view of (3.10) and (3.12), we have

$$\min\{\max\{|\nabla\Psi(x_0)|, |\nabla\Psi(x_0)|^Q\} - \Lambda\Psi(x_0), -\Delta_\infty\Psi(x_0)\} \geq 0. \quad (3.13)$$

On the other hand, if $u_\Lambda(x_0) = \Psi(x_0) = 0$, we either have $\nabla\Psi(x_0) \neq 0$ (if so, we can use very similar arguments to conclude that (3.10) and (3.12) hold), or else $\nabla\Psi(x_0) = 0$, in which case $\Delta_\infty\Psi(x_0) = 0$, and hence (3.10) and (3.12) trivially hold. We deduce that the inequality (3.13) is again true in this case.

To complete the proof it remains to check that u_Λ is also a viscosity subsolution of (1.9). Let us consider a point $x_0 \in \Omega$ and a function $\Psi \in C^2(\bar{\Omega})$ such that $u_\Lambda(x_0) = \Psi(x_0)$ and $u_\Lambda(x) < \Psi(x)$ for every x in a neighborhood of x_0 . We need to show that

$$\min\{\max\{|\nabla\Psi(x_0)|, |\nabla\Psi(x_0)|^Q\} - \Lambda\Psi(x_0), -\Delta_\infty\Psi(x_0)\} \leq 0.$$

First, if $\nabla\Psi(x_0) = 0$, then the previous inequality clearly holds. Hence, we may assume in what follows that $\nabla\Psi(x_0) \neq 0$. It suffices to show that if

$$\max\{|\nabla\Psi(x_0)|, |\nabla\Psi(x_0)|^Q\} - \Lambda\Psi(x_0) > 0 \quad (3.14)$$

then

$$-\Delta_\infty\Psi(x_0) \leq 0. \quad (3.15)$$

Indeed, we may construct, as above, a sequence $x_p \rightarrow x_0$ such that

$$\begin{aligned} & -\frac{|\nabla\Psi(x_p)|^{p-2} + |\nabla\Psi(x_p)|^{q(p)-2}}{(p-2)|\nabla\Psi(x_p)|^{p-4} + (q(p)-2)|\nabla\Psi(x_p)|^{q(p)-4}} \Delta\Psi(x_p) - \Delta_\infty\Psi(x_p) \\ & \leq \frac{\lambda_p\Psi(x_p)^{p-1}}{(p-2)|\nabla\Psi(x_p)|^{p-4} + (q(p)-2)|\nabla\Psi(x_p)|^{q(p)-4}} \\ & = \left(\frac{\lambda_p^{1/p}\Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla\Psi(x_p)|^{p-4} + (q(p)-2)|\nabla\Psi(x_p)|^{q(p)-4}]^{1/p}} \right)^p. \end{aligned}$$

Letting $p \rightarrow \infty$ and using (3.14) and (3.11) we deduce that

$$- \Delta_\infty \Psi(x_0) \leq \liminf_{p \rightarrow \infty} \left[\frac{\lambda_p^{1/p} \Psi(x_p)^{(p-1)/p}}{[(p-2)|\nabla \Psi(x_p)|^{p-4} + (q(p)-2)|\nabla \Psi(x_p)|^{q(p)-4}]^{1/p}} \right]^p = 0,$$

which gives (3.15).

Remark 3.4. The above proof settles the existence part in Theorem 1.3 for $\Lambda \in (\Lambda_\infty, \infty)$ if $Q \in (0, \infty) \setminus \{1\}$ and for $\Lambda = \Lambda_\infty$ if $Q \in (0, 1)$. For the remaining case, $\Lambda = \Lambda_\infty$ and $Q \in (1, \infty)$, first note that we have $|\nabla u(x)|^Q \leq |\nabla u(x)|$ for all $x \in \Omega$ such that $|\nabla u(x)| \leq 1$. Next, let u_0 be a nonnegative and nontrivial viscosity solution of the problem (1.4) with $\Lambda = \Lambda_\infty$ (such a solution exists by [10]). It is well-known that for any positive constant C the function Cu_0 is also a viscosity solution of the problem (1.4) with $\Lambda = \Lambda_\infty$. It follows that $v_0(x) := \frac{u_0(x)}{\|\nabla u_0\|_{L^\infty(\Omega)}}$ is a viscosity solution of (1.4) with $\Lambda = \Lambda_\infty$. Moreover, $|\nabla v_0(x)| \leq 1$ and hence $|\nabla v_0(x)|^Q \leq |\nabla v_0(x)|$ for all $x \in \Omega$. Thus, v_0 is a viscosity solution of (1.9) with $\Lambda = \Lambda_\infty$.

3.2. Nonexistence

The nonexistence part of the proof of Theorem 1.3 (that is, showing that for any $\Lambda \in (-\infty, \Lambda_\infty)$ the problem (1.9) has no nontrivial and nonnegative viscosity solutions) combines ideas from [9] and [10]. We begin with a simple result regarding the problem (1.9).

Lemma 3.5. *If u_Λ is a nonnegative viscosity solution of (1.9) then it is a viscosity supersolution of the equation*

$$- \Delta_\infty u = 0 \quad \text{in } \Omega. \quad (3.16)$$

Moreover, for every constant $C > 0$, the function Cu_Λ is also a supersolution of (3.16).

Proof. Let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $u_\Lambda(x_0) = \Psi(x_0)$ and $u_\Lambda(x) > \Psi(x)$ for all $x \neq x_0$ in a neighborhood of x_0 . We have

$$\min \left\{ \max \{ |\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q \} - \Lambda \Psi(x_0), -\Delta_\infty \Psi(x_0) \right\} \geq 0.$$

It follows that

$$-\Delta_\infty \Psi(x_0) \geq 0.$$

Hence, u_Λ is a viscosity supersolution of (3.16), as claimed.

For the second part, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $Cu_\Lambda(x_0) = \Psi(x_0)$ and $Cu_\Lambda(x) > \Psi(x)$ for all $x \neq x_0$ in a neighborhood of x_0 . Thus, $u_\Lambda(x_0) = \frac{1}{C}\Psi(x_0)$ and $u_\Lambda(x) > \frac{1}{C}\Psi(x)$ for all $x \neq x_0$ in a neighborhood of x_0 . We can use $\frac{1}{C}\Psi$ as a test function in the definition of u_Λ being a nonnegative viscosity supersolution of (1.9), to obtain

$$\min \left\{ \max \left\{ \frac{1}{C} |\nabla \Psi(x_0)|, \frac{1}{C^Q} |\nabla \Psi(x_0)|^Q \right\} - \frac{\Lambda}{C} \Psi(x_0), -\frac{1}{C^3} \Delta_\infty \Psi(x_0) \right\} \geq 0.$$

Since $C > 0$ we get

$$-\Delta_\infty \Psi(x_0) \geq 0,$$

which implies that Cu_Λ is a viscosity supersolution of (3.16). $\square$

Lemma 3.6. *For each $\Lambda \leq 0$ the problem (1.9) has no nontrivial and nonnegative viscosity solutions.*

Proof. We proceed by contradiction. Let $\Lambda \leq 0$, and assume that there exists u , a nonnegative viscosity solution of the problem (1.9). We will show that this implies $u \equiv 0$ in Ω .

Let $x_0 \in \Omega$ be arbitrary, and let $\Psi \in C^2(\Omega)$ be such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ for all $x \neq x_0$ in a neighborhood of x_0 . Since u is in particular a viscosity subsolution of (1.9), we have

$$\min \{ \max \{ |\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q \} - \Lambda \Psi(x_0), -\Delta_\infty \Psi(x_0) \} \leq 0.$$

Thus, either $-\Delta_\infty \Psi(x_0) \leq 0$ or $\max \{ |\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q \} - \Lambda \Psi(x_0) \leq 0$. Since $\Lambda \leq 0$ and $\Psi(x_0) = u(x_0) \geq 0$, we have $\max \{ |\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q \} - \Lambda \Psi(x_0) \geq 0$. We deduce that the second case in the above alternative only holds if $\max \{ |\nabla \Psi(x_0)|, |\nabla \Psi(x_0)|^Q \} = \Lambda \Psi(x_0) = 0$, which gives $\Delta_\infty \Psi(x_0) = 0$. Overall, we must have $-\Delta_\infty \Psi(x_0) \leq 0$. Hence, u is a viscosity subsolution of the ∞ -Laplace equation, $-\Delta_\infty u = 0$.

On the other hand, by Lemma 3.5 we know that u is also a viscosity supersolution of $-\Delta_\infty u = 0$. Therefore, u is a viscosity solution of the ∞ -Laplace equation with homogeneous Dirichlet boundary conditions. By Jensen's uniqueness result [8, Corollary 3.14], we deduce that $u \equiv 0$ in Ω . $\square$

Before we can establish our final lemma, we recall (see [9, Theorem 8.2]) that if u is a nonnegative viscosity supersolution of $-\Delta_\infty u = 0$ in Ω then

$$|\nabla \log(u(x))| \leq |\nabla \log(\delta(x))| \quad \text{for a.e. } x \in \Omega. \quad (3.17)$$

The proof of the next result relies on some ideas introduced in [9, Theorem 8.1].

Lemma 3.7. *For each $\Lambda \in (0, \Lambda_\infty)$ the problem (1.9) has no nontrivial and nonnegative viscosity solutions.*

Proof. Let $\Lambda > 0$ be such that there exists u_Λ a nontrivial and nonnegative viscosity solution of problem (1.9). We show that $\Lambda \geq \Lambda_\infty$. Let $v_\Lambda := \frac{u_\Lambda}{\Lambda \|u_\Lambda\|_{L^\infty(\Omega)}}$. We show that v_Λ satisfies

$$\min \{ |\nabla v_\Lambda| - 1, -\Delta_\infty v_\Lambda \} \leq 0 \quad \text{in } \Omega \quad (3.18)$$

in the viscosity sense. Indeed, let $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ be such that $v_\Lambda(x_0) = \Psi(x_0)$ and $v_\Lambda(x) < \Psi(x)$ for all $x \neq x_0$ in a neighborhood of x_0 . We have

$$\begin{aligned} & \max \{ \Lambda \|u_\Lambda\|_{L^\infty(\Omega)} |\nabla \Psi(x_0)|, (\Lambda \|u_\Lambda\|_{L^\infty(\Omega)})^Q |\nabla \Psi(x_0)|^Q \} \\ & \geq \Lambda \|u_\Lambda\|_{L^\infty(\Omega)} |\nabla \Psi(x_0)|, \end{aligned} \quad (3.19)$$

and

$$-\Lambda^2 \|u_\Lambda\|_{L^\infty(\Omega)} \Psi(x_0) \geq -\Lambda \|u_\Lambda\|_{L^\infty(\Omega)}. \quad (3.20)$$

Thus,

$$\begin{aligned} & \max\{\Lambda\|u_\Lambda\|_{L^\infty(\Omega)}|\nabla\Psi(x_0)|, (\Lambda\|u_\Lambda\|_{L^\infty(\Omega)})^Q|\nabla\Psi(x_0)|^Q\} \\ & - \Lambda^2\|u_\Lambda\|_{L^\infty(\Omega)}\Psi(x_0) \geq \Lambda\|u_\Lambda\|_{L^\infty(\Omega)}|\nabla\Psi(x_0)| - \Lambda\|u_\Lambda\|_{L^\infty(\Omega)}. \end{aligned} \quad (3.21)$$

On the other hand,

$$-\Delta_\infty(\Lambda\|u_\Lambda\|_{L^\infty(\Omega)}\Psi(x_0)) = -(\Lambda\|u_\Lambda\|_{L^\infty(\Omega)})^3\Delta_\infty\Psi(x_0). \quad (3.22)$$

Further, testing with $\Lambda\|u_\Lambda\|_{L^\infty(\Omega)}\Psi$ in the definition of u_Λ being a nonnegative viscosity solution of (1.9), we obtain

$$\begin{aligned} & \min\left\{\max\left\{\Lambda\|u_\Lambda\|_{L^\infty(\Omega)}|\nabla\Psi(x_0)|, (\Lambda\|u_\Lambda\|_{L^\infty(\Omega)})^Q|\nabla\Psi(x_0)|^Q\right\}\right. \\ & \left.- \Lambda^2\|u_\Lambda\|_{L^\infty(\Omega)}\Psi(x_0), -(\Lambda\|u_\Lambda\|_{L^\infty(\Omega)})^3\Delta_\infty\Psi(x_0)\right\} \leq 0. \end{aligned} \quad (3.23)$$

Combining (3.23) with (3.21) and (3.22) we get

$$\min\left\{\Lambda\|u_\Lambda\|_{L^\infty(\Omega)}|\nabla\Psi(x_0)| - \Lambda\|u_\Lambda\|_{L^\infty(\Omega)}, -(\Lambda\|u_\Lambda\|_{L^\infty(\Omega)})^3\Delta_\infty\Psi(x_0)\right\} \leq 0.$$

Since $\Lambda\|u_\Lambda\|_{L^\infty(\Omega)} > 0$ then either $|\nabla\Psi(x_0)| - 1 \leq 0$, or $-\Delta_\infty\Psi(x_0) \leq 0$. Hence,

$$\min\{|\nabla\Psi(x_0)| - 1, -\Delta_\infty\Psi(x_0)\} \leq 0.$$

Thus, we proved that v_Λ satisfies (3.18) in the viscosity sense. On the other hand, by Lemma 3.5 it is clear that v_Λ is a viscosity supersolution of (3.16). In view of [9, Theorem 8.2] (see also (3.17)) we deduce that

$$|\nabla\log(v_\Lambda(x))| \leq |\nabla\log(\delta(x))| \quad \text{for a.e. } x \in \Omega.$$

Taking into account the fact that $|\nabla\delta(x)| = 1$ for a.e. $x \in \Omega$, this becomes

$$|\nabla v_\Lambda(x)| \leq \frac{|v_\Lambda(x)|}{\delta(x)} \quad \text{for a.e. } x \in \Omega. \quad (3.24)$$

Next, recall that δ satisfies

$$\min\{|\nabla\delta| - 1, -\Delta_\infty\delta\} = 0 \quad \text{in } \Omega, \quad (3.25)$$

and that $\delta = v_\Lambda = 0$ on $\partial\Omega$. Since v_Λ is a viscosity solution of (3.18), by Jensen's comparison principle we must have $v_\Lambda(x) \leq \delta(x)$ for all $x \in \Omega$. Combining (3.24) and (3.25) we find $|\nabla v_\Lambda(x)| \leq 1$ for a.e. $x \in \Omega$ and, consequently,

$$\frac{\|\nabla v_\Lambda\|_{L^\infty(\Omega)}}{\|v_\Lambda\|_{L^\infty(\Omega)}} \leq \frac{1}{\|v_\Lambda\|_{L^\infty(\Omega)}} = \Lambda.$$

Thus, in view of (1.6) we obtain that $\Lambda \geq \Lambda_\infty$, as claimed. $\square$

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