A blow-up result in a nonlinear abstract evolution system with delay

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Abstract. In this paper we consider a second-order abstract evolution system with delay. We show that, under suitable conditions on the initial data and the involved functionals, the energy of solutions blows up in finite-time. We also give some concrete examples to illustrate our main result.

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1. Introduction

Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally not only depend on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. For instance, for the system

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) + a_0 u_t(x, t) + a u_t(x, t - \tau) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, t) = 0 & \text{in } \Gamma_0 \times (0, \infty) \\ \frac{\partial u}{\partial \nu}(x, t) = 0 & \text{in } \Gamma_1 \times (0, \infty), \end{cases} \quad (1.1)$$

it is well-known, in the absence of delay ($a = 0, a_0 > 0$), that this system is exponentially stable, see [9, 27]. In the presence of delay ($a > 0$), Nicaise and Pignotti [17] examined system (1.1) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay (i.e., $0 < a < a_0$), that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is unstable. The same results have been obtained for

the case of boundary delay. See also [1] for the treatment to this problem in more general abstract form and [16, 19, 20] for analogous results in the case of time-varying delay. When the delay term in (1.1) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} a(s)u_t(x, t-s)ds,$$

exponential stability results have been obtained in [18] under the condition

$$\int_{\tau_1}^{\tau_2} a(s)ds < a_0.$$

For global nonexistence, Messaoudi [10] considered the following initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t|u_t|^{m-2} = u|u|^{p-2} & \text{in } \Omega \times (0, \infty) \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{in } \Omega, \end{cases} \quad (1.2)$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$, and $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive nonincreasing function. He showed, under suitable conditions on g , that solutions with initial negative energy blow up in finite time if $p > m$ and continue to exist if $m \geq p$. This result has been later pushed, by the same author [14], to certain solutions with positive initial energy. A similar result has been also obtained by Wu [24] using a different method.

In the absence of the viscoelastic term ($g = 0$), problem (1.2) has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$u_{tt} - \Delta u + au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0, \infty), \quad (1.3)$$

$m, \gamma \geq 0$, it is well known that, when $a = 0$, the source term $b|u|^\gamma u$, ($\gamma > 0$) causes finite time blow up of solutions with negative initial energy (see [3]). The interaction between the damping and the source terms was first considered by Levine [6, 7] in the linear damping case ($m = 0$). Using a concavity argument, he showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [4] extended Levine's result to the nonlinear damping case ($m > 0$). In their work, the authors introduced a different method and showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended to solutions with negative initial energy by Messaoudi [11] and others. For results of same nature, we refer the reader to Levine and Serrin [5], and Vitillaro [23], Messaoudi and Said-Houari [13].

For problem (1.3) in $\mathbb{R}^n$, we mention, among others, the work of Levine Serrin and Park [8], Todorova [21, 22], Messaoudi [12], and Zhou [26]. Recently, Autuori et al. [2] investigated a global nonexistence for nonlinear Kirchhoff system. They established their result using a new approach. Motivated by the

above works, Wu and Lin [25], studied the following initial boundary value problem for an integro-differential equation

$$u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(x, s) ds + Q(x, t, u_t) = f(x, u), \quad \text{in } \Omega \times (0, \infty),$$

where $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive nonincreasing function, Q satisfying, for some positive measurable function $d(x, t)$, $m > 1$,

$$Q(x, t, v)v \geq 0 \quad \text{and} \quad Q(x, t, v) \leq [d(x, t)]^{\frac{1}{m}} [Q(x, t, v)v]^{\frac{1}{m'}},$$

and $f(x, u) \in C(\Omega \times R; R)$ satisfies, for $d_1 > 0$, $p > 1$,

$$|f(x, u)| \leq d_1 |u|^{p-1}.$$

They showed, under the above conditions, that the global nonexistence results can be extended to a bigger region.

In this paper we first consider the following abstract system of nonlinear damped second-order evolution equation with delay

$$\begin{cases} u_{tt}(t) + Au(t) + G(u_t(t)) + \mu G(u_t(t - \tau)) = F(u(t)), & \text{in } (0, \infty), \\ u_t(t - \tau) = f_0(t - \tau), & t \in (0, \tau), \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \quad (1.4)$$

where μ is a real constant, with $|\mu| < 1$, $\tau > 0$ represents the time delay, A is a positive self-adjoint operator in a Hilbert H (with norm and inner product denoted by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$). Let $V := D(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$ so that

$$V \subset H \subset V'$$

with continuous embedding. We prove, under suitable conditions on the initial data and the functionals F and G that the energy of the system blows up in finite time. Some specific examples that illustrated our general setting are then presented.

The paper is organized as follows. In Sect. 2, we present our assumptions and establish some preliminary results. In Sect. 3, we state and prove our main result that is also based on a concavity argument. Section 4 is devoted to some concrete examples in the aim to illustrate our abstract result.

2. Assumptions and preliminary results

Similarly to [17], we introduce the new variable

$$z(\rho, t) = u_t(t - \tau\rho), \quad \rho \in (0, 1), \quad t > 0.$$

Thus, we have

$$\tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0.$$

So, problem (1.4) takes the form

$$\begin{cases} u_{tt}(t) + Au(t) + G(u_t(t)) + \mu G(z(1, t)) = F(u(t)), & \text{in } (0, \infty), \\ \tau z_t(\rho, t) + z_\rho(\rho, t) = 0, & \rho \in (0, 1), \quad t > 0, \\ z(\rho, 0) = f_0(-\tau\rho), & \rho \in (0, 1), \\ z(0, t) = u_t(t), & t > 0, \\ u(0) = u_0, & u_t(0) = u_1. \end{cases} \quad (2.1)$$

In order to state and prove our result, we require the following assumptions:

(A1) Assume that F and G are continuous mappings from V into V' such that there exist continuous and Fréchet differentiable mappings ψ, χ from V into $[0, \infty)$ satisfying

$$D\psi = F, \quad D\chi = G$$

and, for $p > m \geq 2$,

$$\langle F(u), u \rangle_{V' \times V} \geq p\psi(u), \quad \forall u \in V, \quad \langle G(u), u \rangle_{V' \times V} \geq m\chi(u), \quad \forall u \in V. \quad (2.2)$$

(A2) For all $\delta > 0$, we have

$$|\langle G(u), v \rangle_{V' \times V}| \leq \delta^m \chi(v) + (m-1)\delta^{-\frac{m}{m-1}} \chi(u), \quad \forall u, v \in V. \quad (2.3)$$

(A3) There exists a positive constant C such that, for all $\theta \in [\frac{2}{p}, 1]$, one has

$$(\psi(u))^\theta \leq C (\psi(u) + \|u\|_V^2), \quad \forall u \in V. \quad (2.4)$$

(A4) There exists a positive constant C_1 such that

$$\begin{aligned} \chi(u) &\leq C_1 (\psi(u))^{\frac{m}{p}}, \quad \forall u \in V, \\ \|u\|_H &\leq C_1 (\psi(u))^{\frac{1}{p}}, \quad \forall u \in V. \end{aligned} \quad (2.5)$$

Remark 2.1. Condition (A2) is equivalent to (A2') For all $\epsilon > 0$, we have

$$|\langle G(u), v \rangle_{V' \times V}| \leq m\epsilon \chi(u) + \left(\frac{\epsilon m}{m-1} \right)^{1-m} \chi(v), \quad \forall u, v \in V. \quad (2.6)$$

Definition 2.1. For $T > 0$ fixed, $u : [0, T) \rightarrow H$ is called a solution of (2.1) if $u \in C^2([0, T); H) \cap C^1([0, T); V) \cap C([0, T); D(A))$, $z \in C^1([0, 1] \times [0, T); H)$, if the identity (2.1)₁ (resp. (2.1)₂) holds in V' (resp. in H) with the initial conditions specified in (2.1), so that

$$z(\rho, t) = u_t(t - \tau\rho), \quad \rho \in (0, 1), \quad t \in [0, T).$$

The energy at time t of a solution u of (2.1) is defined by

$$E(t) = \frac{1}{2} (\|u_t\|_H^2 + \|u\|_V^2) - \psi(u) + \xi \int_0^1 \chi(z(\rho, t)) d\rho,$$

where ξ is fixed such that

$$\tau |\mu| (m-1) < \xi < \tau (m - |\mu|) \quad (2.7)$$

Lemma 2.1. If u is the solution of (2.1), then there exists $C \geq 0$ such that

$$E'(t) \leq -C (\chi(u_t(t)) + \chi(z(1, t))) \leq 0.$$

Proof. Using (2.1)₁, we obtain

$$\begin{aligned} & \langle u_{tt}(t), u_t(t) \rangle_{V' \times V} + \langle Au(t), u_t(t) \rangle_{V' \times V} + \langle G(u_t(t)), u_t(t) \rangle_{V' \times V} \\ & + \mu \langle G(z(1, t)), u_t(t) \rangle_{V' \times V} - \langle F(u), u_t(t) \rangle_{V' \times V} = 0. \end{aligned}$$

This gives, by virtue of (A1),

$$\frac{d}{dt} \left\{ \frac{1}{2} (\|u_t\|_H^2 + \|u\|_V^2) - \psi(u) \right\} + \mu \langle G(z(1, t)), u_t(t) \rangle_{V' \times V} \leq -m \chi(u_t(t)). \quad (2.8)$$

Similarly, by (2.1)₂, we have

$$\tau \left\langle z_t(\rho, t), \frac{\xi}{\tau} G(z(\rho, t)) \right\rangle_{V' \times V} + \left\langle z_\rho(\rho, t), \frac{\xi}{\tau} G(z(\rho, t)) \right\rangle_{V' \times V} = 0,$$

which is

$$\xi \frac{d}{dt} \chi(z(\rho, t)) + \frac{\xi}{\tau} \frac{d}{d\rho} \chi(z(\rho, t)) = 0.$$

Integration over $(0, 1)$, with respect to ρ , yields

$$\xi \frac{d}{dt} \int_0^1 \chi(z(\rho, t)) d\rho = \frac{\xi}{\tau} [\chi(u_t(t)) - \chi(z(1, t))]. \quad (2.9)$$

By summing (2.8) and (2.9), we arrive at

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} (\|u_t\|_H^2 + \|u\|_V^2) - \psi(u) + \xi \int_0^1 \chi(z(\rho, t)) d\rho \right\} \leq \\ & -\mu \langle G(z(1, t)), u_t(t) \rangle_{V' \times V} - \left(m - \frac{\xi}{\tau} \right) \chi(u_t(t)) - \frac{\xi}{\tau} \chi(z(1, t)). \end{aligned} \quad (2.10)$$

Using (2.6), with $\epsilon = (m - 1)/m$, we estimate the first term of the right-hand side of (2.10) as follows:

$$-\mu \langle G(z(1, t)), u_t(t) \rangle_{V' \times V} \leq |\mu| [(m - 1) \chi(z(1, t)) + \chi(u_t(t))].$$

Hence, we get from (2.10),

$$\begin{aligned} \frac{dE(t)}{dt} & \leq - \left(m - \frac{\xi}{\tau} - |\mu| \right) \chi(u_t(t)) - \left(\frac{\xi}{\tau} - (m - 1) |\mu| \right) \chi(z(1, t)) \\ & \leq -C \{ \chi(u_t(t)) + \chi(z(1, t)) \} \leq 0, \end{aligned} \quad (2.11)$$

where

$$C = \min \left\{ m - \frac{\xi}{\tau} - |\mu|, \frac{\xi}{\tau} - (m - 1) |\mu| \right\},$$

which is positive by (2.7). This completes the proof of the Lemma.

Let us introduce the functional

$$H(t) = -E(t) = \psi(u) - \frac{1}{2} (\|u_t\|_H^2 + \|u\|_V^2) - \xi \int_0^1 \chi(z(\rho, t)) d\rho \quad (2.12)$$

□

and prove the following

Lemma 2.3. *Under the above assumptions, we have*

$$(\psi(u))^\theta \leq C \left(3\psi(u) - \|u_t\|_H^2 - 2\xi \int_0^1 \chi(z(\rho, t)) d\rho - 2H(t) \right), \quad \forall u \in V,$$

for all $\theta \in [\frac{2}{p}, 1]$.

Proof. Using (A3), we have, for all $\theta \in [\frac{2}{p}, 1]$,

$$(\psi(u))^\theta \leq C (\psi(u) + \|u\|_V^2), \quad \forall u \in V.$$

Substituting $\|u\|_V^2$ from (2.12), we obtain the desired result. $\square$

3. Blow up

In this section we state and prove our main result.

Theorem 3.1. *Let $(u_0, u_1) \in V \times H$ and suppose that (A1)–(A4) hold. Assume further that $p > m$ and*

$$E(0) := \frac{1}{2} \|u_1\|_H^2 + \frac{1}{2} \|u_0\|_V^2 - \psi(u_0) + \xi \int_0^1 \chi(f_0(-\rho\tau)) d\rho < 0. \quad (3.1)$$

Then the solution of (2.1) blows up in finite time.

Proof. Recalling (2.11), we have

$$E(t) \leq E(0) < 0.$$

Hence, by (2.12), we get

$$0 < H(0) \leq H(t) \leq \psi(u). \quad (3.2)$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon(u_t, u)_H, \quad t \geq 0, \quad (3.3)$$

where $\varepsilon > 0$ will be specified later and

$$0 < \alpha \leq \min \left\{ \frac{p-2}{2p}, \frac{p-m}{p(m-1)} \right\}. \quad (3.4)$$

A direct differentiation of L gives

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|_H^2 - \varepsilon \|u\|_V^2 + \varepsilon \langle F(u), u \rangle_{V' \times V} \\ &\quad - \varepsilon \langle G(u_t(t)), u \rangle_{V' \times V} - \varepsilon \mu \langle G(z(1, t)), u \rangle_{V' \times V}. \end{aligned} \quad (3.5)$$

We then use (A2) to estimate the last two terms:

$$\varepsilon |\mu| \langle G(z(1, t)), u \rangle_{V' \times V} \leq \varepsilon |\mu| \left[(m-1) \delta^{\frac{-m}{m-1}} \chi(z(1, t)) + \delta^m \chi(u) \right]$$

and

$$\varepsilon \langle G(u_t(t)), u \rangle_{V' \times V} \leq \varepsilon \left[(m-1) \delta^{\frac{-m}{m-1}} \chi(u_t(t)) + \delta^m \chi(u) \right].$$

So, from Lemma 2.1, we obtain

$$\begin{aligned} L'(t) &\geq \left[(1-\alpha) H^{-\alpha}(t) - \varepsilon (1+|\mu|) \frac{m-1}{C} \delta^{\frac{-m}{m-1}} \right] H'(t) \\ &\quad + \varepsilon \|u_t\|_H^2 - \varepsilon \|u\|_V^2 + \varepsilon p \psi(u) - \varepsilon \delta^m (1+|\mu|) \chi(u). \end{aligned} \quad (3.6)$$

Of course (3.6) remains valid even if δ is time dependent. Therefore by taking δ so that

$$\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t),$$

for large k to be specified later, and substituting in (3.6) we arrive at

$$\begin{aligned} L'(t) \geq & \left[(1 - \alpha) - \varepsilon (1 + |\mu|) \frac{m-1}{C} k \right] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|_H^2 - \varepsilon \|u\|_V^2 \\ & + \varepsilon p \psi(u) - \varepsilon (1 + |\mu|) k^{1-m} H^{\alpha(m-1)}(t) \chi(u). \end{aligned} \quad (3.7)$$

Using (3.2), we find $H^\alpha(t) \leq (\psi(u))^\alpha$. Thus, exploiting (A4), we obtain

$$H^{\alpha(m-1)} \chi(u) \leq c (\psi(u))^{\alpha(m-1) + \frac{m}{p}}, \quad \text{for some } c > 0. \quad (3.8)$$

Inserting (3.8) in (3.7) and using (2.12), we get, for $0 < a < 1$,

$$\begin{aligned} L'(t) \geq & \left[(1 - \alpha) - \varepsilon (1 + |\mu|) \frac{m-1}{C} k \right] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|_H^2 - \varepsilon \|u\|_V^2 \\ & - \varepsilon (1 + |\mu|) k^{1-m} (\psi(u))^{\alpha(m-1) + \frac{m}{p}} + \varepsilon a p \psi(u) \\ & + \varepsilon (1 - a) p \left[H(t) + \frac{1}{2} (\|u_t\|_H^2 + \|u\|_V^2) + \xi \int_0^1 \chi(z(\rho, t)) d\rho \right] \\ \geq & \left[(1 - \alpha) - \varepsilon (1 + |\mu|) \frac{m-1}{C} k \right] H^{-\alpha}(t) H'(t) + \varepsilon \left(1 + \frac{p(1-a)}{2} \right) \|u_t\|_H^2 \\ & + \varepsilon \left(\frac{p(1-a)}{2} - 1 \right) \|u\|_V^2 + \varepsilon a p \psi(u) + \varepsilon (1 - a) p H(t) \\ & + \varepsilon (1 - a) p \xi \int_0^1 \chi(z(\rho, t)) d\rho - \varepsilon (1 + |\mu|) k^{1-m} (\psi(u))^{\alpha(m-1) + \frac{m}{p}}. \end{aligned}$$

Then we use Lemma 2.3, for $s = \frac{m}{p} + \alpha(m-1) \leq 1$, to deduce that

$$(\psi(u))^{\alpha(m-1) + \frac{m}{p}} \leq C \left[3\psi(u) - \|u_t\|_H^2 - 2\xi \int_0^1 \chi(z(\rho, t)) d\rho - 2H(t) \right].$$

Consequently, we obtain

$$\begin{aligned} L'(t) \geq & \left[(1 - \alpha) - \varepsilon (1 + |\mu|) \frac{m-1}{C} k \right] H^{-\alpha}(t) H'(t) \\ & + \varepsilon \left(1 + \frac{p(1-a)}{2} + c_2 k^{1-m} \right) \|u_t\|_H^2 + \varepsilon [(1-a)p + c_2 k^{1-m}] H(t) \\ & + \varepsilon \xi [(1-a)p + c_2 k^{1-m}] \int_0^1 \chi(z(\rho, t)) d\rho \\ & + \varepsilon \left(\frac{p(1-a)}{2} - 1 \right) \|u\|_V^2 + \varepsilon [ap - c_3 k^{1-m}] \psi(u), \end{aligned} \quad (3.9)$$

for two positive constants c_2 and c_3 .

At this point, we choose $a > 0$ so small that makes

$$\frac{p(1-a)}{2} - 1 > 0.$$

Then, we select k so large that

$$ap - c_3 k^{1-m} > 0.$$

Finally, we pick ε so small so that

$$(1 - \alpha) - \varepsilon(1 + |\mu|) \frac{m-1}{C} k > 0,$$

$$H^{1-\alpha}(0) + \varepsilon \langle u_1, u_0 \rangle_{V' \times V} > 0.$$

Thus, for some $\lambda > 0$, estimation (3.9) becomes

$$L'(t) \geq \lambda \left[\|u_t\|_H^2 + \|u\|_V^2 + H(t) + \psi(u) + \int_0^1 \chi(z(\rho, t)) d\rho \right]. \quad (3.10)$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad t \geq 0.$$

Next, using Cauchy–Schwarz inequality and (A4), we obtain

$$(u_t, u)_H \leq \|u\|_H \|u_t\|_H \leq C_1 (\psi(u))^{\frac{1}{p}} \|u_t\|_H.$$

Young's inequality then yields

$$|(u_t, u)_H|^{\frac{1}{1-\alpha}} \leq C \left((\psi(u))^{\frac{\mu}{(1-\alpha)p}} + \|u_t\|_H^{\frac{\theta}{1-\alpha}} \right), \quad (3.11)$$

for $1/\mu + 1/\theta = 1$. To be able to use (A3), we take $\theta = 2(1 - \alpha)$ which gives $\mu/[p(1 - \alpha)] = 2/[p(1 - 2\alpha)] \leq 1$. Therefore, for $s = 2/[p(1 - 2\alpha)]$, (3.11) becomes

$$|(u_t, u)_H|^{\frac{1}{1-\alpha}} \leq C \left((\psi(u))^s + \|u_t\|_H^2 \right).$$

Lemma 2.3 gives

$$|(u_t, u)_H|^{\frac{1}{1-\alpha}} \leq C \left[H(t) + \|u_t\|_H^2 + \psi(u) \right].$$

Therefore,

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= (H^{1-\alpha}(t) + \varepsilon |(u_t, u)_H|)^{\frac{1}{1-\alpha}} \\ &\leq C \left[H(t) + |(u_t, u)_H|^{\frac{1}{1-\alpha}} \right] \\ &\leq C \left[H(t) + \|u_t\|_2^2 + \psi(u) \right], \quad t \geq 0. \end{aligned} \quad (3.12)$$

Combining (3.10) and (3.12), we arrive at

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \quad t \geq 0. \quad (3.13)$$

where Λ is a positive constant depending only on λ and C .

A simple integration of (3.13) over $(0, t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \Lambda \alpha t / (1 - \alpha)}, \quad t < T^*.$$

Therefore, $L(t)$ blows up in time

$$T \leq T^* = \frac{1 - \alpha}{\Lambda \alpha L^{-\frac{\alpha}{1-\alpha}}(0)}.$$

This completes the proof. $\square$

4. Applications

Our result can be applied to a variety of problems. In this section, we consider only three illustrative problems.

In the whole section, Ω is a bounded and regular domain of $\mathbb{R}^n$, with $n \geq 1$. For shortness, the $L^p(\Omega)$ -norm, with $p > 1$, will be denoted by $\|\cdot\|_p$.

4.1. Problem 1

We consider the following system of nonlinear damped wave equation with delay

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) + \mu_1 |u_t(x, t)|^{m-2} u_t(x, t) + \mu_2 |u_t(x, t - \tau)|^{m-2} u_t(x, t - \tau) \\ = b |u(x, t)|^{p-2} u(x, t), \quad \text{in } \Omega \times (0, +\infty), \\ u_t(x, t - \tau) = f_0(x, t - \tau), \quad t \in (0, \tau), \\ u(x, t) = 0, \quad \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } \Omega, \end{cases} \quad (4.1)$$

where $p > m \geq 2$, b, μ_1 are positive constants, μ_2 is a real number such that $|\mu_2| < \mu_1$, and $\tau > 0$ represents the time delay. For this system, we obtain the next result.

Theorem 4.1. *Let $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ and $f_0 \in C^1([- \tau, 0]; L^2(\Omega))$. Suppose that $2 \leq m < p \leq \frac{2n}{n-2}$, if $n \geq 3$ and $2 \leq m < p \leq +\infty$ if $n = 1, 2$. Assume further that*

$$E(0) := \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{b}{p} \|u_0\|_p^p + \frac{\xi}{m} \int_{\Omega} \int_0^1 |f_0(x, -\rho\tau)|^m d\rho dx < 0, \quad (4.2)$$

where

$$\tau |\mu_2| (m - 1) < \xi < \tau (m\mu_1 - |\mu_2|). \quad (4.3)$$

Then the solution of (4.1) blows up in finite time.

Proof. The operator A is simply the Laplace operator with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and such that

$$Au = -\Delta u, \quad \forall u \in D(A).$$

It is well known that it is a positive selfadjoint operator in $H = L^2(\Omega)$ with $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ and $\|u\|_{D(A^{\frac{1}{2}})} = \|\nabla u\|_2$.

It then remains to verify the assumptions (A1)–(A4), for

$$F(v) = b |v|^{p-2} v, \quad G(v) = \mu_1 |v|^{m-2} v, \quad \psi(v) = \frac{b}{p} \int_{\Omega} |v|^p dx, \quad \chi(v) = \frac{\mu_1}{m} \int_{\Omega} |v|^m dx.$$

In fact, (A1) is trivial, (A2) can be obtained by Young's inequality, and (A4) is a direct result of the Lebesgue embedding. Finally (A3) is proved in the following Lemma. Applying Theorem 3.1, the desired result is established. $\square$

Lemma 4.2. *Suppose that $2 \leq p \leq \frac{2n}{n-2}$, if $n \geq 3$ and $2 \leq p \leq +\infty$ if $n = 1, 2$. Then there exists a positive constant $C > 1$ depending on Ω only such that*

$$\|u\|_p^s \leq C \left[\|u\|_p^p + \|\nabla u\|_2^2 \right],$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof. If $\|u\|_p \geq 1$ then

$$\|u\|_p^s \leq \|u\|_p^p.$$

If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq C \|\nabla u\|_2^2.$$

Note that the condition between p and m in Theorem 4.1 are the same as the case without delay (see [11, 12]), since we do not require local existence. $\square$

4.2. Problem 2

Here we consider the nonlinear damped plate equation with delay.

$$\begin{cases} u_{tt}(x, t) + \Delta^2 u(x, t) + \mu_1 |u_t(x, t)|^{m-2} u_t(x, t) + \mu_2 |u_t(x, t - \tau)|^{m-2} u_t(x, t - \tau) \\ = b |u(x, t)|^{p-2} u(x, t), & \text{in } \Omega \times (0, +\infty), \\ u_t(x, t - \tau) = f_0(x, t - \tau), & t \in (0, \tau), \\ u(x, t) = \frac{\partial u}{\partial n}(x, t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (4.4)$$

where $p > m \geq 2$, b, μ_1 are positive constants, μ_2 is a real number such that $|\mu_2| < \mu_1$, and $\tau > 0$ represents the time delay. Our main result is

Theorem 4.3. *Let $(u_0, u_1) \in [H^4(\Omega) \cap H_0^2(\Omega)] \times H_0^2(\Omega)$ and $f_0 \in C^1([- \tau, 0]; L^2(\Omega))$. Suppose that $2 \leq m < p \leq \frac{2n}{n-4}$, if $n \geq 5$ and $2 \leq m < p \leq +\infty$ if $n \leq 4$. Assume further that*

$$E(0) := \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \|\Delta u_0\|_2^2 - \frac{b}{p} \|u_0\|_p^p + \frac{\xi}{m} \int_{\Omega} \int_0^1 |f_0(x, -\rho\tau)|^m d\rho dx < 0, \quad (4.5)$$

where

$$\tau |\mu_2| (m - 1) < \xi < \tau (m\mu_1 - |\mu_2|). \quad (4.6)$$

Then the solution of (4.4) blows up in finite time.

Proof. The proof is the same as the one of Theorem 4.1 where we here use the continuous embedding $H^2(\Omega) \subset L^p(\Omega)$ for all $1 \leq p \leq \frac{2n}{n-4}$, if $n \geq 5$ and $1 \leq p \leq +\infty$ if $n \leq 4$. Here the operator A is defined as follows $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$ and

$$Au = \Delta^2 u, \quad \forall u \in D(A).$$

It is well known that it is a positive selfadjoint operator in $H = L^2(\Omega)$ with $D(A^{\frac{1}{2}}) = H_0^2(\Omega)$ and $\|u\|_{D(A^{\frac{1}{2}})} = \|\Delta u\|_2$. As before the condition between p

and m in Theorem 4.3 are the same as the case without delay (see [15]) as no local existence is required. $\square$

4.3. Problem 3

We consider the following system of damped wave equation with delay and a compact nonlinearity (compare with example II in [6])

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) \\ = \left(\int_{\Omega} K(x, y) u(y, t)^2 dy \right) u(x, t), & \text{in } \Omega \times (0, +\infty), \\ u_t(x, t - \tau) = f_0(x, t - \tau), & t \in (0, \tau), \\ u(x, t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (4.7)$$

where K is a bounded function from $\Omega \times \Omega \rightarrow (0, \infty)$ such that $K(x, y) = K(y, x)$ and satisfying

$$\alpha \leq K(x, y) \leq \beta, \quad \forall x, y \in \Omega, \quad (4.8)$$

for some positive real numbers α and β . As before μ_1 is a positive constant, μ_2 is a real number such that $|\mu_2| < \mu_1$, and $\tau > 0$ represents the time delay. Our main result is

Theorem 4.4. *Let $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ and $f_0 \in C^1([-\tau, 0]; L^2(\Omega))$. Assume that*

$$\begin{aligned} E(0) &:= \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{1}{4} \int_{\Omega \times \Omega} K(x, y) u_0(x)^2 u_0(y)^2 dx dy \\ &+ \frac{\xi}{2} \int_{\Omega} \int_0^1 |f_0(x, -\rho\tau)|^2 d\rho dx < 0, \end{aligned}$$

where

$$0 < \xi < \tau (\mu_1 - |\mu_2|).$$

Then the solution of (4.7) blows up in finite time.

Proof. The operator A is the same as the one in Sect. 4.1 defined in $H = L^2(\Omega)$. Again it then suffices to verify the assumptions (A1)–(A4), for $G(v) = \mu_1 v$ and

$$F(v)(x) = \left(\int_{\Omega} K(x, y) v(y)^2 dy \right) v(x), \quad \forall x \in \Omega,$$

for any $v \in H_0^1(\Omega)$. As in problem 1, the potential χ is given by

$$\chi(v) = \frac{\mu_1}{2} \int_{\Omega} |v|^2 dx,$$

while according to [6], we have

$$\psi(v) = \frac{1}{4} \int_{\Omega \times \Omega} K(x, y) v(x)^2 v(y)^2 dx dy.$$

Hence, (A1) holds with $m = 2$ and $p = 4$, while (A2) is again obtained by Young's inequality. To check (A3) and (A4), we notice that the assumption (4.8) on K yields

$$\frac{\alpha}{4} \left(\int_{\Omega} v(x)^2 dx \right)^2 \leq \psi(v) \leq \frac{\beta}{4} \left(\int_{\Omega} v(x)^2 dx \right)^2.$$

The first estimate yields directly (A4), while the second one furnishes (A3) by using an argument similar to Lemma 4.2.

Again without delay we recover the result from [6]. $\square$

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References

- [1] Ait Benhassi, E.M., Ammari, K., Boulite, S., Maniar, L.: Feedback stabilization of a class of evolution equations with delay. *J. Evol. Equ.* **9**, 103–121 (2009)
- [2] Autuori, G., Pucci, P., Salvatori, M.C.: Global nonexistence for nonlinear Kirchhoff systems. *Arch. Rat. Mech. Anal.* **196**, 489–516 (2010)
- [3] Ball, J.: Remarks on blow up and nonexistence theorems for nonlinear evolution equations. *Q. J. Math. Oxf.* **28**, 473–486 (1977)
- [4] Georgiev, V., Todorova, G.: Existence of solutions of the wave equation with nonlinear damping and source terms. *J. Differ. Equ.* **109**, 295–308 (1994)
- [5] Levine, H.A., Serrin, J.: A global nonexistence theorem for quasilinear evolution equation with dissipation. *Arch. Rat. Mech. Anal.* **137**, 341–361 (1997)
- [6] Levine, H.A.: Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$. *Trans. Am. Math. Soc.* **192**, 1–21 (1974)
- [7] Levine, H.A.: Some additional remarks on the nonexistence of global solutions to nonlinear wave equation. *SIAM J. Math. Anal.* **5**, 138–146 (1974)
- [8] Levine, H.A., Ro Park, S., Serrin, J.: Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. *J. Math. Anal. Appl.* **228**, 181–205 (1998)
- [9] Liu, K.: Locally distributed control and damping for the conservative systems. *SIAM J. Control Optim.* **35**, 1574–1590 (1997)
- [10] Messaoudi, S.A.: Blow up and global existence in a nonlinear viscoelastic wave equation. *Math. Nachr.* **260**, 58–66 (2003)

- [11] Messaoudi, S.A.: Blow up in a nonlinearly damped wave equation. *Math. Nachr.* **231**, 1–7 (2001)
- [12] Messaoudi, S.A.: Blow up in the Cauchy problem for a nonlinearly damped wave equation. *Commun. Appl. Anal.* **7**(2–3), 379–386 (2003)
- [13] Messaoudi, S.A., Said-Houari, B.: Blow up of solutions of a class of wave equations with nonlinear damping and source terms. *Math. Methods Appl. Sci.* **27**, 1687–1696 (2004)
- [14] Messaoudi, S.A.: Blow up of solutions with positive initial energy in a nonlinear viscoelastic equation. *J. Math. Anal. Appl.* **320**, 902–915 (2006)
- [15] Messaoudi, S.A.: Global existence and nonexistence in a system of Petrovsky. *J. Math. Anal. Appl.* **265**, 296–308 (2002)
- [16] Mustafa, M.I.: Uniform stability for thermoelastic systems with boundary time-varying delay. *J. Math. Anal. Appl.* **383**, 490–498 (2011)
- [17] Nicaise, S., Pignotti, C.: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.* **45**(5), 1561–1585 (2006)
- [18] Nicaise, S., Pignotti, C.: Stabilization of the wave equation with boundary or internal distributed delay. *Differ. Int. Equ.* **21**(9–10), 935–958 (2008)
- [19] Nicaise, S., Pignotti, C., Valein, J.: Exponential stability of the wave equation with boundary time-varying delay. *Discret. Contin. Dyn. Syst. Ser. S* **4**(3), 693–722 (2011)
- [20] Nicaise, S., Valein, J., Fridman, E.: Stability of the heat and the wave equations with boundary time-varying delays. *Discret. Contin. Dyn. Syst.* **2**(3), 559–581 (2009)
- [21] Todorova, G.: Cauchy problem for a nonlinear wave with nonlinear damping and source terms *CR . Acad. Sci. Paris Ser. I* **326**, 191–196 (1998)
- [22] Todorova, G.: Stable and unstable sets for the Cauchy problem for a nonlinear wave with nonlinear damping and source terms. *J. Math. Anal. Appl.* **239**, 213–226 (1999)
- [23] Vitillaro, E.: Global nonexistence theorems for a class of evolution equations with dissipation. *Arch. Rat. Mech. Anal.* **149**, 155–182 (1999)
- [24] Wu, S.T.: Blow-up of solutions for an integro-differential equation with a nonlinear source. *Elect. J. Differ. Equ.* **45**, 1–9 (2006)
- [25] Wu, S.T., Lin, C.Y.: Global nonexistence for an integro-differential equation. *Mathe. Methods Appl. Sci.* , (2011). doi:[10.1002/mma.1535](https://doi.org/10.1002/mma.1535)
- [26] Yong, Z.: A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in Ω . *Appl. Math. Lett.* **18**, 281–286 (2005)
- [27] Zuazua, E.: Exponential decay for the semi-linear wave equation with locally distributed damping. *Commun. Part. Differ. Equ.* **15**, 205–235 (1990)

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