σ_2 -Energy as a polyconvex functional on a space of self-maps of annuli in the multi-dimensional calculus of variations

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Abstract. Let $\mathbb{X} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$\mathbb{F}_{\sigma_2}[u; \mathbb{X}] := \int_{\mathbb{X}} \mathbf{F}(\nabla u) dx,$$

over the space of *admissible* maps

$$\mathcal{A}_{\varphi}(\mathbb{X}) := \{u \in W^{1,4}(\mathbb{X}, \mathbb{R}^n) : \det \nabla u > 0 \text{ for } \mathcal{L}^n\text{-a.e. in } \mathbb{X}, u|_{\partial\mathbb{X}} = \varphi\},$$

where the integrand $\mathbf{F} : \mathbb{M}_{n \times n} \rightarrow \mathbb{R}$ is *quasiconvex* and sufficiently regular. Here our attention is paid to the prototypical case when $\mathbf{F}(\xi) := \frac{1}{2}\sigma_2(\xi) + \Phi(\det \xi)$. The aim of this paper is to discuss the question of multiplicity *versus* uniqueness for *extremals* and *strong* local minimizers of $\mathbb{F}_{\sigma_2}$ and the relation it bears to the domain topology. In contrast, for constructing *explicitly* and directly solutions to the system of Euler–Lagrange equations associated to $\mathbb{F}_{\sigma_2}$, we use a topological class of maps referred to as *generalised* twists and relate the problem to *extremising* an associated energy on the compact Lie group $\mathbf{SO}(n)$. The main result is a surprising discrepancy between *even* and *odd* dimensions. In even dimensions the latter system of equations admits *infinitely* many smooth solutions amongst such maps whereas in odd dimensions this number reduces to *one*.

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1. Introduction

Let $\mathbb{X} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We consider variational integral of the form

$$\mathbb{F}_{\sigma_2}[u; \mathbb{X}] := \int_{\mathbb{X}} \left[\frac{1}{2}\sigma_2(u) + \Phi(\det \nabla u) \right] dx, \quad (1.1)$$

while $\sigma_2(u) := |\wedge^2 \nabla u|^2$, over the space of *admissible* [orientation preserving deformations] maps

$$\mathcal{A}(\mathbb{X}) = \{u \in W^{1,4}(\mathbb{X}, \mathbb{R}^n) : \det \nabla u > 0 \text{ for } \mathcal{L}^n\text{-a.e. in } \mathbb{X}, u|_{\partial\mathbb{X}} = x\}. \quad (1.2)$$

Before going further we pause briefly to comment on the *first* term, $\sigma_2(u)$, appearing in the variational integral described above. It is well-known that the quantity $|\nabla u|$ measures how *lengths* are stretched by the deformation u , and the quantity $|\det \nabla u|$ measures how *volumes* change. Similarly, it is also well-known that $|\operatorname{adj} \nabla u|$ governs the changes in $(n-1)$ -dimensional areas. This can be unified when we consider the *Cauchy–Green* strain tensor $[\nabla u]^t[\nabla u]$, analogously to the case of deformations in nonlinear elasticity (see, e.g., [9]).

In view of the Cauchy–Green tensor being *symmetric* and *positive* semi-definite, let $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denoting the square root of its eigen-values. Recall that $(\lambda_i)_{i=1}^n$ are also called *principal distortion coefficients* of the map u . The elementary symmetric functions in the eigen-values of the endomorphism $[\nabla u]^t \circ [\nabla u] : T\mathbb{X} \rightarrow T\mathbb{X}$ represent a measure of the geometrical distortion induced by the map u . They are called *principal invariants* of ∇u and we denote them as follows¹

$$\sigma_p(u) := |\wedge^p \nabla u|^2 = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n} \lambda_{i_1}^2 \lambda_{i_2}^2 \dots \lambda_{i_p}^2,$$

where $p = 1, 2, \dots, n$. Motivated by the above representation in what follows the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ can alternatively be represented in the following way

$$\begin{aligned} \mathbb{F}_{\sigma_2}[u; \mathbb{X}] &= \int_{\mathbb{X}} \left[\frac{1}{2} \sigma_2(u) + \Phi(\det \nabla u) \right] dx \\ &= \frac{1}{4} \int_{\mathbb{X}} \left[\left(\sum_{i=1}^n \lambda_i^2 \right)^2 - \sum_{i=1}^n \lambda_i^4 + 4\Phi(\det \nabla u) \right] dx \\ &= \frac{1}{4} \int_{\mathbb{X}} \left[|\nabla u|^4 - |[\nabla u][\nabla u]^t|^2 + 4\Phi(\det \nabla u) \right] dx, \end{aligned}$$

where in the *last* implication we have used the fact that $(\lambda_i)_{i=1}^n$ are *singular*-values of the matrix ∇u . In our analysis we shall make the following set of *assumptions* on the function Φ appearing in the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$:

- [A1] Φ is *convex* and *non-negative* on $]0, \infty[$;
- [A2] $\Phi \in C^2(]0, \infty[)$;

¹According to [10] the σ_p -energy is defined as the following functional

$$\mathbb{I}_{\sigma_p}[u; \mathbb{X}] := \frac{1}{2} \int_{\mathbb{X}} \sigma_p(u) dx.$$

In [15] we initiated the task of *extremising* the σ_2 and σ_1 -energy, respectively, over the space of measure preserving map by introducing a class of maps referred to as *generalised* twists and examining them as possible solutions to the associated system of Euler–Lagrange equations. However the variational problem for the σ_p -energy has already been treated in [6, 10, 22], very little is known about its solutions (e.g., multiplicity *versus* uniqueness, existence of non-trivial *strong* local minimisers, etc.) and is not *fully* understood.

[A3] Φ has the two *limiting* behaviours

$$\lim_{t \downarrow 0} \Phi(t) = \lim_{t \uparrow \infty} \frac{\Phi(t)}{t} = \infty;$$

[A4] there exists $\beta > 0$ and $\delta > 0$ such that for all $t \in]0, \infty[$ and $\alpha > 0$ satisfying $|\alpha - 1| < \delta$ we have that

$$|\Phi'(\alpha t)| \leq \beta[\Phi(t) + 1]. \quad (1.3)$$

The primary aim of this paper is to investigate the problem of *extremising* the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over the space of $\mathcal{A}(\mathbb{X})$ and to examine a special class of maps as solutions to the corresponding system of Euler–Lagrange equations which can *formally* be written as

$$\begin{cases} \operatorname{div} \mathfrak{T}[\nabla u(x)] = 0 & x \in \mathbb{X}, \\ \det \nabla(x) > 0 & x \in \mathbb{X}, \\ u(x) = \varphi(x) & x \in \partial \mathbb{X}. \end{cases}$$

Noticing that the divergence operator acts *row-wise* and the tensor field $\mathfrak{T}$ is defined through

$$\mathfrak{T}[\xi] := |\xi|^2 \xi - \xi \xi^t \xi + \Phi'(\det \xi) \operatorname{cof} \xi,$$

for all $\xi \in \mathbb{M}_{n \times n}$ satisfying $\det \xi > 0$.

A motivation for the study of the above energy functional comes from *nonlinear* elasticity where $\mathbb{F}_{\sigma_2}$ represents the elastic energy of a *homogeneous* hyperelastic material and $\mathcal{A}(\mathbb{X})$ represents the space of orientation preserving deformations of $\mathbb{X}$ fixing the boundary *pointwise* and solutions to the above system of equations serve as the corresponding *equilibrium* states (see, e.g., [1, 2]).

The second motivation is *Skyrme* model which is a new model describing strong interactions of quantum fields and σ_2 -energy appears as a *part* of this physical model. Skyrme's idea was to add to the standard Dirichlet energy an additional stabilizing term which is the σ_2 -energy and it would prevent stationary fields from being singular as it happens for harmonic maps (see, e.g., [4, 18, 19]).

While the *linear* map $u = \varphi$ is the *unique* minimizer of $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over $\mathcal{A}(\mathbb{X})$, due to the non-convexity of the energy $\mathbb{F}_{\sigma_2}$ and the singular behaviour of the function Φ near origin, the *structure* and *features* of the solutions *set* to this system of Euler–Lagrange equations [e.g., multiplicity *versus* uniqueness, *regularity*, global *invertibility*, existence of *strong* local minimizers for $\mathbb{F}_{\sigma_2}$, etc. (see, e.g., [3, 5, 12, 20])] is not *fully* understood.²

In this article we contribute towards understanding aspects of these questions by way of presenting *multiple* solutions to the above system of equations.

²A map $\bar{u} \in \mathcal{A}(\mathbb{X})$ is a *strong* local minimizer of $\mathbb{F}_{\sigma_2}$ if and only if there exists $\delta = \delta(\bar{u}) > 0$ such that $\mathbb{F}_{\sigma_2}[\bar{u}, \mathbb{X}] \leq \mathbb{F}_{\sigma_2}[v, \mathbb{X}]$ for all $v \in \mathcal{A}(\mathbb{X})$ satisfying $\|\bar{u} - v\|_{L^1} < \delta$.

Indeed we focus attention on the case where the domain $\mathbb{X} \subset \mathbb{R}^n$ is an n -dimensional *annulus*, i.e., $\mathbb{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$.³ We proceed by introducing a class of maps, referred to as *generalised twists*, characterised and defined by

$$u(x) = \mathbf{G}(r)\theta,$$

with

$$\mathbf{G}(r) := f(r)\mathbf{Q}(r)$$

where $r = |x|$, $\theta = x/|x|$, $\mathbf{Q} \in C([a, b], \mathbf{SO}(n))$ and $f \in C[a, b]$. In addition, to ensure *admissibility*, i.e., $u \in \mathcal{A}(\mathbb{X})$ it suffices to impose a *further* L^2 -summability on $\dot{f} := df/dr$ and $\dot{\mathbf{Q}} := d\mathbf{Q}/dr$ along with $\dot{f} > 0$ $\mathcal{L}^1$ -a.e. on $]a, b[$ while $f(a) = a$, $f(b) = b$ and $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. The main motivation for considering such maps comes partly from geometric topology where *Dehn-twists* are crucial instruments in describing the mapping class groups of *surfaces* (see [7]) and partly nonlinear elasticity where generalised twists have played a curious role in the multiple solution problems for *equilibrium* states and local minimizers (see [15, 21]).

Next by restricting the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ to the space of such *twists* we can write

$$\begin{aligned} \mathbb{E}_{\sigma_2}[\mathbf{Q}, f] := \frac{4}{\omega_n} \mathbb{F}_{\sigma_2}[u; \mathbb{X}] &= \int_a^b \left\{ (n-2)f^4 \left[n(n-1) \frac{1}{r^2} + 2|\dot{\mathbf{Q}}|^2 \right] \right. \\ &\quad \left. + 2n(n-1)f^2\dot{f}^2 + 4nr^2\Phi \left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right\} r^{n-3} dr, \end{aligned}$$

and the corresponding Euler–Lagrange equations can again be *formally* shown to be the *second* order system of ordinary *differential* equations

$$\begin{cases} \frac{d}{dr} \left[r^{n-3} f^4 \mathbf{Q}^t \frac{d}{dr} \mathbf{Q} \right] = \mathbf{0}, \\ \frac{d}{dr} \left[r^{n-3} f^2 \dot{f} + \frac{1}{n-1} f^{n-1} \Phi' \right] = (n-2)r^{n-5} f^3 + r^{n-3} f \dot{f}^2 \\ \quad + \dot{f} f^{n-2} \Phi' + \frac{2(n-2)}{n(n-1)} r^{n-3} f^3 |\dot{\mathbf{Q}}|^2, \end{cases}$$

where $\Phi' = \Phi'(\dot{f}(\frac{f}{r})^{n-1})$ and the equations are hold on $]a, b[$.

Now in order to characterise among solutions to the above system *all* those which grant a solution to the Euler–Lagrange equations associated with $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over $\mathcal{A}(\mathbb{X})$ we are *confronted* with the of task of verifying the *necessary* and *sufficient* condition $\dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$ on $]a, b[$. This *analysis* occupies a major part of the paper and is *fully* settled in Theorems 4.3 and 5.2.

³Recall that for *star-shaped* domains and subject to *linear* boundary conditions there is a *uniqueness* result associated with (*sufficiently* regular) *equilibrium* states in hyperelasticity (see [11, 17]).

The conclusion that this analysis bares on to the *original* Euler–Lagrange equations turns to be a surprising discrepancy between *even* and *odd* dimensions. Indeed it follows that in *even* dimensions the latter system of equations admit *infinitely* many *smooth* solutions, modulo isometries, in the form of *generalised* twists whilst in *odd* dimensions this number *severely* reduces to *one*.

We end this section by pointing out the following *notes*:

- [N1] It is worthwhile noting that, by definition, the functions $\sigma_p(u)$ in terms of ∇u are *polyconvex* but *not* convex in general except for the $\sigma_1(u)$!
- [N2] It is convenient to extend Φ to the *entire* real line by setting $\Phi(t) = \infty$ for $t \in]-\infty, 0]$. Evidently with this convention for any $u \in W_\varphi^{1,2}(\mathbb{X}, \mathbb{R}^n)$ we have that

$$\mathbb{F}[u; \mathbb{X}] < \infty \implies \det \nabla u > 0 \text{ } \mathcal{L}^n\text{-a.e. in } \mathbb{X}.$$

However notice that the *reverse* implication, in general, need *not* be true.

- [N3] The system of Euler–Lagrange equations associated with $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ for any solution of class C^2 can alternatively be expressed as (see Definition 5.1)

$$\begin{cases} [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + \det \nabla u \nabla[\Phi'(\det \nabla u)] = 0 & \text{in } \mathbb{X}, \\ \det \nabla u > 0 & \text{in } \mathbb{X}, \\ u = \varphi & \text{on } \partial\mathbb{X}, \end{cases}$$

where⁴

$$\begin{aligned} \mathfrak{D}_{\sigma_2}[u] &:= \operatorname{div}[[\nabla u]^2 \nabla u - [\nabla u][\nabla u]^t [\nabla u]] \\ &= \Delta_4 u - \operatorname{div}[[\nabla u][\nabla u]^t [\nabla u]]. \end{aligned}$$

2. The energy restricted to the space of twists

We begin the section by first recalling the definition of a *generalised* twist, that is, a *continuous* map u on $\overline{\mathbb{X}}$ into $\overline{\mathbb{X}}$ which can be expressed in *spherical* coordinate in the form

$$u(x) = \mathbf{G}(r)\theta, \quad (2.1)$$

where $x \in \overline{\mathbb{X}}$ and $r = |x|$, $\theta = x/|x|$ with

$$\mathbf{G}(r) = f(r)\mathbf{Q}(r),$$

where $\mathbf{Q} \in C([a, b], \mathbf{SO}(n))$ is a continuous path in $\mathbf{SO}(n)$ and $f \in C[a, b]$.⁵ It can be verified that subject to the additional set of conditions

- [1] Conditions on $\mathbf{Q}$:
 - [1a] $\mathbf{Q} \in W^{1,4}([a, b], \mathbf{SO}(n))$,
 - [1b] $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$,
- [2] Conditions on f :
 - [2a] $f \in W^{1,4}(a, b)$,
 - [2b] $\dot{f} > 0$ $\mathcal{L}^1$ -a.e. on $]a, b[$,

⁴Recall that $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

⁵Notice that as a result of the basic requirement $u \in C(\overline{\mathbb{X}}, \overline{\mathbb{X}})$ built into the definition of a *generalised* twists it follows in particular that $a \leq |f| \leq b$ on $[a, b]$.

$$[2c] \quad f(a) = a \text{ and } f(b) = b,$$

then the associated twist u lies in $\mathcal{A}(\mathbb{X})$. Before proceeding further we encourage the reader to consult the *second* section in [16] where some of the basic results with regards to generalised twists used in the subsequent sections are introduced and proved.

Proposition 2.1. *Suppose that u is a generalised twist with the associated twist path $\mathbf{Q} \in C^1([a, b[, \mathbf{SO}(n))$ and $f \in C^1([a, b[)$. Then the following identity holds.*

$$\begin{aligned} |\nabla u|^2 \nabla u - [\nabla u][\nabla u]^t [\nabla u] &= \alpha \mathbf{Q} + \beta \mathbf{Q} \theta \otimes \theta \\ &\quad + \frac{1}{r} f^2 [\dot{f} \mathbf{Q} + f \dot{\mathbf{Q}}] \theta \otimes \left[\frac{n-2}{r} \mathbf{I}_n - \mathbf{A} \right] \theta, \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} \alpha &= \alpha(r, \theta) := (n-2) \frac{1}{r^3} f^3 + \frac{1}{r} f \dot{f}^2 + \frac{1}{r} f^3 |\dot{\mathbf{Q}} \theta|^2, \\ \beta &= \beta(r, \theta) := r^{-2} f^2 \dot{f} - \alpha \end{aligned}$$

and $\mathbf{A} := \mathbf{Q}^t \dot{\mathbf{Q}}$.

Proof. Indeed this follows by *direct* verification and using the facts that

$$\nabla u = \frac{f}{r} \mathbf{Q} + \left(\dot{f} - \frac{f}{r} \right) \mathbf{Q} \theta \otimes \theta + f \dot{\mathbf{Q}} \theta \otimes \theta, \quad (2.3)$$

and consequently that

$$\begin{aligned} |\nabla u|^2 &= \text{tr}\{[\nabla u][\nabla u]^t\} \\ &= (n-1) \left(\frac{f}{r} \right)^2 + \dot{f}^2 + f^2 |\dot{\mathbf{Q}} \theta|^2. \end{aligned} \quad (2.4)$$

Therefore the details of the proof is omitted. $\square$

Proposition 2.2. *Suppose that u is a generalised twist as described. Then subject to $\mathbf{Q} \in C^2([a, b[, \mathbf{SO}(n))$, $f \in C^2([a, b[)$ and with the notations used in Proposition 2.1, we have*

$$\begin{aligned} \mathfrak{D}_{\sigma_2}[u] &:= \text{div}[|\nabla u|^2 \nabla u - [\nabla u][\nabla u]^t [\nabla u]] \\ &= \mathbf{Q} \left\{ \frac{f^3}{r^2} [(n-5) \mathbf{A}^2 - (n+1) |\mathbf{A} \theta|^2 \mathbf{I}_n] \right. \\ &\quad + \frac{n-1}{r^{n-1}} \left[\frac{d}{dr} (r^{n-3} f^2 \dot{f}) - (n-2) r^{n-5} f^3 \right. \\ &\quad \left. \left. - r^{n-3} f \dot{f}^2 \right] \mathbf{I}_n + \frac{n-2}{r^{n-1} f} \frac{d}{dr} [r^{n-3} f^4 \mathbf{A}] \right\} \theta, \end{aligned} \quad (2.5)$$

where $\mathbf{A} = \mathbf{Q}^t \dot{\mathbf{Q}}$.

Proof. For a *generalised twist* u an application of previous proposition gives

$$\begin{aligned}\mathfrak{D}_{\sigma_2}[u] &:= \operatorname{div}[[\nabla u]^2 \nabla u - [\nabla u][\nabla u]^t[\nabla u]] \\ &= \operatorname{div}\left[\alpha \mathbf{Q} + \beta \mathbf{Q} \theta \otimes \theta + \frac{1}{r} f^2 (\dot{f} \mathbf{Q} + f \dot{\mathbf{Q}}) \theta \otimes \left(\frac{n-2}{r} \mathbf{I}_n - \mathbf{A}\right) \theta\right] \\ &=: \operatorname{div}(\mathbf{I} + \mathbf{II} + \mathbf{III}).\end{aligned}\quad (2.6)$$

We now proceed by evaluating the *divergence* of each term *separately*. Indeed to begin note that a straight-forward *differentiation* gives

$$\begin{aligned}\operatorname{div}(\mathbf{I}) &= \operatorname{div}[\alpha \mathbf{Q}] \\ &= \alpha \operatorname{div}[\mathbf{Q}] + \mathbf{Q} \nabla \alpha \\ &= \alpha \dot{\mathbf{Q}} \theta + \mathbf{Q} \nabla \alpha.\end{aligned}$$

In order to evaluate the divergence of the *second* and *third* terms in (2.6), we *first* observe that for the vector fields v and w , *smooth* enough, the following identity holds

$$\operatorname{div}[v \otimes w] = [\nabla v]w + (\operatorname{div} w)v. \quad (2.7)$$

Now, motivated by the above identity with regards to the second term we can write

$$\begin{aligned}\operatorname{div}(\mathbf{II}) &= \operatorname{div}[\beta \mathbf{Q} \theta \otimes \theta] \\ &= [\nabla(\mathbf{Q} \theta)]\beta \theta + \operatorname{div}(\beta \theta) \mathbf{Q} \theta \\ &= \frac{1}{r} [\mathbf{Q} + (r \dot{\mathbf{Q}} - \mathbf{Q}) \theta \otimes \theta] \beta \theta + \left(\langle \nabla \beta, \theta \rangle + \frac{n-1}{r} \beta\right) \mathbf{Q} \theta \\ &= \beta \dot{\mathbf{Q}} \theta + \left(\langle \nabla \beta, \theta \rangle + \frac{n-1}{r} \beta\right) \mathbf{Q} \theta.\end{aligned}$$

Hence by *combining* the latter we arrived at the identity

$$\begin{aligned}\operatorname{div}(\mathbf{I} + \mathbf{II}) &= \mathbf{Q} \left[\alpha \mathbf{A} \theta + \nabla \alpha + \beta \mathbf{A} \theta + \left(\langle \nabla \beta, \theta \rangle + \frac{n-1}{r} \beta\right) \theta \right] \\ &= \mathbf{Q} \left[\frac{1}{r^2} f^2 \dot{f} \mathbf{A} \theta + \nabla \alpha + \left(\langle \nabla \beta, \theta \rangle + \frac{n-1}{r} \beta\right) \theta \right].\end{aligned}\quad (2.8)$$

In a similar way, again, with the aid of mentioned identity (2.7) we can write

$$\begin{aligned}\operatorname{div}(\mathbf{III}) &= \operatorname{div}\left[(\dot{f} \mathbf{Q} + f \dot{\mathbf{Q}}) \theta \otimes \frac{1}{r} f^2 \left(\frac{n-2}{r} \mathbf{I}_n - \mathbf{A}\right) \theta\right] \\ &= [\nabla(\dot{f} \mathbf{Q} \theta + f \dot{\mathbf{Q}} \theta)] \frac{1}{r} f^2 \left[\frac{n-2}{r} \mathbf{I}_n - \mathbf{A}\right] \theta \\ &\quad + \operatorname{div}\left\{\frac{n-2}{r^2} f^2 \theta - \frac{1}{r} f^2 \mathbf{A} \theta\right\} [\dot{f} \mathbf{Q} + f \dot{\mathbf{Q}}] \theta\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r} f^2 \left[\frac{\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}}{r} + r \frac{d}{dr} \left(\frac{\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}}{r} \right) \theta \otimes \theta \right] \left[\frac{n-2}{r} \mathbf{I}_n - \mathbf{A} \right] \theta \\
&\quad + \left\{ \operatorname{div} \left(\frac{n-2}{r^2} f^2 \theta \right) - \operatorname{div} \left(\frac{1}{r} f^2 \mathbf{A} \theta \right) \right\} [\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}] \theta.
\end{aligned}$$

In order to further simplify the *second* term on the *right* in the last identity, for the sake of convenience, we first introduce the *skew-symmetric* matrix

$$\mathbf{F} = \mathbf{F}(r) := \frac{1}{r} f^2 \mathbf{A}.$$

Then a straight-forward *differentiation* shows that

$$\begin{aligned}
\operatorname{div} \left(\frac{1}{r} f^2 \mathbf{A} \right) &= \operatorname{div}(\mathbf{F}\theta) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left\{ \sum_{j=1}^n \mathbf{F}_{ij} \theta_j \right\} \\
&= \sum_{i=1}^n \left\{ \sum_{j=1}^n \dot{\mathbf{F}}_{ij} \theta_i \theta_j + \frac{1}{r} \sum_{j=1}^n \mathbf{F}_{ij} (\delta_{ij} - \theta_i \theta_j) \right\} \\
&= \langle \dot{\mathbf{F}}\theta, \theta \rangle + \frac{1}{r} \operatorname{tr} \mathbf{F} - \frac{1}{r} \langle \mathbf{F}\theta, \theta \rangle = 0,
\end{aligned}$$

where in the last implication we have used the fact that $\mathbf{F}$ is a *skew-symmetric* matrix. Consequently, a straight-forward calculation gives

$$\begin{aligned}
\operatorname{div} \left(\frac{n-2}{r^2} f^2 \theta \right) &= \frac{n-2}{r^2} f \left(2\dot{f} + (n-3) \frac{f}{r} \right) \\
&= \frac{n-2}{r^{n-1}} \frac{d}{dr} (r^{n-3} f^2).
\end{aligned}$$

Therefore upon substituting these quantities back gives

$$\begin{aligned}
\operatorname{div}(\mathbf{III}) &= \frac{1}{r} f^2 \left[\frac{n-2}{r^2} (\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}) + (n-2) \langle \theta, \theta \rangle \frac{d}{dr} \left(\frac{\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}}{r} \right) \right. \\
&\quad \left. - \frac{1}{r} (\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}) \mathbf{A} - r \langle \mathbf{A}\theta, \theta \rangle \frac{d}{dr} \left(\frac{\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}}{r} \right) \right] \theta \\
&\quad + \frac{n-2}{r^{n-1}} \frac{d}{dr} (r^{n-3} f^2) [\dot{f}\mathbf{Q} + f\dot{\mathbf{Q}}] \theta \\
&= \mathbf{Q} \left[\frac{n-2}{r^2} f^2 \mathbf{Q}^t (\ddot{f}\mathbf{Q} + 2\dot{f}\dot{\mathbf{Q}} + f\ddot{\mathbf{Q}}) - \frac{1}{r^2} f^2 (\dot{f}\mathbf{I}_n + f\mathbf{Q}^t \dot{\mathbf{Q}}) \mathbf{A} \right. \\
&\quad \left. + \frac{n-2}{r^{n-1}} \frac{d}{dr} (r^{n-3} f^2) (\dot{f}\mathbf{I}_n + f\mathbf{Q}^t \dot{\mathbf{Q}}) \right] \theta \\
&= \mathbf{Q} \left[\frac{n-2}{r^{n-1}} \frac{d}{dr} (r^{n-3} f^2 \dot{f}) \mathbf{I}_n - \frac{1}{r^2} f^2 \dot{f} \mathbf{A} \right. \\
&\quad \left. + \frac{n-3}{r^2} f^3 \mathbf{A}^2 + \frac{n-2}{r^{n-1} f} \frac{d}{dr} (r^{n-3} f^4 \mathbf{A}) \right] \theta, \tag{2.9}
\end{aligned}$$

where in concluding the *last* line we have used the substitutions $\dot{\mathbf{Q}} = \mathbf{Q}\mathbf{A}$ and $\ddot{\mathbf{Q}} = \mathbf{Q}[\dot{\mathbf{A}} + \mathbf{A}^2]$. On the other hand, referring to the definition of the functions α and β given in Proposition 2.1, a direct verification reveals that

$$\begin{aligned} \nabla\alpha &= \nabla \left[(n-2)\frac{1}{r^3}f^3 + \frac{1}{r}f\dot{f}^2 + \frac{1}{r}f^3|\dot{\mathbf{Q}}\theta|^2 \right] \\ &= \nabla \left[(n-2)\frac{1}{r^3}f^3 + \frac{1}{r}f\dot{f}^2 + \frac{1}{r}f^3|\mathbf{A}\theta|^2 \right] \\ &= \left\{ \frac{d}{dr} \left(\frac{n-2}{r^3}f^3 + \frac{1}{r}f\dot{f}^2 \right) \mathbf{I}_n + r^2 \frac{d}{dr} \left(\frac{1}{r^3}f^3 \right) |\mathbf{A}\theta|^2 \mathbf{I}_n \right. \\ &\quad \left. + \frac{2}{r^2}f^3[\mathbf{A}^t\mathbf{A} + r\langle\mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\mathbf{I}_n] \right\} \theta \end{aligned}$$

and consequently

$$\begin{aligned} \nabla\alpha + \left(\langle\nabla\beta, \theta\rangle + \frac{n-1}{r}\beta \right) \theta &= \nabla\alpha + \frac{d}{dr} \left(\frac{1}{r^2}f^2\dot{f} \right) \theta - \langle\nabla\alpha, \theta\rangle\theta + \frac{n-1}{r}\beta\theta \\ &= \left\{ \frac{1}{r^{n-1}} \left[(n-1)(2-n)r^{n-5}f^3 \right. \right. \\ &\quad \left. \left. - (n-1)r^{n-3}f\dot{f}^2 + \frac{d}{dr}(r^{n-3}f^2\dot{f}) \right] \mathbf{I}_n \right. \\ &\quad \left. - \frac{2}{r^2}f^3\mathbf{A}^2 - \frac{n-1}{r^2}f^3|\mathbf{A}\theta|^2\mathbf{I}_n \right\} \theta. \end{aligned} \quad (2.10)$$

The required conclusion now follows by substituting (2.10) into (2.8) and combining it with (2.9), respectively. We abbreviate the details and the proof is thus complete. $\square$

The rest of this section is devoted to the evaluation of the first variation of the energy functional (1.1) by restricting it to the space of generalised twists. Here we bring argument in more details, present a precise formulation and examine of the corresponding Euler–Lagrange equations. Before proceeding further, however, as can be easily verified for a generalised twists u , we have

$$|[\nabla u][\nabla u]^t|^2 = \left(\frac{n-1}{r^4} + \frac{2}{r^2} \right) f^4 + [\dot{f}^2 + f^2|\dot{\mathbf{Q}}\theta|^2]^2.$$

Now with the aid of the above identity together with (2.4) the energy functional $\mathbb{F}_{\sigma_2}$ restricted to the space of generalised twist can be described by integral⁶

⁶Note that here $\omega_n = \mathcal{H}^n(\mathbb{B}^n) = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2} + 1)$ and for a generalised twist u we have

$$\det \nabla u = \dot{f} \left(\frac{f}{r} \right)^{n-1}. \quad (2.11)$$

$$\begin{aligned}
\mathbb{F}_{\sigma_2}[u; \mathbb{X}] &:= \frac{1}{4} \int_{\mathbb{X}} [|\nabla u|^4 - |[\nabla u][\nabla u]^t|^2 + 4\Phi(\det \nabla u)] dx \\
&= \frac{1}{4} \int_a^b \int_{\mathbb{S}^{n-1}} \left\{ (n-1)(n-2) \left(\frac{1}{r} f \right)^4 + 2(n-1) \left(\frac{1}{r} f \dot{f} \right)^2 \right. \\
&\quad \left. + 2(n-2) \left(\frac{1}{r} f^2 \right)^2 |\dot{\mathbf{Q}}\theta|^2 \right. \\
&\quad \left. + 4\Phi \left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right\} r^{n-1} d\mathcal{H}^{n-1}(\theta) dr \\
&= \frac{\omega_n}{4} \int_a^b \left\{ (n-2) \frac{1}{r^2} f^4 \left[n(n-1) \frac{1}{r^2} + 2|\dot{\mathbf{Q}}|^2 \right] \right. \\
&\quad \left. + 2n(n-1) \frac{1}{r^2} f^2 \dot{f}^2 + 4n\Phi \left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right\} r^{n-1} dr,
\end{aligned}$$

where in the last identity we have used the advantage of Proposition A.4 in [14]. Now, motivated by the above representation in what follows we introduce the energy functional

$$\begin{aligned}
\mathbb{E}_{\sigma_2}[\mathbf{Q}, f] &:= \int_a^b \left\{ (n-2) f^4 \left[n(n-1) \frac{1}{r^2} + 2|\dot{\mathbf{Q}}|^2 \right] \right. \\
&\quad \left. + 2n(n-1) f^2 \dot{f}^2 + 4nr^2 \Phi \left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right\} r^{n-3} dr.
\end{aligned}$$

Associated with the energy functional $\mathbb{E}_{\sigma_2}$ and those discussed earlier in the beginning of this section we introduce the space of *admissible* loops

$$\mathcal{E} = \mathcal{E}[a, b] := \left\{ \begin{array}{l} \mathbf{Q} \in W^{1,2}([a, b], \mathbf{SO}(n)), \\ \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n, \\ (\mathbf{Q}, f) : f \in W^{1,2}[a, b], \\ \dot{f} > 0 \mathcal{L}^1\text{-a.e. on } (a, b), \\ f(a) = a, f(b) = b. \end{array} \right\}$$

Proposition 2.3. *Let $(\mathbf{Q}, f) \in \mathcal{E}$ with $\mathbf{Q} \in C^2([a, b], \mathbf{SO}(n))$, $f \in C^2([a, b])$ and $\dot{f} > 0$ on $]a, b[$. Then assuming $\mathbb{E}_{\sigma_2}[\mathbf{Q}, f] < \infty$ the Euler–Lagrange equations associated with $\mathbb{E}_{\sigma_2}[\cdot, \cdot]$ over $\mathcal{E}[a, b]$ at $(\mathbf{Q}, f)$ take the form*

$$\mathbb{E}\mathbb{L}_{\sigma_2}[(\mathbf{Q}, f)] = 0,$$

which depending on n can be expressed as:

[1] The case $n = 2$.⁷

$$\frac{d}{dr} \left[\frac{1}{r} f^2 \dot{f} + f \Phi' \right] = \frac{1}{r} f \dot{f}^2 + \dot{f} \Phi'. \quad (2.12)$$

⁷As a matter of fact we will see, in the Theorem 5.2, that $f \equiv r$ is the *only* solution of the Euler–Lagrange equation (2.12).

[2] *The case $n \geq 3$.*

$$\begin{cases} \text{(i)} & \frac{d}{dr} \left[r^{n-3} f^4 \mathbf{Q}^t \frac{d}{dr} \mathbf{Q} \right] = \mathbf{0}, \\ \text{(ii)} & \frac{d}{dr} \left[r^{n-3} f^2 \dot{f} + \frac{1}{n-1} f^{n-1} \Phi' \right] = (n-2) r^{n-5} f^3 + r^{n-3} f \dot{f}^2 \\ & \quad + \dot{f} f^{n-2} \Phi' + \frac{2(n-2)}{n(n-1)} r^{n-3} f^3 |\dot{\mathbf{Q}}|^2, \end{cases}$$

where $\Phi' = \Phi'(f(\frac{f}{r})^{n-1})$ and in either of these cases, above, the equations are hold on $]a, b[$.

Proof. The proof is only a *minor* modification of that given in Proposition 3.2 in [16] and therefore will be abbreviated. $\square$

Remark 2.1. It is also interesting to note that when $n = 2$ then the energy functional $\mathbb{E}_{\sigma_2}[\cdot, \cdot]$ over the space of *admissible* loops $\mathcal{E}[a, b]$ is *independent* of $\mathbf{Q}$. Indeed in this particular case, for every admissible loops $(\mathbf{Q}, f) \in \mathcal{E}[a, b]$ a straight-forward calculation gives

$$\begin{aligned} \mathbb{E}_{\sigma_2}[\mathbf{Q}, f] &= \mathbb{E}_{\sigma_2}[\mathbf{I}_2, f] = 2\pi^{-1} \mathbb{F}_{\sigma_2}[f(r)\theta; \mathbb{X}] \\ &= 2\pi^{-1} \mathbb{F}_{\sigma_2}[f(r)\mathbf{Q}(r)\theta; \mathbb{X}]. \end{aligned}$$

It is instructive to compare this fact with both Corollary 3.1 in [19] and Remark 3.1 in [14].

3. Existence, regularity and multiplicity of σ_2 -stationary loops

Consider as in the previous section the energy functional $\mathbb{E}_{\sigma_2}[\cdot, \cdot]$ over the space of *admissible* loops $\mathcal{E}[a, b]$. In this section we turn to the question of *existence* and multiplicity of solutions to the Euler–Lagrange equations. As the argument and the main ideas here are largely similar to those in [16, pp. 371–379] we shall remain brief and focus mainly on the differences. Indeed we focus attention on the Euler–Lagrange equation and present a class of σ_2 -stationary loops that in turn will prove fruitful in discussing the *existence* of multiple solutions to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over the space $\mathcal{A}(\mathbb{X})$.

On passing we note that in *dimension* $n = 2$, referring to Remark 2.1, the energy functional $\mathbb{E}_{\sigma_2}[\cdot, \cdot]$ over the space of admissible loops $\mathcal{E}[a, b]$ is *independent* of $\mathbf{Q}$. Hence, here, the question of existence and multiplicity of minimizers is interesting in the dimensions $n \geq 3$.

In what follows, depending on the dimension begin *even* or *odd*, we establish the existence of multiple (*infinitely* many) σ_2 -stationary loops $\mathbf{G} = f\mathbf{Q}$ where the pair $(\mathbf{Q}, f) \in \mathcal{E}$ as the following *specific* forms.

[1] ($4 \leq n = 2k$)

$$\mathbf{Q} = \mathbf{Q}[\mathbf{g}] := \mathbf{P} \text{diag}[\mathcal{R}(g_1), \mathcal{R}(g_2), \dots, \mathcal{R}(g_k)] \mathbf{P}^t,$$

[2] ($3 \leq n = 2k + 1$)

$$\mathbf{Q} = \mathbf{Q}[\mathbf{g}] := \mathbf{P} \text{diag}[\mathcal{R}(g_1), \mathcal{R}(g_2), \dots, \mathcal{R}(g_k), 1] \mathbf{P}^t,$$

where $\mathbf{P} \in \mathbf{O}(n)$ is *fixed*, $\mathbf{g} = (g_1, g_2, \dots, g_k) \in \mathcal{G}_{\mathbf{m}}$ (*see below*) and the matrix $\mathcal{R} \in \mathbb{M}_{2 \times 2}$ is given by

$$\mathcal{R}(s) := \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}.$$

Indeed for $\mathbf{m} := (m_1, m_2, \dots, m_k) \in \mathbb{Z}^k$ we put

$$\mathcal{G}_{\mathbf{m}} = \mathcal{G}_{\mathbf{m}}(a, b) := \left\{ \begin{array}{l} \mathbf{g} \in [W^{1,2}(a, b)]^k, \\ \mathbf{g}(a) = \mathbf{0}, \mathbf{g}(b) = 2\pi\mathbf{m}, \\ (\mathbf{g}, f) : f \in W^{1,2}(a, b), \\ \dot{f} > 0 \mathcal{L}^1\text{-a.e.on}(a, b), \\ f(a) = a, f(b) = b. \end{array} \right\}.$$

It is thus evident that for each such $\mathbf{m}$ and *fixed* $\mathbf{P} \in \mathbf{O}(n)$ and with $\mathbf{Q} = \mathbf{Q}[\mathbf{g}]$ we have that

$$(\mathbf{g}, f) \in \mathcal{G}_{\mathbf{m}} \iff (\mathbf{Q}, f) \in \mathcal{E}.$$

Next for $(\mathbf{g}, f) \in \mathcal{G}_{\mathbf{m}}$ as described above and *fixed* $\mathbf{P} \in \mathbf{O}(n)$ denoting again $\mathbf{Q} = \mathbf{Q}[\mathbf{g}]$ we introduce

$$\begin{aligned} \mathbb{G}_{\sigma_2}[\mathbf{g}, f] &:= \mathbb{E}_{\sigma_2}[\mathbf{Q}, f] = \int_a^b \left\{ (n-2)f^4 \left[n(n-1)\frac{1}{r^2} + 4|\dot{\mathbf{g}}|^2 \right] \right. \\ &\quad \left. + 2n(n-1)f^2\dot{f}^2 + 4nr^2\Phi\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right) \right\} r^{n-3} dr \\ &=: \int_a^b \mathbf{E}(r, \dot{\mathbf{g}}, f, \dot{f}) dr, \end{aligned} \quad (3.1)$$

where we have set $|\dot{\mathbf{g}}|^2 = \sum_{j=1}^k \dot{g}_j^2$ whilst

$$\begin{aligned} \mathbf{E}(r, \mathbf{s}, z, p) &:= n \left\{ (n-2)z^4 \left[(n-1)\frac{1}{r^2} + \frac{4}{n}|\mathbf{s}|^2 \right] \right. \\ &\quad \left. + 2(n-1)z^2p^2 + 4r^2\Phi\left(p\left(\frac{z}{r}\right)^{n-1}\right) \right\} r^{n-3}. \end{aligned}$$

We end this discussion by the following theorem which is the main results of this section. This theorem states the existence and regularity of *infinitely* many σ_2 -stationary loops.

Theorem 3.1. (Existence and regularity of multiple σ_2 -stationary loops) *Let $n \geq 3$ and consider the energy functional $\mathbb{G}_{\sigma_2}[\cdot, \cdot]$ over the space $\mathcal{G}_{\mathbf{m}}$. Then for each $\mathbf{m} \in \mathbb{Z}^k$ there exists $(\mathbf{g}, f) \in \mathcal{G}_{\mathbf{m}}$ such that*

$$\mathbb{G}_{\sigma_2}[\mathbf{g}, f] = \inf_{\mathcal{G}_{\mathbf{m}}} \mathbb{G}_{\sigma_2}[\cdot, \cdot].$$

In addition the pair $(\mathbf{g}, f)$ satisfies the corresponding Euler–Lagrange equations

$$\mathbb{EL}_{\sigma_2}[\mathbf{g}, f] = 0,$$

that is,

$$\begin{cases} \frac{d}{dr} \left[r^{n-3} f^4 \dot{\mathbf{g}} \right] = \mathbf{0}, \\ \frac{d}{dr} \left[r^{n-3} f^2 \dot{f} + \frac{1}{n-1} f^{n-1} \Phi' \right] = (n-2) r^{n-5} f^3 + r^{n-3} f \dot{f}^2 \\ \quad + \dot{f} f^{n-2} \Phi' + \frac{4(n-2)}{n(n-1)} r^{n-3} f^3 |\dot{\mathbf{g}}|^2, \end{cases}$$

on $]a, b[$ where $\Phi' = \Phi'(\dot{f}(\frac{f}{r})^{n-1})$ whilst $(\mathbf{g}, f) \in C^2[a, b] \times \cdots \times C^2[a, b]$ and $\dot{f} > 0$ on $[a, b]$.

Proof. The proof here follows by using an adaption of the argument form [16, Theorem 5.3] and hence will be abbreviated. $\square$

Note that the above Euler–Lagrange equations will be shown to be satisfied by the pair $(\mathbf{g}, f)$ as a result of its *minimizing* property. One can then verify that the latter equations result from those in Proposition 2.3 upon making the substitution $(\mathbf{Q}, f) = (\mathbf{Q}[\mathbf{g}], f)$. Thus any such $(\mathbf{g}, f)$ gives rise to an associated σ_2 -stationary loop!

Remark 3.1. A close inspection of the proof of Theorem 2.6(ii) in [8] reveals that the conclusion in previous theorem holds if the assumption **E** being of class C^1 is replaced by $\mathbf{E}_{\dot{f}}$ being of class C^0 . In a similar way the conclusion of Theorem 2.6(iii) holds if the assumption **E** being of class C^2 is replaced by $\mathbf{E}_{\dot{f}}$ being of class C^1 .

4. The restricted versus the full Euler–Lagrange equations

The primary aim of this section is to give a complete *characterisation* of all those σ_2 -stationary loops $(\mathbf{Q}, f) \in \mathcal{E}$ whose resulting *generalised* twist

$$u(x) = f(r)\mathbf{Q}(r)\theta$$

furnishes a solution to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over the space $\mathcal{A}(\mathbb{X})$. Before this, however, we pause to discuss in detail the implications that the *original* Euler–Lagrange equations [see Definition 5.1 below] will exert upon the *twist* loop associated with a *generalised* twist.

Theorem 4.1. *Let $\mathbb{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ and u be a generalised twist as described. Assume in addition that the twist loop $\mathbf{G} = f\mathbf{Q}$ satisfies the following assumptions.*

- [1] $\mathbf{Q} \in C^2(]a, b[, \mathbf{SO}(n))$,
- [2] $f \in C^2(]a, b[)$,
- [3] $\dot{f} > 0$ on $]a, b[$,
- [4] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then, depending on whether the dimension n is two or higher, the following implications hold.

[a] *The case $n = 2$.*

$$\mathbb{E}\mathbb{L}_{\sigma_2}[(\mathbf{Q}, f)] = 0 \implies \mathbb{E}\mathbb{L}_{\sigma_2}[u] := [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + (\det \nabla u) \nabla [\Phi'(\det \nabla u)] = 0,$$

[b] *The case $n \geq 3$.*

$$\begin{aligned} \mathbb{E}\mathbb{L}_{\sigma_2}[(\mathbf{Q}, f)] = 0 \implies \mathbb{E}\mathbb{L}_{\sigma_2}[u] &:= [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + (\det \nabla u) \nabla [\Phi'(\det \nabla u)] \\ &= \frac{f^3}{r^2} \left\{ (n-5) \frac{1}{r^2} f [|\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n - \dot{\mathbf{Q}}^t \dot{\mathbf{Q}}] \right. \\ &\quad \left. + 2(n-2) \dot{f} \left[\frac{1}{n} |\dot{\mathbf{Q}}|^2 - |\dot{\mathbf{Q}}\theta|^2 \right] \mathbf{I}_n \right\} \theta. \end{aligned}$$

Proof. Firstly suppose $n \geq 3$. We proceed by evaluating each of the *expressions* in $\mathbb{E}\mathbb{L}_{\sigma_2}[u]$ separately. Indeed with regards to the *first* term by substituting (i) and (ii), the Euler–Lagrange equations, in Proposition 2.3 into (2.5) in Proposition 2.2 upon simplification we arrive at the identity

$$\begin{aligned} \mathfrak{D}_{\sigma_2}[u] &= \mathbf{Q} \left\{ \frac{f^3}{r^2} \left[(n-5) \mathbf{A}^2 - (n+1) |\mathbf{A}\theta|^2 \mathbf{I}_n \right] \right. \\ &\quad \left. + \left[2(n-2) \frac{1}{nr^2} f^3 |\dot{\mathbf{Q}}|^2 - \left(\frac{f}{r} \right)^{n-1} \frac{d}{dr} (\Phi') \right] \mathbf{I}_n \right\} \theta \\ &=: \mathbf{Q} \left\{ \frac{f^3}{r^2} \left[(n-5) \mathbf{A}^2 - (n+1) |\mathbf{A}\theta|^2 \mathbf{I}_n \right] + \gamma \mathbf{I}_n \right\} \theta, \end{aligned} \quad (4.1)$$

where for the sake of convenience we have introduced

$$\begin{aligned} \gamma &= \gamma(r) \\ &:= 2(n-2) \frac{1}{nr^2} f^3 |\dot{\mathbf{Q}}|^2 - \left(\frac{f}{r} \right)^{n-1} \frac{d}{dr} \left[\Phi' \left(\left(\frac{f}{r} \right)^{n-1} \right) \right]. \end{aligned} \quad (4.2)$$

Now using (2.3) in conjunction with the above equation (4.1) we can write

$$\begin{aligned} [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] &= \left[\frac{f}{r} \mathbf{Q}^t + \left(\dot{f} - \frac{f}{r} \right) \theta \otimes \mathbf{Q} \theta + f \theta \otimes \dot{\mathbf{Q}} \theta \right] \\ &\quad \times \mathbf{Q} \left\{ \frac{f^3}{r^2} \left[(n-5) \mathbf{A}^2 - (n+1) |\mathbf{A}\theta|^2 \mathbf{I}_n \right] + \gamma \mathbf{I}_n \right\} \theta \\ &= \left[\frac{f}{r} \mathbf{I}_n + \left(\dot{f} - \frac{f}{r} \right) \theta \otimes \theta + f \theta \otimes \mathbf{A} \theta \right] \\ &\quad \times \left\{ \frac{f^3}{r^2} \left[(n-5) \mathbf{A}^2 - (n+1) |\mathbf{A}\theta|^2 \mathbf{I}_n \right] + \gamma \mathbf{I}_n \right\} \theta, \end{aligned} \quad (4.3)$$

where in concluding the *last* equation we have made repeated *use* of the notation $\mathbf{A} = \mathbf{Q}^t \dot{\mathbf{Q}}$. Moreover, upon expansion of the above Eq. (4.3) we have that

$$\begin{aligned}
[\nabla u]^t \mathfrak{D}_{\sigma_2}[u] &= \left\{ \frac{f^4}{r^3} \left[(n-5)\mathbf{A}^2 - (n+1)|\mathbf{A}\theta|^2 \mathbf{I}_n \right] + \gamma \frac{f}{r} \mathbf{I}_n \right. \\
&\quad + \left[\frac{f^3}{r^2} (n-5) \langle \theta, \mathbf{A}^2 \theta \rangle - \frac{f^3}{r^2} (n+1) |\mathbf{A}\theta|^2 \langle \theta, \theta \rangle \right. \\
&\quad \left. + \gamma \langle \theta, \theta \rangle \right] \left(\dot{f} - \frac{f}{r} \right) \mathbf{I}_n + \left[\frac{f^3}{r^2} (n-5) \langle \mathbf{A}\theta, \mathbf{A}^2 \theta \rangle \right. \\
&\quad \left. - \frac{f^3}{r^2} (n+1) |\mathbf{A}\theta|^2 \langle \mathbf{A}\theta, \theta \rangle + \gamma \langle \mathbf{A}\theta, \theta \rangle \right] f \mathbf{I}_n \left. \right\}.
\end{aligned}$$

However, in view of $\mathbf{A}$ being *skew*-symmetric it can be easily verified that $\langle \mathbf{A}\theta, \theta \rangle = 0$ and in a similar way $\langle \mathbf{A}\theta, \mathbf{A}^2 \theta \rangle = 0$. Therefore by substituting of these identities to the above equation we are lead, upon simplification, to the identity

$$\begin{aligned}
[\nabla u]^t \mathfrak{D}_{\sigma_2}[u] &= \left\{ (n-5) \frac{1}{r^3} f^4 [\mathbf{A}^2 + |\mathbf{A}\theta|^2 \mathbf{I}_n] \right. \\
&\quad \left. - 2(n-2) \frac{1}{r^2} \dot{f} f^3 |\mathbf{A}\theta|^2 \mathbf{I}_n + \gamma \dot{f} \mathbf{I}_n \right\} \theta. \quad (4.4)
\end{aligned}$$

On the other hand for the *second* term in $\mathbb{E}\mathbb{L}_{\sigma_2}[u]$ with the aid of (2.11) and (4.2), a direct verification reveals that

$$\begin{aligned}
(\det \nabla u) \nabla \left[\Phi'(\det \nabla u) \right] &= \dot{f} \left(\frac{f}{r} \right)^{n-1} \frac{d}{dr} \left[\Phi' \left(\dot{f} \left(\frac{f}{r} \right)^{n-1} \right) \right] \theta \\
&= \left[2(n-2) \frac{1}{nr^2} \dot{f} f^3 |\dot{\mathbf{Q}}|^2 - \dot{f} \gamma \right] \theta. \quad (4.5)
\end{aligned}$$

Therefore, *summarizing*, by combining (4.4) and (4.5) we arrive at

$$\begin{aligned}
\mathbb{E}\mathbb{L}_{\sigma_2}[u] &= [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + (\det \nabla u) \nabla \left[\Phi'(\det \nabla u) \right] \\
&= \left\{ (n-5) \frac{1}{r^3} f^4 [\mathbf{A}^2 + |\mathbf{A}\theta|^2 \mathbf{I}_n] \right. \\
&\quad - 2(n-2) \frac{1}{r^2} \dot{f} f^3 |\mathbf{A}\theta|^2 \mathbf{I}_n + \gamma \dot{f} \mathbf{I}_n \\
&\quad \left. + 2(n-2) \frac{1}{nr^2} \dot{f} f^3 |\dot{\mathbf{Q}}|^2 \mathbf{I}_n - \dot{f} \gamma \mathbf{I}_n \right\} \theta \\
&= \frac{f^3}{r^2} \left\{ (n-5) \frac{1}{r^2} f [\mathbf{A}^2 + |\mathbf{A}\theta|^2 \mathbf{I}_n] \right. \\
&\quad \left. + 2(n-2) \dot{f} \left[\frac{1}{n} |\dot{\mathbf{Q}}|^2 - |\mathbf{A}\theta|^2 \right] \mathbf{I}_n \right\} \theta,
\end{aligned}$$

which is the required conclusion in [b]. We now turn to the case $n = 2$. In order to prove [a] the similar argument, slightly different, we have used to prove implication [b] will give us

$$\mathbb{E}\mathbb{L}_{\sigma_2}[u] = -\frac{3}{r^4} f^4 [\mathbf{A}^2 + |\mathbf{A}\theta|^2 \mathbf{I}_n] \theta.$$

We abbreviate the details as it will follow closely that given in detail in the proof of [b] above. Finally, the assertion [a], *immediately*, follows by utilising the latter identity along with the fact that $\mathbf{A}^2 + |\mathbf{A}\theta|^2 \mathbf{I}_n = \mathbf{0}$. This completes the proof. $\square$

Theorem 4.2. Put $\mathbb{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ and let u be a generalised twist. Assume $\mathbb{F}_{\sigma_2}[u; \mathbb{X}] < \infty$ and that the twist loop $\mathbf{G} = f\mathbf{Q}$ satisfies the following assumptions.

- [1] $\mathbf{Q} \in C^2(]a, b[, \mathbf{SO}(n))$,
- [2] $f \in C^2(]a, b[)$,
- [3] $\dot{f} > 0$ on $]a, b[$,
- [4] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then the following equivalence between the full and the restricted Euler–Lagrange equations holds.

$$\mathbb{EL}_{\sigma_2}[u] = 0 \iff \begin{cases} [a] & (n = 2) & \mathbb{EL}_{\sigma_2}[(\mathbf{Q}, f)] = 0, \\ [b] & (n \geq 3) & \begin{aligned} & \text{(i) } \mathbb{EL}_{\sigma_2}[(\mathbf{Q}, f)] = 0, \\ & \text{(ii) } \dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n), \\ & \text{for all } r \in]a, b[. \end{aligned} \end{cases}$$

Proof. Let u be a generalised twist and let $\mathbf{G} = f\mathbf{Q}$ denote its twist loop. Indeed to begin note that in either of the cases [a] and [b] we have

$$\mathbb{EL}_{\sigma_2}[u] = 0 \implies \mathbb{EL}_{\sigma_2}[(\mathbf{Q}, f)] = 0. \quad (4.6)$$

This follows, e.g., by arguing as in Proposition 2.3 and noting that the equation on the *left* results from taking a larger class of variations in $\mathbb{F}_{\sigma_2}$ than that on the *right*. Furthermore, in view of [1]–[4] above an application of [a] in Theorem 4.1 immediately gives assertion [a]. To justify [b], *again*, an application of [b] in previous theorem gives

$$\begin{aligned} \mathbb{EL}_{\sigma_2}[u] = 0 &\iff [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + (\det \nabla u) \nabla [\Phi'(\det \nabla u)] = 0 \\ &\implies \left[(n-5) \frac{1}{r^2} f [|\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n - \dot{\mathbf{Q}}^t \dot{\mathbf{Q}}] \right. \\ &\quad \left. + 2(n-2) \dot{f} \left[\frac{1}{n} |\dot{\mathbf{Q}}|^2 - |\dot{\mathbf{Q}}\theta|^2 \right] \mathbf{I}_n \right] \theta = 0. \end{aligned} \quad (4.7)$$

With the aid of the *equivalence* and the implications in (4.6) and (4.7) we now proceed by establishing the *two* implications in the conclusion [b] of the theorem separately.

(*Sufficiency* “ $\Leftarrow$ ”)

Fix $r \in]a, b[$ and assume that $\dot{\mathbf{Q}}(r) \in \mathbb{R}\mathbf{SO}(n)$. Then by definition there exists $\rho = \rho(r) \in \mathbb{R}$ and $\mathbf{R} = \mathbf{R}(r) \in \mathbf{SO}(n)$ such that

$$\dot{\mathbf{Q}} = \rho \mathbf{R}.$$

A straight-forward calculation now gives

$$\begin{aligned}
 0 &= \left[(n-5) \frac{1}{r^2} f[\rho^2 - \rho^2] \right. \\
 &\quad \left. + 2(n-2) \dot{f} \left[\frac{1}{n} n \rho^2 - \rho^2 \right] \right] \mathbf{I}_n \\
 &= \left[(n-5) \frac{1}{r^2} f[|\dot{\mathbf{Q}}\theta|^2 \mathbf{I}_n - \dot{\mathbf{Q}}^t \dot{\mathbf{Q}}] \right. \\
 &\quad \left. + 2(n-2) \dot{f} \left[\frac{1}{n} |\dot{\mathbf{Q}}|^2 - |\dot{\mathbf{Q}}\theta|^2 \right] \mathbf{I}_n \right].
 \end{aligned}$$

Therefore if $\mathbb{E}\mathbb{L}_{\sigma_2}[(\mathbf{Q}, f)] = 0$ an application of [b] in Theorem 4.1 immediately gives $\mathbb{E}\mathbb{L}_{\sigma_2}[u] = 0$.

(Necessity “ $\implies$ ”)

Assume that $\mathbb{E}\mathbb{L}_{\sigma_2}[u] = 0$. Fix $r \in]a, b[$ and put $\mathbf{Q} := \mathbf{Q}(r)$ and $\mathbf{F} := \dot{\mathbf{Q}}(r)$. Then referring to (4.7) for every $\theta \in \mathbb{S}^{n-1}$ we have that

$$\begin{aligned}
 0 &= \left\langle \left[(n-5) \frac{1}{r^2} f[|\mathbf{F}\theta|^2 \mathbf{I}_n - \mathbf{F}^t \mathbf{F}] \right. \right. \\
 &\quad \left. \left. + 2(n-2) \dot{f} \left[\frac{1}{n} |\mathbf{F}|^2 - |\mathbf{F}\theta|^2 \right] \mathbf{I}_n \right] \theta, \theta \right\rangle \\
 &= (n-5) \frac{1}{r^2} f[|\mathbf{F}\theta|^2 - |\mathbf{F}\theta|^2] + 2(n-2) \dot{f} \left[\frac{1}{n} |\mathbf{F}|^2 - |\mathbf{F}\theta|^2 \right] \\
 &= 2(n-2) \dot{f} \left[\frac{1}{n} |\mathbf{F}|^2 - |\mathbf{F}\theta|^2 \right].
 \end{aligned}$$

In view of the latter being true for all $\theta \in \mathbb{S}^{n-1}$ and that $(n-2)\dot{f}(r) \neq 0$ it follows that

$$\frac{1}{n} |\mathbf{F}|^2 - |\mathbf{F}\theta|^2 = 0.$$

This gives $\mathbf{F} \in \mathbb{R}\mathbf{O}(n)$. In view of $\mathbf{Q}\mathbf{F}^t$ is *skew-symmetric* it follows that $\mathbf{Q}\mathbf{F}^t \in \mathbb{R}\mathbf{SO}(n)$ and so $\mathbf{F} \in \mathbb{R}\mathbf{SO}(n)$ (see [16, p. 383] for details). This together with (4.6) completes the proof. $\square$

Theorem 4.3. *Let $\mathbb{X} = \{x \in \mathbb{R}^n : a < |x| < b\}$ and let u be a generalised twist. Assume $\mathbb{F}_{\sigma_2}[u; \mathbb{X}] < \infty$ and that the twist loop $\mathbf{G} = f\mathbf{Q}$ satisfies the following assumptions.*

- [1] $\mathbf{Q} \in C^2(]a, b[, \mathbf{SO}(n))$,
- [2] $f \in C^2(]a, b[)$,
- [3] $\dot{f} > 0$ on $]a, b[$,
- [4] $(\mathbf{Q}, f) \in \mathcal{E}$.

Then the following equivalence between the full and the restricted Euler–Lagrange equations hold.

$$\mathbb{E}\mathbb{L}_{\sigma_2}[u] = 0 \iff \left\{ \begin{array}{l} [a] \ (n = 2) \quad \begin{array}{l} (i) \ \mathbb{E}\mathbb{L}_{\sigma_2}[(\mathbf{Q}, f)] = 0, \\ \text{Moreover we have that} \\ (ii) \ f \in C^2[a, b]. \end{array} \\ [b] \ (4 \leq n = 2k) \quad \begin{array}{l} \text{There exist } g = g(r) \in C^2[a, b] \text{ with} \\ g(a), g(b) \in 2\pi\mathbb{Z} \text{ and } \mathbf{P} \in \mathbf{O}(n) \text{ such} \\ \text{that} \\ (i) \ \mathbf{Q} = \mathbf{P} \text{diag}[\mathcal{R}(g), \dots, \mathcal{R}(g)] \mathbf{P}^t, \\ (ii) \ \mathbb{E}\mathbb{L}_{\sigma_2}[g, f] = 0. \\ \text{Moreover we have that} \\ (iii) \ f \in C^2[a, b]. \end{array} \\ [c] \ (n = 2k + 1) \quad \begin{array}{l} (i) \ \mathbf{Q} = \mathbf{I}_n, \\ (ii) \ \mathbb{E}\mathbb{L}_{\sigma_2}[0, f] = 0, \\ \text{Moreover we have that} \\ (iii) \ f \in C^2[a, b]. \end{array} \end{array} \right.$$

Note that in [b](ii) and [c](ii) above we have denoted

$$\mathbb{E}\mathbb{L}_{\sigma_2}[g, f] = 0,$$

as an abbreviation for the second order system

$$\left\{ \begin{array}{l} \frac{d}{dr} [r^{n-3} f^4 \dot{g}] = 0, \\ \frac{d}{dr} \left[r^{n-3} f^2 \dot{f} + \frac{1}{n-1} f^{n-1} \Phi' \right] = (n-2) r^{n-5} f^3 + r^{n-3} f \dot{f}^2 \\ \quad + \dot{f} f^{n-2} \Phi' + \frac{2(n-2)}{(n-1)} r^{n-3} f^3 \dot{g}^2, \end{array} \right.$$

where $\Phi' = \Phi'(\dot{f}(\frac{f}{r})^{n-1})$ on $]a, b[$.

Proof. Although the result is not in any obvious way but the proof is only a minor modification of the arguments in Theorem 6.3 that given in [16] and for this we shall remain brief. $\square$

Remark 4.1. It is interesting to note that as a result of the above theorems we have *uniqueness* up to *isometries* [i.e., the choices of the $\mathbf{P} \in \mathbf{O}(n)$] associated with *minimizers* in all cases $n = 2k \geq 4$.

5. A characterisation of all twist solutions

In the previous section we focused on all those σ_2 -stationary loops $\mathbf{G} = f\mathbf{Q}$ whose resulting *generalised* twist furnished a solution to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ over the space $\mathcal{A}(\mathbb{X})$. The aim of this section is to show how this analysis enables one to give a *complete* characterisation of all such *twist* solutions.

Towards this end we begin by considering the case of *even* dimensions ($n = 2k$). Here for each *fixed* $m \in \mathbb{Z}$ we set

$$\mathcal{J}_m = \mathcal{J}_m(a, b) := \left\{ (g, f) : \begin{array}{l} g \in W^{1,2}(a, b), \\ g(a) = 0, g(b) = 2\pi m, \\ f \in W^{1,2}(a, b), \\ \dot{f} > 0 \mathcal{L}^1\text{-a.e. on } (a, b), \\ f(a) = a, f(b) = b. \end{array} \right\}$$

and

$$\begin{aligned} \mathbb{J}_{\sigma_2}[g, f] &:= \mathbb{G}_{\sigma_2}[\mathbf{g}, f] \\ &= \int_a^b \left\{ (n-2)f^4 \left[n(n-1)\frac{1}{r^2} + 2n\dot{g}^2 \right] \right. \\ &\quad \left. + 2n(n-1)f^2\dot{f}^2 + 4nr^2\Phi\left(\dot{f}\left(\frac{f}{r}\right)^{n-1}\right) \right\} r^{n-3} dr \end{aligned}$$

where $\mathbf{g} = (g, g, \dots, g)$. With the aid of these notations we have the following statement.

Theorem 5.1. (Existence and regularity of special σ_2 -stationary loops) *Consider the energy functional $\mathbb{J}_{\sigma_2}$ over the space $\mathcal{J}_m$. Then for each $m \in \mathbb{Z}$ there exist $g = g(r; a, b, m)$ and $f = f(r; a, b, m)$ with $(g, f) \in \mathcal{J}_m$ such that*

$$\mathbb{J}_{\sigma_2}[g, f] = \inf_{\mathcal{J}_m} \mathbb{J}_{\sigma_2}[\cdot, \cdot].$$

Moreover the pair (g, f) satisfies the corresponding Euler–Lagrange equations

$$\mathbb{EL}_{\sigma_2}[g, f] = 0,$$

which depending on the dimension, n , can be expressed as:

[1] The case $n = 2$,

$$\frac{d}{dr} \left[\frac{1}{r} f^2 \dot{f} + f \Phi' \right] = \frac{1}{r} f \dot{f}^2 + \dot{f} \Phi'. \quad (5.1)$$

[2] The case $4 \leq n = 2k$,

$$\left\{ \begin{array}{l} \frac{d}{dr} [r^{n-3} f^4 \dot{g}] = 0, \\ \frac{d}{dr} \left[r^{n-3} f^2 \dot{f} + \frac{1}{n-1} f^{n-1} \Phi' \right] = (n-2)r^{n-5} f^3 + r^{n-3} f \dot{f}^2 \\ \quad + \dot{f} f^{n-2} \Phi' + \frac{2(n-2)}{n-1} r^{n-3} f^3 \dot{g}^2, \end{array} \right.$$

where $\Phi' = \Phi'(\dot{f}(\frac{f}{r})^{n-1})$ and in either of [1] or [2], the equations are hold on $]a, b[$. Additionally in the case [1] $f \in C^2[a, b]$, in the case [2] $(g, f) \in C^2[a, b] \times C^2[a, b]$ and $\dot{f} > 0$ on $[a, b]$ in the either of these cases.

Proof. The argument here is similar to that given in Theorem 3.1 and hence will be abbreviated. $\square$

We now return to the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$ defined over the space of admissible maps $\mathcal{A}(\mathbb{X})$. For the sake of clarity and future reference we proceed with the definition of *classical* solution.

Definition 5.1. (*Classical solution*) An admissible map $u \in \mathcal{A}(\mathbb{X})$ is referred to as a *classical* solution to the Euler–Lagrange equations associated with the energy functional (1.1) over the space (1.2) if and only if the following hold:

- [1] $\mathbb{F}_{\sigma_2}[u; \mathbb{X}] < \infty$,
- [2] $u \in C^2(\mathbb{X}, \mathbb{R}^n) \cap C(\overline{\mathbb{X}}, \mathbb{R}^n)$,
- [3] u satisfies the *system* of equations⁸

$$\begin{cases} [\nabla u]^t \mathfrak{D}_{\sigma_2}[u] + (\det \nabla u) \nabla [\Phi'(\det \nabla u)] = \mathbf{0} & \text{in } \mathbb{X}, \\ \det \nabla u > 0 & \text{in } \mathbb{X}, \\ u(x) = x & \text{on } \partial \mathbb{X}. \end{cases}$$

Note that when speaking of a *classical* solution in the form of a *generalised* twist [i.e., $u(x) = f(r)\mathbf{Q}(r)\theta$] in connection with [2] above we *implicitly* assume the pair $(\mathbf{Q}, f)$ to be of class C^2 , i.e., that $f \in C^2[a, b[$ and $\mathbf{Q} \in C^2[]a, b[, \mathbf{SO}(n))$. Moreover, in connection with [3] we have $\det \nabla u > 0$ in $\mathbb{X} \iff \dot{f} > 0$ in $]a, b[$ [see (2.11)].

We are now in a position to present the main result of this paper which is a complete characterisation of all *twist* solutions to the Euler–Lagrange equations associated with the energy functional $\mathbb{F}_{\sigma_2}[\cdot; \mathbb{X}]$.

Theorem 5.2. (Characterisation of all twist solutions) *Let $\mathbb{X} := \{x \in \mathbb{R}^n : a < |x| < b\}$ and consider the energy functional $\mathbb{F}[\cdot; \mathbb{X}]$ over the space $\mathcal{A}(\mathbb{X})$. Then the set $\mathfrak{S}$ of all classical solutions in the form of generalised twists to the corresponding Euler–Lagrange equations can be characterised as follows.*

- [1] ($n = 2k$) We consider the case $n = 2$ and $n \geq 4$ separately as follows:
 - [1a] ($n = 2$) $\mathfrak{S}$ is infinite and any generalised twist $u \in \mathfrak{S}$ can be described as

$$u(x) = r\mathcal{R}(g)\theta,$$

where $g \in C([a, b]) \cap C^2(]a, b[)$.

- [1b] ($4 \leq n = 2k$) $\mathfrak{S}$ is infinite and any generalised twist $u \in \mathfrak{S}$ can be described as

$$\begin{aligned} u(x) &= \mathbf{G}(r)\theta \\ &= f(r)\mathbf{P}\text{diag}[\mathcal{R}(g), \mathcal{R}(g), \dots, \mathcal{R}(g)]\mathbf{P}^t\theta, \end{aligned}$$

where $\mathbf{P} \in \mathbf{O}(n)$ and $f, g \in C^2[a, b]$ satisfy the second order system (notation as in Theorem 5.1)

$$\mathbb{EL}_{\sigma_2}[g, f] = 0.$$

- [2] ($n = 2k + 1$) $\mathfrak{S}$ consists of the single map $u \equiv x$.

⁸Recall that $\mathfrak{D}_{\sigma_2}[u] := \Delta_4 u - \text{div}[[\nabla u][\nabla u]^t[\nabla u]]$.

Proof. [1] The reason why $\mathfrak{S}$ is *infinite* immediately follows from Theorem 5.1 and the remaining assertions follow from [a] and [b] in Theorem 4.3. To complete the proof in part [1a] we need to justify that $f(r) = r$ is the only solution to the Euler–Lagrange equation (5.1). To this end we proceed by writing

$$\begin{aligned} \frac{d}{dr} \left[\frac{1}{r} f^2 \dot{f} + f \Phi' \right] &= \frac{1}{r} f \dot{f}^2 + \dot{f} \Phi' \iff \frac{d}{dr} \left[\frac{1}{r} f \dot{f} \right] (1 + \Phi'') f = 0 \\ &\iff \frac{d}{dr} \left[\frac{1}{r} f \dot{f} \right] = 0 \\ &\iff f^2(r) = c_1 r^2 + c_2, \end{aligned} \quad (5.2)$$

where in concluding the second line we have used this fact that $(\Phi'' + 1)f > 0$, since $f > 0$ and Φ is convex. Finally, using the boundary conditions $f(a) = a$ and $f(b) = b$ together with (5.2) reveal that $f(r) = r$ is the *only* solution to the described Euler–Lagrange equations. Putting these conclusions together gives the required result.

- [2] Assume $u \in \mathfrak{S}$. Then referring to [c] in Theorem 4.3 it follows that $\mathbf{Q} = \mathbf{I}_n$ while $f \in C^2[a, b]$ and $\mathbb{E}\mathbb{L}_{\sigma_2}[0, f] = 0$. Evidently $f \equiv r$ is a solution to the latter. An application of the phase-plane argument in [13, pp. 111–117] shows that the latter is indeed the *only* solution. $\square$

We end the paper by a very quick outline of some further results regarding the limiting behaviour of the generalised σ_2 -stationary twists from Theorem 5.2 as the *inner* hole shrinks to a point. This is particularly interesting since in the limit (the punctured ball) *all* components of the function space collapse to a single one and so it is important to have a clear understanding as to how the twist solutions and their energies behave.⁹

Remark 5.1. Let $\Omega_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\}$ where $n = 2k$ and for fixed $m \in \mathbb{Z}$ let (u_ε) denote the family of generalised twist obtained from [1b] in Theorem 5.2. Then $u_\varepsilon \rightarrow x$ in $W^{1,2}(\mathbb{B}, \mathbb{R}^n)$ and $u_\varepsilon \rightarrow x$ *uniformly* on $\mathbb{B}$ as $\varepsilon \downarrow 0$.

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⁹The reader is referred to Section 8 in [16] for a detailed coverage of this limiting property.

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