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Mean-field SDEs with jumps and nonlocal integral-PDEs

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Abstract. Recently Buckdahn et al. (Mean-field stochastic differential equations and associated PDEs, arXiv:1407.1215, 2014) studied a mean-field stochastic differential equation (SDE), whose coefficients depend on both the solution process and also its law, and whose solution process $(X_s^{t,x,P_\xi}, X_s^{t,\xi} = X_s^{t,x,P_\xi}|_{x=\xi})$, $s \in [t,T]$, $(t,x) \in [0,T] \times \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t, \mathbb{R}^d)$, admits the flow property. This flow property is the key for the study of the associated nonlocal partial differential equation (PDE). In this work we extend these studies in a non-trivial manner to mean-field SDEs which, in addition to the driving Brownian motion, are governed by a compensated Poisson random measure. We show that under suitable regularity assumptions on the coefficients of the SDE, the solution X^{t,x,P_ξ} is twice differentiable with respect to x and its law. We establish the associated nonlocal integral-PDE, and we show that $V(t,x,P_\xi) = E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]$ is the unique classical solution $V : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ of this nonlocal integral-PDE with terminal condition Φ.

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1. Introduction

The history of mean-field SDEs, also known as McKean–Vlasov equations, can be traced back to the works by Kac [15] in 1956 and McKean [23] in 1966 on stochastic systems with a large number of interacting particles. Since then the theory of mean-field SDEs has attracted many researchers and has been developing dynamically, see for example, [4,12,13,17,24–27] and the references

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therein. Moreover, in the last decade, more and more researchers have been interested in related topics and problems in this field, such as Kloeden and Lorenz [16], Kotelenez and Kurtz [18], Yong [30], Buckdahn et al. [5], Buckdahn et al. [6] and others. In particular, we remark that with their pioneering paper [19], Lasry and Lions have enlarged considerably the horizon for the applications of mean-field problems. They considered namely a general mean-field approach to problems in economics, finance and, in particular, in game theory.

Recently, inspired by the courses given by Lions at *Collège de France* [22] (see also the notes of these courses made by Cardaliaguet [8]) many works have been done. Among them one has to mention, in particular, those by Carmona and Delarue [10, 11] in which they studied the master equation for large population equilibrium as well as forward–backward stochastic differential equations and controlled McKean Vlasov dynamics. Cardaliaguet [9] proved the existence and uniqueness of a weak solution for first order mean field game systems with local coupling by variational methods, while Buckdahn et al. [7] investigated mean-field stochastic differential equations governed by a Brownian motion and they proved that the value function $V(t, x, P_\xi)$ involving the law is the unique classical solution of an associated PDE. This latter work represents a remarkable progress, since it helps to overcome the partial freezing of initial data, which was done to have a flow property (see, e.g., Buckdahn et al. [6], Hao and Li [14]).

The motivation to study mean-field SDEs with jumps stems on one hand from previous study of mean-field SDEs driven only by a Brownian motion and their applications, but on the other hand also from the fact that, because of different applications, namely, in finance, the study of SDEs with jumps and backward SDEs with jumps has been boosted. Let us refer, for instance, to the large number of works on forward–backward stochastic differential equations (FBSDEs) with jumps, for example, see Barles et al. [1], Bass [2], Björk et al. [3], Li and Peng [20], Li and Wei [21], Tang and Li [28], and Wu [29]. Especially, Barles et al. [1] were the first to study BSDEs with jumps and the associated integral-PDEs. They proved that the cost functional introduced by the solution of the BSDE with jumps was the unique viscosity solution of the associated parabolic integral-PDE.

Inspired by above works, in this paper we will discuss mean-field stochastic differential equations (SDEs) with jumps and the associated integral-PDEs. More precisely, we consider a couple of SDEs with jumps:

$$\begin{aligned} X_s^{t,\xi} &= \xi + \int_t^s b(X_r^{t,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dB_r \\ &\quad + \int_t^s \int_K \beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e)\mu_\lambda(dr, de), \quad s \in [t, T], \end{aligned} \tag{1.1}$$

$$\begin{aligned} X_s^{t,x,\xi} &= x + \int_t^s b(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dB_r \\ &\quad + \int_t^s \int_K \beta(X_{r-}^{t,x,\xi}, P_{X_r^{t,\xi}}, e)\mu_\lambda(dr, de), \quad s \in [t, T], \end{aligned} \tag{1.2}$$

with the initial data $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. Recall that, for the case without jumps, i.e., $\beta = 0$, such Eqs. (1.1) and (1.2) are first studied in [7]. They are there the result of a splitting of the classical McKean–Vlasov SDE, made with the objective to get the flow property which the McKean–Vlasov equation usually does not have. Under suitable assumptions on the coefficients b, σ and β (see Sect. 3 for details), the system (1.1) and (1.2) has a unique adapted square integrable càdlàg solution $(X^{t,\xi}, X^{t,x,\xi})$. From the uniqueness of the solution of Eq. (1.1) we know that $X^{t,\xi} = X^{t,x,\xi}|_{x=\xi}$. We will show that the pair $(X^{t,x,\xi}, X^{t,\xi})$ satisfies flow property, i.e.,

$$\left(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}\right) = (X_r^{t,x,\xi}, X_r^{t,\xi}), \quad r \in [s,T],$$

for all $0 \le t \le s \le T$, $x \in \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, and that $X^{t,x,\xi}$ depends on ξ only through the law of ξ. This flow property is the key for the study of the PDE associated with (1.1) and (1.2). In order to give an idea what we can expect, we first consider a simple case: Let $d = 1$, the coefficients $b = \sigma = 0$ and β only depends on e, i.e., $\beta(x,\mu,e) = \beta(e)$, $x \in \mathbb{R}$, $\mu \in \mathcal{P}_2(\mathbb{R})$, $e \in K$. Moreover, suppose that $f(P_\vartheta) := g(E[h(\vartheta)])$, $\vartheta \in L^2(\mathcal{F},\mathbb{R})$, where g, $h : \mathbb{R} \to \mathbb{R}$ are C^2-functions with bounded derivatives of all order. Then considering that $\partial_\mu f(P_\vartheta, y) = g'(E[h(\vartheta)])h'(y)$, $y \in \mathbb{R}$ (see Sect. 2), a straight-forward application of Itô's formula to $h(X_s^{t,\xi})$ and the fact that $\partial_s f(P_{X_s^{t,\xi}})ds = g'(E[h(X_s^{t,\xi})])E[dh(X_s^{t,\xi})]$, yield

$$\begin{aligned}\partial_s f(P_{X_s^{t,\xi}}) = E\bigg[\int_0^1 \int_K \Big((\partial_\mu f)(P_{X_s^{t,\xi}}, X_s^{t,\xi} + \rho\beta(e))\\ - (\partial_\mu f)(P_{X_s^{t,\xi}}, X_s^{t,\xi})\Big)\beta(e)\lambda(de)d\rho\bigg], \quad s \in [t,T].\end{aligned}$$

We will see that this formula also holds in a more general case, when f defined over the space $\mathcal{P}_2(\mathbb{R})$ of the probability laws over $\mathbb{R}$ with finite second moment is regular enough. The proof of such a formula in a general case is a central element in our approach (see Theorem 7.1). It generalizes the second order Taylor-type expansion in [7] and turns out to be much more suitable because of the presence of jump terms. Let now $\Phi : \mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a function smooth enough and consider the value function $V(t,x,P_\xi) = E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]$, defined over $[0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$. As the formula for $\partial_s f(P_{X_s^{t,\xi}})$ suggests, the function $\Psi(t,y) := \Phi(t,y,P_{X_T^{t,\xi}})$ is $C^{1,2}$ with respect to (t,y), and the classical argument shows that $V(t,x,P_\xi) = E[\Psi(t,X_T^{t,x,P_\xi})]$, $(t,x) \in [0,T]\times\mathbb{R}$, satisfies the following integral-PDE:

$$\begin{aligned}0 = \partial_t V(t,x,P_\xi) + \int_K \big(V(t,x+\beta(e),P_\xi) - V(t,x,P_\xi) - \partial_x V(t,x,P_\xi)\beta(e)\big)\lambda(de)\\ + E\bigg[\int_0^1 \int_K [(\partial_\mu V)(t,x,P_\xi,\xi+\rho\beta(e)) - (\partial_\mu V)(t,x,P_\xi,\xi)]\cdot\beta(e)\lambda(de)d\rho\bigg],\end{aligned}$$

$$(t,x,\xi)\in[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_t;\mathbb{R}),$$
$$V(T,x,P_\xi)=\Phi(x,P_\xi),\ (x,\xi)\in\mathbb{R}\times L^2(\mathcal{F}_t;\mathbb{R}).$$

Inspired by this simple case, we show that, in the general case, the associated integral-PDE is of the form:

$$\begin{aligned}
0&=\partial_t V(t,x,P_\xi)+\sum_{i=1}^d\partial_{x_i}V(t,x,P_\xi)b_i(x,P_\xi)\\
&+\frac{1}{2}\sum_{i,j,k=1}^d\partial^2_{x_ix_j}V(t,x,P_\xi)(\sigma_{i,k}\sigma_{j,k})(x,P_\xi)\\
&+\int_K\big(V(t,x+\beta(x,P_\xi,e),P_\xi)-V(t,x,P_\xi)\\
&-\sum_{i=1}^d\partial_{x_i}V(t,x,P_\xi)\beta_i(x,P_\xi,e)\big)\lambda(de)+E\Bigg[\sum_{i=1}^d(\partial_\mu V)_i(t,x,P_\xi,\xi)b_i(\xi,P_\xi)\\
&+\frac{1}{2}\sum_{i,j,k=1}^d\partial_{y_i}(\partial_\mu V)_j(t,x,P_\xi,\xi)(\sigma_{i,k}\sigma_{j,k})(\xi,P_\xi)\\
&+\sum_{i=1}^d\int_0^1\int_K[(\partial_\mu V)_i(t,x,P_\xi,\xi+\rho\beta(\xi,P_\xi,e)),\\
&\quad-(\partial_\mu V)_i(t,x,P_\xi,\xi)]\cdot\beta_i(\xi,P_\xi,e)\lambda(de)d\rho\Bigg]\\
&(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d),\\
&V(T,x,P_\xi)=\Phi(x,P_\xi),\ (x,\xi)\in\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d).
\end{aligned}\tag{1.3}$$

We show that the value function $V(t,x,P_\xi)$ is the unique classical solution of Eq. (1.3) (see Theorem 7.3). Since the main tool in our proof is a generalization of Itô's formula, the first and second order derivatives of $V(t,x,P_\xi)$ with respect to the law and their estimates are crucial. Their study necessitates the investigation of the first and second order derivatives of X^{t,x,P_ξ} with respect to x and the law P_ξ.

We emphasize that, different from [7] we study the mean-field SDEs with jumps, and give the probabilistic representation to the solution of a new type of nonlocal integral PDE (1.3). For this, we need to prove several new results, in particular a new Itô formula (see Theorem 7.2), whose proof is far from being a direct extension of Itô's formula (for mean-field processes without jumps) in [7]. Indeed, the key for the proof of the Itô formula consists in the study of the derivative of $f(P_{X_s^{t,\xi}})$ with respect to s (see Theorem 7.1). While in the case without jumps in [7] this derivative was obtained as a direct consequence of a kind of second order Taylor expansion of $f(P_{\xi_0+\eta})$ at $\xi_0\in L^2(P)$, as $E[|\eta|^3]\to 0$, this approach is here in the case of jumps not possible anymore: Indeed, while in the case without jumps $E[|X^{t,\xi}_{s+h}-X^{t,\xi}_s|^3]=O(h^{3/2})$, as $0<h\to 0$, in the case with jumps (1.1) we only have $E[|X^{t,\xi}_{s+h}-X^{t,\xi}_s|^3]=O(h)$.

To overcome this difficulty which has its origin in the jump part, we have to consider the filtration generated by the Poisson random measure and enlarged by the full information on the underlying Brownian motion, in order to apply the Itô formula just to the jump part of the Itô process. This, combined with a series of subtle estimates, allows to compute the derivative of $f(P_{X_s^{t,\xi}})$ in Theorem 7.1 under the standard assumption that $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$, and thus later to prove the Itô formula in Theorem 7.2. The "lack in the speed of convergence" of $E[|X_{s+h}^{t,\xi} - X_s^{t,\xi}|^3]$ in the jump case leads also at other places of the manuscript to trickier estimates.

Our paper is organized as follows: In Sect. 2, we introduce our setting and some notations, which are used frequently in what follows. The existence and the uniqueness of the solution of SDE for $X^{t,\xi}$ and that for X^{t,x,P_ξ} are proved and the corresponding estimates are obtained in Sect. 3. Section 5 is devoted to the investigation of the Fréchet derivative of X^{t,x,P_ξ} with respect to P_ξ. The derivative of X^{t,x,P_ξ} with respect to the law is characterized as solution of an SDE with jumps. Based on the first order derivatives, we discuss the second order derivatives of $X^{t,x,\xi}$ in Sect. 6. The regularity of the value function $V(t,x,P_\xi)$ is investigated in Sect. 7. In the last Section, the associated integral-PDE is studied. We prove that the value function $V(t,x,P_\xi)$ is the unique classical solution of this integral-PDE of mean-field type.

2. Preliminaries

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\mathcal{P}_2(\mathbb{R}^d) = \left\{\mu \in \mathcal{P}(\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|^2 \ \mu(dx) < \infty\right\}$. We endow $\mathcal{P}_2(\mathbb{R}^d)$ with the 2-Wasserstein metric: for $\mu,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2(\mu,\nu) := \inf\left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^2 \rho(dxdy) \right)^{\frac{1}{2}},\ \rho \in \mathcal{P}_2(\mathbb{R}^{2d}),\ \text{such that} \right.$$
$$\left. \rho(A\times\mathbb{R}^d) = \mu(A),\ A\in\mathcal{B}(\mathbb{R}^d),\ \rho(\mathbb{R}^d\times B) = \nu(B),\ B\in\mathcal{B}(\mathbb{R}^d) \right\}.$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, which we suppose to be rich enough, i.e., for each $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a random variable $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ ($L^2(\mathcal{F};\mathbb{R}^d)$ for short) such that $P_\xi = \nu$. Following the approach introduced by Lions [8], we define the differentiability of a function $f:\ \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ as follows: If for the lifted function $\tilde{f}(\xi) := f(P_\xi),\ \xi \in L^2(\mathcal{F};\mathbb{R}^d)$, there exists a $\xi_0 \in L^2(\mathcal{F};\mathbb{R}^d)$ with $P_{\xi_0} = \mu$ such that $\tilde{f}:\ L^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ is Fréchet differentiable in ξ_0, then f is said to be differentiable in μ. In other words, if this is the case, then there exists a linear continuous mapping $D\tilde{f}(\xi_0):\ L^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ ($D\tilde{f}(\xi_0) \in L(L^2(\mathcal{F};\mathbb{R}^d);\mathbb{R})$) such that

$$\tilde{f}(\xi_0+\eta) - \tilde{f}(\xi_0) = D\tilde{f}(\xi_0)(\eta) + o(|\eta|_{L^2}), \tag{2.1}$$

with $|\eta|_{L^2} \to 0$, for $\eta \in L^2(\mathcal{F};\mathbb{R}^d)$. By Riesz' Representation Theorem, there exists a unique $\zeta \in L^2(\mathcal{F};\mathbb{R}^d)$ such that $D\tilde{f}(\xi_0)(\eta) = E[\zeta\cdot\eta],\ \eta \in L^2(\mathcal{F};\mathbb{R}^d)$.

Cardaliaguet [8] proved that ζ is of the form $\zeta = h_0(\xi_0)$, P-a.s., where h_0 is a Borel function which only depends on the law P_{ξ_0}, but not on ξ_0 itself. Hence, (2.1) can be rewritten as

$$f(P_{\xi_0+\eta}) - f(P_{\xi_0}) = E[h_0(\xi_0)\cdot\eta] + o(|\eta|_{L^2}),\ \eta \in L^2(\mathcal{F};\mathbb{R}^d).$$

The function $\partial_\mu f(P_{\xi_0}, y) := h_0(y),\ y \in \mathbb{R}^d$, is called the derivative of $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at P_{ξ_0}. Notice that $\partial_\mu f(P_{\xi_0}, y)$ is only $P_{\xi_0}(dy)$-a.e. uniquely determined.

Using the above introduced definition, we define the differentiability of a function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ over $\mathcal{P}_2(\mathbb{R}^d)$ by assuming that $\tilde{f} : L^2(\mathcal{F}, \mathbb{R}^d) \to \mathbb{R}$ is Fréchet differentiable over the whole space $L^2(\mathcal{F};\mathbb{R}^d)$.

Let us introduce two spaces which are used in what follows.

Definition 2.1. (1) $C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$ denotes the space of differentiable functions $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, which satisfy $\partial_\mu f(\cdot,\cdot) : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous, i.e., there exists some positive constant C such that

(i) $|\partial_\mu f(\mu, x)| \le C,\ \mu \in \mathcal{P}_2(\mathbb{R}^d),\ x \in \mathbb{R}^d$,

(ii) $|\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \le C(W_2(\mu, \mu') + |x - x'|),\ \mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^d),\ x,\ x' \in \mathbb{R}^d$.

(2) $C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$ denotes the space of all functions $f \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, such that

(i) $(\partial_\mu f)_j(\cdot, y) \in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for all $y \in \mathbb{R}^d,\ 1 \le j \le d$, and $\partial_\mu^2 f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and Lipschitz,

(ii) $(\partial_\mu f)_j(\mu, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable, for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and the derivative $\partial_y(\partial_\mu f) : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and Lipschitz.

Here we use the notations $\partial_\mu^2 f(\mu, x, y) := (\partial_\mu(\partial_\mu f)_j(\cdot, y)(\mu, x))_{1\le j\le d}$, $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d$, $\partial_\mu f(\mu, y) := ((\partial_\mu f)_j(\mu, y))_{1\le j\le d}$, $(\mu, y) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$.

3. Mean-field stochastic differential equations with jumps

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $B = (B_t)_{t\ge0}$ be a d-dimensional Brownian motion and μ be a Poisson random measure on $\mathbb{R}_+ \times K$ independent of B, where $K \subset \mathbb{R}^l$ is a nonempty open set equipped with its Borel field $\mathcal{K}$, with compensator $\nu(dt, de) = \lambda(de)dt$ such that, $\{\mu_\lambda([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t\ge0}$ is a martingale for all $A \in \mathcal{K}$ satisfying $\lambda(A) < \infty$. Here λ is a given σ-finite Lévy measure on $(K, \mathcal{K})$, i.e., a measure on $(K, \mathcal{K})$ with the property that $\int_K (1 \wedge |e|^2)\lambda(de) < \infty$. Let $T > 0$ be a fixed time horizon and $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge0}$ be the filtration generated by the above mutually independent processes, completed and augmented by a σ-field $\mathcal{G}_0$, i.e.,

$$\begin{aligned} \overset{o}{\mathcal{F}}_t &= \sigma\big\{B_s,\ \mu([0,s] \times A)\big|\ s \le t, A \in \mathcal{K}\big\}, \\ \mathcal{F}_t &:= \overset{o}{\mathcal{F}}_{t+} \vee \mathcal{G}_0\Big(= \Big(\bigcap_{s:s>t} \overset{o}{\mathcal{F}}_s\Big) \vee \mathcal{G}_0\Big), \quad t \in [0, T], \end{aligned}$$

where $\mathcal{G}_0 \subset \mathcal{F}$ is supposed to have the following properties:

(i) The Brownian motion B and the Poisson random measure μ are independent of $\mathcal{G}_0$;
(ii) $\mathcal{G}_0$ is "rich enough", i.e., $\mathcal{P}_2(\mathbb{R}^d) = \{P_\xi, \xi \in L^2(\mathcal{G}_0;\mathbb{R}^d)\}$;
(iii) $\mathcal{G}_0 \supset \mathcal{N}_P$, where $\mathcal{N}_P$ is the set of all P-null subsets.

The following space is used frequently. For $t \in [0,T]$,

$$S^2_{\mathbb{F}}(t,T;\mathbb{R}^d) := \Big\{(X_s)_{s\in[t,T]} \ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-adapted càdlàg process: } E[\sup_{t\le s\le T}|X_s|^2] < +\infty\Big\}.$$

$$\mathcal{H}^2_{\mathbb{F}}(t,T;\mathbb{R}^d) := \Big\{(\psi_s)_{s\in[t,T]} \ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-predictable process: } ||\psi||^2 := E[\int_t^T |\psi_s|^2 ds] < +\infty\Big\}.$$

Let $b : \mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$, $\beta : \mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times K \to \mathbb{R}^d$ satisfy:

(H3.1) (i) b, σ are bounded and Lipschitz;

(ii) The coefficient β is Borel measurable. Moreover, there exists some positive constant L such that, for all $e \in K$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$|\beta(x,\mu,e)| \le L(1\wedge|e|)$, $|\beta(x,\mu,e) - \beta(x',\mu',e)| \le L(1\wedge|e|)(|x-x'| + W_2(\mu,\mu'))$.

For $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, let us now consider two SDEs:

$$\begin{aligned} X_s^{t,\xi} &= \xi + \int_t^s b(X_r^{t,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dB_r \\ &\quad + \int_t^s\int_K \beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e)\mu_\lambda(dr,de), \quad s\in[t,T], \end{aligned} \tag{3.1}$$

$$\begin{aligned} X_s^{t,x,\xi} &= x + \int_t^s b(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dr + \int_t^s \sigma(X_r^{t,x,\xi}, P_{X_r^{t,\xi}})dB_r \\ &\quad + \int_t^s\int_K \beta(X_{r-}^{t,x,\xi}, P_{X_r^{t,\xi}}, e)\mu_\lambda(dr,de), \quad s\in[t,T]. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Under assumption* (H3.1)*, the Eqs.* (3.1) *and* (3.2) *admit unique solutions* $X^{t,\xi} = (X_s^{t,\xi})_{s\in[t,T]}$ *and* $X^{t,x,\xi} = (X_s^{t,x,\xi})_{s\in[t,T]}$ *in* $S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$*. The solution* $X^{t,x,\xi}$ *is independent of* $\mathcal{F}_t$*.*

Proof. For convenience, we suppose $b = 0$. But our argument is also feasible for $b \neq 0$. Let $X^0 \equiv \xi \in S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$. For $i \ge 0$, we consider the following iteration equation:

$$X_s^{i+1} = \xi + \int_t^s \sigma(X_r^i, P_{X_r^i})dB_r + \int_t^s\int_K \beta(X_{r-}^i, P_{X_r^i}, e)\mu_\lambda(dr,de), \quad s\in[t,T]. \tag{3.3}$$

Obviously, $X^{i+1} \in S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$, $i \ge 0$. Denote $\bar{X}^i = X^{i+1} - X^i$, $i \ge 0$, then

$$\begin{aligned} \bar{X}_s^i &= \int_t^s (\sigma(X_r^i, P_{X_r^i}) - \sigma(X_r^{i-1}, P_{X_r^{i-1}}))dB_r \\ &\quad + \int_t^s\int_K \beta(X_{r-}^i, P_{X_r^i}, e) - \beta(X_{r-}^{i-1}, P_{X_r^{i-1}}, e)\mu_\lambda(dr,de), \quad s\in[t,T]. \end{aligned}$$

Thanks to Burkholder–Davis–Gundy inequality and the Lipschitz assumption on β and σ, it follows that for $t \le s \le u \le T$,

$$\begin{aligned}
&E\Big[\sup_{s\in[t,u]}|\bar{X}^i_s|^2\Big]\\
&\quad\le CE\left[\int_t^u |\sigma(X^i_r,P_{X^i_r})-\sigma(X^{i-1}_r,P_{X^{i-1}_r})|^2dr\right]\\
&\qquad+CE\left[\int_t^u\int_K |\beta(X^i_r,P_{X^i_r},e)-\beta(X^{i-1}_r,P_{X^{i-1}_r},e)|^2\lambda(de)dr\right]\\
&\quad\le CE\left[\int_t^u |\bar{X}^{i-1}_r|^2+W_2(P_{X^i_r},P_{X^{i-1}_r})^2dr\right]\\
&\qquad+CE\left[\int_t^u\int_K (1\wedge|e|^2)(|\bar{X}^{i-1}_r|^2+W_2(P_{X^i_r},P_{X^{i-1}_r})^2)\lambda(de)dr\right]. \qquad (3.4)
\end{aligned}$$

Recall that $W_2(P_{X^i_r},P_{X^{i-1}_r})^2 \le E|\bar{X}^{i-1}_r|^2$. From Gronwall's inequality we have $E\Big[\sup_{s\in[t,u]}|\bar{X}^i_s|^2\Big] \le CE\int_t^u|\bar{X}^{i-1}_r|^2dr$, where C depends only on Lipschitz constants of β and σ. This implies

$$\begin{aligned}
E|\bar{X}^i_u|^2 &\le C\int_t^u E[|\bar{X}^{i-1}_r|^2]dr \le C^2\int_t^u\int_t^r E[|\bar{X}^{i-2}_{r_1}|^2]dr_1dr\\
&\le\cdots\le C^i\int_t^u\int_t^r\int_t^{r_1}\cdots\int_t^{r_{i-2}}E[|\bar{X}^0_{r_{i-1}}|^2]dr_{i-1}\cdots dr_1dr.
\end{aligned}$$

Hence, $E\Big[\sup_{s\in[t,T]}|\bar{X}^i_s|^2\Big] \le C_0\frac{(CT)^i}{i!},\ i\ge 0$, where C_0 is independent of i. It follows that

$$\sum_{i\ge 0}\Big(E\Big[\sup_{s\in[t,T]}|\bar{X}^i_s|^2\Big]\Big)^{\frac{1}{2}}<\infty.$$

This means that $(X^i)_{i\ge 0}\subset S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$ is a Cauchy sequence. Consequently, there exists a process $X\in S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$ such that $||X^i-X||_{S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)}\to 0,\ i\to\infty$, and for $r\in[t,T]$,

$$W_2(P_{X^i_r},P_{X_r})^2\le E|X^i_r-X_r|^2\le ||X^i-X||^2_{S^2_{\mathbb{F}}(t,T;\mathbb{R}^d)}\to 0,\quad i\to\infty.$$

Taking $i\to\infty$ in (3.3),

$$X_s=\xi+\int_t^s\sigma(X_r,P_{X_r})dB_r+\int_t^s\int_K\beta(X_{r-},P_{X_r},e)\mu_\lambda(dr,de),\quad s\in[t,T].$$

Hence, it only still remains to prove the uniqueness of the solution of SDE (3.1). We suppose $X^i=(X^i_s)_{s\in[t,T]}\in S^2_{\mathbb{F}}(t,T;\mathbb{R}^d),\ i=1,2$ are two solutions of SDE (3.1). In analogy to (3.4), we have for $t\le u\le T$,

$$E|X^1_u-X^2_u|^2\le C\int_t^u E|X^1_r-X^2_r|^2dr.$$

From Gronwall's inequality, we have $X^1=X^2$.

Once knowing $X^{t,\xi}$ and $P_{X^{t,\xi}}$, Eq. (3.2) can be treated as a classical SDE with $\bar{b}(s,x):=b(x,P_{X^{t,\xi}_s})$, $\bar{\sigma}(s,x):=\sigma(x,P_{X^{t,\xi}_s})$, and $\bar{\beta}(s,x,e):=$

$\beta(x, P_{X_s^{t,\xi}}, e)$. Obviously, $\bar{b}$, $\bar{\sigma}$ and $\bar{\beta}$ are Lipschitz in x. Hence, Eq. (3.2) has a unique solution $X^{t,x,\xi} \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^d)$, and it is independent of $\mathcal{F}_t$. □

Remark 3.1. (i) From the uniqueness of Eq. (3.1) for $X^{t,\xi}$, we have $X_s^{t,\xi} = X_s^{t,x,\xi}\Big|_{x=\xi} = X_s^{t,\xi,\xi},\ s \in [t,T]$.

(ii) The solution of Eq. (3.1) and that of (3.2) satisfy a flow property:

$$\left(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}\right) = (X_r^{t,x,\xi}, X_r^{t,\xi}), \quad r \in [s,T],$$

for all $0 \le t \le s \le T,\ x \in \mathbb{R}^d,\ \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$.

Inspired by the work of Li and Wei [21], we obtain the following lemma. Even if (3.5) was originally proved only for $\varphi \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^d)$, by passing to the limit one extends the estimate easily to all $\varphi \in \mathcal{H}_{\mathbb{F}}^2(t,T;\mathbb{R}^d)$.

Lemma 3.1. *For any $\varphi \in \mathcal{H}_{\mathbb{F}}^2(t,T;\mathbb{R}^d)$, we have for $p \ge 2$,*

$$E\left[\left(\int_t^u \int_K (1\wedge |e|^2)|\varphi_{r-}|^2 \mu(dr,de)\right)^{\frac{p}{2}} |\mathcal{F}_t\right] \le C_p E\left[\int_t^u |\varphi_r|^p dr|\mathcal{F}_t\right], \quad t \le u \le T. \tag{3.5}$$

Making use of the preceding lemma, the following estimates for $X^{t,x,\xi}$ and $X^{t,\xi}$ can be obtained.

Proposition 3.1. *For $p \ge 2$, there exists a positive constant C_p depending on the Lipschitz constants of b, σ and β, such that for all $t \in [0,T]$, x, $\bar{x} \in \mathbb{R}^d$, ξ, $\bar{\xi} \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, P-a.s.,*

(i) $E[\sup_{s\in[t,T]} |X_s^{t,x,\xi}|^p|\mathcal{F}_t] \le C_p(1+|x|^p)$, $E[\sup_{s\in[t,T]} |X_s^{t,\xi}|^p|\mathcal{F}_t] \le C_p(1+|\xi|^p)$,

(ii) $E[\sup_{s\in[t,T]} |X_s^{t,x,\xi} - X_s^{t,\bar{x},\bar{\xi}}|^p|\mathcal{F}_t] \le C_p(|x-\bar{x}|^p + W_2(P_\xi, P_{\bar{\xi}})^p)$,
$E[\sup_{s\in[t,T]} |X_s^{t,\xi} - X_s^{t,\bar{\xi}}|^p|\mathcal{F}_t] \le C_p(|\xi-\bar{\xi}|^p + W_2(P_\xi, P_{\bar{\xi}})^p)$,

(iii) $E[\sup_{s\in[t,t+h]} \big(|X_s^{t,\xi} - \xi|^p + |X_s^{t,x,\xi} - x|^p\big)|\mathcal{F}_t] \le C_p h(1+|\xi|^p)$.

Proof. We will only prove $E[\sup_{s\in[t,T]} |X_s^{t,x,\xi} - X_s^{t,\bar{x},\bar{\xi}}|^p|\mathcal{F}_t] \le C_p(|x-\bar{x}|^p + W_2(P_\xi, P_{\bar{\xi}})^p)$. The other estimates can be proved with a similar argument.

Denote $\Delta X_r^t = X_r^{t,x_1,\xi_1} - X_r^{t,x_2,\xi_2},\ \Delta x = x_1 - x_2$. From Burkholder–Davis–Gundy inequality and the Lipschitz continuity of b, σ, β, we have, for $t \le s \le u \le T$,

$$\begin{aligned} &E\left[\sup_{s\in[t,u]} |\Delta X_s^t|^p|\mathcal{F}_t\right] \\ &\quad\le C_p\left(|\Delta x|^p + E\left[\int_t^u (W_2(P_{X_r^{t,\xi_1}}, P_{X_r^{t,\xi_2}})^p + |\triangle X_r^t|^p)dr|\mathcal{F}_t\right]\right) \\ &\qquad + C_p E\left[\left(\int_t^u \int_K (1\wedge|e|^2)(|\Delta X_{r-}^t|^2 + W_2(P_{X_r^{t,\xi_1}}, P_{X_r^{t,\xi_2}})^2)\mu(dr,de)\right)^{\frac{p}{2}} |\mathcal{F}_t\right]. \end{aligned} \tag{3.6}$$

But, from Lemma 3.1 we have

$$E\Bigg[\bigg(\int_t^u\int_K(1\wedge|e|^2)|\Delta X^t_{r-}|^2\mu(dr,de)\bigg)^{\frac{p}{2}}|\mathcal{F}_t\Bigg]$$
$$\le C_pE\bigg[\int_t^u|\Delta X^t_r|^pdr|\mathcal{F}_t\bigg], \tag{3.7}$$
$$E\Bigg[\bigg(\int_t^u\int_K(1\wedge|e|^2)W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2\mu(dr,de)\bigg)^{\frac{p}{2}}|\mathcal{F}_t\Bigg]$$
$$\le C_p\int_t^uW_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^pdr. \tag{3.8}$$

Combining (3.6)–(3.8),

$$E\bigg[\sup_{s\in[t,u]}|\Delta X^t_s|^p|\mathcal{F}_t\bigg]$$
$$\le C_p\Bigg(|\Delta x|^p+\int_t^uW_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^pdr+E\bigg[\int_t^u|\Delta X^t_r|^pdr|\mathcal{F}_t\bigg]\Bigg).$$

From Gronwall's inequality we have

$$E\bigg[\sup_{s\in[t,u]}|\Delta X^t_s|^p|\mathcal{F}_t\bigg]\le C_p\Bigg(|\Delta x|^p+\int_t^uW_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^pdr\Bigg), \tag{3.9}$$

i.e.,

$$E\bigg[\sup_{s\in[t,u]}|X_s^{t,x_1,\xi_1}-X_s^{t,x_2,\xi_2}|^p|\mathcal{F}_t\bigg]$$
$$\le C_p\Bigg(|x_1-x_2|^p+\int_t^uW_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^pdr\Bigg), \tag{3.10}$$

for all $x_1,x_2\in\mathbb{R}^d$. On the other hand, recall that $X_s^{t,x_i,\xi_i},\ s\in[t,T],\ x_i\in\mathbb{R}^d,\ i=1,2,$ is independent of $\mathcal{F}_t$. Hence, for all $\xi_i\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ with $P_{\xi_i'}=P_{\xi_i}$, the laws of X_s^{t,ξ_i',ξ_i} and $X_s^{t,\xi_i,\xi_i}=X_s^{t,\xi_i}$ (see Remark 3.1 (i)) coincide. Thus, for all $\xi_i'\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ with $P_{\xi_i'}=P_{\xi_i},\ i=1,2,$

$$W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2$$
$$=W_2(P_{X_r^{t,\xi_1',\xi_1}},P_{X_r^{t,\xi_2',\xi_2}})^2\le E[|X_r^{t,\xi_1',\xi_1}-X_r^{t,\xi_2',\xi_2}|^2]$$
$$=E[E[|X_r^{t,x_1,\xi_1}-X_r^{t,x_2,\xi_2}|^2|\mathcal{F}_t]|_{x_1=\xi_1',x_2=\xi_2'}],\ r\in[t,T],$$

and from (3.10), for $p=2$,

$$W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2\le CE[|\xi_1'-\xi_2'|^2]+C\int_t^rW_2(P_{X_u^{t,\xi_1}},P_{X_u^{t,\xi_2}})^2du,\ r\in[t,T].$$

Hence, from Gronwall's inequality, $\sup_{r\in[t,T]}W_2(P_{X_r^{t,\xi_1}},P_{X_r^{t,\xi_2}})^2\le CE[|\xi_1'-\xi_2'|^2]$, and taking into account the arbitrariness of $\xi_i'\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ with $P_{\xi_i'}=P_{\xi_i},\ i=1,2,$ we obtain

$$\sup_{r\in[t,T]} W_2(P_{X_r^{t,\xi_1}}, P_{X_r^{t,\xi_2}})^2 \le C W_2(P_{\xi_1}, P_{\xi_2})^2.$$

This result combined with (3.10) yields

$$E\left[\sup_{r\in[t,T]} |X_r^{t,x_1,\xi_1} - X_r^{t,x_2,\xi_2}|^p |\mathcal{F}_t\right] \le C_p\Big(|x_1-x_2|^p + W_2(P_{\xi_1}, P_{\xi_2})^p\Big),$$

$t\in[0,T],\ x_1,\ x_2\in\mathbb{R}^d,\ \xi_1,\ \xi_2\in L^2(\mathcal{F}_t;\mathbb{R}^d).$ □

From Proposition 3.1, we know that the processes X^{t,x,ξ_1} and X^{t,x,ξ_2} are indistinguishable as long as $\xi_1\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ and $\xi_2\in L^2(\mathcal{F}_t;\mathbb{R}^d)$ have the same law. This means that $X^{t,x,\xi}$ depends on ξ only through its law. Hence, we can define $X^{t,x,P_\xi} := X^{t,x,\xi},\ (t,x)\in[0,T]\times\mathbb{R}^d,\ \xi\in L^2(\mathcal{F}_t;\mathbb{R}^d)$.

4. First order derivatives of X^{t,x,P_ξ}

This section is devoted to study the first order derivatives of X^{t,x,P_ξ} with respect to x and P_ξ. Let us first make an assumption which is frequently used in what follows.

(Assumption A.1) For every $e\in K,\ (b,\sigma,\beta(\cdot,\cdot,e))\in C_b^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d\times\mathbb{R}^{d\times d}\times\mathbb{R}^d)$; the components $b_j,\sigma_{i,j},\beta_j(\cdot,\cdot,e),\ 1\le i,\ j\le d$, are assumed to satisfy:

(i) $\sigma_{i,j}(x,\cdot), b_j(x,\cdot), \beta_j(x,\cdot,e)\in C_b^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$, for all $x\in\mathbb{R}^d$, $e\in K$;
(ii) $\sigma_{i,j}(\cdot,\mu),\ b_j(\cdot,\mu),\ \beta_j(\cdot,\mu,e)\in C_b^1(\mathbb{R}^d)$, for all $\mu\in\mathcal{P}_2(\mathbb{R}^d),\ e\in K$;
(iii) The derivatives $\partial_x\sigma_{i,j},\ \partial_x b_j:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ and $\partial_\mu\sigma_{i,j},\ \partial_\mu b_j:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^d$ are bounded and Lipschitz continuous;
(iv) There is some constant $L\in\mathbb{R}$ such that $\partial_x\beta_j(\cdot,\cdot,e):\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ and $\partial_\mu\beta_j(\cdot,\cdot,e):\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^d$ are bounded by $L(1\wedge|e|)$ and Lipschitz continuous with Lipschitz factor $L(1\wedge|e|)$, $e\in K$, i.e., for all $x,\ \bar{x},\ y,\ \bar{y}\in\mathbb{R}^d,\ \mu,\ \bar{\mu}\in\mathcal{P}_2(\mathbb{R}^d)$, $e\in K$, $1\le j\le d$,
(v) $|\partial_\mu\beta_j(x,\mu,e,y)|\le L(1\wedge|e|)$, $|\partial_x\beta_j(x,\mu,e)|\le L(1\wedge|e|)$,
(vi) $|\partial_x\beta_j(x,\mu,e)-\partial_x\beta_j(\bar{x},\bar{\mu},e)|\le L(1\wedge|e|)(|x-\bar{x}|+W_2(\mu,\bar{\mu}))$, $|\partial_\mu\beta_j(x,\mu,e,y)-\partial_\mu\beta_j(\bar{x},\bar{\mu},e,\bar{y})|\le L(1\wedge|e|)(|x-\bar{x}|+|y-\bar{y}|+W_2(\mu,\bar{\mu}))$.

We now study the L^2-differentiability of X^{t,x,P_ξ} with respect to x.

Theorem 4.1. *Under Assumption* (A.1)*, the L^2-derivative $\partial_x X^{t,x,P_\xi} = (\partial_x X^{t,x,P_\xi,j})_{1\le j\le d}$ of X^{t,x,P_ξ} exists and is characterized by the following SDE:*

$$\begin{aligned}\partial_{x_i}X_s^{t,x,P_\xi,j} &= 1+\sum_{k,l=1}^d\int_t^s \partial_{x_k}\sigma_{j,l}(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\partial_{x_i}X_r^{t,x,P_\xi,k}dB_r^l\\ &\quad+\sum_{k=1}^d\int_t^s \partial_{x_k}b_j(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\partial_{x_i}X_r^{t,x,P_\xi,k}dr\\ &\quad+\sum_{k=1}^d\int_t^s\int_K \partial_{x_k}\beta_j(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e)\partial_{x_i}X_{r-}^{t,x,P_\xi,k}\mu_\lambda(dr,de),\end{aligned} \tag{4.1}$$

$s\in[t,T],\ 1\le i,\ j\le d.$

Considering the coefficients $\bar{\sigma}(r,x) := \sigma(x, P_{X_r^{t,\xi}})$, $\bar{b}(r,x) := b(x, P_{X_r^{t,\xi}})$ and $\bar{\beta}(r,x,e) := \beta(x, P_{X_r^{t,\xi}}, e)$, this result is a classical one and we omit the proof.

From standard estimates for classical SDEs and in analogy to Proposition 3.1 we get

Proposition 4.1. *For all $p \geq 2$, there exists a positive constant C_p only depending on the Lipschitz constants of $\partial_x b$, $\partial_x \sigma$ and $\partial_x \beta$, such that for all $t \in [0,T]$, x, $x' \in \mathbb{R}^d$, P_ξ, $P_{\xi'} \in \mathcal{P}_2(\mathbb{R}^d)$, P-a.s.,*

(i) $E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi}|^p|\mathcal{F}_t\Big] \leq C_p$,

(ii) $E\Big[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi} - \partial_x X_s^{t,x',P_{\xi'}}|^p|\mathcal{F}_t\Big] \leq C_p\big(|x-x'|^p + W_2(P_\xi, P_{\xi'})^p\big)$,

(iii) $E[\sup_{s\in[t,t+h]} |\partial_x X_s^{t,x,P_\xi} - I_{\mathbb{R}^d}|^p|\mathcal{F}_t] \leq C_p h,\ 0 \leq h \leq T-t.$

Remark 4.1. Notice that all terms of (4.1) are independent of $\mathcal{F}_t$. Hence, $\partial_x X^{t,x,P_\xi}$ is independent of $\mathcal{F}_t$. Consequently, $\partial_x X^{t,\xi,P_\xi} (= (\partial_x X^{t,\xi,P_\xi,j})_{1\leq j\leq d}) := \partial_x X^{t,x,P_\xi}\big|_{x=\xi} (= (\partial_x X^{t,x,P_\xi,j})_{1\leq j\leq d}\big|_{x=\xi})$ is well defined and is the unique solution of the following SDE:

$$\begin{aligned}
\partial_{x_i} X_s^{t,\xi,P_\xi,j} &= 1 + \sum_{k,l=1}^{d} \int_t^s \partial_{x_k} \sigma_{j,l}(X_r^{t,\xi}, P_{X_r^{t,\xi}}) \partial_{x_i} X_r^{t,\xi,P_\xi,k} dB_r^l \\
&\quad + \sum_{k=1}^{d} \int_t^s \partial_{x_k} b_j(X_r^{t,\xi}, P_{X_r^{t,\xi}}) \partial_{x_i} X_r^{t,\xi,P_\xi,k} dr \\
&\quad + \sum_{k=1}^{d} \int_t^s \int_K \partial_{x_k} \beta_j(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e) \partial_{x_i} X_{r-}^{t,\xi,P_\xi,k} \mu_\lambda(dr,de). \quad (4.2)
\end{aligned}$$

Moreover, from Proposition 4.1, for $p \geq 2$,

$$E\Bigg[\sup_{s\in[t,T]} |\partial_x X_s^{t,\xi,P_\xi}|^p|\mathcal{F}_t\Bigg] \leq \sup_{x\in\mathbb{R}^d} E\Bigg[\sup_{s\in[t,T]} |\partial_x X_s^{t,x,P_\xi}|^p|\mathcal{F}_t\Bigg] \leq C_p, \text{ P-a.s.,}$$

where C_p is independent of the choice of ξ and in fact, only depends on the Lipschitz constants of $\partial_x b$, $\partial_x \sigma$ and $\partial_x \beta$.

The following theorem shows that the solution $X^{t,x,\xi}$ of (3.2) interpreted as a functional of $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ is Gâteaux differentiable.

Theorem 4.2. *Let (b,σ,β) satisfy Assumption* (A.1)*, and by $X^{t,x,\xi}$ we denote the solution X^{t,x,P_ξ} of* (3.2) *lifted from $\mathcal{P}_2(\mathbb{R}^d)$ to $L^2(\mathcal{F}_t;\mathbb{R}^d)$, $X_s^{t,x,\xi} := X_s^{t,x,P_\xi}$, $s \in [t,T]$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. Then, for all $0 \leq t \leq s \leq T$ and $x \in \mathbb{R}^d$, the mapping $X_s^{t,x,\cdot} : L^2(\mathcal{F}_t;\mathbb{R}^d) \to L^2(\mathcal{F}_s;\mathbb{R}^d)$ is Gâteaux differentiable. Moreover, there exists a stochastic process $N^{t,x,P_\xi}(y) = \big(N_s^{t,x,P_\xi}(y)\big)_{s\in[t,T]} \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$ such that the Gâteaux derivative is of the form:*

$$\partial_\xi X_s^{t,x,\xi}(\eta) = \tilde{E}\big[N_s^{t,x,P_\xi}(\tilde{\xi})\cdot\tilde{\eta}\big] = \left(\tilde{E}\Big[\sum_{j=1}^{d} N_{s,i,j}^{t,x,P_\xi}(\tilde{\xi})\cdot\tilde{\eta}_j\Big]\right)_{1\le i\le d},$$

for all $\eta = (\eta_1, \eta_2, \ldots, \eta_d) \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. *Here,* $(\tilde{\xi}, \tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_d))$ *defined on some probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, *obeys the same law as* $(\xi, \eta = (\eta_1, \ldots, \eta_d))$ *on* $(\Omega, \mathcal{F}, P)$, *and* $\tilde{E}[\cdot]$ *denotes the expectation under* $\tilde{P}$*;* $\tilde{E}[\cdot]$ *concerns here only* $\tilde{\xi}$ *and* $\tilde{\eta}$.

Proof. For simplicity we restrict ourselves only to the one-dimensional case. The proof can be extended to the case $d \ge 1$ by using the same argument.

We suppose that $(\tilde{\xi}, \tilde{\eta}, \tilde{B}, \widetilde{\mu_\lambda})$ is a copy of $(\xi, \eta, B, \mu_\lambda)$ on some complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, and we denote by $\tilde{X}^{t,\tilde{\xi}}$ the solution of SDE (3.1), but now driven by the Brownian Motion $\tilde{B}$ and the compensated Poisson random measure $\widetilde{\mu_\lambda}$ instead of B and μ_λ, and with the initial value $\tilde{\xi}$ instead of ξ; $\tilde{X}^{t,x,\tilde{P}_{\tilde{\xi}}}$ denotes the solution of SDE (3.2) for $X^{t,x,\xi}$ driven by $\tilde{B}$ and $\widetilde{\mu_\lambda}$. Then,

$$\begin{aligned}\tilde{X}_s^{t,\tilde{\xi}} &= \tilde{\xi} + \int_t^s b(\tilde{X}_r^{t,\tilde{\xi}}, \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}})dr + \int_t^s \sigma(\tilde{X}_r^{t,\tilde{\xi}}, \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}})d\tilde{B}_r \\ &\quad + \int_t^s\int_K \beta(\tilde{X}_{r-}^{t,\tilde{\xi}}, \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}}, e)\widetilde{\mu_\lambda}(dr,de), \quad s\in[t,T], \qquad (4.3)\\ \tilde{X}_s^{t,x,\tilde{P}_{\tilde{\xi}}} &= x + \int_t^s b(\tilde{X}_r^{t,x,\tilde{P}_{\tilde{\xi}}}, \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}})dr + \int_t^s \sigma(\tilde{X}_r^{t,x,\tilde{P}_{\tilde{\xi}}}, \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}})d\tilde{B}_r \\ &\quad + \int_t^s\int_K \beta(\tilde{X}_{r-}^{t,x,\tilde{P}_{\tilde{\xi}}},\ \tilde{P}_{\tilde{X}_r^{t,\tilde{\xi}}}, e)\widetilde{\mu_\lambda}(dr,de), \quad s\in[t,T]. \qquad (4.4)\end{aligned}$$

From $\tilde{P}_{\tilde{\xi}} = P_\xi$ we know that $\tilde{X}^{t,x,\tilde{P}_{\tilde{\xi}}} = \tilde{X}^{t,x,P_\xi}$, $x\in\mathbb{R}$, $t\in[0,T]$, and $(\tilde{\xi}, \tilde{\eta}, \tilde{X}^{t,x,P_\xi}, \tilde{B}, \widetilde{\mu_\lambda})$ defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ is an independent copy of $(\xi, \eta, X^{t,x,P_\xi}, B, \mu_\lambda)$. Using the above notations, by $\partial_x\tilde{X}_r^{t,x,P_\xi}$ we denote the derivative of $\tilde{X}_r^{t,x,P_\xi}$ with respect to x, and we define $\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi} := \partial_x\tilde{X}_r^{t,x,P_\xi}\big|_{x=\tilde{\xi}}$.

Let us consider the SDE

$$\begin{aligned}Y_s^{t,\xi}(\eta) &= \int_t^s \partial_x\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})Y_r^{t,\xi}(\eta)dB_r + \int_t^s \partial_x b(X_r^{t,\xi}, P_{X_r^{t,\xi}})Y_r^{t,\xi}(\eta)dr \\ &\quad + \int_t^s\int_K \partial_x\beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e)Y_{r-}^{t,\xi}(\eta)\mu_\lambda(dr,de) \\ &\quad + \int_t^s \tilde{E}[\partial_\mu\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot(\partial_x X_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]dB_r \\ &\quad + \int_t^s \tilde{E}[\partial_\mu b(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot(\partial_x X_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]dr \\ &\quad + \int_t^s\int_K \tilde{E}[\partial_\mu\beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e) \\ &\quad \cdot(\partial_x X_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]\mu_\lambda(dr,de),\ s\in[t,T]. \qquad (4.5)\end{aligned}$$

It is straight-forward to show that (4.5) possesses a unique solution $Y^{t,\xi}(\eta) \in S^2_{\mathbb{F}}(t,T;\mathbb{R})$. This solution $Y^{t,\xi}(\eta)$ of (4.5) allows to consider the following equation for which the existence and the uniqueness of a solution is standard too,

$$\begin{aligned} Y_s^{t,x,P_\xi}(\eta) =& \int_t^s \partial_x\sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})Y_r^{t,x,\xi}(\eta)dB_r \\ &+ \int_t^s \partial_x b(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})Y_r^{t,x,\xi}(\eta)dr \\ &+ \int_t^s\int_K \partial_x\beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e)Y_{r-}^{t,x,\xi}(\eta)\mu_\lambda(dr,de) \\ &+ \int_t^s \tilde{E}[\partial_\mu\sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]dB_r \\ &+ \int_t^s \tilde{E}[\partial_\mu b(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]dr \\ &+ \int_t^s\int_K \tilde{E}[\partial_\mu\beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e) \\ &\cdot(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta}))]\mu_\lambda(dr,de),\ \ s\in[t,T]. \end{aligned} \tag{4.6}$$

Obviously, from (4.5) the mapping $Y_s^{t,\xi}(\cdot): L^2(\mathcal{F}_t;\mathbb{R}) \to L^2(\mathcal{F}_s;\mathbb{R})$ is linear. From standard estimates for SDE it follows that

$$E\left[\sup_{s\in[t,T]}|Y_s^{t,\xi}(\eta)|^2\right] \le CE[|\eta|^2], \quad \eta\in L^2(\mathcal{F}_t;\mathbb{R}).$$

Hence, $Y_s^{t,\xi}(\cdot): L^2(\mathcal{F}_t;\mathbb{R}) \to L^2(\mathcal{F}_s;\mathbb{R})$ is a linear and continuous mapping. Using this for (4.6), we also get $Y_s^{t,x,P_\xi}(\cdot):\ L^2(\mathcal{F}_t;\mathbb{R}) \to L^2(\mathcal{F}_s;\mathbb{R})$ is a linear and continuous mapping.

In order to obtain the concrete expression of $Y_s^{t,x,P_\xi}(\eta)$, we now, for $y\in\mathbb{R}$, consider the following SDE:

$$\begin{aligned} N_s^{t,\xi}(y) =& \int_t^s \partial_x\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})N_r^{t,\xi}(y)dB_r + \int_t^s \partial_x b(X_r^{t,\xi}, P_{X_r^{t,\xi}})N_r^{t,\xi}(y)dr \\ &+ \int_t^s\int_K \partial_x\beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e)N_{r-}^{t,\xi}(y)\mu_\lambda(dr,de) \\ &+ \int_t^s \tilde{E}[\partial_\mu\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi} \\ &+\partial_\mu\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]dB_r \\ &+ \int_t^s \tilde{E}[\partial_\mu b(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi} \\ &+\partial_\mu b(X_r^{t,\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]dr \end{aligned}$$

$$+\int_t^s\int_K \tilde{E}[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\partial_x\tilde{X}_r^{t,y,P_\xi}$$
$$+\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]\mu_\lambda(dr,de), \tag{4.7}$$

$s\in[t,T]$, where $(\tilde{N}^{t,\tilde{\xi}}(y),\tilde{B},\widetilde{\mu_\lambda})$ under $\tilde{P}$ is assumed to have the same law as $(N^{t,\xi}(y),B,\mu_\lambda)$ under P. From our assumption on b, σ and β, as well as the properties of the process ∂_xX^{t,y,P_ξ} (see Proposition 4.1), it is easy to prove that Eq. (4.7) possesses a unique solution $N^{t,\xi}(y)=(N_s^{t,\xi}(y))_{s\in[t,T]}\in S^2_{\mathbb{F}}(t,T)$. Moreover, with SDE standard estimates we can also show that, for all $y\in\mathbb{R}$, $t\in[0,T]$, $\xi\in L^2(\mathcal{F}_t;\mathbb{R})$, $E[\sup_{s\in[t,T]}|N_s^{t,\xi}(y)|^2]\le C$, where C depends only on the bounds of the derivatives of b, σ and β.

Once having $N^{t,\xi}(y)$, we consider the unique solution $N^{t,x,P_\xi}(y)\in S^2_{\mathbb{F}}(t,T)$ of the following SDE:

$$\begin{aligned}N_s^{t,x,P_\xi}(y)=&\int_t^s\partial_x\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})N_r^{t,x,P_\xi}(y)dB_r\\&+\int_t^s\partial_xb(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})N_r^{t,x,P_\xi}(y)dr\\&+\int_t^s\int_K\partial_x\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)N_{r-}^{t,x,P_\xi}(y)\mu_\lambda(dr,de),\\&+\int_t^s\tilde{E}[\partial_\mu\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi}\\&+\partial_\mu\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]dB_r\\&+\int_t^s\tilde{E}[\partial_\mu b(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi}\\&+\partial_\mu b(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]dr\\&+\int_t^s\int_K\tilde{E}[\partial_\mu\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\partial_x\tilde{X}_r^{t,y,P_\xi}\\&+\partial_\mu\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]\mu_\lambda(dr,de).\end{aligned}\tag{4.8}$$

Since $N^{t,x,P_\xi}(y)$ is independent of $\mathcal{F}_t$, and due to the uniqueness of the solution of the Eq. (4.7) we obtain, by substituting ξ for x, $N_s^{t,\xi}(y)=N_s^{t,x,\xi}(y)\big|_{x=\xi}$, $s\in[t,T]$, P-a.s. After substituting $\bar{\xi}$ for y in $N_\cdot^{t,\xi}(y)$, we multiply both sides of such obtained equation by $\bar{\eta}$, and then take the expectation $\bar{E}[\cdot]$ under $\bar{P}$, where $(\bar{\xi},\bar{\eta},\bar{B},\overline{\mu_\lambda})$ defined on some probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ is an independent copy of (ξ,η,B,μ_λ) (and, hence, also of $(\tilde{\xi},\tilde{\eta},\tilde{B},\widetilde{\mu_\lambda})$). This yields

$$\begin{aligned}&\bar{E}[N_s^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]\\&\quad=\int_t^s\partial_x\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]dB_r\\&\qquad+\int_t^s\partial_xb(X_r^{t,\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]dr\end{aligned}$$

$$
\begin{aligned}
&+\int_t^s\int_K \partial_x\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\bar{E}[N_{r-}^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]\mu_\lambda(dr,de)\\
&+\int_t^s \tilde{E}\Big[\bar{E}[\partial_\mu\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\bar{\xi},P_\xi})\partial_x\tilde{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}]\Big]dB_r\\
&+\int_t^s \tilde{E}\Big[\partial_\mu\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}]\Big]dB_r\\
&+\int_t^s \tilde{E}\Big[\bar{E}[\partial_\mu b(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\bar{\xi},P_\xi})\partial_x\tilde{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}]\Big]dr\\
&+\int_t^s \tilde{E}\Big[\partial_\mu b(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}]\Big]dr\\
&+\int_t^s\int_K \tilde{E}\Big[\bar{E}[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\bar{\xi},P_\xi},e)\partial_x\tilde{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}]\Big]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}\Big[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}]\Big]\mu_\lambda(dr,de),\quad s\in[t,T].
\end{aligned}\tag{4.9}
$$

Since $(\bar{\xi},\bar{\eta})$ is independent of (ξ,η,B,μ_λ) and $(\tilde{\xi},\tilde{\eta},\tilde{B},\widetilde{\mu_\lambda})$, and of the same law under $\bar{P}$ as $(\tilde{\xi},\tilde{\eta})$ under $\tilde{P}$, we have for $f=b,\ \sigma$ and for β,

$$
\begin{aligned}
&\tilde{E}\Big[\bar{E}[\partial_\mu f(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\bar{\xi},P_\xi})\partial_x\tilde{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}]\Big]\\
&\quad=\tilde{E}\Big[\partial_\mu f(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi},P_\xi})\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\Big],\\
&\tilde{E}\Big[\bar{E}[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_{r-}^{t,\bar{\xi},P_\xi},e)\partial_x\tilde{X}_{r-}^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}]\Big]\\
&\quad=\tilde{E}\Big[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_{r-}^{t,\tilde{\xi},P_\xi},e)\partial_x\tilde{X}_{r-}^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\Big].
\end{aligned}\tag{4.10}
$$

Combining (4.9) with (4.10), we have

$$
\begin{aligned}
&\bar{E}[N_s^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]\\
&\quad=\int_t^s \partial_x\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]dB_r\\
&\qquad+\int_t^s \partial_x b(X_r^{t,\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]dr\\
&\qquad+\int_t^s\int_K \partial_x\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\bar{E}[N_{r-}^{t,\xi}(\bar{\xi})\cdot\bar{\eta}]\mu_\lambda(dr,de)\\
&\qquad+\int_t^s \tilde{E}\Big[\partial_\mu\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]dB_r\\
&\qquad+\int_t^s \tilde{E}\Big[\partial_\mu b(X_r^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]dr
\end{aligned}
$$

$$+\int_t^s\int_K \tilde{E}\Big[\partial_\mu\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\\ \times(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]\mu_\lambda(dr,de),\quad s\in[t,T]. \tag{4.11}$$

However, Eq. (4.11) is just Eq. (4.5) for $Y^{t,\xi}(\eta)$. Due to the uniqueness of the solution of this equation, we have

$$Y_s^{t,\xi}(\eta)=\bar{E}[N_s^{t,\xi}(\bar{\xi})\cdot\bar{\eta}],\quad s\in[t,T].$$

Using the same argument as before but now for $N^{t,x,P_\xi}(y)$ instead of $N^{t,\xi}(y)$, we obtain

$$\begin{aligned}&\bar{E}[N_s^{t,x,P_\xi}(\bar{\xi})\cdot\bar{\eta}]\\ &=\int_t^s\partial_x\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,x,P_\xi}(\bar{\xi})\cdot\bar{\eta}]dB_r\\ &\quad+\int_t^s\partial_x b(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\bar{E}[N_r^{t,x,P_\xi}(\bar{\xi})\cdot\bar{\eta}]dr\\ &\quad+\int_t^s\int_K\partial_x\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\bar{E}[N_{r-}^{t,x,P_\xi}(\bar{\xi})\cdot\bar{\eta}]\mu_\lambda(dr,de)\\ &\quad+\int_t^s\tilde{E}\Big[\partial_\mu\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]dB_r\\ &\quad+\int_t^s\tilde{E}\Big[\partial_\mu b(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]dr\\ &\quad+\int_t^s\int_K\tilde{E}\Big[\partial_\mu\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\\ &\quad\times(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}+\bar{E}[\tilde{N}_r^{t,\tilde{\xi}}(\bar{\xi})\cdot\bar{\eta}])\Big]\mu_\lambda(dr,de),\ s\in[t,T].\end{aligned} \tag{4.12}$$

Equation (4.12) is nothing else but Eq. (4.6) for $Y^{t,x,P_\xi}(\eta)$. Hence, from the uniqueness for Eq. (4.6) it follows that

$$Y_s^{t,x,P_\xi}(\eta)=\bar{E}[N_s^{t,x,P_\xi}(\bar{\xi})\cdot\bar{\eta}],\quad s\in[t,T],\ \eta\in L^2(\mathcal{F}_t;\mathbb{R}).$$

In the following lemma (Lemma 4.1) we show that for the lifted process $X_s^{t,x,\xi}:=X_s^{t,x,P_\xi}$, $s\in[t,T]$, the directional derivative in any direction $\eta\in L^2(\mathcal{F}_t;\mathbb{R})$ exists, and coincides with $Y_s^{t,x,P_\xi}(\eta)$: $Y_s^{t,x,P_\xi}(\eta)=L^2-\lim_{h\to 0}\frac{1}{h}(X_s^{t,x,\xi+h\eta}-X_s^{t,x,\xi})$, $s\in[t,T]$. Since $Y^{t,x,P_\xi}(\cdot)$ is a linear continuous mapping from $L^2(\mathcal{F}_t;\mathbb{R})$ to $L^2(\mathcal{F}_s;\mathbb{R})$, $\xi\to X_s^{t,x,\xi}$ is Gâteaux differentiable on $L^2(\mathcal{F}_t;\mathbb{R})$. □

Lemma 4.1. *Under the assumptions of Theorem 4.1 we have, for all* $(t,x)\in[0,T]\times\mathbb{R}^d$, ξ, $\eta\in L^2(\mathcal{F}_t;\mathbb{R}^d)$,

$$E\Big[\sup_{s\in[t,T]}|(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})-hY_s^{t,x,P_\xi}(\eta)|^2\Big]\le C|h|^4(E[\eta^2])^2.$$

Proof. For simplicity, but without restriction of the generality of the method, we consider $d=1,\ b=0$ and $\sigma=0$, i.e., we restrict ourselves to

$$\begin{aligned} X_s^{t,x,P_\xi} &= x+\int_t^s\int_K \beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de),\\ X_s^{t,\xi} &= x+\int_t^s\int_K \beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de),\quad s\in[t,T]. \end{aligned}\tag{4.13}$$

From Proposition 3.1, for all $p\geq 2$, there is some $C_p\in\mathbb{R}_+$ such that, for all $\xi,\ \eta\in L^2(\mathcal{F}_t;\mathbb{R}),\ h\in\mathbb{R}$:

$$\begin{aligned} E\Big[\sup_{s\in[t,T]}|X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi}|^p|\mathcal{F}_t\Big] &\leq C_pW_2(P_{\xi+h\eta},P_\xi)^p\leq C_p(E[\eta^2])^{\frac{p}{2}}|h|^p,\\ E\Big[\sup_{s\in[t,T]}|X_s^{t,\xi+h\eta}-X_s^{t,\xi}|^p|\mathcal{F}_t\Big] &\leq C_p|h|^p(|\eta|^p+(E[\eta^2])^{\frac{p}{2}}). \end{aligned}\tag{4.14}$$

Suppose that $h\neq 0$, we have from (4.13), for $s\in[t,T]$,

$$\begin{aligned} &\frac{1}{h}(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})\\ &\quad=\frac{1}{h}\int_t^s\int_K\Big(\beta(X_{r-}^{t,x,P_{\xi+h\eta}},P_{X_r^{t,\xi+h\eta}},e)-\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\Big)\mu_\lambda(dr,de). \end{aligned}$$

But, using the regularity assumptions on β we get

$$\begin{aligned} &\beta(X_{r-}^{t,x,P_{\xi+h\eta}},P_{X_r^{t,\xi+h\eta}},e)-\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\\ &\quad=\beta(X_{r-}^{t,x,P_\xi}+(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi}),P_{X_r^{t,\xi+h\eta}},e)-\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi+h\eta}},e)\\ &\qquad+\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}+(X_r^{t,\xi+h\eta}-X_r^{t,\xi})},e)-\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\\ &\quad=\int_0^1(\partial_x\beta)(X_{r-}^{t,x,P_\xi}+\rho(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi}),P_{X_r^{t,\xi+h\eta}},e)d\rho\\ &\qquad\cdot(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi})+\int_0^1\tilde{E}[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}+\rho(X_r^{t,\xi+h\eta}-X_r^{t,\xi})},\tilde{X}_r^{t,\tilde{\xi}}\\ &\qquad+\rho(\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}}-\tilde{X}_r^{t,\tilde{\xi}}),e)(\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}}-\tilde{X}_r^{t,\tilde{\xi}})]d\rho. \end{aligned}$$

As concerns the notations $\tilde{X}^{t,\tilde{\xi}}$, $\tilde{X}^{t,\tilde{\xi}+h\tilde{\eta}}$, we refer to the first part of the proof of Theorem 4.2. Hence,

$$\begin{aligned} &\frac{1}{h}(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})\\ &\quad=\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\frac{1}{h}(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi})\mu_\lambda(dr,de)\\ &\qquad+\int_t^s\int_K\tilde{E}[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}}-\tilde{X}_r^{t,\tilde{\xi}})]\mu_\lambda(dr,de)\\ &\qquad+R_s^{t,x,\xi,\eta}(h),\ s\in[t,T], \end{aligned}\tag{4.15}$$

where

$$
\begin{aligned}
R_s^{t,x,\xi,\eta}(h) = &\int_t^s \int_K \int_0^1 \Big((\partial_x \beta)(X_{r-}^{t,x,P_\xi} + \rho(X_{r-}^{t,x,P_{\xi+h\eta}} - X_{r-}^{t,x,P_\xi}), P_{X_r^{t,\xi+h\eta}}, e) \\
&- (\partial_x \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e)\Big) d\rho \frac{1}{h}(X_{r-}^{t,x,P_{\xi+h\eta}} - X_{r-}^{t,x,P_\xi})\mu_\lambda(dr,de) \\
&+ \int_t^s \int_K \int_0^1 \tilde{E}\Big[\Big((\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}+\rho(X_r^{t,\xi+h\eta}-X_r^{t,\xi})}, \tilde{X}_r^{t,\tilde{\xi}} \\
&+ \rho(\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}} - \tilde{X}_r^{t,\tilde{\xi}}), e) - (\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e)\Big) \\
&\times \frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}} - \tilde{X}_r^{t,\tilde{\xi}})\Big] d\rho \mu_\lambda(dr,de).
\end{aligned}
$$

Thus, in virtue of the Lipschitz property of $\partial_x \beta$ and $\partial_\mu \beta$,

$$
\begin{aligned}
&E\Big[\sup_{s\in[t,T]} |R_s^{t,x,\xi,\eta}(h)|^2 | \mathcal{F}_t\Big] \\
&\le C\int_t^s \int_K E\Big[(1\wedge |e|^2)\big(|X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi}|^2 \\
&\quad + W_2(P_{X_r^{t,\xi+h\eta}}, P_{X_r^{t,\xi}})^2\big)\frac{1}{|h|^2}|X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi}|^2 | \mathcal{F}_t\Big]\lambda(de)dr \\
&\quad + C\int_t^s \int_K \Big(\tilde{E}\big[(1\wedge |e|)\big(W_2(P_{X_r^{t,\xi}+\rho(X_r^{t,\xi+h\eta}-X_r^{t,\xi})}, P_{X_r^{t,\xi}}) \\
&\quad + |\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}} - \tilde{X}_r^{t,\tilde{\xi}}|\big)\frac{1}{|h|}|\tilde{X}_r^{t,\tilde{\xi}+h\tilde{\eta}} - \tilde{X}_r^{t,\tilde{\xi}}|\big]\Big)^2 \lambda(de)dr \\
&\le C\int_t^s \Big(\frac{1}{|h|^2}E[|X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi}|^4 | \mathcal{F}_t] \\
&\quad + \frac{1}{|h|^2}E[|X_r^{t,x,P_{\xi+h\eta}} - X_r^{t,x,P_\xi}|^2 | \mathcal{F}_t]\cdot E[|X_r^{t,\xi+h\eta} - X_r^{t,\xi}|^2]\Big)dr \\
&\quad + C\int_t^s \frac{1}{|h|^2}\big(E[|X_r^{t,\xi+h\eta} - X_r^{t,\xi}|^2]\big)^2 dr,
\end{aligned}
$$

and from (4.14) we get for $C\in\mathbb{R}_+$ independent of $t,\ x,\ \xi,\ \eta$,

$$
E\Big[\sup_{s\in[t,T]} |R_s^{t,x,\xi,\eta}(h)|^2 | \mathcal{F}_t\Big] \le C|h|^2(E[\eta^2])^2, \quad h\in\mathbb{R}.
$$

On the other hand, notice that

$$
\begin{aligned}
&\frac{1}{h}(X_r^{t,\xi+h\eta} - X_r^{t,\xi}) \\
&\quad = \frac{1}{h}(X_r^{t,\xi+h\eta,P_{\xi+h\eta}} - X_r^{t,\xi,P_{\xi+h\eta}}) + \frac{1}{h}(X_r^{t,\xi,P_{\xi+h\eta}} - X_r^{t,\xi,P_\xi}) \\
&\quad = \int_0^1 \partial_x X_r^{t,\xi+\rho h\eta,P_{\xi+h\eta}} d\rho\cdot\eta + \frac{1}{h}(X_r^{t,\xi,P_{\xi+h\eta}} - X_r^{t,\xi,P_\xi}), \quad r\in[t,T],
\end{aligned}
$$

and

$$E\bigg[\int_t^T\int_K |\tilde{E}\Big[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\times\int_0^1(\partial_x\tilde{X}_r^{t,\tilde{\xi}+\rho h\tilde{\eta},P_{\xi+h\eta}}-\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi})d\rho\cdot\tilde{\eta}\Big]|^2\lambda(de)dr\bigg]$$
$$\le C\int_K(1\wedge|e|^2)\lambda(de)\int_0^1E\Big[\sup_{r\in[t,T]}|\partial_xX_r^{t,\xi+\rho h\eta,P_{\xi+h\eta}}-\partial_xX_r^{t,\xi,P_\xi}|^2\Big]d\rho\cdot E[\eta^2]$$
$$\le C|h|^2(E[\eta^2])^2,$$

we have, from (4.15),

$$\begin{aligned}&\frac{1}{h}(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})\\&=\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\frac{1}{h}(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi})\mu_\lambda(dr,de)\\&\quad+\int_t^s\int_K\tilde{E}\Big[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\\&\quad+\frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi},P_{\xi+h\eta}}-\tilde{X}_r^{t,\tilde{\xi},P_\xi}))\Big]\mu_\lambda(dr,de)+\tilde{R}_s^{t,x,\xi,\eta}(h),\end{aligned}\tag{4.16}$$

where

$$E\Big[\sup_{s\in[t,T]}|\tilde{R}_s^{t,x,\xi,\eta}(h)|^2|\mathcal{F}_t\Big]\le C|h|^2(E[\eta^2])^2,\quad h\in\mathbb{R},\ x\in\mathbb{R}.$$

Substituting ξ for x in (4.16), we get

$$\begin{aligned}&\frac{1}{h}(X_s^{t,\xi,P_{\xi+h\eta}}-X_s^{t,\xi,P_\xi})\\&=\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\frac{1}{h}(X_{r-}^{t,\xi,P_{\xi+h\eta}}-X_{r-}^{t,\xi,P_\xi})\mu_\lambda(dr,de)\\&\quad+\int_t^s\int_K\tilde{E}\Big[(\partial_\mu\beta)(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\Big(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\\&\quad+\frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi},P_{\xi+h\eta}}-\tilde{X}_r^{t,\tilde{\xi},P_\xi})\Big)\Big]\mu_\lambda(dr,de)+\tilde{R}_s^{t,\xi,\xi,\eta}(h),\end{aligned}\tag{4.17}$$

with $E[\sup_{s\in[t,T]}|\tilde{R}_s^{t,\xi,\xi,\eta}(h)|^2]\le E[\sup_{x\in\mathbb{R}}E[\sup_{s\in[t,T]}|\tilde{R}_s^{t,x,\xi,\eta}(h)|^2|\mathcal{F}_t]]\le C|h|^2(E[\eta^2])^2$.

Combining (4.17) with (4.5) (of course, for $\sigma=0,\ b=0$), we obtain

$$\begin{aligned}&\frac{1}{h}(X_s^{t,\xi,P_{\xi+h\eta}}-X_s^{t,\xi,P_\xi})-Y_s^{t,\xi}(\eta)\\&=\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\Big(\frac{1}{h}(X_{r-}^{t,\xi,P_{\xi+h\eta}}-X_{r-}^{t,\xi,P_\xi})-Y_{r-}^{t,\xi}(\eta)\Big)\mu_\lambda(dr,de)\\&\quad+\int_t^s\int_K\tilde{E}\Big[(\partial_\mu\beta)(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\\&\quad\times\Big(\frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi},P_{\xi+h\eta}}-\tilde{X}_r^{t,\tilde{\xi},P_\xi})-\tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta})\Big)\Big]\mu_\lambda(dr,de)+\tilde{R}_s^{t,\xi,\xi,\eta}(h),\quad s\in[t,T],\end{aligned}$$

and moreover, an SDE standard estimate yields:

$$E\Big[\sup_{s\in[t,T]}|\frac{1}{h}(X_s^{t,\xi,P_{\xi+h\eta}}-X_s^{t,\xi,P_\xi})-Y_s^{t,\xi}(\eta)|^2\Big]$$
$$\leq CE\left[\sup_{s\in[t,T]}|\tilde{R}_s^{t,\xi,\xi,\eta}(h)|^2\right]\leq C|h|^2(E[\eta^2])^2\to 0,\quad \text{as } h\to 0. \tag{4.18}$$

Combining now (4.16) and (4.6), we have

$$\frac{1}{h}(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})-Y_s^{t,x,P_\xi}(\eta)$$
$$=\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)$$
$$\times\Big(\frac{1}{h}(X_{r-}^{t,x,P_{\xi+h\eta}}-X_{r-}^{t,x,P_\xi})-Y_{r-}^{t,x,P_\xi}(\eta)\Big)\mu_\lambda(dr,de)$$
$$+\int_t^s\int_K\tilde{E}\Big[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)$$
$$\times\Big(\frac{1}{h}(\tilde{X}_r^{t,\tilde{\xi},P_{\xi+h\eta}}-\tilde{X}_r^{t,\tilde{\xi},P_\xi})-\tilde{Y}_r^{t,\tilde{\xi}}(\tilde{\eta})\Big)\Big]\mu_\lambda(dr,de)+\tilde{R}_s^{t,x,\xi,\eta}(h),\quad s\in[t,T],$$

and from (4.18) and Gronwall's inequality it follows that

$$E\Big[\sup_{s\in[t,T]}\Big|\frac{1}{h}(X_s^{t,x,P_{\xi+h\eta}}-X_s^{t,x,P_\xi})-Y_s^{t,x,P_\xi}(\eta)\Big|^2\Big]\leq C|h|^2(E[\eta^2])^2.$$

□

We have finished the proof for the Gâteaux derivative.

Lemma 4.2. *Let Assumption* (A.1) *hold true. By* N^{t,x,P_ξ}, $N^{t,\xi}$ *we denote the unique solutions of* (4.8) *and* (4.7)*, respectively. Then, for all* $p\geq 2$*, there exists a positive constant* C_p *such that, for* $t\in[0,T]$*,* $x,\ x',\ y,\ y'\in\mathbb{R}^d$ *and* $\xi,\ \xi'\in L^2(\mathcal{F}_t;\mathbb{R}^d)$*,*

(i) $E\left[\sup_{s\in[t,T]}(|N_s^{t,x,P_\xi}(y)|^p+|N_s^{t,\xi}(y)|^p)\right]\leq C_p,$

(ii) $E\left[\sup_{s\in[t,T]}(|N_s^{t,x,P_\xi}(y)-N_s^{t,x',P_{\xi'}}(y')|^p+|N_s^{t,\xi}(y)-N_s^{t,\xi'}(y')|^p)\right]$
$\leq C_p\Big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\Big),$

(iii) $E\Big[\sup_{s\in[t,t+h]}|N_s^{t,x,P_\xi}(y)|^p\Big]\leq C_ph,\ 0\leq h\leq T-t.$

Proof. For convenience we only consider $d=1$ and set $b=0$. The proof is standard. However, for the reader's convenience we shall give a sketch of it here.

From the boundedness of $\partial_xb,\ \partial_x\sigma,\ \partial_x\beta$ and the fact that $E\Big[\sup_{s\in[t,T]}|\partial_xX_s^{t,x,P_\xi}|^p|\mathcal{F}_t\Big]\leq C_p$ (see Proposition 4.1), it is standard to prove (i), (iii) by using the Eqs. (4.7) and (4.8).

Concerning the proof of (ii), we notice that its main ingredient are the following two estimates, which make use of the Lipschitz property and the boundedness of $\partial_\mu\sigma,\ \partial_\mu\beta$, as well as of Propositions 3.1 and 4.1:

$$\begin{aligned}
&E\Big[|\tilde{E}[(\partial_\mu\sigma)(X_r^{t,x,P_\xi},\ P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi}\\
&\quad-(\partial_\mu\sigma)(X_r^{t,x',P_{\xi'}},\ P_{X_r^{t,\xi'}},\tilde{X}_r^{t,y',P_{\xi'}})\partial_x\tilde{X}_r^{t,y',P_{\xi'}}]|^p\Big]\\
&\le C_p\big(E[|X_r^{t,x,P_\xi}-X_r^{t,x',P_{\xi'}}|^p]+W_2(P_{X_r^{t,\xi}},P_{X_r^{t,\xi'}})^p\\
&\quad+E[|X_r^{t,y,P_\xi}-X_r^{t,y',P_{\xi'}}|^p+|\partial_xX_r^{t,y,P_\xi}-\partial_xX_r^{t,y',P_{\xi'}}|^p]\big),\quad r\in[t,T],
\end{aligned}\tag{4.19}$$

and for $t\le u\le T$,

$$\begin{aligned}
&E\Big[\Big|\int_t^u\int_K\tilde{E}[|\partial_\mu\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\partial_x\tilde{X}_r^{t,y,P_\xi}\\
&\quad-\partial_\mu\beta(X_{r-}^{t,x',P_{\xi'}},P_{X_r^{t,\xi'}},\tilde{X}_r^{t,y',P_{\xi'}},e)\partial_x\tilde{X}_r^{t,y',P_{\xi'}}|]\mu_\lambda(dr,de)\Big|^p\Big]\\
&\le C_pE\Bigg[\Bigg(\int_t^u\int_K(1\wedge|e|^2)(|X_{r-}^{t,x,P_\xi}-X_{r-}^{t,x',P_{\xi'}}|^2+W_2(P_{X_r^{t,\xi}},P_{X_r^{t,\xi'}})^2\\
&\quad+\tilde{E}\big[|\tilde{X}_r^{t,y,P_\xi}-\tilde{X}_r^{t,y',P_{\xi'}}|^2+|\partial_x\tilde{X}_r^{t,y,P_\xi}-\partial_x\tilde{X}_r^{t,y',P_{\xi'}}|^2\big])\mu(dr,de)\Bigg)^{\frac{p}{2}}\Bigg]\\
&\le C_p\Bigg(E\Bigg[\int_t^u|X_r^{t,x,P_\xi}-X_r^{t,x',P_{\xi'}}|^pdr\Bigg]+\int_t^uW_2(P_{X_r^{t,\xi}},P_{X_r^{t,\xi'}})^pdr\\
&\quad+\int_t^uE\big[|X_r^{t,y,P_\xi}-X_r^{t,y',P_{\xi'}}|^p+|\partial_xX_r^{t,y,P_\xi}-\partial_xX_r^{t,y',P_{\xi'}}|^p\big]dr\Bigg).
\end{aligned}\tag{4.20}$$

Applying Propositions 3.1 and 4.1, we immediately have

$$\begin{aligned}
&E\Big[|\tilde{E}[\partial_\mu\sigma(X_r^{t,x,P_\xi},\ P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\partial_x\tilde{X}_r^{t,y,P_\xi}\\
&\quad-\partial_\mu\sigma(X_r^{t,x',P_{\xi'}},\ P_{X_r^{t,\xi'}},\tilde{X}_r^{t,y',P_{\xi'}})\partial_x\tilde{X}_r^{t,y',P_{\xi'}}]|^p\Big]\\
&\le C_p\Big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\Big),
\end{aligned}\tag{4.21}$$

and

$$\begin{aligned}
&E\Bigg[\Bigg|\int_t^u\int_K\tilde{E}[|\partial_\mu\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_{r-}^{t,y,P_\xi},e)\partial_x\tilde{X}_{r-}^{t,y,P_\xi}\\
&\quad-\partial_\mu\beta(X_{r-}^{t,x',P_{\xi'}},P_{X_r^{t,\xi'}},\tilde{X}_{r-}^{t,y',P_{\xi'}},e)\partial_x\tilde{X}_{r-}^{t,y',P_{\xi'}}|]\mu_\lambda(dr,de)\Bigg|^p\Bigg]\\
&\le C_p(u-t)\Big(|x-x'|^p+|y-y'|^p+W_2(P_\xi,P_{\xi'})^p\Big).
\end{aligned}\tag{4.22}$$

With the help of (4.21), (4.22) and Gronwall's inequality we deduce from (4.8) and (4.7) the wished result. □

Theorem 4.3. *We make the same assumptions as in Theorem 4.2. Then, for all* $0 \le t \le T$, $x \in \mathbb{R}^d$, *the lifted mapping* $X^{t,x,\cdot} : L^2(\mathcal{F}_t;\mathbb{R}^d) \to L^2(\mathcal{F}_s;\mathbb{R}^d)$ *is Fréchet differentiable and its Fréchet derivative is just the Gâteaux derivative, i.e.,*

$$DX_s^{t,x,\xi}(\eta) = \partial_\xi X_s^{t,x,\xi}(\eta) = \tilde{E}\big[N_s^{t,x,P_\xi}(\tilde{\xi}) \cdot \tilde{\eta}\big], \quad s \in [t,T],\ \eta \in L^2(\mathcal{F}_t;\mathbb{R}^d),$$

where for $y \in \mathbb{R}^d$, $N^{t,x,P_\xi}(y) = ((N_{s,i,j}^{t,x,P_\xi}(y))_{s\in[t,T]})_{1\le i,\ j\le d} \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$ *is the unique solution of* (4.8).

Proof. We suppose $d = 1$. It is sufficient to show the continuity of the Gâteaux derivative $\partial_\xi X_s^{t,x,\xi}$ with respect to $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$. The Gâteaux derivative $\partial_\xi X_s^{t,x,\xi}$ as a continuous linear mapping from $L^2(\mathcal{F}_t;\mathbb{R})$ to $L^2(\mathcal{F}_s;\mathbb{R})$, i.e., it belongs to $L(L^2(\mathcal{F}_t;\mathbb{R}), L^2(\mathcal{F}_s;\mathbb{R}))$, and the operator norm of $\partial_\xi X_s^{t,x,\xi} - \partial_\xi X_s^{t,x,\xi'}$, $s \in [t,T]$, $(t,x) \in [0,T]\times\mathbb{R}$, $\xi,\ \xi' \in L^2(\mathcal{F}_t;\mathbb{R})$, in this space can be estimated as follows: from Lemma 4.2 we have

$$\begin{aligned}
&\Big|\partial_\xi X_s^{t,x,\xi} - \partial_\xi X_s^{t,x,\xi'}\Big|^2_{L(L^2(\mathcal{F}_t;\mathbb{R}),L^2(\mathcal{F}_s;\mathbb{R}))} \\
&= \sup_{\substack{\eta\in L^2(\mathcal{F}_t;\mathbb{R})\\ |\eta|_{L^2}\le 1}} \Big|\partial_\xi X_s^{t,x,\xi}(\eta) - \partial_\xi X_s^{t,x,\xi'}(\eta)\Big|^2_{L^2} \\
&= \sup_{\substack{\eta\in L^2(\mathcal{F}_t;\mathbb{R})\\ |\eta|_{L^2}\le 1}} E\Big[|\partial_\xi X_s^{t,x,\xi}(\eta) - \partial_\xi X_s^{t,x,\xi'}(\eta)|^2\Big] \\
&= \sup_{\substack{\eta\in L^2(\mathcal{F}_t;\mathbb{R})\\ |\eta|_{L^2}\le 1}} E\Big[|\tilde{E}[(N_s^{t,x,P_\xi}(\tilde{\xi}) - N_s^{t,x,P_{\xi'}}(\tilde{\xi}')) \cdot \tilde{\eta}]|^2\Big] \\
&\le \sup_{\substack{\eta\in L^2(\mathcal{F}_t;\mathbb{R})\\ |\eta|_{L^2}\le 1}} E\Big[(\tilde{E}[|(N_s^{t,x,P_\xi}(\tilde{\xi}) - N_s^{t,x,P_{\xi'}}(\tilde{\xi}')|^2]) \cdot (\tilde{E}|\tilde{\eta}|^2)\Big] \\
&\le E\Big[\tilde{E}[|N_s^{t,x,P_\xi}(\tilde{\xi}) - N_s^{t,x,P_{\xi'}}(\tilde{\xi}')|^2]\Big] \\
&\le \tilde{E}\Big[E[|N_s^{t,x,P_\xi}(y) - N_s^{t,x,P_{\xi'}}(y')|^2]\big|_{y=\tilde{\xi},\ y'=\tilde{\xi}'}\Big] \\
&\le C(\tilde{E}|\tilde{\xi} - \tilde{\xi}'|^2 + W_2(P_\xi, P_{\xi'})^2) \le 2CE|\xi - \xi'|^2, \quad t \le s \le T.
\end{aligned}$$

The proof is complete. □

Remark 4.2. From Theorem 4.3 we know that $X^{t,x,\xi}$ is Fréchet differentiable with respect to ξ. In extension of the definition of the derivative of function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ over $\mathcal{P}_2(\mathbb{R}^d)$, this allows to consider X^{t,x,P_ξ} as differentiable with respect to the law, and the derivative is just $N^{t,x,P_\xi}(y)$, i.e., $\partial_\mu X_s^{t,x,P_\xi}(y) = N_s^{t,x,P_\xi}(y)$, $s \in [t,T]$, $y \in \mathbb{R}^d$, $0 \le t \le T$, $x \in \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$.

5. Second order derivatives of X^{t,x,P_ξ}

The existence and the properties of the second order derivatives of X^{t,x,P_ξ} are studied in this section.

(Assumption A.2) Suppose that for all $e \in K$, $(b, \sigma, \beta(\cdot,\cdot,e))$ belongs to $C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^d)$, i.e., $(b, \sigma, \beta(\cdot,\cdot,e)) \in C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^d)$ and the derivatives of the components $\sigma_{i,j},\ b_j,\ \beta_j, 1 \le i,\ j \le d$, have the following properties:

- (i) $\partial_{x_k}\sigma_{i,j}(\cdot,\cdot),\ \partial_{x_k} b_j(\cdot,\cdot)$ and $\partial_{x_k}\beta_j(\cdot,\cdot,e)$ belong to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, for all $1 \le k \le d$, $e \in K$;
- (ii) $\partial_\mu \sigma_{i,j}(\cdot,\cdot,\cdot),\ \partial_\mu b_j(\cdot,\cdot,\cdot),\ \partial_\mu \beta_j(\cdot,\cdot,\cdot,e)$ belong to $C_b^{1,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d)$, for all $1 \le i,\ j \le d$, $e \in K$;
- (iii) All the derivatives of $\sigma_{i,j}, b_j$ up to order 2 are bounded and Lipschitz continuous, and all the derivatives of $\beta_j(\cdot,\cdot,e)$ up to order 2 are bounded by $L(1 \wedge |e|)$ and Lipschitz continuous with a Lipschitz factor $L(1 \wedge |e|)$, where the constant L is independent of $e \in K$.

The following lemma states the equality of the second order mixed derivatives for functions from $C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$:

Lemma 5.1. *Suppose $g \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Then, for all $1 \le l \le d$,*

$$\partial_{x_l}\big(\partial_\mu g(x,\mu,y)\big) = \partial_\mu\big(\partial_{x_l} g(x,\mu)\big)(y),\ (x,\mu,y) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d.$$

The proof is standard. The reader is referred to Buckdahn et al. [7].

Let us now state the main result of this section.

Proposition 5.1. *Let Assumption* (A.2) *hold true. Then the first order derivatives $\partial_{x_i} X_s^{t,x,P_\xi}$ and $\partial_\mu X_s^{t,x,P_\xi}(y)$, which are interpreted as functional of $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, are Fréchet differentiable with respect to ξ, and also L^2-differentiable with respect to x and y. Furthermore, for all $t \in [0,T]$, $x,\ y,\ z \in \mathbb{R}^d$ and $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, there exist two stochastic processes $\partial_\mu(\partial_{x_i} X^{t,x,P_\xi})(y) \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$, $1 \le i \le d$, and $\partial_\mu^2 X^{t,x,P_\xi}(y,z) = \partial_\mu(\partial_\mu X^{t,x,P_\xi}(y))(z) \in S_{\mathbb{F}}^2(t,T;\mathbb{R}^{d\times d})$ such that the Fréchet derivatives in ξ of $\partial_{x_i} X_s^{t,x,P_\xi}$ and $\partial_\mu X_s^{t,x,P_\xi}(y)$, satisfy*

$$\begin{aligned} &\text{(i) } D_\xi[\partial_{x_i} X_s^{t,x,P_\xi}](\eta) = \hat{E}[\partial_\mu(\partial_{x_i} X_s^{t,x,P_\xi})(\hat{\xi}) \cdot \hat{\eta}],\\ &\text{(ii) } D_\xi[\partial_\mu X_s^{t,x,P_\xi}(y)](\eta) = \hat{E}[\partial_\mu^2 X_s^{t,x,P_\xi}(y,\hat{\xi}) \cdot \hat{\eta}], \end{aligned} \tag{5.1}$$

$s \in [t,T],\ y \in \mathbb{R}^d,\ \eta \in L^2(\mathcal{F}_t;\mathbb{R}^d)$. Moreover, the mixed second order derivatives $\partial_{x_i}(\partial_\mu X^{t,x,P_\xi}(y))$ and $\partial_\mu(\partial_{x_i} X^{t,x,P_\xi})(y)$ coincide, and the process

$$\begin{aligned} &Q_{s,i,j}^{t,x,P_\xi}(y,z)\\ &:= \Big(\partial_{x_i x_j}^2 X_s^{t,x,P_\xi}, \partial_\mu(\partial_{x_i} X_s^{t,x,P_\xi})(y), \partial_\mu^2 X_s^{t,x,P_\xi}(y,z), \partial_{y_i}(\partial_\mu X_s^{t,x,P_\xi}(y))\Big), \end{aligned}$$

$1 \le i,\ j \le d$, has the following properties: For all $p \ge 2$, there exists a positive real constant C_p such that, for all $t \in [0,T]$, $x,\ x',\ y,\ y'\ z,\ z' \in \mathbb{R}^d$ and $\xi,\ \xi' \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $1 \le i,\ j \le d$,

$$\text{(iii)}\ E\left[\sup_{s\in[t,T]}|Q^{t,x,P_\xi}_{s,i,j}(y,z)|^p\right]\le C_p,$$

$$\text{(iv)}\ E\left[\sup_{s\in[t,T]}|Q^{t,x,P_\xi}_{s,i,j}(y,z)-Q^{t,x',P_{\xi'}}_{s,i,j}(y',z')|^p\right]$$
$$\le C_p\Big(|x-x'|^p+|y-y'|^p+|z-z'|^p+W_2(P_\xi,P_{\xi'})^p\Big),$$

$$\text{(v)}\ E\Big[\sup_{s\in[t,t+h]}|Q^{t,x,P_\xi}_{s,i,j}(y,z)|^p\Big]\le C_ph,\quad 0\le h\le T-t. \tag{5.2}$$

Proof. For convenience we suppose $d=1$ and $b=0$. Let us recall the equation for $\partial_xX^{t,x,P_\xi}_s$:

$$\begin{aligned}\partial_xX^{t,x,P_\xi}_s&=1+\int_t^s\partial_x\sigma(X^{t,x,P_\xi}_r,P_{X^{t,\xi}_r})\partial_xX^{t,x,P_\xi}_r dB_r\\&\quad+\int_t^s\int_K\partial_x\beta(X^{t,x,P_\xi}_{r-},P_{X^{t,\xi}_r},e)\partial_xX^{t,x,P_\xi}_{r-}\mu_\lambda(dr,de),\quad s\in[t,T].\end{aligned}\tag{5.3}$$

A straight-forward extension of the approach developed in the proof of Lemma 4.1 for the directional differentiability of $L^2(\mathcal{F}_t;\mathbb{R})\ni\xi\to X^{t,x,\xi}_s(:=X^{t,x,P_\xi}_s)\in L^2(\mathcal{F}_s;\mathbb{R})$, and in the proof of Theorem 4.2 for its Gâteaux differentiability, allow to show that the mapping $L^2(\mathcal{F}_t;\mathbb{R})\ni\xi\mapsto\partial_xX^{t,x,\xi}\ (=\partial_xX^{t,x,P_\xi})\in L^2(\mathcal{F}_s;\mathbb{R})$ is differentiable in any direction $\eta\in L^2(\mathcal{F}_t;\mathbb{R})$, and its directional derivative $\partial_\xi[\partial_xX^{t,x,P_\xi}_s](\eta)$ in direction $\eta\in L^2(\mathcal{F}_t;\mathbb{R})$ satisfies the following equation:

$$\begin{aligned}\bar Y_s(\eta)&=\int_t^s\bar Y_r(\eta)\cdot\partial_x\sigma(X^{t,x,P_\xi}_r,P_{X^{t,\xi}_r})dB_r\\&\quad+\hat E\Big[\int_t^s\partial_xX^{t,x,P_\xi}_r\cdot\partial_x^2\sigma(X^{t,x,P_\xi}_r,P_{X^{t,\xi}_r})\partial_\mu X^{t,x,P_\xi}_r(\hat\xi)dB_r\cdot\hat\eta\Big]\\&\quad+\hat E\Big[\int_t^s\partial_xX^{t,x,P_\xi}_r\cdot\tilde E[\partial_\mu(\partial_x\sigma)(X^{t,x,P_\xi}_r,P_{X^{t,\xi}_r},\tilde X^{t,\hat\xi,P_\xi}_r)\cdot\partial_x\tilde X^{t,\hat\xi,P_\xi}_r\\&\quad+\partial_\mu(\partial_x\sigma)(X^{t,x,P_\xi}_r,P_{X^{t,\xi}_r},\tilde X^{t,\tilde\xi}_r)\cdot\partial_\mu\tilde X^{t,\tilde\xi,P_\xi}_r(\hat\xi)]dB_r\cdot\hat\eta\Big]\\&\quad+\int_t^s\int_K\bar Y_{r-}(\eta)\cdot\partial_x\beta(X^{t,x,P_\xi}_{r-},P_{X^{t,\xi}_r},e)\mu_\lambda(dr,de)\\&\quad+\hat E\Big[\int_t^s\int_K\partial_xX^{t,x,P_\xi}_{r-}\cdot\partial_x^2\beta(X^{t,x,P_\xi}_{r-},P_{X^{t,\xi}_r},e)\partial_\mu X^{t,x,P_\xi}_{r-}(\hat\xi)\mu_\lambda(dr,de)\cdot\hat\eta\Big]\\&\quad+\hat E\Big[\int_t^s\int_K\partial_xX^{t,x,P_\xi}_{r-}\cdot\tilde E[\partial_\mu(\partial_x\beta)(X^{t,x,P_\xi}_{r-},P_{X^{t,\xi}_r},\tilde X^{t,\hat\xi,P_\xi}_r,e)\cdot\partial_x\tilde X^{t,\hat\xi,P_\xi}_r\\&\quad+\partial_\mu\partial_x\beta(X^{t,x,P_\xi}_{r-},P_{X^{t,\xi}_r},\tilde X^{t,\tilde\xi}_r,e)\partial_\mu\tilde X^{t,\tilde\xi,P_\xi}_r(\hat\xi)]\mu_\lambda(dr,de)\cdot\hat\eta\Big].\end{aligned}\tag{5.4}$$

Here $(\hat{\xi}, \hat{\eta})$ denotes an independent copy of (ξ, η) (and, also $(\tilde{\xi}, \tilde{\eta})$) and is defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$; $\hat{E}[\cdot]$ denotes the expectation under probability measure $\hat{P}$.
From the structure of (5.4) it yields

$$\partial_\xi[\partial_x X_s^{t,x,P_\xi}](\eta) = \hat{E}[\partial_\mu \partial_x X_s^{t,x,P_\xi}(\hat{\xi}) \cdot \hat{\eta}], \quad s \in [t,T],$$

where $\partial_\mu \partial_x X_s^{t,x,P_\xi}(y)$ is the unique solution in $S^2_{\mathbb{F}}(t,T)$ of the following equation:

$$\begin{aligned}
&\partial_\mu \partial_x X_s^{t,x,P_\xi}(y)\\
&\quad= \int_t^s \partial_x \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \cdot \partial_\mu \partial_x X_r^{t,x,P_\xi}(y) dB_r\\
&\qquad+ \int_t^s \partial_x^2 \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \cdot \partial_x X_r^{t,x,P_\xi} \cdot \partial_\mu X_r^{t,x,P_\xi}(y) dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_\mu(\partial_x \sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}) \cdot \partial_x \tilde{X}_r^{t,y,P_\xi}] \partial_x X_r^{t,x,P_\xi} dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_\mu(\partial_x \sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}) \cdot \partial_\mu \tilde{X}_r^{t,\tilde{\xi},P_\xi}(y)] \partial_x X_r^{t,x,P_\xi} dB_r\\
&\qquad+ \int_t^s \int_K \partial_x \beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e) \cdot \partial_\mu \partial_x X_{r-}^{t,x,P_\xi}(y) \mu_\lambda(dr, de)\\
&\qquad+ \int_t^s \int_K \partial_x^2 \beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e) \cdot \partial_x X_{r-}^{t,x,P_\xi} \cdot \partial_\mu X_{r-}^{t,x,P_\xi}(y) \mu_\lambda(dr, de)\\
&\qquad+ \int_t^s \int_K \tilde{E}[\partial_\mu(\partial_x \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}, e) \cdot \partial_x \tilde{X}_r^{t,y,P_\xi}] \partial_x X_{r-}^{t,x,P_\xi} \mu_\lambda(dr, de)\\
&\qquad+ \int_t^s \int_K \tilde{E}[\partial_\mu(\partial_x \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e) \cdot \partial_\mu \tilde{X}_r^{t,\tilde{\xi},P_\xi}(y)] \partial_x X_{r-}^{t,x,P_\xi} \mu_\lambda(dr, de),
\end{aligned} \tag{5.5}$$

$s \in [t,T]$. Indeed, substituting $\hat{\xi}$ for y and multiplying both sides of (5.5) with $\hat{\eta}$ before taking the expectation $\hat{E}[\cdot]$, we get Eq. (5.4), but solved by $\hat{E}[\partial_\mu \partial_x X_s^{t,x,P_\xi}(\hat{\xi})\hat{\eta}]$, $s \in [t,T]$. Thus, the equality stated above follows from the uniqueness of the solution of SDE (5.4). Moreover, applying Propositions 3.1 and 4.1 it follows from standard SDE estimates that, for all $p \geq 2$, there exists a positive real constant C_p such that, for all $t \in [0,T]$, $x,\ x',\ y,\ y' \in \mathbb{R}$ and $\xi,\ \xi' \in L^2(\mathcal{F}_t;\mathbb{R})$,

$$\begin{aligned}
&\text{(i) } E\Big[\sup_{s\in[t,T]} |\partial_\mu \partial_x X_s^{t,x,P_\xi}(y)|^p\Big] \leq C_p,\\
&\text{(ii) } E\Big[\sup_{s\in[t,T]} |\partial_\mu \partial_x X_s^{t,x,P_\xi}(y) - \partial_\mu \partial_x X_s^{t,x',P_{\xi'}}(y')|^p\Big]\\
&\quad \leq C_p\Big(|x-x'|^p + |y-y'|^p + W_2(P_\xi, P_{\xi'})^p\Big),\\
&\text{(iii) } E\Bigg[\sup_{s\in[t,t+h]} |\partial_\mu \partial_x X_s^{t,x,P_\xi}(y)|^p\Bigg] \leq C_p h, \quad 0 \leq h \leq T-t.
\end{aligned}$$

Using the second estimate, we can prove that the Gâteaux derivative of $L^2(\mathcal{F}_t;\mathbb{R}) \ni \xi \mapsto \partial_x X_s^{t,x,\xi} \in L^2(\mathcal{F}_s;\mathbb{R})$ is in fact a Fréchet derivative (see the proof of Theorem 4.3 for the corresponding proof for $\xi \to X_s^{t,x,\xi}$), and the Fréchet derivative $D_\xi[\partial_x X_s^{t,x,\xi}]$ satisfies the relation

$$D_\xi[\partial_x X_s^{t,x,\xi}](\eta) = \hat{E}[\partial_\mu \partial_x X_s^{t,x,P_\xi}(\hat{\xi}) \cdot \hat{\eta}],$$

$s \in [t,T], \eta \in L^2(\mathcal{F}_t;\mathbb{R})$. This means that $\partial_\mu \partial_x X_s^{t,x,P_\xi}(y)$ is the derivative of $\partial_x X_s^{t,x,P_\xi}$ with respect to the measure P_ξ. Now recall Eq. (4.8) for $\partial_\mu X_s^{t,x,P_\xi}(y) = N_s^{t,x,P_\xi}(y)$. Standard arguments allow to show that, due to our assumptions on coefficients σ and β, $x \mapsto N^{t,x,P_\xi}(y) \in L^2(\mathcal{F}_t;\mathbb{R})$ is differentiable (in L^2-sense) and the derivative satisfies the equation (recall that in this proof $b = 0$):

$$\begin{aligned}
&\partial_x N_s^{t,x,P_\xi}(y)\\
&= \int_t^s \Big(\partial_x \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \cdot \partial_x N_r^{t,x,P_\xi}(y)\\
&\quad + (\partial_x^2 \sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \cdot \partial_x X_r^{t,x,P_\xi} \cdot N_r^{t,x,P_\xi}(y)\Big) dB_r\\
&\quad + \int_t^s \tilde{E}[\partial_x(\partial_\mu \sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}) \cdot \partial_x \tilde{X}_r^{t,y,P_\xi}]\partial_x X_r^{t,x,P_\xi} dB_r\\
&\quad + \int_t^s \tilde{E}[\partial_x(\partial_\mu \sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}) \cdot \tilde{N}_r^{t,\tilde{\xi}}(y)]\partial_x X_r^{t,x,P_\xi} dB_r\\
&\quad + \int_t^s \int_K \Big(\partial_x \beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e) \cdot \partial_x N_{r-}^{t,x,P_\xi}(y)\\
&\quad + (\partial_x^2 \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e) \cdot \partial_x X_{r-}^{t,x,P_\xi} \cdot N_{r-}^{t,x,P_\xi}(y)\Big) \mu_\lambda(dr,de)\\
&\quad + \int_t^s \int_K \tilde{E}[\partial_x(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}, e)\\
&\quad \cdot \partial_x \tilde{X}_r^{t,y,P_\xi}]\partial_x X_{r-}^{t,x,P_\xi} \mu_\lambda(dr,de)\\
&\quad + \int_t^s \int_K \tilde{E}[\partial_x(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e) \cdot \tilde{N}_r^{t,\tilde{\xi}}(y)]\partial_x X_{r-}^{t,x,P_\xi} \mu_\lambda(dr,de),\\
&\quad s \in [t,T]
\end{aligned} \tag{5.6}$$

(see Theorem 4.1 for the equation for $\partial_x X_s^{t,x,P_\xi}$). Due to Lemma 5.1, $\partial_x(\partial_\mu \sigma) = \partial_\mu(\partial_x \sigma)$ and $\partial_x(\partial_\mu \beta)(\cdot,\cdot,\cdot,e) = \partial_\mu(\partial_x \beta)(\cdot,\cdot,\cdot,e)$, $e \in K$. Consequently, as $\partial_\mu X_s^{t,x,P_\xi}(y) = N_s^{t,x,P_\xi}(y)$ (see Remark 4.2), (5.5) and (5.6) coincide, and from the uniqueness of the solution it follows $\partial_x(\partial_\mu X_s^{t,x,P_\xi}(y)) = \partial_x N_s^{t,x,P_\xi}(y) = \partial_\mu(\partial_x X_s^{t,x,P_\xi})(y)$, $s \in [t,T]$, x, $y \in \mathbb{R}$ and $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$.

Let us now study the second order derivative of X_s^{t,x,P_ξ} with respect to the measure P_ξ, $\partial_\mu^2 X_s^{t,x,P_\xi}(y,z)$, $y,z \in \mathbb{R}$. For this, recall the Eq. (4.8) for $N^{t,x,P_\xi}(y) = \partial_\mu X^{t,x,P_\xi}(y)$ and that for $N^{t,\xi}(y)\big(:= N^{t,x,P_\xi}(y)\Big|_{x=\xi}\big)$. Under our assumptions on coefficients b, σ and β, following the argument of the proof of Lemma 4.1, we can show that $L^2(\mathcal{F}_t;\mathbb{R}) \ni \xi \mapsto N_s^{t,x,\xi}(y)$ $(:= N_s^{t,x,P_\xi}(y)) \in$

$L^2(\mathcal{F}_s;\mathbb{R})$ is differentiable in any direction $\eta \in L^2(\mathcal{F}_t;\mathbb{R})$, and the directional derivative $\partial_\xi[N_s^{t,x,\xi}(y)](\eta)$ satisfies the following equation:

$$
\begin{aligned}
&\partial_\xi[N_s^{t,x,P_\xi}(y)](\eta)\\
&\quad= \int_t^s (\partial_x\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\partial_\xi[N_r^{t,x,P_\xi}(y)](\eta)dB_r\\
&\qquad+ \int_t^s (\partial_x^2\sigma)(X_r^{t,x,P_\xi} P_{X_r^{t,\xi}})N_r^{t,x,P_\xi}(y)\hat{E}[N_r^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_\mu(\partial_x\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})N_r^{t,x,P_\xi}(y)\\
&\qquad\times(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \hat{E}[\tilde{N}_r^{t,\tilde{\xi}}(\hat{\xi})\cdot\hat{\eta}])]dB_r\\
&\qquad+ \int_t^s \tilde{E}[(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi})\cdot\hat{E}[\partial_\mu\partial_x\tilde{X}_r^{t,y,P_\xi}(\hat{\xi})\cdot\hat{\eta}]]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_x(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}]\cdot\hat{E}[N_r^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}, \bar{X}_r^{t,\bar{\xi}})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\\
&\qquad\times(\partial_x\bar{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta} + \hat{E}[\bar{N}_r^{t,\bar{\xi}}(\hat{\xi})\cdot\hat{\eta}])]]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_y(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\cdot\hat{E}[\tilde{N}_r^{t,y,P_\xi}(\hat{\xi})\cdot\hat{\eta}]]dB_r\\
&\qquad+ \int_t^s \tilde{E}[(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot\partial_\xi[\tilde{N}_r^{t,\tilde{\xi}}(y)](\tilde{\eta})]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_x(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]\cdot\hat{E}[N_r^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, \bar{X}_r^{t,\bar{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\\
&\qquad\times(\partial_x\bar{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta} + \hat{E}[\bar{N}_r^{t,\bar{\xi}}(\hat{\xi})\cdot\hat{\eta}])]]dB_r\\
&\qquad+ \int_t^s \tilde{E}[\partial_y(\partial_\mu\sigma)(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\\
&\qquad\cdot(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta} + \hat{E}[\tilde{N}_r^{t,\tilde{\xi}}(\hat{\xi})\cdot\hat{\eta}])]dB_r\\
&\qquad+ \int_t^s\int_K (\partial_x\beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e)\partial_\xi[N_{r-}^{t,x,P_\xi}(y)](\eta)\mu_\lambda(dr,de)\\
&\qquad+ \int_t^s\int_K (\partial_x^2\beta)(X_{r-}^{t,x,P_\xi} P_{X_r^{t,\xi}}, e)N_{r-}^{t,x,P_\xi}(y)\hat{E}[N_{r-}^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]\mu_\lambda(dr,de)\\
&\qquad+ \int_t^s\int_K \tilde{E}[\partial_\mu(\partial_x\beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e)N_{r-}^{t,x,P_\xi}(y)(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\\
&\qquad+ \hat{E}[\tilde{N}_r^{t,\tilde{\xi}}(\hat{\xi})\cdot\hat{\eta}])]\mu_\lambda(dr,de)\\
&\qquad+ \int_t^s\int_K \tilde{E}[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}, e)\\
&\qquad\cdot\hat{E}[\partial_\mu\partial_x\tilde{X}_r^{t,y,P_\xi}(\hat{\xi})\cdot\hat{\eta}]]\mu_\lambda(dr,de)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^s\int_K \tilde{E}[\partial_x(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}]\\
&\cdot\hat{E}[N_{r-}^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[\bar{E}[(\partial_\mu^2\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},\bar{X}_r^{t,\bar{\xi}},e)\\
&\times\partial_x\tilde{X}_r^{t,y,P_\xi}(\partial_x\bar{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}+\hat{E}[\bar{N}_r^{t,\bar{\xi}}(\hat{\xi})\cdot\hat{\eta}])]]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[\partial_y(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\\
&\cdot\hat{E}[\tilde{N}_r^{t,y,P_\xi}(\hat{\xi})\cdot\hat{\eta}]]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\partial_\xi[\tilde{N}_r^{t,\tilde{\xi}}(y)](\tilde{\eta})]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[\partial_x(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\\
&\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]\cdot\hat{E}[N_{r-}^{t,x,P_\xi}(\hat{\xi})\cdot\hat{\eta}]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[\bar{E}[(\partial_\mu^2\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},\bar{X}_r^{t,\bar{\xi}},e)\\
&\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)(\partial_x\bar{X}_r^{t,\bar{\xi},P_\xi}\cdot\bar{\eta}+\hat{E}[\bar{N}_r^{t,\bar{\xi}}(\hat{\xi})\cdot\hat{\eta}])]]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K \tilde{E}[\partial_y(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\cdot(\partial_x\tilde{X}_r^{t,\tilde{\xi},P_\xi}\cdot\tilde{\eta}\\
&+\hat{E}[\tilde{N}_r^{t,\tilde{\xi}}(\hat{\xi})\cdot\hat{\eta}])]\mu_\lambda(dr,de), \qquad (5.7)
\end{aligned}
$$

$s\in[t,T]$. Here $(\bar{X}^{t,\bar{\xi}},\bar{N}^{t,\bar{\xi}}(y))$ obeys the same law, but under $\bar{P}$ on $(\bar{\Omega},\bar{\mathcal{F}})$, as the corresponding processes endowed with $\tilde{}$ or $\hat{}$ under $\tilde{P}$ and $\hat{P}$, respectively: The quadruple $(\bar{\xi},\bar{\eta},\bar{B},\overline{\mu_\lambda})$ is an independent copy of (ξ,η,B,μ_λ), $(\tilde{\xi},\tilde{\eta},\tilde{B},\widetilde{\mu_\lambda})$ and $(\hat{\xi},\hat{\eta},\hat{B},\widehat{\mu_\lambda})$, and $\bar{X}^{t,\bar{\xi}}$ is the solution of the SDE for $X^{t,\xi}$, and $\bar{N}^{t,\bar{\xi}}$ that of the SDE for $N^{t,\xi}$, but both with the data $(\bar{\xi},\bar{B},\overline{\mu_\lambda})$ under $\bar{P}$, instead of (ξ,B,μ_λ) under P. We remark that the equation for $\partial_\xi[N^{t,\xi}(y)]$ is obtained by substituting $x=\xi$ in the above equation for $\partial_\xi[N^{t,x,P_\xi}(y)]$.

Let us consider the process $S^{t,x,P_\xi}(y,z)=\big(S_s^{t,x,P_\xi}(y,z)\big)_{s\in[t,T]}$, which is defined as the unique solution in $S_{\mathbb{F}}^2(t,T)$ of the following SDE:

$$
\begin{aligned}
&S_s^{t,x,P_\xi}(y,z)\\
&=\int_t^s(\partial_x\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})S_r^{t,x,P_\xi}(y,z)dB_r\\
&\quad+\int_t^s(\partial_x^2\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})N_r^{t,x,P_\xi}(y)N_r^{t,x,P_\xi}(z)dB_r\\
&\quad+\int_t^s\tilde{E}[\partial_\mu(\partial_x\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,z,P_\xi})N_r^{t,x,P_\xi}(y)\cdot\partial_x\tilde{X}_r^{t,z,P_\xi}]dB_r\\
&\quad+\int_t^s\tilde{E}[\partial_\mu(\partial_x\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})N_r^{t,x,P_\xi}(y)\tilde{N}_r^{t,\tilde{\xi}}(z)]dB_r\\
&\quad+\int_t^s\tilde{E}[(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\cdot\partial_\mu\partial_x\tilde{X}_r^{t,y,P_\xi}(z)]dB_r
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^s \tilde{E}[\partial_x(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}]\cdot N_r^{t,x,P_\xi}(z)dB_r\\
&+\int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},\bar{X}_r^{t,z,P_\xi})\\
&\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\cdot\partial_x\bar{X}_r^{t,z,P_\xi}]]dB_r\\
&+\int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},\bar{X}_r^{t,\bar{\xi}})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\cdot\bar{N}_r^{t,\bar{\xi}}(z)]]dB_r\\
&+\int_t^s \tilde{E}[\partial_y(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi})\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\cdot\tilde{N}_r^{t,y,P_\xi}(z)]dB_r\\
&+\int_t^s \tilde{E}[(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{S}_r^{t,\tilde{\xi}}(y,z)]dB_r\\
&+\int_t^s \tilde{E}[\partial_x(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)]\cdot N_r^{t,x,P_\xi}(z)dB_r\\
&+\int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},\bar{X}_r^{t,z,P_\xi})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\cdot\partial_x\bar{X}_r^{t,z,P_\xi}]]dB_r\\
&+\int_t^s \tilde{E}[\bar{E}[(\partial_\mu^2\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},\bar{X}_r^{t,\bar{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\cdot\bar{N}_r^{t,\bar{\xi}}(z)]]dB_r\\
&+\int_t^s \tilde{E}[\partial_y(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,z,P_\xi})\cdot\tilde{N}_r^{t,z,P_\xi}(y)\cdot\partial_x\tilde{X}_r^{t,z,P_\xi}]dB_r\\
&+\int_t^s \tilde{E}[\partial_y(\partial_\mu\sigma)(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})\cdot\tilde{N}_r^{t,\tilde{\xi}}(y)\cdot\tilde{N}_r^{t,\tilde{\xi}}(z)]dB_r\\
&+\int_t^s\int_K(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)S_{r-}^{t,x,P_\xi}(y,z)\mu_\lambda(dr,de)\\
&+\int_t^s\int_K(\partial_x^2\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)N_{r-}^{t,x,P_\xi}(y)N_{r-}^{t,x,P_\xi}(z)\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[\partial_\mu(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,z,P_\xi},e)N_{r-}^{t,x,P_\xi}(y)\\
&\cdot\partial_x\tilde{X}_r^{t,z,P_\xi}]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[\partial_\mu(\partial_x\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}},e)N_{r-}^{t,x,P_\xi}(y)\tilde{N}_r^{t,\tilde{\xi}}(z)]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\cdot\partial_\mu\partial_x\tilde{X}_r^{t,y,P_\xi}(z)]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[\partial_x(\partial_\mu\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},e)\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}]\\
&\cdot N_{r-}^{t,x,P_\xi}(z)\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[\bar{E}[(\partial_\mu^2\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},\bar{X}_r^{t,z,P_\xi},e)\\
&\cdot\partial_x\tilde{X}_r^{t,y,P_\xi}\cdot\partial_x\bar{X}_r^{t,z,P_\xi}]]\mu_\lambda(dr,de)\\
&+\int_t^s\int_K\tilde{E}[\bar{E}[(\partial_\mu^2\beta)(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,y,P_\xi},\bar{X}_r^{t,\bar{\xi}},e)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \partial_x \tilde{X}_r^{t,y,P_\xi} \cdot \bar{N}_r^{t,\bar{\xi}}(z)]]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\partial_y(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,y,P_\xi}, e)\\
&\cdot \partial_x \tilde{X}_r^{t,y,P_\xi} \cdot \tilde{N}_r^{t,y,P_\xi}(z)]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e)\cdot \tilde{S}_r^{t,\tilde{\xi}}(y,z)]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\partial_x(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e)\cdot \tilde{N}_r^{t,\tilde{\xi}}(y)]\\
&\cdot N_{r-}^{t,x,P_\xi}(z)\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\bar{E}[(\partial_\mu^2 \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, \bar{X}_r^{t,z,P_\xi}, e)\\
&\cdot \tilde{N}_r^{t,\tilde{\xi}}(y)\cdot \partial_x \bar{X}_r^{t,z,P_\xi}]]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\bar{E}[(\partial_\mu^2 \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, \bar{X}_r^{t,\bar{\xi}}, e)\\
&\cdot \tilde{N}_r^{t,\tilde{\xi}}(y)\cdot \bar{N}_r^{t,\bar{\xi}}(z)]]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\partial_y(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,z,P_\xi}, e)\\
&\cdot \tilde{N}_r^{t,z,P_\xi}(y)\cdot \partial_x \tilde{X}_r^{t,z,P_\xi}]\mu_\lambda(dr,de)\\
&+\int_t^s \int_K \tilde{E}[\partial_y(\partial_\mu \beta)(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}, e)\\
&\cdot \tilde{N}_r^{t,\tilde{\xi}}(y)\cdot \tilde{N}_r^{t,\tilde{\xi}}(z)]\mu_\lambda(dr,de),\ s\in[t,T]. \qquad (5.8)
\end{aligned}
$$

Here $S^{t,\xi}(y,z) = (S_s^{t,\xi}(y,z))_{s\in[t,T]}$ is the unique solution of the equation obtained by substituting $x=\xi$ in that for $S^{t,x,P_\xi}(y,z)$. Multiplying (5.8) with $\hat{\eta}$ after having substituted $\hat{\xi}$ for z, and taking the expectation $\hat{E}[\cdot]$ with respect to $\hat{P}$ on both sides of the equation, we see that $\hat{E}[S^{t,x,P_\xi}(y,\hat{\xi})\cdot\hat{\eta}]$ solves the same equation as $\partial_\xi[N_s^{t,x,P_\xi}(y)](\eta)$ and $\hat{E}[S^{t,\xi}(y,\hat{\xi})\cdot\hat{\eta}]$ the same as $\partial_\xi[N_s^{t,x,P_\xi}(y)](\eta)|_{x=\xi}$. Hence, due to the uniqueness of the solution of (5.7), we get

$$\partial_\xi[N_s^{t,x,P_\xi}(y)](\eta) = \hat{E}[S_s^{t,x,P_\xi}(y,\hat{\xi})\cdot\hat{\eta}], \quad s\in[t,T],\ y\in\mathbb{R}.$$

Moreover, standard SDE estimates in (5.8) yield that, for $p\geq 2$, there exists some $C_p\in\mathbb{R}^+$ such that, for all $t\in[0,T]$, $x,\ x',\ y,\ y',\ z,\ z'\in\mathbb{R}$ and $\xi,\ \xi'\in L^2(\mathcal{F}_t;\mathbb{R})$,

$$\text{(iv) } E\left[\sup_{s\in[t,T]}|S_s^{t,x,P_\xi}(y,z)|^p\right]\leq C_p,$$

$$\text{(v) } E\left[\sup_{s\in[t,T]}|S_s^{t,x,P_\xi}(y,z)-S_s^{t,x',P_{\xi'}}(y',z')|^p\right]$$

$$\leq C_p\Big(|x-x'|^p+|y-y'|^p+|z-z'|^p+W_2(P_\xi,P_{\xi'})^p\Big),$$

$$\text{(vi) } E\Big[\sup_{s\in[t,t+h]}|S_s^{t,x,P_\xi}(y,z)|^p\Big]\leq C_p h,\quad 0\leq h\leq T-t, \tag{5.9}$$

(see also Propositions 3.1 and 4.1). Estimate (v) allows to show that $\partial_\mu X^{t,x,\xi}$ is Fréchet differentiable (we use the same argument as that in the proof of Theorem 4.3), and the Fréchet derivative satisfies

$$D_\xi[\partial_\mu X_s^{t,x,P_\xi}(y)](\eta)=D_\xi[N_s^{t,x,P_\xi}(y)](\eta)=\hat{E}[S_s^{t,x,P_\xi}(y,\hat{\xi})\cdot\hat{\eta}].$$

Consequently, $\partial_\mu X_s^{t,x,P_\xi}$ is differentiable with respect to the law P_ξ, and

$$\partial_\mu^2 X_s^{t,x,P_\xi}(y,z):=\partial_\mu(\partial_\mu X_s^{t,x,P_\xi}(y))(z)=S_s^{t,x,P_\xi}(y,z),\quad s\in[t,T],$$

for all $t\in[0,T]$, x, y, $z\in\mathbb{R}$, $\xi\in L^2(\mathcal{F}_t;\mathbb{R})$. The other both second order derivatives $\partial_x^2 X^{t,x,P_\xi}$ and $\partial_y(\partial_\mu X^{t,x,P_\xi}(y))$ of X^{t,x,P_ξ} and their estimates can be investigated by using arguments developed above. The proof is complete. □

6. Regularity of the value function

Our purpose in this section is to study the regularity of the function

$$V(t,x,P_\xi):=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})],\ (t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d),$$

associated with the SDEs (3.1) and (3.2).

Proposition 6.1. *Let* $\Phi\in C_b^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ *and let Assumption* (A.1) *hold true. Then* $V(t,\cdot,\cdot)\in C_b^{1,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, $t\in[0,T]$ *and its derivatives* $\partial_x V(t,x,P_\xi)=(\partial_{x_i}V(t,x,P_\xi))_{1\leq i\leq d}$, $\partial_\mu V(t,x,P_\xi,y)=((\partial_\mu V)_i(t,x,P_\xi,y))_{1\leq i\leq d}$ *satisfy, respectively,*

$$\partial_{x_i}V(t,x,P_\xi)=\sum_{j=1}^d E\left[(\partial_{x_j}\Phi)(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})\cdot\partial_{x_i}X_{T,j}^{t,x,P_\xi}\right], \tag{6.1}$$

$$\begin{aligned}(\partial_\mu V)_i(t,x,P_\xi,y)=\sum_{j=1}^d E\Big[&(\partial_{x_j}\Phi)(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})(\partial_\mu X_{T,j}^{t,x,P_\xi})_i(y)\\&+\tilde{E}[(\partial_\mu\Phi)_j(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\tilde{X}_T^{t,y,P_\xi})\cdot\partial_{x_i}\tilde{X}_{T,j}^{t,y,P_\xi}\\&+(\partial_\mu\Phi)_j(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}},\tilde{X}_T^{t,\tilde{\xi}})\cdot(\partial_\mu\tilde{X}_{T,j}^{t,\tilde{\xi},P_\xi})_i(y)]\Big].\end{aligned} \tag{6.2}$$

Furthermore, for

$$F_i(t,x,P_\xi,y):=(\partial_{x_i}V(t,x,P_\xi),\ (\partial_\mu V)_i(t,x,P_\xi,y)),\quad 1\leq i\leq d,$$

there is a constant $C>0$ *such that, for all* $t,\ t_1,\ t_2\in[0,T]$, $x,\ x_1,\ x_2,\ y_1,\ y_2\in\mathbb{R}^d$ *and* $\xi_1,\ \xi_2\in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $1\le i\le d$,

$$\begin{aligned}&\text{(i) } |F_i(t,x,P_\xi,y)|\le C,\\&\text{(ii) } |F_i(t,x_1,P_{\xi_1},y_1)-F_i(t,x_2,P_{\xi_2},y_2)|\\&\quad\le C(|x_1-x_2|+|y_1-y_2|+W_2(P_{\xi_1},P_{\xi_2})),\\&\text{(iii) } |V(t_1,x,P_\xi)-V(t_2,x,P_\xi)|+|F_i(t_1,x,P_\xi,y)-F_i(t_2,x,P_\xi,y)|\\&\quad\le C|t_1-t_2|^{\frac{1}{2}}.\end{aligned}\tag{6.3}$$

Proof. The proof of (6.1), (6.2) and (6.3)(i) and (ii) of our preceding results are standard (see also Lemma 5.1 in [7] for the case without jumps). We only prove that V is $\frac{1}{2}$-Hölder continuous with respect to t. The proof of $\frac{1}{2}$-Hölder continuity for the first order derivatives of V uses an argument similar to that for V and is therefore omitted. As before, for simplicity we consider only the one dimensional case and let $b=0$.

We choose $(t,x)\in[0,T]\times\mathbb{R}$ arbitrarily but fix it. Recall that $\mathcal{G}_0$ is rich enough, such that we can find for each $\xi\in L^2(\mathcal{F}_t;\mathbb{R})$, $\xi'\in L^2(\mathcal{G}_0;\mathbb{R})$ such that $P_{\xi'}=P_\xi$. This implies $X_s^{t,x,P_\xi}=X_s^{t,x,P_{\xi'}},\ s\in[t,T]$. We define $B_s^t:=B_{t+s}-B_t$, $\mu^t([0,s]\times A):=\mu([0,s+t]\times A)-\mu([0,t]\times A),\ s\ge 0,\ A\in\mathcal{K}$, and we see that $X^{t,\xi'}$ and $X^{t,x,\xi'}$ can be regarded as the solutions of the following two SDEs:

$$\begin{aligned}X_{s+t}^{t,\xi'}&=\xi'+\int_0^s\sigma(X_{r+t}^{t,\xi'},P_{X_{r+t}^{t,\xi'}})dB_r^t+\int_0^s\int_K\beta(X_{r+t}^{t,\xi'},P_{X_{r+t}^{t,\xi'}},e)\mu_\lambda^t(dr,de),\\X_{s+t}^{t,x,P_{\xi'}}&=x+\int_0^s\sigma(X_{r+t}^{t,x,P_{\xi'}},P_{X_{r+t}^{t,\xi'}})dB_r^t\\&\quad+\int_0^s\int_K\beta(X_{r+t}^{t,x,P_{\xi'}},P_{X_{r+t}^{t,\xi'}},e)\mu_\lambda^t(dr,de),\quad s\in[0,T-t],\end{aligned}\tag{6.4}$$

where $\mu_\lambda^t([0,s]\times A):=\mu^t([0,s]\times A)-\lambda(de)d(s),\ s\in[0,T-t]$, is a martingale for all $A\in\mathcal{K}$. Since $(B_s^t,\ \mu_\lambda^t([0,s]\times A))$ obeys the same law as $(B_s,\ \mu_\lambda([0,s]\times A))$, and ξ' is $\mathcal{G}_0$-measurable and, thus, independent of B and μ, the process $(X_{\cdot+t}^{t,x,P_{\xi'}},X_{\cdot+t}^{t,P_{\xi'}})$ has the same law as $(X^{0,x,P_{\xi'}},X^{0,P_{\xi'}})$, where $(X^{0,x,P_{\xi'}},X^{0,P_{\xi'}})$ is the solution of (6.4) but driven by B and μ. Consequently,

$$V(t,x,P_\xi)=V(t,x,P_{\xi'})=E[\Phi(X_T^{t,x,P_{\xi'}},P_{X_T^{t,\xi'}})]=E[\Phi(X_{T-t}^{0,x,P_{\xi'}},P_{X_{T-t}^{0,\xi'}})].$$

Hence, thanks to the Lipschitz continuity of Φ, for $t_1,\ t_2\in[0,T]$,

$$\begin{aligned}&|V(t_1,x,P_\xi)-V(t_2,x,P_\xi)|\\&\quad\le E[|\Phi(X_{T-t_1}^{0,x,P_{\xi'}},P_{X_{T-t_1}^{0,\xi'}})-\Phi(X_{T-t_2}^{0,x,P_{\xi'}},P_{X_{T-t_2}^{0,\xi'}})|]\\&\quad\le C(E[|X_{T-t_1}^{0,x,P_{\xi'}}-X_{T-t_2}^{0,x,P_{\xi'}}|]+W_2(P_{X_{T-t_1}^{0,\xi'}},P_{X_{T-t_2}^{0,\xi'}}))\\&\quad\le C\big(E[|X_{T-t_1}^{0,x,P_{\xi'}}-X_{T-t_2}^{0,x,P_{\xi'}}|^2]\big)^{\frac{1}{2}}+C\big(E[|X_{T-t_1}^{0,\xi'}-X_{T-t_2}^{0,\xi'}|^2]\big)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, letting without loss of generality $0 \le t_1 \le t_2 \le T$, i.e., $T-t_2 \le T-t_1$, we have from the flow property of $(X_s^{t,x,P_\xi}, X_s^{t,\xi})$ (see Remark 3.1 (ii))

$$\begin{aligned}
&\Big(E[|X_{T-t_1}^{0,x,P_{\xi'}} - X_{T-t_2}^{0,x,P_{\xi'}}|^2]\Big)^{\frac{1}{2}} + \Big(E[|X_{T-t_1}^{0,\xi'} - X_{T-t_2}^{0,\xi'}|^2]\Big)^{\frac{1}{2}} \\
&= \Big(E[|X_{T-t_1}^{T-t_2, X_{T-t_2}^{0,x,P_{\xi'}}, X_{T-t_2}^{0,\xi'}} - X_{T-t_2}^{0,x,P_{\xi'}}|^2]\Big)^{\frac{1}{2}} \\
&\quad + \Big(E[|X_{T-t_1}^{T-t_2, X_{T-t_2}^{0,\xi'}} - X_{T-t_2}^{0,\xi'}|^2]\Big)^{\frac{1}{2}},
\end{aligned}$$

and from Proposition 3.1 (iii),

$$\begin{aligned}
&\Big(E[|X_{T-t_1}^{0,x,P_{\xi'}} - X_{T-t_2}^{0,x,P_{\xi'}}|^2]\Big)^{\frac{1}{2}} + \Big(E[|X_{T-t_1}^{0,\xi'} - X_{T-t_2}^{0,\xi'}|^2]\Big)^{\frac{1}{2}} \\
&\le C\Big(E\Big[E\Big[\sup_{s\in[T-t_2,T-t_1]} \big(|X_s^{T-t_2,y,P_\eta} - y|^2 \\
&\quad + |X_s^{T-t_2,\eta} - \eta|^2\big)|\mathcal{F}_{T-t_2}\Big]\Big|_{y=X_{T-t_2}^{0,x,P_{\xi'}}, \eta=X_{T-t_2}^{0,\xi'}}\Big]\Big)^{\frac{1}{2}} \\
&\le C(t_2-t_1)^{\frac{1}{2}}.
\end{aligned}$$

Consequently, we have

$$|V(t_1,x,P_\xi) - V(t_2,x,P_\xi)| \le C|t_1-t_2|^{\frac{1}{2}}.$$

□

Using now the regularity results obtained for $(X_s^{t,x,P_\xi}, X_s^{t,\xi})$ in the preceding section we can proceed similarly as for Proposition 6.1, in order to show the following result.

Proposition 6.2. *Suppose that b, σ and β satisfy Assumption* (A.2) *and $\Phi \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Then $V(t,\cdot,\cdot) \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, $t \in [0,T]$ and*

$$\partial_{x_i}(\partial_\mu V(t,x,P_\xi,y)) = \partial_\mu(\partial_{x_i} V(t,x,P_\xi))(y), \tag{6.5}$$

$(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $1 \le i \le d$, and for

$$\begin{aligned}
W_{i,j}(t,x,P_\xi,y,z) = (&\partial_\mu^2 V(t,x,P_\xi,y,z), \partial^2_{x_ix_j} V(t,x,P_\xi), \\
&\partial_{x_i}(\partial_\mu V)(t,x,P_\xi,y), \partial_{y_i}(\partial_\mu V)(t,x,P_\xi,y))
\end{aligned}$$

there exists a constant $C > 0$ such that, for all $t,\ t_1,\ t_2 \in [0,T]$, $x,\ x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2 \in \mathbb{R}^d$ and $\xi_1,\ \xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $1 \le i,\ j \le d$,

$$\begin{aligned}
&\text{(i) } |W_{i,j}(t,x,P_\xi,y,z)| \le C, \\
&\text{(ii) } |W_{i,j}(t,x_1,P_{\xi_1},y_1,z_1) - W_{i,j}(t,x_2,P_{\xi_2},y_2,z_2)| \\
&\qquad \le C(|x_1-x_2| + |y_1-y_2| + |z_1-z_2| + W_2(P_{\xi_1},P_{\xi_2})), \\
&\text{(iii) } |W_{i,j}(t_1,x,P_\xi,y,z) - W_{i,j}(t_2,x,P_\xi,y,z)| \le C|t_1-t_2|^{\frac{1}{2}}.
\end{aligned} \tag{6.6}$$

7. Associated nonlocal integral-PDE

The objective of this section is to study the nonlocal PDE associated with the mean-field SDE. The Itô's formula which is established through the following both theorems will play an important role in our approach.

Theorem 7.1. *Suppose $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}^d))$. Then, under Assumption* (A.2)*, for all* $0 \le t \le s \le T,\ \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, *we have*

$$\begin{aligned}\partial_s f(P_{X_s^{t,\xi}}) = E\Bigg[&\sum_{i=1}^d (\partial_\mu f)_i(P_{X_s^{t,\xi}}, X_s^{t,\xi}) b_i(X_s^{t,\xi}, P_{X_s^{t,\xi}})\\ &+\frac{1}{2}\sum_{i,j,k=1}^d \partial_{y_i}((\partial_\mu f)_j)(P_{X_s^{t,\xi}}, X_s^{t,\xi})(\sigma_{i,k}\sigma_{j,k})(X_s^{t,\xi}, P_{X_s^{t,\xi}})\\ &+\sum_{i=1}^d\int_0^1\int_K \big[(\partial_\mu f)_i(P_{X_s^{t,\xi}}, X_s^{t,\xi}+\rho\beta(X_s^{t,\xi}, P_{X_s^{t,\xi}}, e))\\ &-(\partial_\mu f)_i(P_{X_s^{t,\xi}}, X_s^{t,\xi})\big]\beta_i(X_s^{t,\xi}, P_{X_s^{t,\xi}}, e)\lambda(de)d\rho\Bigg].\end{aligned}$$

Proof. Without loss of generality we restrict ourselves to the one dimensional case and prove first the existence of $\partial_s f(P_{X_s^{t,\xi}})\big|_{s=t}$. The general case will be obtained by using the flow property of $X_s^{t,\xi}$.

We first observe that under our assumptions on f the mapping $\rho \to f(P_{\xi+\rho(X_s^{t,\xi}-\xi)})$ is differentiable on $[0,1]$, and $\partial_\rho[f(P_{\xi+\rho(X_s^{t,\xi}-\xi)})] = E[(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)}, \xi+\rho(X_s^{t,\xi}-\xi)))\cdot(X_s^{t,\xi}-\xi)],\ \rho\in[0,1]$. Hence,

$$\begin{aligned}&f(P_{X_s^{t,\xi}})-f(P_\xi)\\ &\quad=\int_0^1 \partial_\rho f(P_{\xi+\rho(X_s^{t,\xi}-\xi)})d\rho\\ &\quad=\int_0^1 E\Big[\Big(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)}, \xi+\rho(X_s^{t,\xi}-\xi))\Big)\cdot(X_s^{t,\xi}-\xi)\Big]d\rho\\ &\quad=E\big[\partial_\mu f(P_\xi,\xi)(X_s^{t,\xi}-\xi)\big]\\ &\qquad+\int_0^1 E\Big[\Big(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)},\xi)-\partial_\mu f(P_\xi,\xi)\Big)(X_s^{t,\xi}-\xi)\Big]d\rho \quad (:= I_1)\\ &\qquad+\int_0^1 E\Big[\Big(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)}, \xi+\rho(X_s^{t,\xi}-\xi))\\ &\qquad-\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)},\xi)\Big)(X_s^{t,\xi}-\xi)\Big]d\rho. \qquad (:= I_2)\end{aligned}$$

Notice $\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)},\xi)-\partial_\mu f(P_\xi,\xi)$ is $\mathcal{F}_t$-measurable, b is bounded and $\partial_\mu f$ is Lipschitz-continuous. Hence, from the estimates of Proposition 3.1,

$$
\begin{aligned}
|I_1| &= \Big| \int_0^1 E\Big[\Big(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)},\xi) - \partial_\mu f(P_\xi,\xi)\Big)\Big(\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr \\
&\quad + \int_t^s \sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r + \int_t^s\int_K \beta(X_r^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big)\Big]d\rho\Big| \\
&= \Big| \int_0^1 E\Big[\Big(\partial_\mu f(P_{\xi+\rho(X_s^{t,\xi}-\xi)},\xi) - \partial_\mu f(P_\xi,\xi)\Big)\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr\Big]d\rho\Big| \\
&\le C(s-t)W_2(P_{\xi+\rho(X_s^{t,\xi}-\xi)},P_\xi) \le C(s-t)E[|X_s^{t,\xi}-\xi|^2]^{\frac12} \le C(s-t)^{\frac32}.
\end{aligned}
$$

As concerns I_2, putting $\eta = \xi + \rho(X_s^{t,\xi}-\xi)$, we have

$$
\begin{aligned}
I_2 &= \int_0^1 E\Big[\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t) \\
&\quad + \rho\int_t^s\int_K \beta(\xi,P_\xi,e)\mu_\lambda(dr,de)) - (\partial_\mu f)(P_\eta,\xi)\Big)(X_s^{t,\xi}-\xi)\Big]d\rho + R_1,
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 = \int_0^1 E\Bigg[\Bigg\{&(\partial_\mu f)(P_\eta,\xi+\rho(X_s^{t,\xi}-\xi)) \\
&-(\partial_\mu f)\Bigg(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t) \\
&+\rho\int_t^s\int_K \beta(\xi,P_\xi,e)\mu_\lambda(dr,de)\Bigg)\Bigg\}(X_s^{t,\xi}-\xi)\Bigg]d\rho.
\end{aligned}
$$

Thanks to our assumptions that b is bounded and $\partial_\mu f$, σ and β are Lipschitz, we have,

$$
\begin{aligned}
|R_1| &\le C\int_0^1 \rho E\Bigg[|X_s^{t,\xi}-\xi|\Bigg(\Big|\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr\Big| \\
&\quad +\Big|\int_t^s\Big(\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})-\sigma(\xi,P_\xi)\Big)dB_r\Big| \\
&\quad +|\int_t^s\int_K\Big(\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)-\beta(\xi,P_\xi,e)\Big)\mu_\lambda(dr,de)|\Bigg)\Bigg]d\rho \\
&\le C\Big\{(s-t)E|X_s^{t,\xi}-\xi| + (E|X_s^{t,\xi}-\xi|^2)^{\frac12} \\
&\quad \times (E|\int_t^s\big(\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})-\sigma(\xi,P_\xi)\big)dB_r|^2)^{\frac12} \\
&\quad + (E|X_s^{t,\xi}-\xi|^2)^{\frac12}\Big(E\int_t^s\int_K|\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)-\beta(\xi,P_\xi,e)|^2\lambda(de)dr\Big)^{\frac12}\Big\} \\
&\le C(s-t)E|X_s^{t,\xi}-\xi| + C(s-t)^{\frac12}\Big(\Big\{E\int_t^s\big(|X_r^{t,\xi}-\xi|^2+E[|X_r^{t,\xi}-\xi|^2]\big)dr\Big\}^{\frac12}
\end{aligned}
$$

$$+\Big\{E\int_t^s\int_K(1\wedge|e|^2)(|X_r^{t,\xi}-\xi|^2+E[|X_r^{t,\xi}-\xi|^2])\lambda(de)dr\Big\}^{\frac{1}{2}}\Big)$$
$$\le C(s-t)^{\frac{3}{2}}.$$

On the other hand, we notice

$$\int_0^1 E\Big[\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de))$$
$$-(\partial_\mu f)(P_\eta,\xi)\Big)(X_s^{t,\xi}-\xi)\Big]d\rho=I_{2,1}+I_{2,2}+I_{2,3},$$

where

$$I_{2,1}=\int_0^1 E\Big[\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)$$
$$+\rho\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de))-\ (\partial_\mu f)(P_\eta,\xi)\Big)\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr\Big]d\rho,$$
$$I_{2,2}=\int_0^1 E\Big[\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)$$
$$+\rho\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de))-\ (\partial_\mu f)(P_\eta,\xi)\Big)\int_t^s \sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r\Big]d\rho,$$
$$I_{2,3}=\int_0^1 E\Big[\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)$$
$$+\rho\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de))-\ (\partial_\mu f)(P_\eta,\xi)\Big)$$
$$\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho.$$

For $I_{2,1}$, due to the boundedness of $b,\ \sigma$ and the Lipschitz continuity of $\partial_\mu f$, we get

$$|I_{2,1}|\le\int_0^1 E\Big[|(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de))$$
$$-(\partial_\mu f)(P_\eta,\xi)|\cdot|\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr|\Big]d\rho$$
$$\le C\int_0^1\rho E\Big[\Big(|B_s-B_t|+|\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de)|\Big)$$
$$\cdot|\int_t^s b(X_r^{t,\xi},P_{X_r^{t,\xi}})dr|\Big]d\rho$$
$$\le C(s-t)\Big((E|B_s-B_t|^2)^{\frac{1}{2}}+(E|\int_t^s\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de)|^2)^{\frac{1}{2}}\Big)$$
$$\le C(s-t)^{\frac{3}{2}}.$$

We put $M_s=\int_t^s\int_E\beta(\xi,P_\xi,e)\mu_\lambda(dr,de)$. Then

$$I_{2,2}=\frac{1}{2}(s-t)E[\partial_y(\partial_\mu f)(P_\xi,\xi)\sigma(\xi,P_\xi)^2]+R_2+R_3,$$

where

$$R_2 = \frac{1}{2}E\left[\partial_y(\partial_\mu f)(P_\xi,\xi)\int_t^s \sigma(\xi,P_\xi)(\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})-\sigma(\xi,P_\xi))dr\right],$$

$$R_3 = \int_0^1 E\left[\left((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_s)-(\partial_\mu f)(P_\eta,\xi)\right)\right.$$
$$\left.\times\int_t^s \sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r\right]d\rho$$
$$-\frac{1}{2}E\left[\partial_y(\partial_\mu f)(P_\xi,\xi)\int_t^s \sigma(\xi,P_\xi)\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dr\right].$$

From the boundedness of $\partial_y(\partial_\mu f)$ and σ, and the Lipschitz continuity of σ it follows

$$|R_2| \le C\left(\int_t^s (E|X_r^{t,\xi}-\xi|+(E|X_r^{t,\xi}-\xi|^2)^{\frac{1}{2}})dr\right) \le C(s-t)^{\frac{3}{2}}.$$

Let $\mathcal{F}_s^\mu := \sigma\{\mu([0,r]\times A),\ r\le s,\ A\in\mathcal{K}\}$. Then, since $\partial_\mu f(P_\eta,\xi)$ is $\mathcal{F}_t$-measurable and M_s is $\mathcal{F}_t\vee\mathcal{F}_s^\mu$-measurable, while the increments of the Brownian Motion B after t are independent of $\mathcal{F}_t\vee\mathcal{F}_s^\mu$, then

$$\int_0^1 E\left[\left((\partial_\mu f)(P_\eta,\xi+\rho M_s)-(\partial_\mu f)(P_\eta,\xi)\right)\int_t^s \sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r\right]d\rho$$
$$=\int_0^1 E\left[\left((\partial_\mu f)(P_\eta,\xi+\rho M_s)-(\partial_\mu f)(P_\eta,\xi)\right)E\right.$$
$$\left.\times\left[\int_t^s \sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r|\mathcal{F}_t\vee\mathcal{F}_s^\mu\right]\right]d\rho=0.$$

Consequently,

$$R_3 = \int_0^1\int_0^1 E\left[\partial_y(\partial_\mu f)(P_\eta,\xi+\rho M_s+\rho\gamma\sigma(\xi,P_\xi)(B_s-B_t))\right.$$
$$\left.\cdot\rho\int_t^s\sigma(\xi,P_\xi)dB_r\cdot\int_t^s\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r\right]d\gamma d\rho$$
$$-\frac{1}{2}E\left[\partial_y(\partial_\mu f)(P_\xi,\xi)\int_t^s\sigma(\xi,P_\xi)\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dr\right]$$
$$=\int_0^1\int_0^1\rho E\left[\left(\partial_y(\partial_\mu f)(P_\eta,\xi+\rho M_s+\rho\gamma\sigma(\xi,P_\xi)(B_s-B_t))\right.\right.$$
$$\left.\left.-\partial_y(\partial_\mu f)(P_\xi,\xi)\right)\cdot\int_t^s\sigma(\xi,P_\xi)dB_r\cdot\int_t^s\sigma(X_r^{t,\xi},P_{X_r^{t,\xi}})dB_r\right]d\gamma d\rho.$$

Hence,

$$|R_3| \le CE\Big[\Big|\int_t^s \sigma(\xi, P_\xi)dB_r\Big|\cdot\Big|\int_t^s \sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})dB_r\Big| \cdot\Big((E|X_s^{t,\xi}-\xi|^2)^{\frac{1}{2}} + |M_s| + |B_s - B_t|\Big)\Big]$$

$$\le C\Big(E|\int_t^s |\sigma(\xi, P_\xi)|^2 dr|^2\Big)^{\frac{1}{4}}\cdot\Big(E|\int_t^s |\sigma(X_r^{t,\xi}, P_{X_r^{t,\xi}})|^2 dr|^2\Big)^{\frac{1}{4}} \cdot\Big((E|X_s^{t,\xi}-\xi|^2)^{\frac{1}{2}} + (E|M_s|^2)^{\frac{1}{2}} + (E|B_s - B_t|^2)^{\frac{1}{2}}\Big)$$

$$\le C(s-t)^{\frac{3}{2}}.$$

We now compute $I_{2,3}$. Let

$$\mathcal{F}_r^\mu = \sigma\{\mu([0,u]\times A)|A\in\mathcal{K},\ u\le r\},\quad 0\le r\le T,$$

$$\mathcal{G}_r = \mathcal{F}_{r+}^\mu\left(:= \bigcap_{s\ge r}\mathcal{F}_s^\mu \vee \sigma\{B_u,\ 0\le u\le T\}\vee\mathcal{G}_0\right),\quad 0\le r\le T,$$

where $\mathcal{G}_0$ is defined in Sect. 3. We set

$$\varphi(z') := (\partial_\mu f)(P_\eta,\ z'),\ \delta_s := \xi + \rho\sigma(\xi,\ P_\xi)(B_s - B_t),$$
$$\theta_u^{\delta_s,e} := \varphi(\delta_s + \rho M_u + \rho\beta(\xi, P_\xi, e)) - \varphi(\delta_s + \rho M_u) - \varphi'(\delta_s + \rho M_u)\cdot\rho\beta(\xi, P_\xi, e).$$

Applying Itô's formula to $\varphi(\delta_s + \rho M_u)$ with respect to u over $[t,s]$, we obtain (recall that $M = (M_u)_{u\in[t,T]}$ is a $\mathbb{G} := (\mathcal{G}_r)$- martingale)

$$\varphi(\delta_s+\rho M_s) = \varphi(\delta_s) + \int_t^s \rho\varphi'(\delta_s + \rho M_{r-})dM_r + \int_t^s\int_K \theta_{r-}^{\delta_s,e}\mu(dr,de)$$
$$= \varphi(\delta_s) + \int_t^s\int_K \theta_r^{\delta_s,e}\lambda(de)dr + \int_t^s\int_K(\varphi(\delta_s+\rho M_{r-}+\rho\beta(\xi,P_\xi,e)) - \varphi(\delta_s+\rho M_{r-}))\mu_\lambda(dr,de).$$

Consequently,

$$I_{2,3} = \int_0^1 E\Big[\Big\{\varphi(\delta_s + \rho M_s) - (\partial_\mu f)(P_\eta, \xi)\Big\} \cdot\int_t^s\int_K \beta(X_{r-}^{t,\xi}, P_{X_r^{t,\xi}}, e)\mu_\lambda(dr,de)\Big]d\rho$$
$$= \int_0^1 E\Big[\Big\{\int_t^s\int_K\Big((\partial_\mu f)(P_\eta, \xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_{r-}+\rho\beta(\xi,P_\xi,e)) - (\partial_\mu f)(P_\eta, \xi + \rho\sigma(\xi, P_\xi)(B_s - B_t) + \rho M_{r-})\Big)\mu_\lambda(dr,de) + \varphi(\delta_s)$$

$$
\begin{aligned}
&+\int_t^s\int_K\theta_r^{\delta_s,e}\lambda(de)dr-(\partial_\mu f)(P_\eta,\xi)\Big\}\\
&\cdot\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho.
\end{aligned}
$$

Since $(\int_t^s\int_E\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de))_{s\geq t}$ is $\mathbb{G}=(\mathcal{G}_r)_{r\geq 0}$-martingale and $\varphi(\delta_s)$, $(\partial_\mu f)(P_\eta,\xi)$ are $\mathcal{G}_t$-measurable, we have

$$
E[(\varphi(\delta_s)-(\partial_\mu f)(P_\eta,\xi))\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)]=0.
$$

Consequently,

$$
\begin{aligned}
I_{2,3}=\int_0^1E\Big[\Big\{\int_t^s\int_K\Big(&(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_{r-}+\rho\beta(\xi,P_\xi,e))\\
&-(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_{r-})\Big)\mu_\lambda(dr,de)\Big\}\\
&\cdot\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho+R_4,
\end{aligned}
$$

where

$$
R_4=\int_0^1E\Big[\int_t^s\int_K\theta_r^{z,e}\lambda(de)dr\cdot\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho.
$$

From the Lipschitz continuity of $\partial_y(\partial_\mu f)$, Hölder inequality as well as Burkholder–Davis–Gundy inequality we have

$$
\begin{aligned}
|R_4|=\Big|\int_0^1E\Big[\int_t^s\int_K\Big\{&\partial_\mu f(P_\eta,\delta_s+\rho M_r+\rho\beta(\xi,P_\xi,e))-\partial_\mu f(P_\eta,\delta_s+\rho M_r)\\
&-\partial_y(\partial_\mu f)(P_\eta,\delta_s+\rho M_r)\rho\beta(\xi,P_\xi,e)\Big\}\lambda(de)dr\\
&\cdot\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho\Big|\\
=\Big|\int_0^1E\Big[\int_t^s\int_K\int_0^\rho\Big\{&\partial_y(\partial_\mu f)(P_\eta,\delta_s+\rho M_r+\gamma\beta(\xi,P_\xi,e))\\
&-\partial_y(\partial_\mu f)(P_\eta,\delta_s+\rho M_r)\Big\}\beta(\xi,P_\xi,e)d\gamma\lambda(de)dr\\
&\cdot\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(dr,de)\Big]d\rho\Big|\\
\leq C\int_0^1\Big(E\Big[\Big|\int_t^s\int_K\int_0^\rho\Big\{&\partial_y(\partial_\mu f)(P_\eta,\delta_s+\rho M_r+\gamma\beta(\xi,P_\xi,e))\\
&-\partial_y(\partial_\mu f)(P_\eta,\delta_s+\rho M_r)\Big\}\beta(\xi,P_\xi,e)d\gamma\lambda(de)dr\Big|^2\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\cdot\bigg(E\bigg[\bigg|\int_t^s\int_K\beta(X_{r-}^{t,\xi},P_{X_r^{t,\xi}},e)\mu_\lambda(de,dr)\bigg|^2\bigg]\bigg)^{\frac{1}{2}}d\rho$$
$$\leq C(s-t)^{\frac{3}{2}}.$$

Hence, $I_{2,3}$ can be written as

$$\begin{aligned}I_{2,3}=&\int_0^1E\bigg[\int_t^s\int_K\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_r+\rho\beta(\xi,P_\xi,e))\\&-(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_r)\Big)\beta(X_r^{t,\xi},P_{X_r^{t,\xi}},e)\lambda(de)dr\bigg]d\rho+\rho(s)\\=&\,(s-t)\int_0^1E\bigg[\int_K\Big((\partial_\mu f)(P_\xi,\xi+\rho\beta(\xi,P_\xi,e))-(\partial_\mu f)(P_\xi,\xi)\Big)\beta(\xi,P_\xi,e)\lambda(de)\bigg]d\rho\\&+\rho(s)+R_5+R_6,\end{aligned}$$

where $|\rho(s)|\leq C(s-t)^{\frac{3}{2}}$ and

$$\begin{aligned}R_5=&\int_0^1E\bigg[\int_t^s\int_K\bigg\{\Big((\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_r+\rho\beta(\xi,P_\xi,e))\\&-(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_r)\Big)\\&-\Big((\partial_\mu f)(P_\eta,\xi+\rho\beta(\xi,P_\xi,e))-(\partial_\mu f)(P_\eta,\xi)\Big)\bigg\}\cdot\beta(X_r^{t,\xi},P_{X_r^{t,\xi}},e)\lambda(de)dr\bigg]d\rho,\\R_6=&\int_0^1E\bigg[\int_t^s\int_K\bigg\{\Big((\partial_\mu f)(P_\eta,\xi+\rho\beta(\xi,P_\xi,e))-(\partial_\mu f)(P_\eta,\xi)\Big)\beta(X_r^{t,\xi},P_{X_r^{t,x}},e)\\&-\Big((\partial_\mu f)(P_\xi,\xi+\rho\beta(\xi,P_\xi,e))-(\partial_\mu f)(P_\xi,\xi)\Big)\beta(\xi,P_\xi,e)\bigg\}\lambda(de)dr\bigg]d\rho.\end{aligned}$$

However, using the Lipschitz property of $\partial_y(\partial_\mu f)$, with the help of Burkholder–Davis–Gundy inequality we obtain

$$\begin{aligned}|R_5|\leq&\int_0^1E\bigg[\int_t^s\int_K\int_0^\rho\Big|\partial_y(\partial_\mu f)(P_\eta,\xi+\rho\sigma(\xi,P_\xi)(B_s-B_t)+\rho M_r+\gamma\beta(\xi,P_\xi,e))\\&-\partial_y(\partial_\mu f)(P_\eta,\xi+\gamma\beta(\xi,P_\xi,e))\Big|d\gamma\cdot|\beta(X_r^{t,\xi},P_{X_r^{t,\xi}},e)|\cdot|\beta(\xi,P_\xi,e)|\lambda(de)dr\bigg]d\rho\\\leq&\,CE\bigg[\int_t^s\int_K|\sigma(\xi,P_\xi)(B_s-B_t)+M_r|\cdot|\beta(X_r^{t,\xi},P_{X_r^{t,\xi}},e)|\cdot|\beta(\xi,P_\xi,e)|\lambda(de)dr\bigg]\\\leq&\,C\bigg\{E\int_t^s|B_s-B_t|dr+E\int_t^s\bigg|\int_t^r\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de)\bigg|dr\bigg\}\cdot\int_K(1\wedge|e|^2)\lambda(de)\\\leq&\,C(s-t)^{\frac{3}{2}}+C(s-t)^{\frac{1}{2}}E\bigg(\int_t^s\bigg|\int_t^r\int_K\beta(\xi,P_\xi,e)\mu_\lambda(dr,de)\bigg|^2dr\bigg)^{\frac{1}{2}}\\\leq&\,C(s-t)^{\frac{3}{2}}.\end{aligned}$$

Moreover, again from the Lipschitz continuity of $\partial_y(\partial_\mu f)$ and that of β (see (H3.1) (ii)),

$$\begin{aligned}|R_6| = \Big| \int_0^1 E\Big[\int_t^s \int_K \int_0^\rho \Big(\partial_y(\partial_\mu f)(P_\eta, \xi + \gamma\beta(\xi, P_\xi, e)) - \partial_y(\partial_\mu f) \\ \times (P_\xi, \xi + \gamma\beta(\xi, P_\xi, e)) \Big) \beta(\xi, P_\xi, e)\beta(X_r^{t,\xi}, P_{X_r^{t,\xi}}, e) d\gamma \lambda(de) dr \Big] d\rho \\ + \int_0^1 E\Big[\int_t^s \int_K \Big((\partial_\mu f)(P_\xi, \xi + \rho\beta(\xi, P_\xi, e)) - (\partial_\mu f)(P_\xi, \xi) \Big) \\ \times \Big(\beta(X_r^{t,\xi}, P_{X_r^{t,\xi}}, e) - \beta(\xi, P_\xi, e) \Big) \lambda(de) dr \Big] d\rho \Big| \\ \le C\Big\{ \int_t^s (E|X_s^{t,\xi} - \xi|^2)^{\frac{1}{2}} dr + \int_t^s \Big(E|X_r^{t,\xi} - \xi| + (E|X_r^{t,\xi} - \xi|^2)^{\frac{1}{2}} \Big) dr \Big\} \\ \le C(s-t)^{\frac{3}{2}}.\end{aligned}$$

Summing up the above estimates, we obtain

$$\begin{aligned}I_2 = \frac{1}{2}(s-t)\Big\{ E[(\partial_y\partial_\mu f)(P_\xi, \xi)\sigma(\xi, P_\xi)^2] \\ + \int_0^1 E\Big[\int_K \Big((\partial_\mu f)(P_\xi, \xi + \rho\beta(\xi, P_\xi, e)) \\ - (\partial_\mu f)(P_\xi, \xi) \Big) \beta(\xi, P_\xi, e)\lambda(de) \Big] d\rho \Big\} + \rho(s),\end{aligned}$$

where $|\rho(s)| \le C(s-t)^{\frac{3}{2}}$. Hence,

$$\begin{aligned}f(P_{X_s^{t,\xi}}) - f(P_\xi) \\ = (s-t)\Big\{ E[(\partial_\mu f)(P_\xi, \xi)b(\xi, P_\xi)] + \frac{1}{2}E[\partial_y(\partial_\mu f)(P_\xi, \xi)\sigma(\xi, P_\xi)^2] \\ + \int_0^1 E\Big[\int_K \Big[(\partial_\mu f)(P_\xi, \xi + \rho\beta(\xi, P_\xi, e)) \\ - (\partial_\mu f)\Big(P_\xi, \xi) \Big] \beta(\xi, P_\xi, e)\lambda(de) \Big] d\rho \Big\} + \rho_1(s) + \rho(s),\end{aligned}$$

where $|\rho(s)| \le C(s-t)^{\frac{3}{2}}$ and

$$|\rho_1(s)| = \Big| E\Big[\int_t^s (\partial_\mu f)(P_\xi, \xi)[b(X_r^{t,\xi}, P_{X_r^{t,\xi}}) - b(\xi, P_\xi)] dr \Big] \Big| \le C(s-t)^{\frac{3}{2}}.$$

Consequently, $s \to f(P_{X_s^{t,\xi}})$ is differentiable from the right at $s = t$, and the right-derivative is given by

$$
\begin{aligned}
&\partial_s^+ f(P_{X_s^{t,\xi}})\Big|_{s=t}\\
&\quad= E\Big[(\partial_\mu f)(P_\xi,\xi)b(\xi,P_\xi)\Big] + \frac{1}{2}E[\partial_y(\partial_\mu f)(P_\xi,\xi)\sigma(\xi,P_\xi)^2]\\
&\qquad + E\Big[\int_0^1\int_K \Big[(\partial_\mu f)(P_\xi,\xi+\rho\beta(\xi,P_\xi,e))-(\partial_\mu f)(P_\xi,\xi)]\beta(\xi,P_\xi,e)\lambda(de)d\rho\Big].
\end{aligned}
$$

From the arbitrariness of ξ and the flow property of $X_s^{t,\xi}$ we get the existence of the right-derivative

$$
\begin{aligned}
\partial_s^+ f(P_{X_s^{t,\xi}}) &= \lim_{0<h\downarrow 0}\frac{1}{h}(f(P_{X_{s+h}^{t,\xi}}) - f(P_{X_s^{t,\xi}}))\\
&= \lim_{0<h\downarrow 0}\frac{1}{h}(f(P_{X_{s+h}^{s,X_s^{t,\xi}}}) - f(P_{X_s^{t,\xi}}))\\
&= E[(\partial_\mu f)(P_{X_s^{t,\xi}},X_s^{t,\xi})b(X_s^{t,\xi},P_{X_s^{t,\xi}})]\\
&\quad+\frac{1}{2}E[\partial_y(\partial_\mu f)(P_{X_s^{t,\xi}},X_s^{t,\xi})\sigma(X_s^{t,\xi},P_{X_s^{t,\xi}})^2]\\
&\quad+E\Big[\int_0^1\int_K[((\partial_\mu f)(P_{X_s^{t,\xi}},X_s^{t,\xi}+\rho\beta(X_s^{t,\xi},P_{X_s^{t,\xi}},e))\\
&\quad-(\partial_\mu f)(P_{X_s^{t,\xi}},X_s^{t,\xi}))\beta(X_s^{t,\xi},P_{X_s^{t,\xi}},e)]\lambda(de)d\rho\Big],\quad s\in[t,T).
\end{aligned}
$$

Finally, from the continuity of $s\to\partial_s^+ f(X_s^{t,\xi}),\ s\in[t,T]$, it follows that the function $\psi(s) := f(P_{X_s^{t,\xi}}) - \int_t^s \partial_r^+ f(P_{X_s^{t,\xi}})dr,\ s\in[t,T)$, is continuous, right-differentiable at $[t,T)$, and $\partial_s^+\psi(s)=0,\ s\in[t,T)$. Consequently, $\psi(s)=\psi(t)=f(P_\xi),\ s\in[t,T)$, i.e., $s\to f(P_{X_s^{t,\xi}})$ is differentiable: $\partial_s f(P_{X_s^{t,\xi}}) = \partial_s^+ f(P_{X_s^{t,\xi}}),\ s\in[t,T]$. □

Definition 7.1. We say a function $F:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is in $C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, if

(i) $F(t,\cdot,\cdot)\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, for all $t\in[0,T]$;
(ii) $F(\cdot,x,\mu)\in C_b^1([0,T])$, for all $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$;
(iii) All derivatives, that in t of first order, and those in (x,μ) of first and second order, are uniformly bounded over $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ and Lipschitz in (x,μ), uniformly with respect to t.

Theorem 7.2. *Let* $\Psi\in C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. *Under Assumption* (A.2), *we have the following Itô's formula: for* $0\le t\le s\le T,\ x\in\mathbb{R}^d,\ \xi\in L^2(\mathcal{F}_t;\mathbb{R}^d)$,

$$
\begin{aligned}
&\Psi(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}) - \Psi(t,x,P_\xi)\\
&\quad=\int_t^s\Big(\partial_t\Psi(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})+\sum_{i=1}^d\partial_{x_i}\Psi(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})b_i(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})
\end{aligned}
$$

$$+\frac{1}{2}\sum_{i,j,k=1}^{d}\partial^2_{x_ix_j}\Psi(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})(\sigma_{i,k}\sigma_{j,k})(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\Big)dr$$
$$+\int_t^s\int_K\Big(\Psi(r,X_{r-}^{t,x,P_\xi}+\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e),P_{X_r^{t,\xi}})-\Psi(r,X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}})$$
$$-\sum_{i=1}^{d}\partial_{x_i}\Psi(r,X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}})\cdot\beta_i(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\Big)\lambda(de)dr$$
$$+\int_t^s\tilde{E}\Big[\sum_{i=1}^{d}(\partial_\mu\Psi)_i(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})b_i(\tilde{X}_r^{t,\tilde{\xi}},P_{X_r^{t,\xi}})$$
$$+\frac{1}{2}\sum_{i,j,k=1}^{d}\partial_{y_i}(\partial_\mu\Psi)_j(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})(\sigma_{i,k}\sigma_{j,k})(\tilde{X}_r^{t,\tilde{\xi}},P_{X_r^{t,\xi}})$$
$$+\sum_{i=1}^{d}\int_0^1\int_K[(\partial_\mu\Psi)_i(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}}+\rho\beta(\tilde{X}_r^{t,\tilde{\xi}},P_{X_r^{t,\xi}},e))$$
$$-(\partial_\mu\Psi)_i(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}},\tilde{X}_r^{t,\tilde{\xi}})]\cdot\beta_i(\tilde{X}_r^{t,\tilde{\xi}},P_{X_r^{t,\xi}},e)\lambda(de)d\rho\Big]dr$$
$$+\int_t^s\sum_{i,j=1}^{d}\partial_{x_i}\Psi(r,X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\sigma_{i,j}(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})dB_r^j$$
$$+\int_t^s\int_K\Big(\Psi(r,X_{r-}^{t,x,P_\xi}+\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e))$$
$$-\Psi(r,X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}})\Big)\mu_\lambda(dr,de). \tag{7.1}$$

Proof. As before in other proofs let us restrict ourselves to the dimension $d=1$. Define $F(s,x):=\Psi(s,x,P_{X_s^{t,\xi}})$. From Theorem 7.1 we know that $F\in C_b^{1,2}([0,T]\times\mathbb{R})$ and

$$\begin{aligned}\partial_sF(s,x)=&(\partial_s\Psi)(s,x,P_{X_s^{t,\xi}})+E[(\partial_\mu\Psi)(s,x,P_{X_s^{t,\xi}},X_s^{t,\xi})b(X_s^{t,\xi},P_{X_s^{t,\xi}})]\\&+\frac{1}{2}E[\partial_y(\partial_\mu\Psi)(s,x,P_{X_s^{t,\xi}},X_s^{t,\xi})\sigma(X_s^{t,\xi},P_{X_s^{t,\xi}})^2]\\&+E\Big[\int_0^1\int_K\Big((\partial_\mu\Psi)(s,x,P_{X_s^{t,\xi}},X_s^{t,\xi}+\rho\beta(X_s^{t,\xi},P_{X_s^{t,\xi}},e))\\&-(\partial_\mu\Psi)(s,x,P_{X_s^{t,\xi}},X_s^{t,\xi})\Big)\beta(X_s^{t,\xi},P_{X_s^{t,\xi}},e)\lambda(de)d\rho\Big].\end{aligned}$$

Applying Itô's formula to $F(s,X_s^{t,x,P_\xi})$, we get

$$\begin{aligned}&F(s,X_s^{t,x,P_\xi})-F(t,x)\\&=\int_t^s\Big[\partial_sF(r,X_r^{t,x,P_\xi})+\partial_xF(r,X_r^{t,x,P_\xi})b(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})\\&\quad+\frac{1}{2}\partial_x^2F(r,X_r^{t,x,P_\xi})\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})^2\Big]dr\end{aligned}$$

$$
\begin{aligned}
&+\int_t^s\int_K\bigg(F(r,X_{r-}^{t,x,P_\xi}+\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e))-F(r,X_{r-}^{t,x,P_\xi})\\
&-\partial_xF(r,X_{r-}^{t,x,P_\xi})\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e)\bigg)\lambda(de)dr\\
&+\int_t^s\partial_xF(r,X_r^{t,x,P_\xi})\cdot\sigma(X_r^{t,x,P_\xi},P_{X_r^{t,\xi}})dB_r\\
&+\int_t^s\int_K\bigg(F(r,X_{r-}^{t,x,P_\xi}+\beta(X_{r-}^{t,x,P_\xi},P_{X_r^{t,\xi}},e))\\
&-F(r,X_{r-}^{t,x,P_\xi})\bigg)\mu_\lambda(dr,de),\ \ s\in[t,T].
\end{aligned}
$$

Consequently, substituting the representation formula for $\partial_sF(s,x)$, we get (7.1) for $d=1$ (recall that $\tilde{X}^{t,\tilde{\xi}}$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ is an independent copy of the process $X^{t,\xi}$ on $(\Omega,\mathcal{F},P)$). □

Lemma 7.1. *Suppose* $\Phi\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ *and Assumption* (A.2) *holds true. Then* $V(t,x,P_\xi):=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]$, $(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d)$, *belongs to* $C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. *Moreover, there is some positive real constant* C *such that, for all* $t\in[0,T]$, $x,\ x'\in\mathbb{R}^d$, *and* $\xi,\ \xi'\in L^2(\mathcal{F}_t;\mathbb{R}^d)$,

(i) $|\partial_tV(t,x,P_\xi)|\le C$;

(ii) $|\partial_tV(t,x,P_\xi)-\partial_tV(t,x',P_{\xi'})|\le C(|x-x'|+W_2(P_\xi,P_{\xi'}))$;

(iii) $|\partial_tV(t,x,P_\xi)-\partial_tV(t',x,P_\xi)|\le C|t-t'|^{\frac{1}{2}}$.

Proof. From Proposition 6.1 we know already that $V(t,\cdot,\cdot)\in C_b^{2,1}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. Thus, we have still to prove the differentiability of V with respect to t. As the σ-field $\mathcal{G}_0$ is rich enough, in the sense that $\mathcal{P}_2(\mathbb{R}^d)=\{P_\xi|\xi\in L^2(\mathcal{G}_0;\mathbb{R}^d)\}$ it suffices to study the function $V(t,x,P_\xi)$ for ξ running $L^2(\mathcal{G}_0;\mathbb{R}^d)$. But, as $\xi\in L^2(\mathcal{G}_0;\mathbb{R}^d)$ the processes $X_{\cdot+t}^{t,x,P_\xi}=(X_{s+t}^{t,x,P_\xi})_{s\in[0,T-t]}$ and $X^{0,x,P_\xi}=(X_s^{0,x,P_\xi})_{s\in[0,T-t]}$ obey the same law, for all $x\in\mathbb{R}^d$ and so do $X_{\cdot+t}^{t,\xi}=(X_{s+t}^{t,\xi}=X_{s+t}^{t,\xi,P_\xi})_{s\in[0,T-t]}$ and $X^{0,\xi}=(X_s^{0,\xi}=X_s^{0,\xi,P_\xi})_{s\in[0,T-t]}$. Consequently,

$$V(t,x,P_\xi)=E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})]=E[\Phi(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})]$$

(see the proof of Lemma 5.1 in [7], or Proposition 6.1). Applying Itô's formula in Theorem 7.2 to $\Phi(X_s^{0,x,P_\xi},P_{X_s^{0,\xi}})$, $s\in[0,T-t]$, and taking the expectation on both sides, we obtain that

$$
\begin{aligned}
&V(t,x,P_\xi)-V(T,x,P_\xi)\\
&\quad=\int_0^{T-t}E\bigg[\sum_{i=1}^d\partial_{x_i}\Phi(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})\cdot b_i(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})\\
&\quad\quad+\frac{1}{2}\sum_{i,j,k=1}^d\partial^2_{x_ix_j}\Phi(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})(\sigma_{i,k}\sigma_{j,k})(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})
\end{aligned}
$$

$$
\begin{aligned}
&+\int_K \Big(\Phi(X_r^{0,x,P_\xi}+\beta(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},e),P_{X_r^{0,\xi}})-\Phi(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})\\
&-\sum_{i=1}^d \partial_{x_i}\Phi(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}})\beta_i(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},e)\Big)\lambda(de)\\
&+\tilde{E}\Big[\sum_{i=1}^d(\partial_\mu\Phi)_i(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},\tilde{X}_r^{0,\tilde{\xi}})b_i(\tilde{X}_r^{0,\tilde{\xi}},P_{X_r^{0,\xi}})\\
&+\frac{1}{2}\sum_{i,j,k=1}^d \partial_{y_i}(\partial_\mu\Phi)_j(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},\tilde{X}_r^{0,\tilde{\xi}})(\sigma_{i,k}\sigma_{j,k})(\tilde{X}_r^{0,\tilde{\xi}},P_{X_r^{0,\xi}})\\
&+\sum_{i=1}^d\int_0^1\int_K\Big[(\partial_\mu\Phi)_i(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},\tilde{X}_r^{0,\tilde{\xi}}+\rho\beta(\tilde{X}_r^{0,\tilde{\xi}},P_{X_r^{0,\xi}},e))\\
&-(\partial_\mu\Phi)_i(X_r^{0,x,P_\xi},P_{X_r^{0,\xi}},\tilde{X}_r^{0,\tilde{\xi}})\Big]\cdot\beta_i(\tilde{X}_r^{0,\tilde{\xi}},P_{X_r^{0,\xi}},e)\lambda(de)d\rho\Big]\Big]dr. \quad (7.2)
\end{aligned}
$$

From the continuity in r of the integrand in (7.2) it follows that $V(t,x,P_\xi)$ is continuously differentiable with respect to t, and

$$
\begin{aligned}
&\partial_t V(t,x,P_\xi)\\
&=-E\Big[\sum_{i=1}^d \partial_{x_i}\Phi(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})\cdot b_i(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})\\
&+\frac{1}{2}\sum_{i,j,k=1}^d \partial^2_{x_ix_j}\Phi(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})(\sigma_{i,k}\sigma_{j,k})(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})\\
&+\int_K\Big(\Phi(X_{T-t}^{0,x,P_\xi}+\beta(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},e),P_{X_{T-t}^{0,\xi}})-\Phi(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})\\
&-\sum_{i=1}^d \partial_{x_i}\Phi(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}})\beta_i(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},e)\Big)\lambda(de)\\
&+\tilde{E}\Big[\sum_{i=1}^d(\partial_\mu\Phi)_i(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\tilde{X}_{T-t}^{0,\tilde{\xi}})b_i(\tilde{X}_{T-t}^{0,\tilde{\xi}},P_{X_{T-t}^{0,\xi}})\\
&+\frac{1}{2}\sum_{i,j,k=1}^d \partial_{y_i}(\partial_\mu\Phi)_j(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\tilde{X}_{T-t}^{0,\tilde{\xi}})(\sigma_{i,k}\sigma_{j,k})(\tilde{X}_{T-t}^{0,\tilde{\xi}},P_{X_{T-t}^{0,\xi}})\\
&+\sum_{i=1}^d\int_0^1\int_K\Big[(\partial_\mu\Phi)_i(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\tilde{X}_{T-t}^{0,\tilde{\xi}}+\rho\beta(\tilde{X}_{T-t}^{0,\tilde{\xi}},P_{X_{t-t}^{0,\xi}},e))\\
&-(\partial_\mu\Phi)_i(X_{T-t}^{0,x,P_\xi},P_{X_{T-t}^{0,\xi}},\tilde{X}_{T-t}^{0,\tilde{\xi}})\Big]\cdot\beta_i(\tilde{X}_{T-t}^{0,\tilde{\xi}},P_{X_{T-t}^{0,\xi}},e)\lambda(de)d\rho\Big]\Big]. \quad (7.3)
\end{aligned}
$$

Furthermore, from (7.3) the estimates (i)–(iii) of $\partial_t V$ stated in the lemma can be obtained directly from the estimates got in the preceding sections, namely in Proposition 3.1. The proof is complete. □

Let us now consider the nonlocal integral-PDE:

$$
\begin{aligned}
0 = {} & \partial_t V(t,x,P_\xi) + \sum_{i=1}^d \partial_{x_i} V(t,x,P_\xi) b_i(x,P_\xi) \\
& + \frac{1}{2} \sum_{i,j,k=1}^d \partial^2_{x_i x_j} V(t,x,P_\xi)(\sigma_{i,k}\sigma_{j,k})(x,P_\xi) \\
& + \int_K \Big(V(t, x+\beta(x,P_\xi,e), P_\xi) - V(t,x,P_\xi) \\
& - \sum_{i=1}^d \partial_{x_i} V(t,x,P_\xi)\beta_i(x,P_\xi,e) \Big) \lambda(de) \\
& \quad + E\Big[\sum_{i=1}^d (\partial_\mu V)_i(t,x,P_\xi,\xi) b_i(\xi,P_\xi) \\
& + \frac{1}{2} \sum_{i,j,k=1}^d \partial_{y_i}(\partial_\mu V)_j(t,x,P_\xi,\xi)(\sigma_{i,k}\sigma_{j,k})(\xi,P_\xi) \\
& + \sum_{i=1}^d \int_0^1 \int_K [(\partial_\mu V)_i(t,x,P_\xi,\xi+\rho\beta(\xi,P_\xi,e)) - (\partial_\mu V)_i(t,x,P_\xi,\xi)] \\
& \cdot \beta_i(\xi,P_\xi,e)\lambda(de)d\rho \Big], (t,x,\xi) \in [0,T]\times\mathbb{R}^d \times L^2(\mathcal{F}_t;\mathbb{R}^d), \\
& V(T,x,P_\xi) = \Phi(x,P_\xi), \qquad (x,\xi)\in\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d).
\end{aligned} \tag{7.4}
$$

Theorem 7.3. *Let* $\Phi \in C_b^{2,1}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ *and let Assumption* (A.2) *hold true. Then the function* $V(t,x,P_\xi) := E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]$, $(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times L^2(\mathcal{F}_t;\mathbb{R}^d)$, *is the unique solution in* $C_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ *of the PDE* (7.4).

Proof. As before in other proofs, we restrict ourselves again to the one dimensional case. From the definition of V, the flow property and the $\mathcal{F}_s$-measurability of X_s^{t,x,P_ξ}, we have

$$
\begin{aligned}
V(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}) &= V(s,y,P_\eta)\Big|_{y=X_s^{t,x,P_\xi},\eta=X_s^{t,\xi}} \\
&= E[\Phi(X_T^{s,y,P_\eta},P_{X_T^{s,\eta}})]\Big|_{y=X_s^{t,x,P_\xi},\eta=X_s^{t,\xi}} \\
&= E[\Phi(X_T^{s,y,P_\eta},P_{X_T^{s,\eta}})|\mathcal{F}_s]\Big|_{y=X_s^{t,x,P_\xi},\eta=X_s^{t,\xi}} \\
&= E[\Phi(X_T^{s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}}},P_{X_T^{s,X_s^{t,\xi}}})|\mathcal{F}_s] \\
&= E[\Phi(X_T^{t,x,P_\xi},P_{X_T^{t,\xi}})|\mathcal{F}_s],\ s\in[0,T].
\end{aligned}
$$

But this means that $V(s,X_s^{t,x,P_\xi},P_{X_s^{t,\xi}})$ is a martingale.

On the other hand, due to Proposition 6.2 and Lemma 7.1, the function $V \in C_b^{1,(2,1)}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ satisfies the necessary regularity conditions for Itô's formula. Hence, by applying Itô's formula to $V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}})$, we obtain the following semimartingale decomposition:

$$\begin{aligned}
&V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}) - V(t,x,P_\xi) \\
&= \int_t^s \Big\{ \partial_t V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) + \partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) b(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \\
&\quad + \frac{1}{2}\partial_x^2 V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})^2 \\
&\quad + \int_K \Big(V(r, X_r^{t,x,P_\xi} + \beta(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e), P_{X_r^{t,\xi}}) - V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \\
&\quad - \partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \cdot \beta(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e) \Big) \lambda(de) \\
&\quad + \tilde{E}\Big[\partial_\mu V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}) b(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}}) \\
&\quad + \frac{1}{2}\partial_y(\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}) \sigma(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}})^2 \\
&\quad + \int_0^1 \int_K \Big[(\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}} + \rho\beta(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}}, e)) \\
&\quad - (\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}) \Big] \beta(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}}, e) \lambda(de) d\rho \Big] \Big\} dr \\
&\quad + \int_t^s \partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) dB_r \\
&\quad + \int_t^s \int_K \Big(V(r, X_{r-}^{t,x,P_\xi} + \beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e), P_{X_r^{t,\xi}}) \\
&\quad - V(r, X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \Big) \mu_\lambda(dr, de),
\end{aligned}$$

$s \in [t,T]$. Consequently, as $V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}),\ s \in [t,T]$, is a martingale,

$$\begin{aligned}
&V(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}}) - V(t,x,P_\xi) \\
&= \int_t^s \partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) dB_r \\
&\quad + \int_t^s \int_K \Big(V(r, X_{r-}^{t,x,P_\xi} + \beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e), P_{X_r^{t,\xi}}) \\
&\quad - V(r, X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \Big) \mu_\lambda(dr, de), s \in [t,T],
\end{aligned}$$

and

$$\begin{aligned}
0 &= \int_t^s \Big\{ \partial_t V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) + \partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) b(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \\
&\quad + \frac{1}{2}\partial_x^2 V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}) \sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})^2
\end{aligned}$$

$$
\begin{aligned}
&+\int_K \bigg(V(r, X_r^{t,x,P_\xi}+\beta(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e), P_{X_r^{t,\xi}})-V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\\
&-\partial_x V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\cdot\beta(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e)\bigg)\lambda(de)\\
&+\tilde{E}\bigg[\partial_\mu V(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})b(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}})\\
&+\frac{1}{2}\partial_y(\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\sigma(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}})^2\\
&+\int_0^1\int_K\Big[(\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}}+\rho\beta(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}}, e))\\
&-(\partial_\mu V)(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}}, \tilde{X}_r^{t,\tilde{\xi}})\Big]\beta(\tilde{X}_r^{t,\tilde{\xi}}, P_{X_r^{t,\xi}}, e)\lambda(de)d\rho\bigg]\bigg\}dr,\ \ s\in[t,T].
\end{aligned}
$$

Letting $s\downarrow t$, we obtain the wished PDE.

It remains to prove the uniqueness of the solution of the PDE. We suppose $W\in C_b^{1,(2,1)}([0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}))$ is a solution of the PDE. Then, applying Itô's formula to $W(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}})$, we have

$$
\begin{aligned}
&W(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}})-W(t,x,P_\xi)\\
&\quad=\int_t^s \partial_x W(r, X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})\sigma(X_r^{t,x,P_\xi}, P_{X_r^{t,\xi}})dB_r\\
&\qquad+\int_t^s\int_K\Big(W(r, X_{r-}^{t,x,P_\xi}+\beta(X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}}, e), P_{X_r^{t,\xi}})\\
&\qquad-W(r, X_{r-}^{t,x,P_\xi}, P_{X_r^{t,\xi}})\Big)\mu_\lambda(dr,de),\quad s\in[t,T],
\end{aligned}
$$

where we have taken into account that W satisfies (7.4) (for $d=1$). But this means that $W(s, X_s^{t,x,P_\xi}, P_{X_s^{t,\xi}})-W(t,x,P_\xi),\ s\in[t,T]$, is a martingale. Hence, for all $(t,x,\xi)\in[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_t;\mathbb{R})$,

$$
\begin{aligned}
W(t,x,P_\xi)&=E[W(T, X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})|\mathcal{F}_t]=E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})|\mathcal{F}_t]\\
&=E[\Phi(X_T^{t,x,P_\xi}, P_{X_T^{t,\xi}})]=V(t,x,P_\xi).
\end{aligned}
$$

The proof is complete. □

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