Space-time analytic smoothing effect for the pseudo-conformally invariant Schrödinger equations

Gaku Hoshino and Tohru Ozawa

Abstract. We study the global Cauchy problem for the mass critical nonlinear Schrödinger equations. We prove the global existence of analytic solutions in both space and time variables for sufficiently small and exponentially decaying Cauchy data. The method of proof depends on the Leibniz rule for the generator of pseudo-conformal transforms at the L^2 critical level.

Mathematics Subject Classification. 35Q55.

Keywords. Nonlinear Schrödinger equation, Analytic smoothing effect, Global existence.

1. Introduction

We study the Cauchy problem for the pseudo-conformally invariant Schrödinger equations in one and two space dimensions

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{4/n}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0) = \phi, & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $n = 1, 2$, $u : \mathbb{R} \times \mathbb{R}^n \ni (t, x) \mapsto u(t, x) \in \mathbb{C}$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$, $\lambda \in \mathbb{C}$. There is a large literature on the Cauchy problem for the nonlinear Schrödinger equations (see [1, 2, 4, 13, 24, 26, 27] and reference therein). We refer the reader to [3, 5, 9–12, 14–18, 20–22, 25] for the available results on the analytic solutions to the nonlinear Schrödinger equations and the other evolution equations.

In this paper, a special attention is given on the analyticity in both space and time variables of solutions to (1.1). In particular, we emphasize that the

available literature has treated only local properties with respect to the time variable. For instance, in [5, 18] space-time analytic smoothing effect for non-linear Schrödinger equations has been proved on the basis of the following estimate

$$\sum_{l=0}^{\infty} \frac{a^l}{l!} \|(K + \alpha t)^l f; X_T\| \leq C \frac{1}{1 - abT} \sum_{l=0}^{\infty} \frac{a^l}{l!} \|K^l f; X_T\|,$$

where $K(t) = |x|^2 + nit + 2it(t\partial_t + x \cdot \nabla)$, $\alpha \in \mathbb{C}$, $abT < 1$, $a, T > 0$, $b > 2$ and X_T is an appropriate Banach space of functions of space-time variables, such as space of functions on $[0, T]$ with values in Banach spaces. To overcome difficulty in constructing time global solutions, we prove the Leibniz rule for $K(t) + 4it$ in the L^2 critical case. To be more specific, we have obtained the Leibniz rule with respect to the generator of pseudo-conformal transforms in the pseudo-conformally invariant case (see Lemma 4 below), which is essential in our argument in this paper.

To state our main result precisely, we introduce the following notation. $L^p = L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, denotes the usual Lebesgue space. $U(t) = \exp(i\frac{t}{2}\Delta)$, $t \in \mathbb{R}$, is the free Schrödinger propagator and we use the notation such as $(U\phi)(t) = U(t)\phi$. We put the generator of Galilei transform by

$$J(t) = x + it\nabla = U(t)xU(-t), \quad t \in \mathbb{R},$$

which has another representation:

$$J(t) = M(t)it\nabla M(-t), \quad t \neq 0,$$

where $M(t) = e^{i\frac{|x|^2}{2t}}$, $t \neq 0$. As in [6, 11, 15] we put

$$A_\delta(t) = U(t)e^{\delta \cdot x}U(-t), \quad t \in \mathbb{R}, \quad \delta \in \mathbb{R}^n,$$

which has another representation:

$$A_\delta(t) = M(t)e^{it\delta \cdot \nabla}M(-t), \quad t \neq 0,$$

where $e^{it\delta \cdot \nabla} = \mathcal{F}^{-1}e^{-t\delta \cdot \xi}\mathcal{F}$ and $\mathcal{F}$ denotes the Fourier transform defined by $\mathcal{F}[\varphi](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) dx$. We put the generator of dilations by

$$P(t) = t\partial_t + x \cdot \nabla, \quad t \in \mathbb{R}.$$

We put

$$\tilde{K}(t) = |x|^2 + 2itP(t), \quad t \in \mathbb{R},$$

which has another representation:

$$\tilde{K}(t) = 2itM(t)P(t)M(-t), \quad t \neq 0.$$

We put the generator of pseudo-conformal transforms by

$$K(t) = \tilde{K}(t) + nit, \quad t \in \mathbb{R}.$$

We introduce the following basic function spaces:

$$\begin{aligned} \mathcal{X}_1 &= L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty), \quad n = 1, \\ \mathcal{X}_2 &= L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^4), \quad n = 2, \end{aligned}$$

where $\|u; \mathcal{A} \cap \mathcal{B}\| = \|u; \mathcal{A}\| + \|u; \mathcal{B}\|$ for Banach spaces $\mathcal{A}, \mathcal{B}$. Let $D \subset \mathbb{R}^n$ be a domain and $a > 0$. We define the following function spaces:

$$\begin{aligned} G^D(x; L^2) &\equiv \left\{ \phi \in L^2; \|\phi; G^D(x; L^2)\| \equiv \sup_{\delta \in D} \|e^{\delta \cdot x} \phi; L^2\| < \infty \right\}, \\ G^D(J; \mathcal{X}_n) &\equiv \left\{ u \in \mathcal{X}_n; \|u; G^D(J; \mathcal{X}_n)\| \equiv \sup_{\delta \in D} \|A_\delta u; \mathcal{X}_n\| < \infty \right\}, \\ G^{D,a}(x, |x|^2; L^2) &\equiv \left\{ \phi \in G^D(x; L^2); \|\phi; G^{D,a}(x, |x|^2; L^2)\| \right. \\ &\quad \left. \equiv \sum_{l=0}^{\infty} \frac{a^l}{l!} \| |x|^{2l} \phi; G^D(x; L^2) \| < \infty \right\}, \\ G^{D,a}(J, K; \mathcal{X}_n) &\equiv \left\{ u \in G^D(J; \mathcal{X}_n); \|u; G^{D,a}(J, K; \mathcal{X}_n)\| \right. \\ &\quad \left. \equiv \sum_{l=0}^{\infty} \frac{a^l}{l!} \|K^l u; G^D(J; \mathcal{X}_n)\| < \infty \right\}. \end{aligned}$$

For $\varepsilon > 0$, we define $B_\varepsilon^{D,a}$ as follows:

$$B_\varepsilon^{D,a} = \left\{ \phi \in G^{D,a}(x, |x|^2; L^2); \|\phi; G^{D,a}(x, |x|^2; L^2)\| \leq \varepsilon \right\}.$$

We consider the following integral equation associated with the Cauchy problem (1.1):

$$u = U\phi - i\lambda V(|u|^{4/n}u)$$

where $(Vf)(t) = \int_0^t U(t-s)f(s)ds$. We state our main result:

Theorem 1. *Let $n = 1, 2$. There exists an $\varepsilon > 0$ such that for any $a > 0$, any domain $D \subset \mathbb{R}^n$ satisfying $0 \in D$, $-D = D$, and any $\phi \in B_\varepsilon^{D,a}$ (1.1) has a unique solution $u \in G^{D,a}(J, K; \mathcal{X}_n)$.*

Remark 1. As regards analyticity, the operators J and K correspond analyticity in space and time, respectively.

Remark 2. The main feature of the result above is that the time interval in question is the whole interval $\mathbb{R}$.

Remark 3. As stated in the theorem, a and D may be taken independent of ε .

2. Preliminaries

In this section, we introduce the some basic lemmas. We say that (r, q) is an admissible pair if (r, q) satisfies

$$\frac{2}{r} = n \left(\frac{1}{2} - \frac{1}{q} \right),$$

with $2 \leq q \leq \infty$ ($n = 1$), $2 \leq q < \infty$ ($n = 2$).

Lemma 1. ([1, 24, 27]) For any admissible pairs $(r, q), (\sigma, \rho)$ the following inequalities hold:

$$\begin{aligned}\|U\phi; L^r(\mathbb{R}; L^q)\| &\leq C\|\phi; L^2\|, \\ \|Vf; L^r(\mathbb{R}; L^q)\| &\leq C\|f; L^{\sigma'}(\mathbb{R}; L^{\rho'})\|,\end{aligned}$$

where p' denotes the Hölder conjugate of $1 \leq p \leq \infty$ defined by $1/p + 1/p' = 1$.

Lemma 2. ([5]) We put $L = i\partial_t + \frac{1}{2}\Delta$. Then

$$[K(t), L] = -4itL$$

for $t \in \mathbb{R}$, where $[A, B] = AB - BA$ is the commutator.

Proof. See [5]. □

Lemma 3. Let $t \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned}& (K(t) + 4it) \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\&= \sum_{k=1}^{\sigma} \left\{ K(t) u_{2k-1} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2l-1} \overline{u_{2j}} \right) - \overline{K(t) u_{2k}} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2j-1} \overline{u_{2l}} \right) \right\} u_{2\sigma+1} \\&+ \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) K(t) u_{2\sigma+1} + nit \left(1 + \frac{4}{n} - (2\sigma + 1) \right) \\&\times \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1}.\end{aligned}$$

Proof. By the definition and the Leibniz rule of $\tilde{K}(t) = 2itM(t)P(t)M(-t)$, we have

$$\begin{aligned}& (K(t) + 4it) \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\&= (\tilde{K}(t) + (n+4)it) \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\&= \tilde{K}(t) \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} + (n+4)it \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1}\end{aligned}$$

$$\begin{aligned}
&= 2itM(t)P(t) \left(\prod_{j=1}^{\sigma} M(-t)u_{2j-1} \overline{M(-t)u_{2j}} \right) M(-t)u_{2\sigma+1} \\
&\quad + (n+4)it \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\
&= \sum_{k=1}^{\sigma} \left\{ \tilde{K}(t)u_{2k-1} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2l-1} \overline{u_{2j}} \right) - \overline{\tilde{K}(t)u_{2k}} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2j-1} \overline{u_{2l}} \right) \right\} u_{2\sigma+1} \\
&\quad + \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) \tilde{K}(t)u_{2\sigma+1} + (n+4)it \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\
&= \sum_{k=1}^{\sigma} \left\{ \left(\tilde{K}(t) + nit \right) u_{2k-1} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2l-1} \overline{u_{2j}} \right) - \overline{\left(\tilde{K}(t) + nit \right) u_{2k}} \right. \\
&\quad \times \left. \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2j-1} \overline{u_{2l}} \right) \right\} u_{2\sigma+1} \\
&\quad + \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) \left(\tilde{K}(t) + nit \right) u_{2\sigma+1} + (n+4)it \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\
&\quad - n(2\sigma+1)it \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\
&= \sum_{k=1}^{\sigma} \left\{ K(t)u_{2k-1} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2l-1} \overline{u_{2j}} \right) - \overline{K(t)u_{2k}} \left(\prod_{\substack{j=1 \\ l \neq k}}^{\sigma} u_{2j-1} \overline{u_{2l}} \right) \right\} u_{2\sigma+1} \\
&\quad + \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) K(t)u_{2\sigma+1} + nit \left(1 + \frac{4}{n} - (2\sigma+1) \right) \\
&\quad \times \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1}.
\end{aligned}$$

□

Remark 4. For $1 < p < \infty$, as in the proof of the Lemma 3, we see that

$$\begin{aligned}
&(K(t) + 4it)|u|^{p-1}u \\
&= \frac{p+1}{2}|u|^{p-1}K(t)u - \frac{p-1}{2}u^2|u|^{p-3}\overline{K(t)u} + nit\left(1 + \frac{4}{n} - p\right)|u|^{p-1}u.
\end{aligned}$$

Lemma 4. Let $t \in \mathbb{R}$, $n = 1, 2$ and $\sigma = \frac{2}{n}$. Then we have

$$\begin{aligned} & (K(t) + 4it)^l \left(\prod_{j=1}^{\sigma} u_{2j-1} \overline{u_{2j}} \right) u_{2\sigma+1} \\ &= \sum_{l_1+l_2+\dots+l_{2\sigma+1}=l} \frac{l!}{l_1!l_2!\dots l_{2\sigma+1}!} \left(\prod_{j=1}^{\sigma} K^{l_{2j-1}}(t) u_{2j-1} (-1)^{l_{2j}} \overline{K^{l_{2j}}(t) u_{2j}} \right) \\ & \quad \times K^{l_{2\sigma+1}}(t) u_{2\sigma+1}, \end{aligned}$$

for all $l \in \mathbb{Z}_{\geq 0}$.

Proof. The result follows immediately by the Lemma 3. $\square$

3. Proof of Theorem 1.

Let $\phi \in B_{\varepsilon}^{D,a}$, we define Φ by

$$\Phi u = U\phi - i\lambda V(|u|^{4/n}u)$$

for $u \in G^{D,a}(J, K; \mathcal{X}_n)$, $n = 1, 2$. For $R > 0$, we define a metric space $(X_n(R), d)$ by

$$\begin{aligned} X_n(R) &= \left\{ u \in G^{D,a}(J, K; \mathcal{X}_n); \|u; G^{D,a}(J, K; \mathcal{X}_n)\| \leq R \right\}, \\ d(u, v) &= \|u - v; G^{D,a}(J, K; \mathcal{X}_n)\|. \end{aligned}$$

We see that $(X_n(R), d)$ is a complete metric space. We prove that $\Phi : u \mapsto \Phi u$ is a contraction mapping on $G^{D,a}(J, K; \mathcal{X}_n)$. By Lemma 1 and Hölder's inequality, we estimate

$$\begin{aligned} \|\Phi u; \mathcal{X}_n\| &\leq C\|\phi; L^2\| + \| |u|^{4/n}u; L^{\frac{2n+4}{n+4}}(\mathbb{R} \times \mathbb{R}^n) \| \\ &\leq C\|\phi; L^2\| + C\|u; \mathcal{X}_n\|^{1+4/n}, \end{aligned}$$

where we have used the fact $\mathcal{X}_1 \subset L^6(\mathbb{R}; L^6)$ with inequality

$$\|u; L^6(\mathbb{R}; L^6)\| \leq \|u; L^\infty(\mathbb{R}; L^2)\|^{1/3} \|u; L^4(\mathbb{R}; L^\infty)\|^{2/3} \leq \|u; \mathcal{X}_1\|.$$

Hence Φ is a mapping in $\mathcal{X}_n$ for $n = 1, 2$. By Lemmas 2 and 4, we have for any $l \in \mathbb{Z}_{\geq 0}$

$$K^l \Phi u$$

$$\begin{aligned} &= U|x|^{2l}\phi - i\lambda V\left((K + 4is)^l |u|^{4/n}u\right) \\ &= U|x|^{2l}\phi - i\lambda V\left(\sum_{l_1+l_2+l_3+l_4+l_5=l} \frac{(-1)^{l_2+l_4}l!}{l_1!l_2!l_3!l_4!l_5!} \left(\prod_{j=1}^2 K^{l_{2j-1}}u \overline{K^{l_{2j}}u}\right) K^{l_5}u\right) \\ &= U|x|^{2l}\phi - i\lambda \sum_{l_1+l_2+l_3+l_4+l_5=l} \frac{(-1)^{l_2+l_4}l!}{l_1!l_2!l_3!l_4!l_5!} V\left(\left(\prod_{j=1}^2 K^{l_{2j-1}}u \overline{K^{l_{2j}}u}\right) K^{l_5}u\right) \end{aligned}$$

for $n = 1$ and

$$\begin{aligned}
 K^l \Phi u &= U|x|^{2l} \phi - i\lambda V \left((K + 4is)^l |u|^2 u \right) \\
 &= U|x|^{2l} \phi - i\lambda V \left(\sum_{l_1+l_2+l_3=l} \frac{(-1)^{l_2} l!}{l_1! l_2! l_3!} K^{l_1} u \overline{K^{l_2} u} K^{l_3} u \right) \\
 &= U|x|^{2l} \phi - i\lambda \sum_{l_1+l_2+l_3=l} \frac{(-1)^{l_2} l!}{l_1! l_2! l_3!} V \left(K^{l_1} u \overline{K^{l_2} u} K^{l_3} u \right)
 \end{aligned}$$

for $n = 2$. We remark that for any $t \neq 0$ $e^{it\delta \cdot \nabla}$ gives an analytic continuation with formula $e^{it\delta \cdot \nabla} v = v(\cdot + it\delta)$ for all v satisfying $\|e^{it\delta \cdot \nabla} v; L^2\| < \infty$, and all $\delta \in D$ ([6, 11, 15, 19, 23]). Hence, we have

$$\begin{aligned}
 A_\delta(t) v_1 \overline{v_2} v_3 &= M(t) e^{it\delta \cdot \nabla} (M(-t) v_1 \overline{M(-t) v_2} M(-t) v_3) \\
 &= A_\delta(t) v_1 \overline{A_{-\delta}(t) v_2} A_\delta(t) v_3, \\
 A_\delta(t) \left(\prod_{j=1}^2 v_{2j-1} \overline{v_{2j}} \right) v_5 &= M(t) e^{it\delta \cdot \nabla} \left(\left(\prod_{j=1}^2 M(-t) v_{2j-1} \overline{M(-t) v_{2j}} \right) M(-t) v_5 \right) \\
 &= \left(\prod_{j=1}^2 A_\delta(t) v_{2j-1} \overline{A_{-\delta}(t) v_{2j}} \right) A_\delta(t) v_5.
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 A_\delta K^l \Phi u &= U e^{\delta \cdot x} |x|^{2l} \phi - i\lambda \sum_{l_1+l_2+l_3+l_4+l_5=l} \frac{(-1)^{l_2+l_4} l!}{l_1! l_2! l_3! l_4! l_5!} V \\
 &\quad \times \left(A_\delta \left(\left(\prod_{j=1}^2 K^{l_{2j-1}} u \overline{K^{l_{2j}} u} \right) K^{l_5} u \right) \right) \\
 &= U e^{\delta \cdot x} |x|^{2l} \phi - i\lambda \sum_{l_1+l_2+l_3+l_4+l_5=l} \frac{(-1)^{l_2+l_4} l!}{l_1! l_2! l_3! l_4! l_5!} V \\
 &\quad \times \left(\left(\prod_{j=1}^2 A_\delta K^{l_{2j-1}} u \overline{A_{-\delta} K^{l_{2j}} u} \right) A_\delta K^{l_5} u \right)
 \end{aligned}$$

for $n = 1$ and

$$\begin{aligned}
 A_\delta K^l \Phi u &= U e^{\delta \cdot x} |x|^{2l} \phi - i\lambda \sum_{l_1+l_2+l_3=l} \frac{(-1)^{l_2} l!}{l_1! l_2! l_3!} V \left(A_\delta \left(K^{l_1} u \overline{K^{l_2} u} K^{l_3} u \right) \right) \\
 &= U e^{\delta \cdot x} |x|^{2l} \phi - i\lambda \sum_{l_1+l_2+l_3=l} \frac{(-1)^{l_2} l!}{l_1! l_2! l_3!} V \left(A_\delta K^{l_1} u \overline{A_{-\delta} K^{l_2} u} A_\delta K^{l_3} u \right)
 \end{aligned}$$

for $n = 2$. Therefore by Lemma 1 and Hölder's inequality, and make use of the assumption $-D = D$ we have

$$\begin{aligned}
& \|K^l \Phi u; G^D(J; \mathcal{X}_1)\| \leq C \| |x|^{2l} \phi; G^D(x; L^2) \| \\
& + C \sum_{l_1+l_2+l_3+l_4+l_5=l} \frac{l!}{l_1!l_2!l_3!l_4!l_5!} \prod_{j=1}^2 \|K^{l_{2j-1}} u; G^D(J; \mathcal{X}_1)\| \\
& \times \|K^{l_{2j}} u; G^D(J; \mathcal{X}_1)\| \|K^{l_5} u; G^D(J; \mathcal{X}_1)\|, \\
& \|K^l \Phi u; G^D(J; \mathcal{X}_2)\| \leq C \| |x|^{2l} \phi; G^D(x; L^2) \| \\
& + C \sum_{l_1+l_2+l_3=l} \frac{l!}{l_1!l_2!l_3!} \|K^{l_1} u; G^D(J; \mathcal{X}_2)\| \|K^{l_2} u; G^D(J; \mathcal{X}_2)\| \\
& \times \|K^{l_3} u; G^D(J; \mathcal{X}_2)\|.
\end{aligned}$$

We multiply $\frac{a^l}{l!}$ both sides of the above inequalities, take the summation over all $l \geq 0$ we obtain

$$\|\Phi u; G^{D,a}(J, K; \mathcal{X}_n)\| \leq C \|\phi; G^{D,a}(x, |x|^2; L^2)\| + C \|u; G^{D,a}(J, K; \mathcal{X}_n)\|^{1+4/n}.$$

Similarly, we have

$$\begin{aligned}
& \|\Phi u - \Phi v; G^{D,a}(J, K; \mathcal{X}_n)\| \\
& \leq C \left(\|u; G^{D,a}(J, K; \mathcal{X}_n)\|^{4/n} + \|v; G^{D,a}(J, K; \mathcal{X}_n)\|^{4/n} \right) \\
& \|u - v; G^{D,a}(J, K; \mathcal{X}_n)\|.
\end{aligned}$$

Hence Φ is a mapping in $G^{D,a}(J, K; \mathcal{X}_n)$ for $n = 1, 2$. Let $u, v \in X_n(R)$, we have

$$\begin{aligned}
& \|\Phi u; G^{D,a}(J, K; \mathcal{X}_n)\| \leq C \|\phi; G^{D,a}(x, |x|^2; L^2)\| + CR^{1+4/n}, \\
& \|\Phi u - \Phi v; G^{D,a}(J, K; \mathcal{X}_n)\| \leq CR^{4/n} \|u - v; G^{D,a}(J, K; \mathcal{X}_n)\|.
\end{aligned}$$

Therefore, the contraction argument goes through, if ε and R satisfy

$$\begin{cases} C\varepsilon + CR^{1+4/n} \leq R, \\ CR^{4/n} < 1, \end{cases}$$

for $n = 1, 2$. This completes the proof of Theorem 1.

Acknowledgments

This work was supported by Grant-in-Aid for JSPS Fellows.

References

- [1] Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Math., vol. 10. Amer. Math. Soc. (2003)
- [2] Cazenave, T., Weissler, F.B.: Some remarks on the nonlinear Schrödinger equation in the critical case. Lecture notes in Math, vol. 1394, pp. 18–29. Springer, Berlin (1989)

- [3] DeBouard, A.: Analytic solution to non elliptic non linear Schrödinger equations. *J. Differ. Equ.* **104**, 196–213 (1993)
- [4] Ginibre, J.: Introduction aux équations de Schrödinger non linéaires, Paris Onze Edition, L161. Université Paris-Sud(1998)
- [5] Hayashi, N., Kato, K.: Analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations. *Commun. Math. Phys.* **184**, 273–300 (1997)
- [6] Hayashi, N., Ozawa, T.: On the derivative nonlinear Schrödinger equation. *Physica D.* **55**, 14–36 (1992)
- [7] Hayashi, N., Saitoh, S.: Analyticity and smoothing effect for the Schrödinger equation. *Ann. Inst. Henri Poincaré, Phys. Théor.* **52**, 163–173 (1990)
- [8] Hayashi, N., Saitoh, S.: Analyticity and global existence of small solutions to some nonlinear Schrödinger equations. *Commun. Math. Phys.* **129**, 27–41 (1990)
- [9] Hoshino, G., Ozawa, T.: Analytic smoothing effect for a system of nonlinear Schrödinger equations. *Differ. Equ. Appl.* **5**, 395–408 (2013)
- [10] Hoshino, G., Ozawa, T.: Analytic smoothing effect for nonlinear Schrödinger equation in two space dimensions. *Osaka J. Math.* **51**, 609–618 (2014)
- [11] Hoshino, G., Ozawa, T.: Analytic smoothing effect for nonlinear Schrödinger equations with quintic nonlinearity. *J. Math. Anal. Appl.* **419**, 285–297 (2014)
- [12] Kato, K., Taniguchi, K.: Gevrey regularizing effect for nonlinear Schrödinger equations. *Osaka J. Math.* **33**, 863–880 (1996)
- [13] Kato, T.: On nonlinear Schrödinger equations. *Ann. Inst. Henri Poincaré, Phys. Théor.* **46**, 113–129 (1987)
- [14] Kato, T., Masuda, K.: Nonlinear evolution equations and analyticity . *Ann. Inst. Henri Poincaré Analyse Non linéaire* **3**, 455–467 (1986)
- [15] Nakamitsu, K.: Analytic finite energy solutions of the nonlinear Schrödinger equation. *Commun. Math. Phys.* **260**, 117–130 (2005)
- [16] Ozawa, T., Yamauchi, K.: Remarks on analytic smoothing effect for the Schrödinger equation. *Math. Z.* **261**, 511–524 (2009)
- [17] Ozawa, T., Yamauchi, K.: Analytic smoothing effect for global solutions to nonlinear Schrödinger equation. *J. Math. Anal. Appl.* **364**, 492–497 (2010)
- [18] Ozawa, T., Yamauchi, K., Yamazaki, Y.: Analytic smoothing effect for solutions to Schrödinger equations with nonlinearity of integral type. *Osaka J. Math.* **42**, 737–750 (2005)
- [19] Reed, M., Simon, B.: *Method of Mathematical Physics II*. Academic Press (1975)
- [20] Robbiano, L., Zuily, C.: Microlocal analytic smoothing effect for the Schrödinger equation. *Duke Math. J.* **100**, 93–129 (1999)

- [21] Robbiano, L., Zuily, C.: Effect régularisant microlocal analytique pour l'équation de Schrödinger : le cas données oscillantes. Commun. Part. Differ. Equ. **25**, 1891–1906 (2000)
- [22] Simon, J.C.H., Taflin, E.: Wave operators and analytic solutions of nonlinear Klein-Gordon equations and of nonlinear Schrödinger equations. Commun. Math. Phys. **99**, 541–562 (1985)
- [23] Stein, E.M. Weiss G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ. Press, Princeton(1971)
- [24] Sulem, C., Sulem P.-L.: The Nonlinear Schrödinger Equation. Self-focusing and Wave Collapse. Appl. Math. Sci., vol. 139. Springer (1999)
- [25] Takuwa, H.: Analytic smoothing effects for a class of dispersive equations. Tsukuba J. Math. **28**, 1–34 (2004)
- [26] Tsutsumi, Y.: L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups. Funkcial Ekvac. **30**, 115–125 (1987)
- [27] Yajima, K.: Existence of solutions for Schrödinger evolution equations. Commun. Math. Phys. **110**, 415–426 (1987)

Gaku Hoshino and Tohru Ozawa
Department of Applied Physics
Waseda University
Tokyo 169-8555
Japan
e-mail: gaku-hoshino@ruri.waseda.jp

Tohru Ozawa
e-mail: txozawa@waseda.jp

Received: 15 April 2015.

Accepted: 7 October 2015.