Limit problems for a Fractional p -Laplacian as $p \rightarrow \infty$

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Abstract. The purpose of this work is the analysis of the solutions to the following problems related to the fractional p -Laplacian in a Lipschitzian bounded domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{\alpha p}} dy = f(x, u) & x \in \Omega, \\ u = g(x) & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\alpha \in (0, 1)$ and the exponent p goes to infinity. In particular we will analyze the cases:

- (i) $f = f(x)$.
- (ii) $f = f(u) = |u|^{\theta(p)-1}u$ with $0 < \theta(p) < p - 1$ and $\lim_{p \rightarrow \infty} \frac{\theta(p)}{p-1} = \Theta < 1$ with $g \geq 0$.

We show the convergence of the solutions to certain limit as $p \rightarrow \infty$ and identify the limit equation. In both cases, the limit problem is closely related to the Infinity Fractional Laplacian:

$$\mathcal{L}_\infty v(x) = \mathcal{L}_\infty^+ v(x) + \mathcal{L}_\infty^- v(x),$$

where

$$\mathcal{L}_\infty^+ v(x) = \sup_{y \in \mathbb{R}^N} \frac{v(y) - v(x)}{|y - x|^\alpha}, \quad \mathcal{L}_\infty^- v(x) = \inf_{y \in \mathbb{R}^N} \frac{v(y) - v(x)}{|y - x|^\alpha}.$$

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1. Introduction

In this work we are interested in describing the behaviour of the solutions to the Dirichlet problem for the fractional p -Laplacian operator, as the exponent $p \rightarrow \infty$. More precisely, given a bounded domain $\Omega \subset \mathbb{R}^N$, whose boundary, $\partial\Omega$, is Lipschitz continuous, we consider

$$\begin{cases} -\mathcal{L}_p u(x) = f(x, u) & x \in \Omega, \\ u(x) = g(x) & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $\alpha \in (0, 1)$, $p > N/\alpha$,

$$\mathcal{L}_p u(x) = \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|y - x|^{\alpha p}} dy, \quad (1.2)$$

and the boundary data $g \in W^{s,p}(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N)$ with $s = \alpha - N/p$, see Sect. 2. As function f we are specially interested in a couple of choices. We will analyze in detail the cases

$$f = f(x) \quad \text{and} \quad f = f(u) = |u|^{\theta(p)-1} u, \quad (1.3)$$

with

$$0 < \theta(p) < p - 1 \quad \text{such that} \quad \lim_{p \rightarrow \infty} \frac{\theta(p)}{p - 1} = \Theta < 1. \quad (1.4)$$

The requirements regarding the regularity of the function f will be specified in each of the statements. Nevertheless, our techniques are applicable to more general functions f . For instance, if $f(x, u)$ is a continuous function, $f(x, \cdot)$ nondecreasing and there exist two exponents $\theta_i(p)$, satisfying (1.4) and two positive constants $c_i > 0$ such that

$$c_1 |u|^{\theta_1-1} u \leq f(x, u) \leq c_2 |u|^{\theta_2-1} u.$$

More precisely, we study the limit as $p \rightarrow \infty$ of the weak solutions to (1.1), u_p , and we prove their convergence, up to subsequences, to certain function u_∞ . The main goal of this paper is to identify the limit problem for u_∞ .

It is well known (see for instance [2, 5, 20]) that for a nonnegative f , the solutions to the corresponding local problem for the p -Laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u_p|^{p-2} \nabla u_p) = f(x), & \text{in } \Omega, \\ u_p = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

converge, up to subsequences, to some function u_∞ , which satisfies, in viscosity sense, the following problem

$$\begin{cases} |\nabla u_\infty| = 1, & \text{in the set } \{f > 0\}, \\ \Delta_\infty u_\infty = 0, & \text{in the set } \overline{\{f > 0\}}^c. \end{cases}$$

The infinity Laplacian operator is defined as

$$\Delta_\infty u = \langle \nabla u D^2 u, (\nabla u)^T \rangle = \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}.$$

One of the most important motivations to analyze this kind of problems is the Lipschitz extension problem of a function $g \in W^{1,\infty}(\partial\Omega)$. Such extension is obtained as limit of solutions to problem (1.5) with $f(x) = 0$ and $u_p = g(x)$ on the boundary. Furthermore, this is the unique Lipschitz extension with best Lipschitz constant, $L_g(\partial\Omega)$, being stable for subdomains. It is known as AMLE, *Absolutely Minimizing Lipschitz Extension*, concept introduced by Aronsson

in the sixties, see for example [2, 14, 22, 26]. For an overview on these matters we suggest reference [3].

The equivalent nonlocal version for the fractional p -Laplacian, namely problem

$$\begin{cases} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\alpha p}} dx = 0, & \text{in } \Omega, \\ u_p = g(x), & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

is studied in [8]. The limit problem as $p \rightarrow \infty$ turns out to be

$$\begin{cases} \sup_{y \in \bar{\Omega}} \frac{u_{\infty}(y) - u_{\infty}(x)}{|y - x|^{\alpha}} + \inf_{y \in \bar{\Omega}} \frac{u_{\infty}(y) - u_{\infty}(x)}{|y - x|^{\alpha}} = 0, & \text{in } \Omega, \\ u_{\infty}(x) = g(x), & \text{on } \partial\Omega. \end{cases}$$

Moreover, if $g \in C^{0,\alpha}(\partial\Omega)$ the function u_{∞} is the optimal Hölder extension to $\bar{\Omega}$ of the Hölder boundary data g , in the sense that the Hölder seminorm for u_{∞} in Ω is always less than or equal to the one for the boundary data given on $\partial\Omega$. That is why this operator is named as Hölder Infinity Laplacian.

In [6] the authors also obtain the best Hölder extension of a function g defined in $\mathbb{R}^N \setminus \Omega$. Such extension solves the Dirichlet problem related to the Infinity Fractional Laplacian. This operator arises invoking nonlocal Tug-of-War games theory, instead of the variational approach used in [8].

We refer also to [15], where the authors study a problem related to (1.6), with the integrals performed over the ball $B(x, \rho(x))$ for some choice of $\rho \in C(\bar{\Omega})$.

We observe that in the references above, the integration set is bounded, while for our problem we integrate over the whole space. For this reason we need to impose the boundary condition in $\mathbb{R}^N \setminus \Omega$. The operator in (1.6) and our operator are related as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\alpha p}} dx &= \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\alpha p}} dx \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\alpha p}} dx. \end{aligned}$$

This additional term makes more delicate the analysis of our problem.

Regarding the choice $f = f(x)$, we see that, in addition, the limit function u_{∞} maximizes the functional

$$I(v) = \int_{\Omega} f v,$$

in a certain set of functions, related to the operator under consideration, see Lemma 6.2 below. This is a natural feature for this kind of limits, see for instance the pioneer work [5]. However, while we wrote this paper we found the article [16], where a different proof of Lemma 6.2 is given under another point of view. In [16] the maximizing problem is identified as the dual of certain mass transportation problem.

There exists also a precedent work for the local p -Laplacian when f is a power of u satisfying (1.4), see [10]. The limit problem is given by

$$\begin{cases} \min\{|\nabla u_\infty(x)| - u_\infty^\Theta, -\Delta_\infty u_\infty(x)\} = 0, & \text{in } \Omega, \\ u_\infty = 0, & \text{on } \partial\Omega. \end{cases}$$

In the passage to the limit the authors use the positivity of the eigenfunctions related to the infinity Laplacian. More precisely, in [18] the limit as p goes to infinity in the eigenvalue problem for the (local) p -Laplacian is studied, that is $f = \lambda_p u^{p-1}$. They show that the eigenfunctions u_p converge to some limit u_∞ , which is strictly positive in Ω and satisfies

$$\begin{cases} \min\left\{\frac{|\nabla u_\infty(x)|}{u_\infty(x)} - \Lambda_\infty^L, -\Delta_\infty u_\infty(x)\right\} = 0, & \text{in } \Omega, \\ u_\infty = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$\Lambda_\infty^L = \lim_{p \rightarrow \infty} \lambda_p^{1/p} = \frac{1}{\max\{\text{dist}(x, \partial\Omega), x \in \Omega\}}.$$

The corresponding non-local eigenvalue problem is treated in [21] and the limit problem is given by

$$\begin{cases} \min\{-\mathcal{L}_\infty^- u_\infty(x) - (\Lambda_\infty)^\alpha u_\infty(x), -\mathcal{L}_\infty u_\infty(x)\} = 0, & \text{in } \Omega, \\ u_\infty(x) = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Incidentally it happens that

$$(\Lambda_\infty)^\alpha = \frac{1}{(\max\{\text{dist}(x, \partial\Omega), x \in \Omega\})^\alpha} = (\Lambda_\infty^L)^\alpha,$$

so that both infinity eigenvalues (local and nonlocal) coincide when $\alpha = 1$. However, the equations are obviously not equivalent.

Making use of the problem above we determine the limit equation for the non-local problem (1.1), when f satisfies (1.4), see Theorem 6.10 below.

The rest of the paper is organized as follows. In the next section we introduce some notation and preliminary results about fractional Sobolev spaces. Section 3 is devoted to state the precise definition of weak and viscosity solutions. We see that for p large these weak solutions are in fact viscosity solutions. In Sects. 4 and 5 we show the existence of weak solutions to (1.1) with each of the choices in (1.3), using variational and sub-super solution methods, respectively. We also deduce some a priori estimates in the fractional Sobolev spaces, ensuring that u_p converges to some function, u_∞ as p goes to infinity. In Sect. 6 we determine the equations satisfied by this limit. We will show that when $f = f(x)$ the limit, u_∞ , turns out to be a maximizer of a certain problem that we mentioned in the introduction. This fact will help to find the limit u_∞ explicitly when f has a strict sign.

2. Preliminaries

The natural frame for problem (1.1) is within the context of fractional Sobolev spaces, (also known as Aronszajn, Gagliardo or Slobodeckij), $W^{s,p}(\mathbb{R}^N)$, with $0 < s < 1$ and $1 < p < \infty$. This space is defined as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(y) - u(x)|^p}{|y - x|^{sp+N}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = (\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_p)^{\frac{1}{p}},$$

where

$$\|u\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}$$

and

$$[u]_p = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|y - x|^{sp+N}} dx dy \right)^{\frac{1}{p}}$$

are the norm in the L^p space and the Gagliardo seminorm, respectively.

Notice that $W^{s,p}(\mathbb{R}^N)$ is an intermediary Banach space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$, see for instance [1, 13]. In fact to recover $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ as $s \rightarrow 0$ and $s \rightarrow 1$, respectively, we must consider a correcting factor in front of the semi-norm:

$$\left(s(1-s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|y - x|^{sp+N}} dx dy + \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}},$$

see [7, 23]. However, for fixed s both norms are equivalent and for simplicity we use the usual one.

For further literature we refer also to [11], where the fractional Sobolev spaces are studied without the use of interpolation theory. As in the usual Sobolev spaces, $W^{s,p}(\mathbb{R}^N)$ embeds into the Hölder-continuous functions for sufficiently large exponent, see for instance [11]. This result will be essential to pass to the limit in our problems.

Theorem 2.1. (Hölder embedding) *Let $sp > N$. For any $u \in W^{s,p}(\mathbb{R}^N)$ there exists a positive constant $\widehat{C}$ such that*

$$\|u\|_{C^{0,\beta}(\mathbb{R}^N)} \leq \widehat{C} \|u\|_{W^{s,p}(\mathbb{R}^N)},$$

where $\beta = \frac{sp-N}{p}$.

Remark 2.2. We notice here that we can follow the constants in the proof of the Hölder embedding (Lemma 2.2 in [12] and Theorem 8.2 in [11]) to see that, in fact, $\widehat{C}(N, p, s) \leq \widehat{C}(N, s)$ for p large. Hence, the constant related to Hölder embedding is uniformly bounded for p large.

Remark 2.3. The following Sobolev embeddings hold:

- if $sp < N$ then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ for $p^* = Np/(N - sp)$;
- if $sp = N$ then $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $r \in [p, \infty)$.

However, the restriction $sp \leq N$ is not applicable for our purposes of letting $p \rightarrow \infty$.

For us it is more convenient to denote $s = \alpha - \frac{N}{p}$. With this notation α has to verify that $\alpha p > N$ and $\frac{\alpha p - N}{p} < 1$, so as to ensure that $s \in (0, 1)$. When p is large we can take an arbitrary $\alpha \in (0, 1)$. Moreover, the solution of (1.1) is known outside Ω , then the natural space for problem (1.1) is

$$W_g^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : s = \alpha - \frac{N}{p}, u = g \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

In order to obtain a Poincaré inequality in $W_0^{s,p}(\Omega)$, valid for p large, we introduce the minimizing problem

$$\lambda_p = \inf_{\phi \in X_0} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(y) - \phi(x)|^p}{|y - x|^{\alpha p}} dx dy}{\int_{\mathbb{R}^N} |\phi|^p(x) dx}$$

and the space $X_0 = \{\phi \in C_0^\infty(\Omega) : \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$.

This minimization problem has been studied in [21]. In this work we can find the following result:

Lemma 2.4. *There exists a minimizer $\psi_p \in W_0^{s,p}(\Omega)$, which is positive in Ω . Moreover,*

(a) *the first eigenvalue satisfies*

$$(\lambda_p)^{1/p} \rightarrow \frac{1}{R^\alpha} \quad \text{as } p \rightarrow \infty, \quad (2.1)$$

with $R = \max\{\text{dist}(x, \mathbb{R}^N \setminus \Omega)\}$ being the radius of the largest ball inscribed in Ω ;

(b) *if ψ_p is normalized such that $\|\psi_p\|_{L^\infty(\mathbb{R}^N)} = 1$, then*

$$\psi_p \rightarrow \psi_\infty \quad \text{uniformly as } p \rightarrow \infty,$$

where the function ψ_∞ is continuous, positive in Ω and $\psi_\infty = 0$ in $\mathbb{R}^N \setminus \Omega$.

Corollary 2.5. (Poincaré inequality) *For all $v \in W_g^{s,p}(\Omega)$ and p large enough, there exists a positive constant, independent of p , such that*

$$\|v\|_{L^p(\mathbb{R}^N)} \leq \tilde{C}([v]_p + [g]_p) + \|g\|_{L^p(\mathbb{R}^N)}$$

Proof. In the case $g \equiv 0$, the inequality for $C = (1/\lambda_p)^{1/p}$ follows straightforward from the density of X_0 in $W_0^{s,p}(\Omega)$. Furthermore, using (2.1), for p large we can take $\tilde{C} = 2R^\alpha$.

For the general case, we observe that $v - g \in W_0^{s,p}(\Omega)$, then

$$\|v\|_{L^p(\mathbb{R}^N)} - \|g\|_{L^p(\mathbb{R}^N)} \leq \|v - g\|_{L^p(\mathbb{R}^N)} \leq C[v - g]_p \leq C([v]_p + [g]_p).$$

□

3. Weak and viscosity solutions

Let us specify the different notions of solutions that we use throughout this work. For a fixed value of p we consider weak solutions. However, in order to pass to the limit as $p \rightarrow \infty$, we need to introduce the concept of viscosity solutions. We will show that, for p large, weak solutions are indeed viscosity solutions.

Definition 3.1. We say that $u \in W_g^{s,p}(\Omega)$ is a weak subsolution (supersolution) to (1.1) if

- (i) $u \leq (\geq) g$ in $\mathbb{R}^N \setminus \Omega$.
- (ii) For all nonnegative $\phi \in W_0^{s,p}(\Omega)$ it holds that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|y - x|^{\alpha p}} (\phi(y) - \phi(x)) \, dx \, dy \\ & \leq (\geq) \int_{\mathbb{R}^N} f(x, u) \phi(x) \, dx. \end{aligned}$$

If u is a weak supersolution and weak subsolution, then we say that u is a weak solution to (1.1).

Remark 3.2. Note that from Theorem 2.1 if $p > 2N/\alpha$ we get that $u \in C^{0,\beta}(\mathbb{R}^N)$ with $\beta = (\alpha p - 2N)/p$.

Now we introduce the concept of viscosity solution.

Definition 3.3. An upper semicontinuous function u such that $u \leq g$ in $\mathbb{R}^N \setminus \Omega$ is a viscosity subsolution to (1.1) if whenever $x_0 \in \Omega$ and $\varphi \in C_c^1(\mathbb{R}^N)$ are such that

- (i) $\varphi(x_0) = u(x_0)$
- (ii) $u(x) \leq \varphi(x)$ for $x \neq x_0$

then

$$-\mathcal{L}_p \varphi(x_0) \leq f(x_0, \varphi(x_0)),$$

where $\mathcal{L}_p$ is defined in (1.2).

Definition 3.4. A lower semicontinuous function u such that $u \geq g$ in $\mathbb{R}^N \setminus \Omega$ is a viscosity supersolution to (1.1) if whenever $x_0 \in \Omega$ and $\phi \in C_c^1(\mathbb{R}^N)$ are such that

- (i) $u(x_0) = \phi(x_0)$
- (ii) $u(x) \geq \phi(x)$ for all $x \neq x_0$,

then

$$-\mathcal{L}_p \phi(x_0) \geq f(x_0, \phi(x_0)).$$

Definition 3.5. A continuous function u is a viscosity solution to (1.1) if it is a viscosity supersolution and a viscosity subsolution.

Remark 3.6. Since $\mathcal{L}_p(\phi + C) = \mathcal{L}_p\phi$ and $f(x, u)$ is non-decreasing in the second variable, the previous definitions are equivalent if the function $\phi(x) + C$ (or $\psi(x) - C$) touches u from below (from above, respectively) at x_0 . Indeed, for the definition of the supersolution we get

$$-\mathcal{L}_p\phi(x_0) = -\mathcal{L}_p(\phi(x_0) + C) \geq f(x_0, \phi(x_0) + C) \geq f(x_0, \phi(x_0)).$$

Remark 3.7. In the previous definitions we may assume that the test function touches u strictly. Indeed, for a test function ϕ touching u from below, we consider the function $h(x) = \phi(x) - \eta(x)$, where $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfies $\eta(x_0) = 0$ and $\eta(x) > 0$ for $x \neq x_0$. Notice that h touches u strictly. Moreover, since the function $s \mapsto |s|^{p-2}s$ is increasing we get $\mathcal{L}_ph(x_0) \leq \mathcal{L}_p\phi(x_0)$. Then,

$$-\mathcal{L}_ph(x_0) \geq -\mathcal{L}_p\phi(x_0) \geq f(x_0, \phi(x_0)) \geq f(x_0, h(x_0)),$$

where the last inequality follows from the fact that f is non-decreasing in the second variable.

For further details about general theory of viscosity solutions we refer to [9, 17, 19] for viscosity solutions related to the (local) ∞ -Laplacian and the p -Laplacian operators. Regarding the approach of viscosity solutions to integro-differential equations with singular kernels, we refer to the article [4].

Our next aim is to see that continuous weak solutions to (1.1) are also viscosity solutions to (1.1). The proof is standard, see for instance [19] and also [21]. We include it for convenience of the reader.

The first step is the following comparison principle, which is proved in [21].

Lemma 3.8. (Comparison Principle) *Let $\Omega \in \mathbb{R}^N$ be a domain and $u, v \in W^{s,p}(\mathbb{R}^N)$ with $s = \alpha - p/N$ be two continuous functions satisfying*

- (i) $v \geq u$ in $\mathbb{R}^N \setminus \Omega$.
- (ii) *For any non-negative continuous function $\psi \in W_0^{s,p}(\Omega)$*

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(x)|^{p-2}(v(y) - v(x))}{|y - x|^{\alpha p}} (\psi(y) - \psi(x)) dx dy \\ & \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|y - x|^{\alpha p}} (\psi(y) - \psi(x)) dx dy. \end{aligned}$$

Then $v \geq u$ in $\mathbb{R}^N$.

Lemma 3.9. *Let $f(x, u)$ be a continuous function such that $f(x, \cdot)$ is non-decreasing. Let $u \in W_g^{s,p}(\Omega)$ be a corresponding weak solution to (1.1) and $\alpha p > N$. If u is continuous then u is a viscosity solution.*

Proof. Let us show that if u is a continuous weak supersolution then, it is a viscosity supersolution. Assume, *ad contrarium*, that this is not the case. Namely, there exists ϕ and $x_0 \in \Omega$ such that $u(x_0) = \phi(x_0)$, $u - \phi$ has a strict minimum at x_0 , and ϕ verifies

$$-\mathcal{L}_p\phi(x_0) < f(x_0, \phi(x_0)).$$

By continuity, there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\mathcal{L}_p\phi(x) < f(x, \phi(x)) \quad \text{for all } x \in B(x_0, r).$$

We modify the test function as follows. Set $\Phi_\varepsilon(x) = \phi(x) + \varepsilon h(x)$ with $h \in C_c^1(\mathbb{R}^N)$ such that $h(x_0) > 0$ and $h \equiv 0$ in $\mathbb{R}^N \setminus B(x_0, r)$. Notice that the regularity of ϕ and h (in particular both are bounded and Lipschitz continuous) allows us to prove that

$$|\mathcal{L}_p \Phi_\varepsilon(x) - \mathcal{L}_p \phi(x)| \leq C\varepsilon. \quad (3.1)$$

Indeed, the identity

$$|b|^{p-2}b - |a|^{p-2}a = (p-1)(b-a) \int_0^1 |a + t(b-a)|^{p-2} dt$$

with $a = \phi(y) - \phi(x)$ and $b = \Phi_\varepsilon(y) - \Phi_\varepsilon(x)$ gives us

$$|\mathcal{L}_p \Phi_\varepsilon(x) - \mathcal{L}_p \phi(x)| = (p-1)\varepsilon I(x),$$

where

$$I(x) = \int_{\mathbb{R}^N} \frac{h(y) - h(x)}{|y-x|^{\alpha p}} \int_0^1 |\phi(y) - \phi(x) + t\varepsilon(h(y) - h(x))|^{p-2} dt dy.$$

In order to bound this integral for $\varepsilon < 1$, we observe that the functions ϕ and h are bounded, then

$$\begin{aligned} & \int_{B^c(x,1)} \frac{h(y) - h(x)}{|y-x|^{\alpha p}} \int_0^1 |\phi(y) - \phi(x) + t\varepsilon(h(y) - h(x))|^{p-2} dt dy \\ & \leq C \int_{B^c(x,1)} \frac{dy}{|y-x|^{\alpha p}} = \frac{C}{\alpha p - N}. \end{aligned}$$

On the other hand, since ϕ and h are Lipschitz continuous we get

$$\begin{aligned} & \int_{B(x,1)} \frac{h(y) - h(x)}{|y-x|^{\alpha p}} \int_0^1 |\phi(y) - \phi(x) + t\varepsilon(h(y) - h(x))|^{p-2} dt dy \\ & \leq C \int_{B(x,1)} \frac{|y-x|^{p-1}}{|y-x|^{\alpha p}} dy = \frac{C}{(1-\alpha)p + N - 1}. \end{aligned}$$

This shows that the integral $I(x)$ is uniformly bounded and the estimate (3.1) follows.

Therefore, for ε small enough we obtain that

$$-\mathcal{L}_p \Phi_\varepsilon(x) < f(x, \phi(x)) \quad \text{for all } x \in B(x_0, r).$$

Multiplying by a non-negative continuous function $\psi \in W_0^{s,p}(\Omega)$ supported in $B(x_0, r)$ and integrating we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\Phi_\varepsilon(y) - \Phi_\varepsilon(x)|^{p-2} (\Phi_\varepsilon(y) - \Phi_\varepsilon(x))}{|y-x|^{\alpha p}} (\psi(y) - \psi(x)) dx dy \\ & < \int_{\mathbb{R}^N} f(x, \phi(x)) \psi(x) dx. \end{aligned}$$

Since f is non-decreasing in the second variable and $\phi \leq u$, we get

$$f(x, \phi(x)) - f(x, u(x)) < 0.$$

Now, using that u is a weak supersolution, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\Phi_\varepsilon(y) - \Phi_\varepsilon(x)|^{p-2} (\Phi_\varepsilon(y) - \Phi_\varepsilon(x))}{|y - x|^{\alpha p}} (\psi(y) - \psi(x)) \, dx \, dy \\ & < \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|y - x|^{\alpha p}} (\psi(y) - \psi(x)) \, dx \, dy. \end{aligned}$$

This inequality and the fact that $\Phi_\varepsilon \leq u$ in $\mathbb{R}^N \setminus B(x_0, r)$ implies by Lemma 3.8 that $\Phi_\varepsilon \leq u$ in $B(x_0, r)$. But this contradicts that $\Phi_\varepsilon(x_0) > u(x_0)$.

We have proved that u is a viscosity supersolution. Observe that the above arguments also include the case $f = f(x)$. The proof of the fact that u is a viscosity subsolution runs as above and we omit the details. $\square$

Remark 3.10. Notice that if $\alpha p > 2N$, the Hölder embedding (Theorem 2.1) guarantees that u is continuous. In this case we can remove the continuity assumption in the previous Lemma.

4. Existence and a priori estimates. The case $f = f(x)$

The existence of weak solutions to (1.1) follows from standard variational arguments. The energy functional associated to our problem is

$$H(v) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(x)|^p}{|y - x|^{\alpha p}} \, dx \, dy - \int_{\mathbb{R}^N} f_0(x) v \, dx. \quad (4.1)$$

where $f_0(x) = f(x)$ for $x \in \Omega$ and it vanishes outside of Ω .

Theorem 4.1. *Let $\alpha \in (0, 1)$, $p > N/\alpha$ and $f = f(x) \in L^r(\Omega)$ with $\frac{1}{p} + \frac{1}{r} \leq 1$. Then, there exists a unique weak solution to problem (1.1), which is a minimizer of the functional in (4.1), that is,*

$$H(u) = \inf_{v \in W_g^{s,p}(\Omega)} H(v).$$

Proof. Note that the energy functional (4.1) is coercive. Indeed, we apply the Hölder and the Poincaré inequalities to obtain

$$\begin{aligned} \int_{\mathbb{R}^N} f_0(x) v \, dx & \leq |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \|v\|_{L^p(\Omega)} \\ & \leq |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} (\tilde{C}([v]_p + [g]_p) + \|g\|_{L^p(\mathbb{R}^N)}). \end{aligned}$$

Therefore, as $\|v\|_{W_g^{s,p}(\Omega)} \rightarrow \infty$

$$H(v) \geq \frac{1}{2p} [v]_p^p - |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} (\tilde{C}([v]_p + [g]_p) + \|g\|_{L^p(\mathbb{R}^N)}) \rightarrow \infty.$$

Clearly, $H(v)$ is bounded below and the lower semicontinuity trivially holds. Therefore, we have ensured the existence of a minimizer.

The uniqueness follows from the strict convexity of the functional. Differentiating the first term in H , it gives

$$\langle L(v), w \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(x)|^{p-2} (v(y) - v(x))}{|y - x|^{\alpha p}} (w(y) - w(x)) \, dx \, dy.$$

Note that the function $h(t) = |t|^{p-2}t$ is increasing and it only vanishes at $t = 0$. Then, for all $a, b \in \mathbb{R}$

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq 0, \quad \text{and} \quad (|a|^{p-2}a - |b|^{p-2}b)(a - b) = 0 \Rightarrow a = b,$$

the strict monotonicity of L follows straight forward, i.e. $\langle L(v) - L(w), v - w \rangle \geq 0$ and $\langle L(v) - L(w), v - w \rangle = 0$ implies that $w = v$. This shows that the first term is strict convex. Since the second one is linear we obtain the desired strict convexity.

Finally, we note that the constructed solution is trivial only in the case $f \equiv 0$ and $g \equiv 0$. Indeed, if $g \not\equiv 0$ it is clear that the solution is non-trivial, while for $g \equiv 0$ and $f \neq 0$ we can take a function v such that $\int_{\Omega} f(x)v(x) dx > 0$, thus $H(\delta v) < 0$ for δ small. Therefore, $u \equiv 0$ (which yields $H(u) = 0$) cannot be the minimizer. $\square$

We conclude this section finding a priori estimates that guarantee the convergence as $p \rightarrow \infty$ of the solutions to certain continuous limit. We recall that the constants in the Hölder embedding and in the Poincaré inequality, $\hat{C}$ and $\tilde{C}$ respectively, are independent of p . Moreover, the Gagliardo seminorm of g converges to $[g]_{\infty}$ defined as

$$[g]_{\infty} := \sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|g(y) - g(x)|}{|y - x|^{\alpha}}. \quad (4.2)$$

Lemma 4.2. *Let u_p be the solution to (1.1) given in Theorem 4.1. Then, there exists a positive constant C_p such that*

$$[u_p]_p = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^p}{|x - y|^{\alpha p}} dx dy \right)^{\frac{1}{p}} \leq C_p.$$

Moreover,

$$\lim_{p \rightarrow \infty} C_p = C_{\infty} = \max \left\{ 1, \sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|g(y) - g(x)|}{|y - x|^{\alpha}} \right\}. \quad (4.3)$$

Proof. Taking $u_p - g$ as test function, we get

$$\begin{aligned} [u_p]_p^p &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))}{|y - x|^{\alpha p}} (g(y) - g(x)) dx dy \\ &= \int_{\mathbb{R}^N} f_0(x) (u_p - g)(x) dx. \end{aligned}$$

We estimate the second term on the left hand side, with the use of the Hölder inequality, as follows

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))}{|y - x|^{\alpha p}} (g(y) - g(x)) dx dy \leq [u_p]_p^{p-1} [g]_p.$$

For the right hand side, we apply Hölder and Poincaré inequalities to obtain

$$\begin{aligned}
 \int_{\mathbb{R}^N} f_0(x)(u_p - g) dx &= \int_{\Omega} f(x)(u_p - g) dx \\
 &\leq |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \|u_p - g\|_{L^p(\Omega)} \\
 &\leq |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \tilde{C}[u_p - g]_p \\
 &\leq |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \tilde{C}([u_p]_p + [g]_p).
 \end{aligned}$$

Summing up, we have shown that the function

$$h_p(t) = t^p - [g]_p t^{p-1} - |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \tilde{C} t$$

satisfies

$$h_p([u_p]_p) \leq \tilde{C} |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} [g]_p.$$

In consequence,

$$[u_p]_p \leq C_p := \sup\{t : h_p(t) \leq \tilde{C} |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} [g]_p\}.$$

The fact that h goes to infinity as $t \rightarrow \infty$ ensures that $C_p < \infty$ and concludes the first part of the Lemma.

We study now the behaviour of C_p as p goes to infinity. Observe that

$$[g]_p \rightarrow [g]_{\infty} \quad \text{and} \quad |\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \rightarrow |\Omega|^{1-\frac{1}{r}} \|f\|_{L^r(\Omega)}.$$

In conclusion, the limit of C_p is given by

$$C_{\infty} = \sup \left\{ t : \lim_{p \rightarrow \infty} h_p(t) \leq \tilde{C} |\Omega|^{1-\frac{1}{r}} \|f\|_{L^r(\Omega)} [g]_{\infty} \right\}.$$

Notice that $h_p(t) \leq 0$ for $0 < t \leq \max\{[g]_p, (|\Omega|^{1-\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(\Omega)} \tilde{C})^{\frac{1}{p-1}}\}$. This implies that

$$\lim_{p \rightarrow \infty} h_p(t) \leq 0 \quad \text{for} \quad 0 < t \leq \max\{1, [g]_{\infty}\}.$$

On the other hand, it is easy to see that

$$\lim_{p \rightarrow \infty} h_p(t) = +\infty \quad \text{for} \quad t > \max\{1, [g]_{\infty}\},$$

which shows (4.3). □

Theorem 4.3. *Let $\alpha \in (0, 1)$, $\alpha p > 2N$ and $f = f(x) \in L^r(\Omega)$ with $\frac{1}{p} + \frac{1}{r} \leq 1$. There exists a subsequence of solutions to problem (1.1) such that $u_p \rightarrow u_{\infty}$ uniformly in $\mathbb{R}^N$ as $p \rightarrow \infty$. In addition, the limit $u_{\infty} \in C(\mathbb{R}^N)$ is bounded, $u_{\infty} = g$ in $\mathbb{R}^N \setminus \Omega$ and it satisfies*

$$\sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_{\infty}(y) - u_{\infty}(x)|}{|y - x|^{\alpha}} \leq C_{\infty},$$

where C_{∞} is given in Lemma 4.2.

Proof. From Lemma 4.2 we derive the following estimate on the Gagliardo seminorm,

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^p}{|x - y|^{\alpha p}} dx dy \right)^{\frac{1}{p}} \leq C_p \rightarrow C_\infty, \quad \text{as } p \rightarrow \infty. \quad (4.4)$$

Now, applying Hölder embedding we obtain that for $\alpha p > 2N$

$$\|u_p\|_{C^{0,\beta}(\mathbb{R}^N)} \leq \widehat{C}(1 + \widetilde{C}^p)^{\frac{1}{p}} C_p \rightarrow \widehat{C} \widetilde{C} C_\infty \quad \text{as } p \rightarrow \infty. \quad (4.5)$$

The Arzelà–Ascoli Theorem and the fact that $u_p = g$ in $\mathbb{R}^N \setminus \Omega$, ensures the existence of a subsequence such that

$$\lim_{p \rightarrow \infty} u_p = u_\infty \quad \text{uniformly in } \mathbb{R}^N.$$

Notice that, since $u_p = g$ in $\mathbb{R}^N \setminus \Omega$ and (4.5) holds, the limit satisfies

$$u_\infty \in C(\mathbb{R}^N), \quad u_\infty = g \text{ in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad \|u_\infty\|_{L^\infty(\mathbb{R}^N)} \leq \widehat{C} \widetilde{C} C_\infty. \quad (4.6)$$

Finally, we observe that we can write

$$\sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_\infty(y) - u_\infty(x)|}{|y - x|^\alpha} = \max\{S_1 \ S_2\},$$

where

$$S_1 = \sup_{\substack{x \in \Omega, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_\infty(y) - u_\infty(x)|}{|y - x|^\alpha}, \quad S_2 = \sup_{\substack{x, y \in \mathbb{R}^N \setminus \Omega \\ y \neq x}} \frac{|g(y) - g(x)|}{|y - x|^\alpha}.$$

Recalling the definition of C_∞ , see (4.3), we just need to study the first supremum. Observe that (4.5) ensures that $\|u\|_{L^\infty(\mathbb{R}^N)}$ is uniformly bounded for p large enough, hence we can define

$$M := \sup_{x, y \in \mathbb{R}^N} (u_p(y) - u_p(x)). \quad (4.7)$$

On the other hand, by Fatou's Lemma

$$\begin{aligned} S_1 &= \lim_{q \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_\infty(y) - u_\infty(x)|^q}{|x - y|^{\alpha q}} dy dx \right)^{1/q} \\ &\leq \lim_{q \rightarrow \infty} \liminf_{p \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx \right)^{1/q}. \end{aligned}$$

In the sequel we take $p > q \gg 1$. Now, we consider the set $E = \{x \in \mathbb{R}^N : d(x, \Omega) \leq M^{1/\alpha}\}$, with M given in (4.7), and rewrite

$$\int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx = I_p + II_p, \quad (4.8)$$

where

$$\begin{aligned} I_p &= \int_{\Omega} \int_E \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx, \\ II_p &= \int_{\Omega} \int_{E^c} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx. \end{aligned}$$

For the first integral, we apply Hölder inequality and (4.4) to see that

$$\begin{aligned} I_p &\leq \left(\int_{\Omega} \int_E \left(\frac{|u_p(y) - u_p(x)|}{|x - y|^{\alpha}} \right)^p dx dy \right)^{\frac{q}{p}} (|\Omega||E|)^{1 - \frac{q}{p}} \\ &\leq C_p^q (|\Omega||E|)^{1 - \frac{q}{p}} \rightarrow C_{\infty}^q |\Omega||E| \quad \text{as } p \rightarrow \infty. \end{aligned}$$

To analyze the second integral, we note that for $x \in \Omega$ we get $E^c \subset \{y \in \mathbb{R}^N : |x - y| \geq M^{1/\alpha}\}$. Hence

$$\begin{aligned} II_p &= \int_{\Omega} \int_{E^c} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx \leq M^q \int_{\Omega} \int_{|x - y| > M^{1/\alpha}} \frac{1}{|x - y|^{\alpha q}} dy dx \\ &= M^q |\Omega| |S_{N-1}| \frac{(M^{1/\alpha})^{(N - \alpha q)}}{\alpha q - N} = \frac{M^{\frac{N}{\alpha}} |\Omega| |S_{N-1}|}{\alpha q - N}. \end{aligned} \quad (4.9)$$

Summing up, we have shown that

$$\liminf_{p \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx \right)^{1/q} \leq \left(C_{\infty}^q |\Omega||E| + \frac{M^{\frac{N}{\alpha}} |\Omega| |S_{N-1}|}{\alpha q - N} \right)^{1/q}.$$

Therefore,

$$\lim_{q \rightarrow \infty} \liminf_{p \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^q}{|x - y|^{\alpha q}} dy dx \right)^{1/q} \leq C_{\infty},$$

which gives us

$$\sup_{\substack{x \in \Omega, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_{\infty}(y) - u_{\infty}(x)|}{|y - x|^{\alpha}} \leq C_{\infty}.$$

for (x, y) , a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, $y \neq x$. The continuity of u_{∞} implies that the above estimate holds for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $y \neq x$. $\square$

5. Existence and a priori estimates. The case $f = |u|^{\theta-1}u$.

To ensure the existence of a non negative weak solution if $f = |u|^{\theta-1}u$ with $\theta < p-1$, we can also apply the variational argument above, though uniqueness is not guaranteed in this case. The energy functional is given by

$$H(v) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(x)|^p}{|y - x|^{\alpha p}} dx dy - \frac{1}{\theta + 1} \int_{\mathbb{R}^N} |v|^{\theta+1} dx.$$

The non-negativity of the minimizer follows from the fact that $|v(y) - v(x)| > ||v(y)| - |v(x)||$ when $v(y)v(x) < 0$, while $|v(y) - v(x)| = ||v(y)| - |v(x)||$ if $v(y)v(x) \geq 0$. Thus, $H(|u|) < H(u)$.

However, when we identify the limit equation as $p \rightarrow \infty$, for this choice of f we need the solutions to be bounded from below by a strict positive function in Ω . For this reason, we argue alternatively in this case and we just use the comparison principle given in Lemma 3.8 and construct an ordered subsolution and supersolution. The sub-supersolution method gives us then the existence of a solution between them, and the subsolution that we find is positive.

Theorem 5.1. *Let $\alpha \in (0, 1)$, $p > 2N/\alpha$ and $f = |u|^{\theta-1}u$ with $\theta < p - 1$. Then, there exists a positive weak solution, u_p , to problem (1.1). In particular u_p verifies*

$$\varepsilon \psi_p \leq u_p,$$

where ψ_p is the positive first eigenfunction in Lemma 2.4 verifying $\|\psi_p\|_\infty = 1$.

Proof. Thanks to the comparison principle given in Lemma 3.8, to prove the existence of a solution, it is enough to find a subsolution, $\underline{u}$, and a supersolution, $\bar{u}$, such that $\underline{u} \leq \bar{u}$.

In order to construct the subsolution, let us consider the first eigenfunction ψ_p normalized such that $\|\psi_p\|_\infty = 1$. We define $\underline{u} = \varepsilon \psi_p$. Notice that $\underline{u} = 0$ in $\mathbb{R}^N \setminus \Omega$ and for all nonnegative $\phi \in W_0^{s,p}(\Omega)$, it holds that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\underline{u}(y) - \underline{u}(x)|^{p-2} (\underline{u}(y) - \underline{u}(x))}{|y - x|^{\alpha p}} (\phi(y) - \phi(x)) dx dy \\ &= \lambda_p \int_{\mathbb{R}^N} \underline{u}^{p-1}(x) \phi(x) dx \\ &\leq \lambda_p (\varepsilon)^{p-1-\theta} \int_{\mathbb{R}^N} \underline{u}^\theta(x) \phi(x) dx. \end{aligned}$$

Then, for ε small enough $\underline{u}$ is a (positive) weak subsolution.

Finally to construct the supersolution, we choose a larger domain $\tilde{\Omega} \supset \Omega$, and the corresponding eigenfunction $\tilde{\psi}_p$, normalized such that $\min_{x \in \Omega} \psi_p(x) = 1$.

We declare $\bar{u} = K(\tilde{\psi}_p + 1)$. Observe that $\bar{u} \geq K$, then taking K sufficiently large we ensure that $\bar{u} \geq \underline{u}$ and $\bar{u} \geq g$ (notice that since $p > 2N/\alpha$ by Theorem 2.1 $g \in L^\infty(\mathbb{R}^N)$).

On the other hand, for all nonnegative $\phi \in W_0^{s,p}(\Omega)$, it holds that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{u}(y) - \bar{u}(x)|^{p-2} (\bar{u}(y) - \bar{u}(x))}{|y - x|^{\alpha p}} (\phi(y) - \phi(x)) dx dy \\ &= \tilde{\lambda}_p K^{p-1} \int_{\mathbb{R}^N} \tilde{\psi}^{p-1}(x) \phi(x) dx = \tilde{\lambda}_p K^{p-1} \int_{\Omega} \tilde{\psi}^{p-1}(x) \phi(x) dx \\ &\geq \tilde{\lambda}_p K^{p-1} \int_{\Omega} \tilde{\psi}^\theta(x) \phi(x) dx \geq \tilde{\lambda}_p K^{p-1} \int_{\Omega} \frac{(\tilde{\psi}(x) + 1)^\theta}{2^\theta} \phi(x) dx \\ &= \frac{\lambda_p K^{p-1-\theta}}{2^\theta} \int_{\mathbb{R}^N} \bar{u}^\theta(x) \phi(x) dx \end{aligned}$$

If K is large enough this proves that $\bar{u}$ is a weak supersolution. $\square$

Remark 5.2. We observe that, for p large and $\theta = \theta(p)$ satisfying (1.4), we can choose both ε and K independent of p . In fact, thanks to (2.1), we can take

$$\varepsilon = \frac{1}{2} R^{\frac{\alpha}{1-\theta}}, \quad K \geq 2(2^\theta R^\alpha)^{\frac{1}{1-\theta}}.$$

Theorem 5.3. *Let $\alpha \in (0, 1)$, $\alpha p > 2N$ and $f = |u|^{\theta(p)-1}u$ with $\theta(p)$ satisfying (1.4). There exists a subsequence of solutions to problem (1.1) such that $u_p \rightarrow$*

u_∞ uniformly in $\mathbb{R}^N$ as $p \rightarrow \infty$. In addition, the limit $u_\infty \in C(\mathbb{R}^N)$ is bounded, $u_\infty = g$ in $\mathbb{R}^N \setminus \Omega$ and it satisfies

1. There exists a positive constant C_∞ such that

$$\sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_\infty(y) - u_\infty(x)|}{|y - x|^\alpha} \leq C_\infty.$$

2. There exists a positive constant ε such that

$$u_\infty(x) \geq \varepsilon \psi_\infty(x),$$

where ψ_∞ is given in Lemma 2.4.

Proof. We claim that the Gagliardo seminorm satisfies

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^p}{|x - y|^{\alpha p}} dx dy \right)^{\frac{1}{p}} \leq C_p \rightarrow C_\infty < \infty \quad \text{as } p \rightarrow \infty.$$

Assuming this claim we can argue as in Theorem 4.3 to obtain that there exists a subsequence u_p such that

$$u_p \rightarrow u_\infty \quad \text{uniformly in } \mathbb{R}^N \text{ as } p \rightarrow \infty,$$

where the limit $u_\infty \in C(\mathbb{R}^N)$ satisfies that $u_\infty = g$ in $\mathbb{R}^N \setminus \Omega$, $\|u_\infty\|_{L^\infty(\mathbb{R}^N)} \leq \widehat{C} \widetilde{C} C_\infty$ and

$$\sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|u_\infty(y) - u_\infty(x)|}{|y - x|^\alpha} \leq C_\infty.$$

Furthermore, the lower estimate for u_∞ follows by Theorem 5.1 and Remark 5.2.

Finally we prove the claim. Again, taking $u_p - g$ as a test function yields

$$\begin{aligned} [u_p]_p^p &- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))}{|y - x|^{\alpha p}} (g(y) - g(x)) dx dy \\ &= \int_{\mathbb{R}^N} u_p^{\theta(p)}(x) (u_p - g)(x) dx. \end{aligned}$$

By Hölder inequality the right hand side satisfies

$$\begin{aligned} [u_p]_p^p &- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))}{|y - x|^{\alpha p}} (g(y) - g(x)) dx dy \\ &\geq [u_p]_p^p - [g]_p [u_p]_p^{p-1}, \end{aligned}$$

while for the left hand side we use the fact that u_p and g are non-negative functions and the Hölder and Poincaré inequalities to deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} u_p^{\theta(p)}(x)(u_p - g)(x)dx &= \int_{\Omega} u_p^{\theta(p)}(x)(u_p - g)(x)dx \leq \int_{\Omega} u_p^{\theta(p)+1}(x)dx \\ &\leq |\Omega|^{\frac{p-1-\theta(p)}{p}} (\|u_p\|_{L^p(\Omega)})^{\theta(p)+1} \\ &\leq |\Omega|^{\frac{p-1-\theta(p)}{p}} (\|u_p\|_{L^p(\mathbb{R}^N)})^{\theta(p)+1} \\ &\leq |\Omega|^{\frac{p-1-\theta(p)}{p}} \left(\tilde{C}([u_p]_p + [g]_p) + \|g\|_{L^p(\mathbb{R}^N)} \right)^{\theta(p)+1}. \end{aligned}$$

We have proved that the function

$$h_p(t) = t^p - [g]_p t^{p-1} - \|u_p\|_{L^{q_p}(\mathbb{R}^N)} \left(\tilde{C}(t + [g]_p) + \|g\|_{L^p(\mathbb{R}^N)} \right)^{\theta(p)+1}$$

satisfies that $h_p([u_p]_p) \leq 0$, which implies

$$[u_p]_p \leq C_p := \sup\{t : h_p(t) = 0\}.$$

As before, the fact that $h_p(t)$ goes to infinity as $t \rightarrow \infty$ guarantees that $C_p < \infty$. Furthermore,

$$\lim_{p \rightarrow \infty} h_p(t) = +\infty, \quad \text{if } t > \max\{1, [g]_{\infty}, t_0\},$$

while

$$\lim_{p \rightarrow \infty} h_p(t) \leq 0, \quad \text{for } 0 < t < \max\{1, [g]_{\infty}, t_0\},$$

where $[g]_{\infty}$ is defined in (4.2) and

$$t_0 = \sup \left\{ t : t^{\frac{1}{\theta}} - \tilde{C}(t + [g]_{\infty}) - \|g\|_{L^{\infty}(\mathbb{R}^N)} \leq 0 \right\}.$$

Then,

$$\lim_{p \rightarrow \infty} C_p = C_{\infty} \leq \max\{1, [g]_{\infty}, t_0\}$$

and the claim is proved. $\square$

6. The limit problem as $p \rightarrow \infty$.

In the previous sections we proved the existence of a subsequence of solutions to (1.1), u_{p_i} , such that $u_{p_i} \rightarrow u_{\infty}$ uniformly as p_i goes to infinity. We focus now on identifying the limit problem verified by u_{∞} . We treat separately the different cases for the function f .

However, let us first introduce some notation from [8, 21], to be used in both cases. The following operator will be essential to deduce the limit equation and it is known as infinity fractional Laplacian,

$$\mathcal{L}_{\infty} u(x) = \sup_{y \in \mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^{\alpha}} + \inf_{y \in \mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^{\alpha}}.$$

We can decompose this operator as follows

$$\mathcal{L}_{\infty} u(x) = \mathcal{L}_{\infty}^{+} u(x) + \mathcal{L}_{\infty}^{-} u(x),$$

where

$$\mathcal{L}_\infty^+ u(x) = \sup_{y \in \mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^\alpha} \quad \text{and} \quad \mathcal{L}_\infty^- u(x) = \inf_{y \in \mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^\alpha}. \quad (6.1)$$

The next result will be useful in our arguments. Let us introduce the space

$$Z = \left\{ \phi \in C^1(\mathbb{R}^N) : \lim_{|y| \rightarrow \infty} \phi(y) = 0 \text{ and } \phi(\tilde{y}) = 0 \text{ for some } \tilde{y} \in \mathbb{R}^N \right\}.$$

Notice that the test functions $C_c^1(\mathbb{R}^N)$ are included in Z .

Lemma 6.1. *Let $\phi \in Z$ and $\{x_p\} \rightarrow x_0$ as $p \rightarrow \infty$. Then,*

$$A_p(\phi(x_p)) \rightarrow \mathcal{L}_\infty^+ \phi(x_0), \quad \text{and} \quad B_p(\phi(x_p)) \rightarrow -\mathcal{L}_\infty^- \phi(x_0),$$

where

$$A_p^{p-1}(\phi(x_p)) = \int_{\mathbb{R}^N} \frac{|\phi(y) - \phi(x_p)|^{p-2} (\phi(y) - \phi(x_p))_+}{|y - x_p|^{\alpha p}} dy,$$

$$B_p^{p-1}(\phi(x_p)) = \int_{\mathbb{R}^N} \frac{|\phi(y) - \phi(x_p)|^{p-2} (\phi(y) - \phi(x_p))_-}{|y - x_p|^{\alpha p}} dy,$$

and $\mathcal{L}_\infty^+ \phi(x_0)$, $\mathcal{L}_\infty^- \phi(x_0)$ are given in (6.1). Here $g_\pm = \max\{\pm g, 0\}$.

Proof. We just examine the convergence of the operator A_p , since analogue arguments can be used to analyze the convergence of B_p . We adapt the proof of Lemma 6.5 in [8], where the same result is proved integrating in Ω instead of $\mathbb{R}^N$. Let us define,

$$h(x, y) = \frac{(\phi(y) - \phi(x))_+}{|y - x|^\alpha}.$$

As a first step we assume that $\mu := \sup_{y \in \mathbb{R}^N} h(x_0, y) > 0$. Then we can take $0 < t < \mu$ and $y_0 \in \mathbb{R}^N$ such that $h(x_0, y_0) > t$. Since $\phi \in C_0^1(\mathbb{R}^N)$ the function $h(x_0, y)$ is continuous and there exists $\delta > 0$ such that $B_\delta(y_0) \subset \{h(x_0, y) > t\}$. Furthermore, $h(x_p, y) \rightarrow h(x_0, y)$ as $p \rightarrow \infty$, thus for p large $B_\delta(y_0) \subset \{h(x_p, y) > t\}$ and $|x_p - x_0| \leq \delta$, we have that

$$\begin{aligned} A_p^{p-1}(\phi(x_p)) &= \int_{\mathbb{R}^N} h(x_p, y)^{p-1} \frac{dy}{|y - x_p|^\alpha} \geq \int_{B_\delta(y_0)} h(x_p, y)^{p-1} \frac{dy}{|y - x_p|^\alpha} \\ &\geq t^{p-1} \int_{B_\delta(y_0)} \frac{dy}{|y - x_p|^\alpha}. \end{aligned}$$

The fact that $y \in B_\delta(y_0)$ and $x_p \in B_\delta(x_0)$ easily yields that $|y - x_p| \leq |y_0 - x_0| + 2\delta$ and hence

$$A_p(\phi(x_p)) \geq t C^{\frac{1}{p-1}} \rightarrow t \quad \text{as } p \rightarrow \infty.$$

In order to find an upper estimate, we observe that $\lim_{|y| \rightarrow \infty} \phi(y) = 0 = \phi(\tilde{y})$. This ensures the existence of a point $y_1 \in \mathbb{R}^N$ such that

$$\sup_{y \in \mathbb{R}^N} (\phi(y) - \phi(x_p))_+ = (\phi(y_1) - \phi(x_p))_+.$$

Let us denote $R = |y_1 - x_p|$ and split A_p as follows

$$\begin{aligned} A_p^{p-1}(\phi(x_p)) &= \int_{B_R(x_p)} h(x_p, y)^{p-1} \frac{dy}{|y - x_p|^\alpha} \\ &\quad + \int_{B_R^c(x_p)} \frac{(\phi(y) - \phi(x_p))_+^{p-1}}{|y - x_p|^{\alpha p}} dy = I_p + II_p. \end{aligned}$$

For I_p we have

$$\begin{aligned} I_p &\leq \left(\sup_{y \in B_R(x_p)} h(x_p, y) \right)^{p-1} \int_{B_R(x_p)} \frac{dy}{|y - x_p|^\alpha} \\ &\leq \left(\sup_{y \in \mathbb{R}^N} h(x_p, y) \right)^{p-1} |S_{N-1}| \frac{R^{N-\alpha}}{N-\alpha}, \end{aligned}$$

while for II_p ,

$$\begin{aligned} II_p &\leq \sup_{y \in \mathbb{R}^N} (\phi(y) - \phi(x_p))_+^{p-1} \int_{y \in B_R^c(x_p)} \frac{dy}{|y - x_p|^{\alpha p}} \\ &= (\phi(y_1) - \phi(x_p))_+^{p-1} \frac{|S_{N-1}|}{(\alpha p - N) R^{\alpha p - N}} \\ &= \left(\frac{(\phi(y_1) - \phi(x_p))_+}{R^\alpha} \right)^{p-1} \frac{|S_{N-1}|}{\alpha p - N} R^{N-\alpha}. \end{aligned}$$

Recalling that $R = |y_1 - x_p|$, we rewrite the estimate on II_p

$$II_p \leq h(x_p, y_1)^{p-1} \frac{|S_{N-1}|}{\alpha p - N} R^{N-\alpha} \leq \left(\sup_{y \in \mathbb{R}^N} h(x_p, y) \right)^{p-1} \frac{|S_{N-1}|}{\alpha p - N} R^{N-\alpha}.$$

Furthermore,

$$A_p(\phi(x_p)) \leq C_p \sup_{y \in \mathbb{R}^N} h(x_p, y) \rightarrow \sup_{y \in \mathbb{R}^N} h(x_0, y), \quad \text{as } p \rightarrow \infty, \quad (6.2)$$

since

$$C_p = \left(|S_{N-1}| \frac{R^{N-\alpha}}{N-\alpha} + \frac{|S_{N-1}|}{\alpha p - N} R^{N-\alpha} \right)^{\frac{1}{p-1}} \rightarrow 1.$$

Summing up, we have shown that

$$t \leq \liminf_{p \rightarrow \infty} A_p(\phi(x_p)) \leq \limsup_{p \rightarrow \infty} A_p(\phi(x_p)) \leq \sup_{y \in \mathbb{R}^N} h(x_0, y)$$

Taking $t \rightarrow \sup_{y \in \mathbb{R}^N} h(x_p, y)$, we obtain the desired result.

Finally, if $\sup_{y \in \mathbb{R}^N} h(x_0, y) = 0$, note that $\mathcal{L}_\infty^+ \phi(x_0) = 0$ and $A_p(\phi(x_p)) \geq 0$. On the other hand, the upper bound (6.2) follows as in the previous case and hence,

$$0 \leq \liminf_{p \rightarrow \infty} A_p(\phi(x_p)) \leq \limsup_{p \rightarrow \infty} A_p(\phi(x_p)) \leq \sup_{y \in \mathbb{R}^N} h(x_0, y) = 0.$$

□

6.1. The limit problem as $p \rightarrow \infty$: $f = f(x)$.

We start by analyzing the case in which the function f just depends on the spatial variable. The case $f \equiv 0$ in Ω has been analyzed in [8]. Also existence and uniqueness for the limit equation, together with some results on regularity of the solutions are shown. We include these results when analyzing the limit at the points where our arbitrary function f vanishes, for completeness of the paper. We will see that the influence of this function on the limit is only through its support and sign.

Let us define the space

$$Y = \left\{ \phi \in C^{0,\alpha}(\mathbb{R}^N) : \phi = g \text{ in } \mathbb{R}^N \setminus \Omega, \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq \widehat{C}\widetilde{C} \right. \\ \left. \text{and } \sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|\phi(y) - \phi(x)|}{|y - x|^\alpha} \leq 1 \right\},$$

where $\widehat{C}$ and $\widetilde{C}$ are the constants corresponding to the Hölder embedding and the Poincaré inequality, respectively. Notice that if

$$\sup_{\substack{x, y \in \mathbb{R}^N \\ y \neq x}} \frac{|g(y) - g(x)|}{|y - x|^\alpha} \leq 1, \quad (6.3)$$

we can take $C_\infty = 1$ in Theorem 4.3 and then (4.6) ensures that $u_\infty \in Y$.

Theorem 6.2. *Let $f = f(x) \in L^r(\Omega)$ with $\frac{1}{r} + \frac{1}{p} \leq 1$, $g \in W^{s,p}(\mathbb{R}^N)$ verifying (6.3) and u_p the corresponding weak solution to (1.1). Then, u_∞ obtained as a uniform limit of a subsequence of $\{u_p\}$, satisfies*

$$\max_{\psi \in Y} \int_{\mathbb{R}^N} f\psi dx = \int_{\mathbb{R}^N} f u_\infty dx. \quad (6.4)$$

Proof. Since the solutions to (1.1) are minimizers of the energy functional (4.1), for any $\psi \in Y$

$$\begin{aligned} & \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_p(y) - u_p(x)|^p}{|y - x|^{\alpha p}} dx dy - \int_{\mathbb{R}^N} f_0(x) u_p(x) dx \\ & \leq \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy - \int_{\mathbb{R}^N} f_0(x) \psi(x) dx. \end{aligned}$$

The idea now is to pass to the limit as p goes to infinity in the previous inequality. In the left hand side, we can neglect the first integral while

$$\int_{\Omega} f(x) u_p(x) dx \rightarrow \int_{\Omega} f(x) u_\infty(x) dx, \quad \text{as } p \rightarrow \infty.$$

For the right hand side we claim that the first integral goes to zero and then (6.4) holds.

To show the claim we argue as in the proof of Theorem 4.3. Let M defined in (4.7) and $E = \{x \in \mathbb{R}^N : d(x, \Omega) \leq M^{1/\alpha}\}$. Observe that

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy &= \int_E \int_E \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy \\ &\quad + 2 \int_{E^c} \int_E \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy. \end{aligned}$$

Since $\psi \in Y$, it holds that

$$\int_E \int_E \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy \leq |E|^2,$$

while, as in (4.9) we have

$$\int_{E^c} \int_E \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy \leq M^p \int_{E^c} \int_E \frac{1}{|y - x|^{\alpha p}} dx dy \leq \frac{M^{\frac{N}{\alpha}} |E|}{\alpha p - N}.$$

In conclusion,

$$\frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(y) - \psi(x)|^p}{|y - x|^{\alpha p}} dx dy \leq \frac{1}{2p} \left(|E|^2 + \frac{M^{\frac{N}{\alpha}} |E|}{\alpha p - 1} \right) \rightarrow 0 \quad \text{as } p \rightarrow \infty,$$

and the claim is proved. $\square$

Remark 6.3. From this Lemma it is clear that u_∞ is not the trivial function.

Our next goal is to identify the limit equation in viscosity sense. As first step, we observe that, since $u_\infty \in Y$ we obtain the following result.

Lemma 6.4. *Let $f = f(x) \in C(\bar{\Omega})$ and $g \in W^{s,p}(\mathbb{R}^N)$ satisfying (6.3). A function u_∞ , obtained as a uniform limit of a subsequence of $\{u_p\}$, verifies the following estimates in viscosity sense*

$$-\mathcal{L}_\infty^- u_\infty \leq 1 \quad \text{and} \quad -\mathcal{L}_\infty^+ u_\infty \geq -1. \quad (6.5)$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^N)$ be such that $u_\infty(x_0) = \varphi(x_0)$ and $\varphi(y) \geq u_\infty(y)$ for all $y \in \mathbb{R}^N$. Then

$$\frac{\varphi(x_0) - \varphi(y)}{|y - x_0|^\alpha} \leq \frac{u_\infty(x_0) - u_\infty(y)}{|y - x_0|^\alpha} \leq 1, \quad \text{for any } y \in \mathbb{R}^N, y \neq x_0,$$

where the last inequality follows from the fact that $u_\infty \in Y$. Therefore,

$$\sup_{\substack{y \in \mathbb{R}^N \\ y \neq x_0}} \frac{\varphi(x_0) - \varphi(y)}{|y - x_0|^\alpha} = - \inf_{\substack{y \in \mathbb{R}^N \\ y \neq x_0}} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha} = -\mathcal{L}_\infty^- \varphi(x_0) \leq 1,$$

as we wanted to show.

On the other hand, we take $\phi \in C_c^1(\mathbb{R}^N)$ such that $u_\infty(x_0) = \phi(x_0)$ and $\phi(y) \leq u_\infty(y)$ for every $y \in \mathbb{R}^N$. Then,

$$\phi(y) - \phi(x_0) \leq u_\infty(y) - u_\infty(x_0) \leq |y - x_0|^\alpha, \quad \text{for any } y \in \mathbb{R}^N.$$

This implies that

$$\sup_{\substack{y \in \mathbb{R}^N \\ y \neq x_0}} \frac{\phi(y) - \phi(x_0)}{|y - x_0|^\alpha} \leq 1,$$

which gives us

$$-\mathcal{L}_\infty^+ \phi(x_0) \geq -1.$$

□

Theorem 6.5. *Let $f = f(x) \in C(\overline{\Omega})$ and $g \in W^{s,p}(\mathbb{R}^N)$ satisfying (6.3). A function $u_\infty \in Y$ obtained as a uniform limit of a subsequence of $\{u_p\}$, is a viscosity solution of the problem*

$$\begin{aligned} -\mathcal{L}_\infty^- u_\infty &= 1, & \text{in } \{f > 0\}, \\ -\mathcal{L}_\infty^+ u_\infty &= -1, & \text{in } \{f < 0\}, \\ -\mathcal{L}_\infty u_\infty &= 0, & \text{in } \{\Omega \setminus \text{supp } f\}^o, \\ -\mathcal{L}_\infty u_\infty &\geq 0, & \text{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}, \\ -\mathcal{L}_\infty u_\infty &\leq 0, & \text{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}. \end{aligned}$$

Proof. We determine the different equations in viscosity sense depending on the sign of the function f .

1. $\{f > 0\}$. From Lemma 6.4 we have that u_∞ is a viscosity subsolution to $-\mathcal{L}_\infty^- u_\infty = 1$.

In order to prove that u_∞ is a viscosity supersolution we consider a test function, ϕ touching u_∞ from below at a point $x_0 \in \{f > 0\}$. Notice that by Remarks 3.6 and 3.7 we can assume that the function $u_\infty - \phi$ is strictly positive (adding some constant to the test function) having a strict minimum at x_0 . Thanks to the uniform convergence in Theorem 4.3 and the compactness of Ω we can extract, up to subsequences, $x_p \rightarrow x_0$ being x_p points at which the positive function $u_p - \phi$ attains a minimum. On the other hand by Lemma 3.9 we know that, in particular u_p are viscosity supersolutions. Hence

$$-\mathcal{L}_p \phi(x_p) = -(A_p^{p-1}(\phi(x_p)) - B_p^{p-1}(\phi(x_p))) \geq f(x_p) > 0.$$

Since $A_p^{p-1}(\phi(x_p)), B_p^{p-1}(\phi(x_p)) \geq 0$, it yields

$$B_p(\phi(x_p)) \geq (f(x_p))^{\frac{1}{p-1}}.$$

Then, thanks to the continuity of f and Lemma 6.1 we can take the limit $p \rightarrow \infty$ to conclude that $-\mathcal{L}_\infty^- \phi(x_0) \geq 1$, which implies that u_∞ is a viscosity supersolution to $-\mathcal{L}_\infty^- u_\infty = 1$.

2. $\{f < 0\}$. In this case, Lemma 6.4 gives us that u_∞ is a viscosity supersolution to $-\mathcal{L}_\infty^+ u_\infty = -1$.

Let $x_0 \in \{f < 0\}$ and φ be such that $u_\infty - \varphi < 0$ and it reaches a strict maximum at x_0 . As before, there exists $\{x_p\} \rightarrow x_0$ where the negative function $u_p - \varphi < 0$ attains its maximum. Using that u_p are in particular subsolutions, we have that

$$-(A_p^{p-1}(\phi(x_p)) - B_p^{p-1}(\phi(x_p))) \leq f(x_p) < 0.$$

This means that

$$-A_p(\phi(x_p)) \leq -(-f(x_p))^{\frac{1}{p-1}},$$

which by Lemma 6.1 and the fact that f is continuous yields that $-\mathcal{L}_\infty^+ \varphi(x_0) \leq -1$. We have proved that u_∞ is a viscosity subsolution to $-\mathcal{L}_\infty^+ u_\infty = -1$.

3. $\{\Omega \setminus \text{supp } f\}^\circ$. Let us show that u_∞ is a viscosity supersolution to $-\mathcal{L}_\infty u_\infty = 0$. The fact that u_∞ is a viscosity subsolution runs similarly.

We take once again $\{x_p\} \rightarrow x_0$ being minima for the positive function $u_p - \phi$ and such that $u_\infty - \phi > 0$ reaches a strict minimum at x_0 . Since we are considering the interior set, for p sufficiently large $f(x_p) = 0$, and hence

$$-(A_p^{p-1}(\phi(x_p)) - B_p^{p-1}(\phi(x_p))) \geq 0.$$

Therefore $B_p(\phi(x_p)) \geq A_p(\phi(x_p))$ and invoking once more Lemma 6.1, in the limit we get $-\mathcal{L}_\infty^+ \phi(x_0) \geq \mathcal{L}_\infty^- \phi(x_0)$, namely,

$$-\mathcal{L}_\infty \phi(x_0) \geq 0.$$

4. $\Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}$. Notice that if $x_0 \in \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}$ it means that $f(x_0) = 0$ and x_0 is approximated by points x_n such that $f(x_n) = 0$ or $f(x_n) > 0$. Then, we can argue as in the previous step to obtain that $-\mathcal{L}_\infty u_\infty \geq 0$ in viscosity sense.

In this case, we just know for subsolutions that $-\mathcal{L}_\infty^- \varphi(x_0) \leq 1$, invoking the general estimates (6.5).

5. $\Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}$. If $x_0 \in \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}$ then $f(x_0) = 0$, with x_0 approached by points x_n such that $f(x_n) = 0$ or $f(x_n) \leq 0$. In this case we can proceed as in step 3 to show that $-\mathcal{L}_\infty u_\infty \leq 0$. Nevertheless, for the supersolutions we only deduce the general estimate $-\mathcal{L}_\infty^+ \phi(x_0) \geq -1$. $\square$

When $f(x) > 0$ (resp. $f(x) < 0$) for all $x \in \Omega$, we can identify the limit function u_∞ in terms of the distance to the boundary $\partial\Omega$. This also occurs for the local p -Laplacian and some related operators, see [5, 24, 25] and references therein.

Lemma 6.6. *If $f(x) > 0$ in Ω and $g \equiv 0$, then there exists a unique maximizer of problem (6.4) given by*

$$u_\infty = \begin{cases} (d(x, \partial\Omega))^\alpha, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proof. For any $\alpha \in (0, 1)$ it holds that

$$|(d(x, \partial\Omega))^\alpha - (d(y, \partial\Omega))^\alpha| \leq |d(x, \partial\Omega) - d(y, \partial\Omega)|^\alpha \leq |x - y|^\alpha,$$

where we have used that $d(x, \partial\Omega)$ is a Lipschitz function with Lipschitz constant equal to one.

On the other hand, since f is positive and u_∞ satisfies (6.4), it suffices to see that for any $v \in Y$, it holds that $v(x) \leq (d(x, \partial\Omega))^\alpha$. Indeed, as $v \in Y$, we get

$$\frac{|v(x)|}{|x - y|^\alpha} \leq 1, \quad \forall y \in \partial\Omega.$$

Thus,

$$|v(x)| \leq \inf_{y \in \partial\Omega} |x - y|^\alpha = (d(x, \partial\Omega))^\alpha.$$

□

Remark 6.7. We note that u_∞ , the unique maximizer of problem (6.4), is in fact a solution of the limit problem given in Theorem 6.5. For $x \in \Omega$ there exists $y \in \partial\Omega$ such that

$$d(x, \partial\Omega) = |x - y|.$$

Then,

$$\frac{u_\infty(y) - u_\infty(x)}{|x - y|^\alpha} = -1.$$

Since $u_\infty \in Y$, we conclude that $-\mathcal{L}_\infty^- u_\infty(x) = 1$.

Remark 6.8. If $f(x) < 0$ in Ω and $g \equiv 0$, the unique maximizer is obviously given by

$$u_\infty(x) = \begin{cases} -(d(x, \partial\Omega))^\alpha, & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

6.2. The limit problem as $p \rightarrow \infty$: $f = f(u) = |u|^{\theta(p)-1}u$.

We focus now on the study of the limit as $p \rightarrow \infty$ in the sublinear problem (1.1) for the positive solutions given in Theorem 5.1. In this case we prove that the limit problem is the following

$$\begin{cases} \min\{-\mathcal{L}_\infty^- u_\infty - u_\infty^\Theta, -\mathcal{L}_\infty u_\infty\} = 0 & \text{in } \Omega \\ u_\infty = g & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (6.6)$$

Definition 6.9. An upper semicontinuous function u such that $u \leq g$ in $\mathbb{R}^N \setminus \Omega$ is a viscosity subsolution to (6.6) if the conditions

$$\min\{-\mathcal{L}_\infty^- \varphi - \varphi^\Theta, -\mathcal{L}_\infty \varphi\} \leq 0$$

holds for all test function φ touching u from above at point x_0 .

We say that a lower semicontinuous function u such that $u \geq g$ in $\mathbb{R}^N \setminus \Omega$ is a viscosity supersolution to (1.1) if whenever $x_0 \in \Omega$ and $\phi \in C_c^1(\mathbb{R}^N)$ touching u from below, the following condition holds

$$\min\{-\mathcal{L}_\infty^- \phi - \phi^\Theta, -\mathcal{L}_\infty \phi\} \geq 0.$$

Finally, a continuous function u is a viscosity solution to (1.1) if it is a viscosity supersolution and a viscosity subsolution.

Theorem 6.10. Let u_p be the positive solution to (1.1) obtained in Theorem 5.1. A function u_∞ obtained as a uniform limit of a subsequence of $\{u_p\}$, is a viscosity solution of the problem (6.6).

Proof. First we observe that, thanks to Theorem 5.3

$$u_\infty(x) \geq \varepsilon \psi_\infty(x) > 0, \quad x \in \Omega.$$

In order to show that u_∞ is a viscosity supersolution to (6.6) we take $x_0 \in \Omega$ and a test function ϕ such that the function $u_\infty - \phi$ is positive and reaches a strict minimum at x_0 . Hence, up to subsequences, $x_p \rightarrow x_0$ being x_p points at which the positive function $u_p - \phi$ attains a minimum. Thanks to the Hölder embedding (Theorem 2.1) the solution u_p is continuous for p large and by Lemma 3.9 it is a viscosity solution. Then, it holds that

$$-(A_p^{p-1}\phi(x_p) - B_p^{p-1}\phi(x_p)) \geq \phi^{\theta(p)}(x_p) > 0.$$

Recall that both A_p and B_p are non-negative, thus

$$B_p\phi(x_p) > A_p\phi(x_p) \quad \text{and} \quad B_p\phi(x_p) \geq \phi^{\frac{\theta(p)}{p-1}}(x_p).$$

From Lemma 6.1 we can take limits to obtain,

$$-\mathcal{L}_\infty^-\phi(x_0) \geq \mathcal{L}_\infty^+\phi(x_0), \quad \text{and} \quad -\mathcal{L}_\infty^-\phi(x_0) \geq \phi^\Theta(x_0).$$

Therefore, $\min\{-\mathcal{L}_\infty^-\phi(x_0) - \phi^\Theta(x_0), -\mathcal{L}_\infty^-\phi(x_0)\} \geq 0$ and u_∞ is a viscosity supersolution to (6.6).

To prove that u_∞ is a viscosity subsolution, we consider $x_0 \in \Omega$ and φ such that $u_\infty - \varphi$ is a negative function which reaches a strict maximum at x_0 . As before consider $x_p \rightarrow x_0$ such that the negative function $u_p - \varphi$ attains a maximum.

We note that if $-\mathcal{L}_\infty^-\varphi(x_0) - \varphi^\Theta(x_0) \leq 0$ then u_∞ is a viscosity subsolution and we are done. Thus we assume that $-\mathcal{L}_\infty^-\varphi(x_0) - \varphi^\Theta(x_0) > 0$. Lemma 6.1 yields

$$B_p\varphi(x_p) > \varphi^{\frac{\theta(p)}{p-1}}(x_p) \geq 0,$$

for p large enough.

Now we use that u_p are viscosity subsolutions to get

$$-(A_p^{p-1}\varphi(x_p) - B_p^{p-1}\varphi(x_p)) \leq \varphi^{\theta(p)}(x_p),$$

which can be rewritten as

$$1 \leq \left[\frac{\varphi^{\frac{\theta(p)}{p-1}}(x_p)}{B_p\varphi(x_p)} \right]^{p-1} + \left[\frac{A_p\varphi(x_p)}{B_p\varphi(x_p)} \right]^{p-1}.$$

The first term on the righthand side goes to zero as $p \rightarrow \infty$. Therefore, the last term cannot tend to zero as well. This fact and Lemma 6.1 imply that

$$\mathcal{L}_\infty^+\varphi(x_0) = \lim_{p \rightarrow \infty} A_p\varphi(x_p) \geq \lim_{p \rightarrow \infty} B_p\varphi(x_p) = -\mathcal{L}_\infty^-\varphi(x_0).$$

Namely, $-\mathcal{L}_\infty^-\varphi(x_0) \leq 0$ which shows that u_∞ is a viscosity subsolution to (6.6). $\square$

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