On a variational problem involving critical Sobolev growth in dimension three

Rabeh Ghoudi

Abstract. In this paper we consider the following nonlinear problem: $-\Delta u = Ku^5$, $u > 0$ in Ω , $u = 0$ on $\partial\Omega$, where $K > 0$ in Ω , $K = 0$ on $\partial\Omega$ and Ω is a bounded domain of $\mathbb{R}^3$. We prove a version of a Morse lemma at infinity for this problem, which allows us to describe the critical points at infinity of the associated variational functional. Using a topological argument, we prove an existence result.

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1. Introduction and the Main Results

In this paper, we consider the following nonlinear elliptic problem:

$$(P) \quad \begin{cases} -\Delta u = \frac{n-2}{4(n-1)} K(x) u^{(n+2)/(n-2)}, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, and K is a smooth positive function in Ω .

The interest in this equation comes from the resemblance to the scalar curvature problem in differential geometry, which consists in finding suitable conditions on a given function K defined on M such that K is the scalar curvature for a metric $\tilde{g}$ conformally equivalent to g , where (M, g) is a n -dimensional Riemannian manifold without boundary.

The special nature of problem (P) appears when we consider it from variational viewpoint. The Euler functional associated to (P) does not satisfy the Palais–Smale condition. This means that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This is due to the noncompactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$. In the case of manifolds without boundary, this problem has been widely studied

in various works (see for example the monographs [1, 15] and the references therein). A phenomenon appears in dimension $n \geq 5$, due the fact that the self-interaction of the functions failing the Palais–Smale condition dominates the interaction of two masses of those functions, while in dimension 3 the reverse happens [4]. In dimension 4, we have a balance phenomenon; that is, the self-interaction and the interaction are of the same size.

In contrast with manifolds, there are few known results of (P) . In dimension greater than 4, Hebey [13] studied the case where Ω is a ball and $K = K(|x|)$; see also [14] for the dimension 4. Also, in dimension 4, Ben Ayed–Hammami [9] studied the case where Ω is a bounded domain. This result implies that there is some kind of compactness near the boundary. However, in [12], the authors studied the perturbed subcritical problem with $\Omega = B(O; 1)$ and $K(x) = |x|^\gamma$ and they proved that the ground state solutions blow up at a point on the boundary.

In this paper, we will focus our attention on the case where $n = 3$ and we will characterize the critical points at infinity of the associated variational functional. We will use this characterization in order to give some existence results.

In order to state our results, we need to introduce some notation, recall some known facts, and state the assumptions that we are using in our results. This problem has a variational structure. The related functional is

$$J(u) = \left(\int_{\Omega} K u^6 \right)^{-1/3}$$

defined on $\Sigma = \{u \in H_0^1(\Omega) : \|u\|_{H_0^1} := \int_{\Omega} |\nabla u|^2 = 1\}$. As we said, the Palais–Smale condition fails for J on $\Sigma^+ := \{u \in \Sigma : u > 0\}$. Its failure has been analyzed throughout the works of [11, 16, 18]. This analysis is carried out here with virtually no change. It has led to the characterization of the sequences failing the Palais–Smale condition. In order to describe this characterization, we need to introduce some notation.

For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = \sqrt[4]{3} \frac{\lambda^{1/2}}{(1 + \lambda^2|x - a|^2)^{1/2}}, \quad (1.1)$$

$\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$-\Delta u = u^5, \quad u > 0 \quad \text{in } \mathbb{R}^3. \quad (1.2)$$

Notice that, the family $\delta_{(a,\lambda)}$ achieves the best Sobolev constant

$$S := \inf \{ \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \|u\|_{L^6(\mathbb{R}^3)}^{-2} : u \not\equiv 0, \nabla u \in (L^2(\mathbb{R}^3))^3 \text{ and } u \in L^6(\mathbb{R}^3) \}.$$

We denote by $P\delta_{(a,\lambda)}$ the projection of the $\delta_{(a,\lambda)}$ onto $H_0^1(\Omega)$, defined by

$$\Delta P\delta_{(a,\lambda)} = \Delta \delta_{(a,\lambda)} \text{ in } \Omega, \quad P\delta_{(a,\lambda)} = 0 \text{ on } \partial\Omega. \quad (1.3)$$

We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part, that is,

$$\begin{aligned} G(x, y) &= |x - y|^{-1} - H(x, y) \quad \text{for } (x, y) \in \Omega^2, \\ \Delta_x H &= 0 \text{ in } \Omega^2, \quad G = 0 \quad \text{on } \partial(\Omega^2). \end{aligned}$$

In order to state our result, we need to introduce the assumption.

(H₁) $K > 0$ in Ω and $K = 0$ on $\partial\Omega$.

Thanks to (H₁), we can prove that the concentration points of the sequences failing the Palais–Smale condition are far away of the boundary.

Theorem 1.1. *Assume that J has no critical point in Σ^+ and let (u_k) be a sequence in Σ^+ such that $J(u_k) \leq B$ and $\nabla J(u_k)$ goes to 0 as k goes to infinity. Then there exist an integer p , a sequence ε_k and $\eta > 0$ such that $u_k \in V(p, \varepsilon_k, \eta)$, where, for $\varepsilon > 0$ and $p \in \mathbb{N}^*$, $V(p, \varepsilon, \eta)$ is defined by*

$$\begin{aligned} V(p, \varepsilon, \eta) &= \left\{ u \in \Sigma / \exists (a_1, \dots, a_p) \in \Omega^p, \exists (\lambda_1, \dots, \lambda_p) \in ((\varepsilon^{-1}, +\infty))^p, \right. \\ &\quad \left. \exists (\alpha_1, \dots, \alpha_p) \in ((0, +\infty))^p \text{ such that } d(a_i, \partial\Omega) > \eta, \right. \\ &\quad \left. \left\| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \right\|_{H_0^1} < \varepsilon, \left| \frac{\alpha_i^4 K(a_i)}{\alpha_j^4 K(a_j)} - 1 \right| < \frac{\varepsilon}{p}, \varepsilon_{ij} < \varepsilon \right\}. \end{aligned}$$

Here, $\forall i \neq j$

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{-\frac{1}{2}}.$$

Remark 1.2. I believe that the condition $K = 0$ on $\partial\Omega$ can be replaced by a positive function K on $\bar{\Omega}$ with $\frac{\partial K}{\partial \nu} < 0$ on $\partial\Omega$. A technical detail blocks the proof but we think that the result remains true.

Thinking of these sequences which do not satisfy the Palais–Smale condition as critical points, a natural idea is to try to find suitable parameters in order to complete a Morse lemma at infinity. For manifolds without boundary, this program has been completed in [3, 8]. We would here extend the proof of the existence of a Morse lemma at infinity to the case of Dirichlet boundary conditions. We introduce the following parametrization of the set $V(p, \varepsilon, \eta)$.

Proposition 1.3. *For any $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that, if $0 < \varepsilon < \varepsilon_p$ and $u \in V(p, \varepsilon, \eta)$, then the following minimization problem*

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \right\|, \quad \alpha_i > 0, \quad \lambda_i > 0, \quad a_i \in \Omega \right\}$$

has a unique solution $(\alpha, a, \lambda) = (\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$ (up to permutation). In particular, we can write $u \in V(p, \varepsilon, \eta)$ as follows

$$u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + v,$$

where $v \in H_0^1(\Omega)$ such that, for any $i = 1, \dots, p$

$$(V_0) : \quad \langle v, \varphi_i \rangle = 0 \text{ for } \varphi_i \\ \times \in \{P\delta_{(a_i, \lambda_i)}, \partial P\delta_{(a_i, \lambda_i)}/\partial \lambda_i, \partial P\delta_{(a_i, \lambda_i)}/\partial (a_i)_j\} \quad \forall j = 1, \dots, n, \quad (1.4)$$

and the α_i 's satisfy

$$\frac{\alpha_i^4 K(a_i)}{\alpha_j^4 K(a_j)} \in [1 - \varepsilon, 1 + \varepsilon] \quad \forall i, \forall j.$$

The proof of Proposition 1.3 is similar, up to minor modifications, to the corresponding statements in [3, 5].

To state our main results, we need to add some assumptions for the critical points of K .

(H₂) In Ω , K has only nondegenerate critical points $y_1, \dots, y_m$.

For $s \in \{1, \dots, m\}$ and for any s -tuple $\tau_s = (i_1, \dots, i_s) \in \{1, \dots, m\}^s$ such that $i_p \neq i_q$ if $p \neq q$, we define a matrix $M(\tau_s) = (M_{pq})_{1 \leq p, q \leq s}$, by

$$M_{pp} = \frac{H(y_{i_p}, y_{i_p})}{(K(y_{i_p}))^{1/2}}, \quad M_{pq} = -\frac{G(y_{i_p}, y_{i_q})}{(K(y_{i_p})K(y_{i_q}))^{1/4}} \quad \text{for } p \neq q, \quad (1.5)$$

and we denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$.

(H₃) For any $s \in \{1, \dots, m\}$, we assume that $M(\tau_s)$ is nondegenerate.

Theorem 1.4. *Under assumptions (H₁), (H₂) and (H₃), if*

$$1 \neq \sum_{s=1}^m \sum_{\tau_s=(i_1, \dots, i_s)/M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^s k_{i_j}}, \quad (1.6)$$

where $k_{i_j} = \text{index}(K, y_{i_j})$, then (P) has a solution.

The proof of Theorem 1.4 is based on the characterization of the critical points at infinity of the associated variational structure J . Assuming that (P) has no solution, using [7], we know that Σ^+ retracts by deformation onto some set X which is the union of the unstable manifolds of the critical points at infinity. Since Σ^+ is a contractible set, we derive that the Euler–Poincaré characteristic of Σ^+ is equal to 1. Furthermore, under the assumptions (H₁) – (H₃), we prove that the contribution of the critical points at infinity in the Euler–Poincaré characteristic is equal to the right hand side of (1.6). Hence we obtain a contradiction with (1.6).

The remainder of this paper is organized as follows. In section 2, we give the characterization of the Palais–Smale sequences. In section 3, we give the expansions of the functional J and its gradient. In section 4, we construct a pseudo gradient, which will be useful in the proof of Theorem 1.4. In section 5, we give the proof of Theorem 1.4.

2. Palais–Smale sequences

In this section, we will prove that the concentration points of the Palais–Smale sequences are far away of the boundary. Recall:

Definition 2.1. A sequence (u_k) is said to be a Palais–Smale sequence for J in $[a, b]$ if it satisfies: $\partial J(u_k) \rightarrow 0$ as $k \rightarrow \infty$ and $a \leq J(u_k) \leq b$ for each k .

It is well know, for $K \equiv 1$, by the works of [18], that if (u_k) is a Palais–Smale sequences for J in $[a, b]$ then it has the form

$$u_k = \sum_{i=1}^p P\delta_{(a_{i,k}, \lambda_{i,k})} + v_k,$$

where $\|v_k\|_{H_0^1} \rightarrow 0$, $\lambda_{i,k}d(a_{i,k}, \partial\Omega) \rightarrow \infty$ and $a_{i,k} \rightarrow a_i \in \bar{\Omega}$.

This program remains true for K a C^2 positive function on $\bar{\Omega}$.

Our goal is to extend this result when $K = 0$ on $\partial\Omega$ and to prove that the concentration points are far away from the boundary.

Theorem 2.2. Assume that J has no critical point in Σ^+ . Let $B > 0$ and (u_k) be a sequence in Σ^+ such that $J(u_k) \leq B$ and $\partial J(u_k)$ goes to 0. Then, replacing (u_k) by a subsequence if necessary, there exist an integer p , sequences $(\lambda_{i,k})$ in $(0, \infty)$, $(a_{i,k})$ in Ω , $1 \leq i \leq p$ and $\eta > 0$ such that,

(i) $\lambda_{i,k} \rightarrow \infty$, and $a_{i,k} \rightarrow a_i \in \Omega_{\eta/2}$ as $k \rightarrow \infty$, where $\Omega_{\eta} = \{x \in \Omega : d(x, \partial\Omega) \geq \eta\}$.

(ii) $\|u_k - c(p) \sum_{i=1}^p (K(a_i))^{-\frac{1}{4}} P\delta_{(a_{i,k}, \lambda_{i,k})}\| \rightarrow 0$ as $k \rightarrow \infty$.

The proof of this theorem will be divided in serval lemmas. We start by defining

$$I(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \frac{1}{6} \int_{\Omega} K v^6, \quad v \in H_0^1(\Omega).$$

For $u \in \Sigma^+$, we have $I(\lambda(u)u) = \frac{1}{3}J(u)^{3/2}$, where $\lambda(u) = (\int_{\Omega} K u^6)^{-1/4} = (J(u))^{3/4}$.

Lemma 2.3. Let (u_k) be as in Theorem 2.2 and $v_k = \lambda(u_k)u_k$. Then (v_k) is bounded in $H_0^1(\Omega)$ and $v_k \rightarrow 0$.

Proof. Since $I(v_k) = \frac{1}{3}J(u_k)^{3/2}$, we get that $I(v_k) \leq \frac{1}{3}B^{3/2}$. Furthermore, it is easy to obtain

$$\int_{\Omega} |\nabla v_k|^2 = \lambda(u_k)^2 \int_{\Omega} K v_k^6, \quad \forall k. \quad (2.1)$$

Thus, we derive

$$\int_{\Omega} |\nabla v_k|^2 \leq B^{\frac{3}{2}}. \quad (2.2)$$

It follows that (v_k) is bounded. Since we are assumed that J does not have critical point in Σ^+ , we get $v_k \rightarrow 0$ weakly in $H_0^1(\Omega)$. $\square$

Lemma 2.4. We have

$$\int_{\Omega} K v_k^6 \geq S^{3/2}(\max_{\Omega} K)^{-1/2} \quad \text{and} \quad \int_{\Omega} |\nabla v_k|^2 \geq S^{3/2}(\max_{\Omega} K)^{-1/2}.$$

Proof. We have

$$J(u_k) \geq S(\max_{\Omega} K)^{-1/3},$$

this implies that

$$\lambda(u_k) \geq S^{3/4}(\max_{\Omega} K)^{-1/4}.$$

The lemma follows from (2.1). $\square$

Lemma 2.5. *For each $\mu > 0$ there exists $\eta(\mu, B) > 0$ such that*

$$\int_{\Omega \setminus \Omega_{\eta}} K v_k^6 \leq \mu,$$

where $\Omega_{\eta} = \{x \in \Omega : d(x, \partial\Omega) \geq \eta\}$.

Proof. Let $\mu > 0$. Since K is a continuous functions and $K = 0$ on $\partial\Omega$, then there exists $\eta > 0$ such that for any $x \in \{x \in \Omega : d(x, \partial\Omega) < \eta\}$, we have $K(x) < \mu S^3/B^{9/2}$.

Now, using the Sobolev's inequality, we obtain

$$\int_{\Omega \setminus \Omega_{\eta}} K v_k^6 \leq \mu \int_{\Omega} v_k^6 \leq \mu S^{-3} \left(\int_{\Omega} |\nabla v_k|^2 \right)^3 \leq \mu.$$

$\square$

As in [18], Theorem 2.2 follows inductively from the following proposition:

Proposition 2.6. *Let (v_k) be a sequence such that $v_k \rightharpoonup 0$ in $H_0^1(\Omega)$ and $I(v_k) \rightarrow c > 0$. Then, replacing (v_k) by a subsequence if necessary, there exist sequences (λ_k) in $(0, \infty)$, (a_k) in Ω and a Palais–Smale sequence $(\bar{v}_k)$ for I such that,*

- (i) $\lambda_k \rightarrow \infty$, and $a_k \rightarrow a \in \Omega_{\eta/2}$ as $k \rightarrow \infty$,
- (ii) $\bar{v}_k = v_k - K(a_k)^{-\frac{1}{4}} P\delta_{(a_k, \lambda_k)} + o(1)$ in $H_0^1(\Omega)$.

The proof is done in [18] for $K \equiv 1$ and it still hold (and it is very used in many papers) for $K \not\equiv 1$ but with $K > 0$ in $\bar{\Omega}$. In our case, $K = 0$ on $\partial\Omega$ and we want to use this information to prove that the concentration points are not near the boundary.

Proof. We will give the idea of the proof and we will justify the claim that $d(a_k, \partial\Omega) \geq c > 0$. Let $\bar{B}(x, r)$ denote the closed ball in $\mathbb{R}^3$ with center x and radius r . The Levy concentration function

$$\Phi_k(r) := \sup_{x \in \mathbb{R}^3} \int_{\bar{B}(x, r)} K v_k^6$$

satisfies $\Phi_k(0) = 0$ and $\Phi_k(\infty) > \delta := \left(\frac{S}{2}\right)^{3/2} (\max_{\Omega} K)^{-1/2}$. Hence we may choose $\widetilde{a}_k \in \Omega$ and $\varepsilon_k > 0$ such that

$$\sup_{x \in \mathbb{R}^3} \int_{\bar{B}(x, \varepsilon_k)} K v_k^6 = \int_{\bar{B}(\widetilde{a}_k, \varepsilon_k)} K v_k^6 = \delta. \quad (2.3)$$

Observe that, since Ω is bounded, the sequence (ε_k) is bounded. We define

$$w_k(z) = \varepsilon_k^{1/2} v_k(\varepsilon_k z + \widetilde{a_k}) \quad \text{for } z \in \Omega_k := \{z \in \mathbb{R}^3 : \varepsilon_k z + \widetilde{a_k} \in \Omega\}.$$

Since, $v_k \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_k \not\rightarrow 0$ in $L^6(\Omega)$, we derive that $\varepsilon_k \rightarrow 0$ (see [18]).

Note that from Lemma 2.5, we see that the point $\widetilde{a_k}$ cannot be near the boundary that is $d(\widetilde{a_k}, \partial\Omega) \geq \eta/2$.

Furthermore, from [18], the sequence $w_k \rightarrow w$ in $H^1(\Omega')$, for any $\Omega' \subset \subset \mathbb{R}^3$. Since w_k is a Palais–Smale sequence for I . We get that w satisfies

$$-\Delta w = K(\lim \widetilde{a_k}) w^5, \quad w > 0 \text{ in } \mathbb{R}^3,$$

which implies that there exist $\alpha \in \mathbb{R}^3$ and $\beta > 0$ such that $w = K(\lim \widetilde{a_k})^{-\frac{1}{4}} \delta_{(\alpha, \beta)}$.

In the second step, we define

$$a_k := \widetilde{a_k} + \varepsilon_k \alpha, \quad \lambda_k := \beta / \varepsilon_k \quad \text{and} \quad \bar{v}_k := v_k - K(a_k)^{-\frac{1}{4}} P\delta_{(a_k, \lambda_k)}.$$

The function $\bar{v}_k$ will converge in $H_0^1(\Omega)$ or it will be a Palais–Smale sequence for I (see [18] for more details).

For the second case, we will repeat the program for $\bar{v}_k$ (the first and the second step). Hence our result is completed. $\square$

3. Expansion of the functional and its gradient

In this section, we will give the expansion of $J(u)$, $(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i})$ and $(\partial J(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i})$, where u belongs to $V(p, \varepsilon, \eta)$ and δ_i denotes $\delta_{(a_i, \lambda_i)}$.

Proposition 3.1. [2] *For ε small enough and $u = \sum_{i=1}^p \alpha_i P\delta_i + v \in V(p, \varepsilon, \eta)$, v satisfying (V_0) , we have the following expansion*

$$\begin{aligned} J(u) = & \frac{(\sum_{i=1}^p \alpha_i^2) S^{2/3}}{(\sum_{i=1}^p \alpha_i^6 K(a_i))^{1/3}} \left[1 + \frac{1}{S \sum_{i=1}^p K(a_i)^{-1/2}} \left(c_2 \sum_{i=1}^p \frac{H(a_i, a_i)}{\lambda_i K(a_i)^{1/2}} \right. \right. \\ & - \sum_{i \neq j} \frac{c_1}{(K(a_i)K(a_j))^{1/4}} \left(\varepsilon_{ij} - \frac{H(a_i, a_j)}{\sqrt{\lambda_i \lambda_j}} \right) \Big) - f(v) \\ & \left. + \frac{1}{\sum_{i=1}^p \alpha_i^2 S} Q(v, v) + o \left(\sum_{i \neq j} \varepsilon_{ij} + \sum \frac{1}{\lambda_i} + \|v\|^2 \right) \right], \end{aligned}$$

where $c_1 = 3^{3/2} \int_{\mathbb{R}^3} \frac{dy}{(1+|y|^2)^{5/2}}$, $c_2 = \frac{3^{3/2}}{6} \int_{\mathbb{R}^3} |y|^2 \delta_{(0,1)}^6$ and $S = \int_{\mathbb{R}^3} \delta_{(0,1)}^6$,

$$\begin{aligned} Q(v, v) = & \|v\|^2 - \frac{5 \sum_{i=1}^p \alpha_i^2}{\sum_{i=1}^p \alpha_i^6 K(a_i)} \int_{\Omega} K \left(\sum_{i=1}^p \alpha_i P\delta_i \right)^4 v^2 \quad \text{and} \\ f(v) = & \frac{2}{\sum_{j=1}^p \alpha_j^6 K(a_j) S} \int_{\Omega} K \left(\sum_{i=1}^p \alpha_i P\delta_i \right)^5 v. \end{aligned}$$

(Here and in the sequel $P\delta_i$ denotes $P\delta_{(a_i, \lambda_i)}$).

As usual in this type of problem, we first deal with the v -part of u . Let us introduce the following set

$$E = \{v/v \text{ satisfies } (V_0) \text{ and } \|v\| < \varepsilon\},$$

where (V_0) is defined in (1.4).

Proposition 3.2. [2] *For any $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon, \eta)$ given, there exists a unique $\bar{v} = \bar{v}(a, \alpha, \lambda)$ which minimizes $J(u + v)$ with respect to $v \in E$. Moreover, we have the following estimate*

$$\|\bar{v}\| \leq c\|f\| \leq c \left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{1/3} \right).$$

Now regarding the gradient of J , we have the following expansions

Proposition 3.3. [2] *For any $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon, \eta)$, we have*

$$\begin{aligned} \left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) &= 2J(u) \left[-\frac{c_1 \alpha_i}{2} \frac{H(a_i, a_i)}{\lambda_i} (1 + o(1)) \right. \\ &\quad \left. - c_1 \sum_{j \neq i} \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{1}{2} \frac{H(a_i, a_j)}{\sqrt{\lambda_i \lambda_j}} \right) (1 + o(1)) + R_1 \right], \end{aligned}$$

where R_1 satisfies

$$R_1 = O \left(\sum_{k \neq r} \varepsilon_{kr}^{7/4} (\log \varepsilon_{kr}^{-1})^{7/12} + \sum_{j=1}^p \frac{1}{\lambda_j^2} \right).$$

Proposition 3.4. [2] *For any $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon, \eta)$, we have*

$$\begin{aligned} \left(\partial J(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right) &= 2J(u) \left[-c_3 \alpha_i^5 J(u)^3 \frac{\nabla K(a_i)}{\lambda_i} (1 + o(1)) \right. \\ &\quad \left. + (1 - J(u)^3 (\alpha_j^4 K(a_j) + \alpha_i^4 K(a_i))) 2c'_1 \sum_{j \neq i} \alpha_j \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right] \\ &\quad + O \left(\sum_j \frac{1}{\lambda_j^2} + \sum_{j \neq i} \lambda_j |a_i - a_j| \varepsilon_{ij}^4 \right), \end{aligned}$$

where c_3 and c'_1 are positive constants.

4. Construction of a Pseudo gradient Flow in $V(p, \varepsilon, \eta)$

In this section, we construct a pseudo gradient W near infinity as in Proposition A.2 of [3], Lemma 3.3 of [8], and Theorem 4.1 of [10]. In the general case, to decrease the functional J , we use $-\partial J$ as pseudo gradient. In this paper, we need other claims to be satisfied. For this effect, we will define another vector field which decreases J and satisfies other properties (see Theorem 4.1 below).

Theorem 4.1. *There exists a pseudo gradient W so that the following holds: There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon, \eta)$ so that*

$$(i) \quad (-\partial J(u), W) \geq c \sum_{i=1}^p \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + \frac{|\nabla K(a_i)|}{\lambda_i} \right)$$

$$(ii) \quad \left(-\partial J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) \right) \geq c \sum_{i=1}^p \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + \frac{|\nabla K(a_i)|}{\lambda_i} \right)$$

(iii) $|W|$ is bounded. Furthermore, the only case where the maximum of the λ_i 's is not bounded is each point a_i in Ω is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $\rho(y_{j_1}, \dots, y_{j_s})$ denotes the least eigenvalue of $M(y_{j_1}, \dots, y_{j_s})$.

Proof. As said before, the construction is based on moving the points a_i if they are away from the critical points of K . In the other cases the vector field is basically defined on the variables λ_i . For this reason, we divide $V(p, \varepsilon, \eta)$ into some regions $V_i(p, \varepsilon, \eta)$, and we will build a vector field W_i on each $V_i(p, \varepsilon, \eta)$. Our global vector field W will be built using a convex combination of the vector fields W_i . Let

$$V_1(p, \varepsilon, \eta) := \{u \in V(p, \varepsilon, \eta) : \forall i \ a_i \in \cup_k B(y_k, \mu)\}$$

$$V_2(p, \varepsilon, \eta) := \{u \in V(p, \varepsilon, \eta) : \exists i \ a_i \notin \cup_k B(y_k, \mu)\}.$$

4.1. Construction of W_1 on $V_1(p, \varepsilon, \eta)$

We divide the set $V_1(p, \varepsilon, \eta)$ onto three sets:

$$V_1(p, \varepsilon, \eta, \sigma) = \{u/a_i \in B(y_{j_i}, 2\sigma), j_i \neq j_k \text{ for } i \neq k \text{ and for } \\ \tau = (j_1, \dots, j_p), \rho(\tau) > 0\}$$

$$V_2(p, \varepsilon, \eta, \sigma) = \{u/a_i \in B(y_{j_i}, 2\sigma), j_i \neq j_k \text{ for } i \neq k, \rho(j_1, \dots, j_p) < 0\}$$

$$V_3(p, \varepsilon, \eta, \sigma) = \{u/a_i \in B(y_{j_i}, 2\sigma), \exists i \neq k \text{ such that } j_i = j_k\}$$

where ρ is the least eigenvalue of the matrix M .

We will define the pseudo gradient depending on the sets V_i to which u belongs.

First, we consider the case of $u = \sum_{i=1}^p \alpha_i P\delta_i \in V_1(p, \varepsilon, \eta, \sigma)$, we have for any $i \neq j$, $|a_i - a_j| > c$ and therefore

$$\varepsilon_{ij} = \frac{1}{\sqrt{\lambda_i \lambda_j} |a_i - a_j|} (1 + o(1)).$$

Thus

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{1}{2} \varepsilon_{ij} (1 + o(1)) = -\frac{1}{2\sqrt{\lambda_i \lambda_j} |a_i - a_j|} (1 + o(1)).$$

Observe that, since $u \in V(p, \varepsilon, \eta)$, we have $\alpha_i^4 J(u)^3 K(a_i) = 1 + o(1)$. Thus, Proposition 3.3 becomes

$$\begin{aligned} \left(\partial J(u), \alpha_i \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) &= \frac{c_1}{J(u)^{\frac{1}{2}}} \\ &\times \left[-\frac{H(a_i, a_i)}{K(a_i)^{\frac{1}{2}} \lambda_i} + \sum_{j \neq i} \frac{G(a_i, a_j)}{(K(a_i)K(a_j))^{\frac{1}{4}}} \frac{1}{\sqrt{\lambda_i \lambda_j}} + o\left(\sum_k \frac{1}{\lambda_k}\right) \right]. \end{aligned} \quad (4.1)$$

We define Z_1 by $Z_1 = \sum_{i=1}^p \alpha_i \lambda_i \partial P \delta_i / \partial \lambda_i$. Thus, we derive that

$$(-\partial J(u), Z_1) = c^T \Lambda M \Lambda + o\left(\sum_k \frac{1}{\lambda_k}\right) \geq c \sum_k \frac{1}{\lambda_k} \geq c \sum_k \frac{1}{\lambda_k} + c \sum_{i \neq j} \varepsilon_{ij}, \quad (4.2)$$

where M is the matrix defined by (1.5) and $\Lambda =^T (1/\sqrt{\lambda_1}, \dots, 1/\sqrt{\lambda_p})$. We also define

$$Z_a = \sum_{i=1}^p \varphi(\lambda_i |\nabla K(a_i)|) \frac{\alpha_i}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \quad \text{and} \quad X_1 = C Z_1 + Z_a,$$

where C is a large constant and where φ is a C^∞ function which satisfies $\varphi(t) = 0$ if $t \leq 1$, $\varphi(t) = 1$ if $t \geq 2$ and $\varphi(t) \in [0, 1]$ for each $t \in (0, \infty)$. Using Proposition 3.4 and (4.2), we derive that

$$(-\partial J(u), X_1) \geq c \left(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k} + \sum_{i \neq j} \varepsilon_{ij} \right). \quad (4.3)$$

Secondly, we study the case of $u = \sum_{i=1}^p \alpha_i P \delta_i \in V_2(p, \varepsilon, \eta, \sigma)$. Let ρ be the least eigenvalue of M . Then, there exists an eigenvector e associated to ρ such that $\|e\| = 1$ with $e_i > 0$ for all i (see [6, 8]). Let $\gamma > 0$ such that for any $x \in B(e, \gamma) = \{y \in S^{p-1} / \|y - e\| \leq \gamma\}$, we have ${}^T x M x < (1/2)\rho$. Two cases may occur.

Case 1 $|\Lambda|^{-1} \Lambda \in B(e, \gamma)$. In this case, we define $X'_2 = -C Z_1 + Z_a$. As in (4.2) and (4.3), we derive that

$$(-\partial J(u), X'_2) \geq c \left(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k} + \sum_{i \neq j} \varepsilon_{ij} \right). \quad (4.4)$$

Case 2 $|\Lambda|^{-1} \Lambda \notin B(e, \gamma)$. In this case, we define

$$Z_2 = - \sum_{i=1}^p |\Lambda| \alpha_i \lambda_i^2 \frac{\partial P \delta_i}{\partial \lambda_i} \left[\frac{|\Lambda| e_i - \Lambda_i}{\|y(0)\|} - \frac{y_i(0)}{\|y(0)\|^3} (y(0), |\Lambda| e - \Lambda) \right],$$

where $y(t) = (1-t)\Lambda + t|\Lambda|e$. Define $\Lambda(t) = y(t)/\|y(t)\|$. Using Proposition 3.3, as in [8], we derive that

$$(-\partial J(u), Z_2) = -c |\Lambda|^2 \frac{\partial}{\partial t} ({}^T \Lambda(t) M \Lambda(t)) + o\left(\sum_k \frac{1}{\lambda_k}\right).$$

Observe that

$${}^T\Lambda(t)M\Lambda(t) = \rho + \frac{(1-t)^2}{\|y(t)\|^2}({}^T\Lambda M\Lambda - \rho\|\Lambda\|^2).$$

Thus, we derive $(\partial)/(\partial t)({}^T\Lambda(t)M\Lambda(t)) < -c$. Therefore for $X_2'' = CZ_2 + Z_a$, we obtain

$$(-\partial J(u), X_2'') \geq c \left(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

The vector field defined on $V_2(p, \varepsilon, \eta, \sigma)$ will be a convex combination of X_2' and X_2'' .

Finally, we will study the case of $u = \sum_{i=1}^p \alpha_i P\delta_i \in V_3(p, \varepsilon, \eta, \sigma)$. Let $B_i = \{j/a_j \in B(y_i, \sigma)\}$. In this case, there is at least one B_i which contains at least two indices. Without loss of generality, we can assume that $1, \dots, q$ are the indices such that the set B_i ($1 \leq i \leq q$) contains at least two indices. We will decrease the λ_i 's for $i \in B_i$ with different speed. For this purpose, let χ be a smooth cutoff function such that $\chi \geq 0$, $\chi = 0$ if $t \leq \gamma'$ and $\chi = 1$ if $t \geq 1$, where γ' is a small constant. For $j \in B_k$, set $\bar{\chi}(\lambda_j) = \sum_{i \neq j, i \in B_k} \chi(\lambda_j/\lambda_i)$. Define

$$Z_3 = - \sum_{k=1}^q \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}.$$

Using Proposition 3.3, we obtain

$$\begin{aligned} (-\partial J(u), Z_3) &= 2J(u) \sum_{k=1}^q \sum_{j \in B_k} \alpha_j \bar{\chi}(\lambda_j) \left[-c_1 \sum_{i \neq j} \alpha_j \left(\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{1}{2} \frac{H(a_i, a_j)}{\sqrt{\lambda_i \lambda_j}} \right) \right. \\ &\quad \left. + O\left(\frac{1}{\lambda_j}\right) + o\left(\sum_r \frac{1}{\lambda_r} + \sum_{i \neq r} \varepsilon_{ir}\right) \right]. \end{aligned}$$

For $j \in B_k$, with $k \leq q$, if $\bar{\chi}(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $\lambda_j^{-1} = o(\varepsilon_{ij})$ (for σ small enough).

Furthermore, for $j \in B_k$, if $i \notin B_k$ or $i \in B_k$ with λ_i and λ_j are of the same order, that is, $\gamma' < \lambda_i/\lambda_j < 1/\gamma'$, then we have $-\lambda_r(\partial \varepsilon_{ij})/(\partial \lambda_r) = \frac{1}{2}\varepsilon_{ij}(1 + o(1))$, for $r = i, j$. In the case where $i \in B_k$ with (assuming that $\lambda_i < \lambda_j$) $\lambda_i/\lambda_j < \gamma'$, we have $\bar{\chi}(\lambda_j) - \bar{\chi}(\lambda_i) \geq 1$. Thus

$$\begin{aligned} &-\bar{\chi}(\lambda_j) \lambda_j (\partial \varepsilon_{ij})/(\partial \lambda_j) - \bar{\chi}(\lambda_i) \lambda_i (\partial \varepsilon_{ij})/(\partial \lambda_i) \\ &\geq -\lambda_j (\partial \varepsilon_{ij})/(\partial \lambda_j) = \frac{1}{2} \varepsilon_{ij} (1 + o(1)). \end{aligned}$$

Thus, we derive that

$$(-\partial J(u), Z_3) \geq c \sum_{k=1}^q \sum_{j \in B_k, \bar{\chi}(\lambda_j) \neq 0} \left(\frac{1}{\lambda_j} + \sum_{i \neq j} \varepsilon_{ij} \right) + o\left(\sum_r \frac{1}{\lambda_r} + \sum_{i \neq r} \varepsilon_{ir}\right). \quad (4.5)$$

The lower bound does not contain all the indices. We need to add some terms. Let

$$\lambda_{i_0} = \inf\{\lambda_i, i = 1, \dots, p\}. \quad (4.6)$$

Now, we distinguish two subcases.

Subcase 1 There exists j such that $\bar{\chi}(\lambda_j) \neq 0$ and $\lambda_{i_0}/\lambda_j > \gamma'$, then we can appear on the lower bound $1/\lambda_{i_0}$ and therefore all the $1/\lambda_i$ and the ε_{ik} . Thus, we can define $X_3 = CZ_3 + Z_a$ where C is a large constant.

Subcase 2 For each j such that $\bar{\chi}(\lambda_j) \neq 0$, we have $\lambda_{i_0}/\lambda_j \leq \gamma'$. In this case, we define

$$D = (\{i/\bar{\chi}(\lambda_i) = 0\} \cup (\cup_{k=1}^q B_k)^c) \cap \{i : \lambda_i/\lambda_{i_0} < 1/\gamma'\}.$$

It is easy to see that $\{i/\bar{\chi}(\lambda_i) = 0\}$ contains at most one index from each B_j for $1 \leq j \leq q$ and therefore for $i, r \in D$ such that $i \neq r$ we have $a_i \in B(y_{j_i}, \eta)$ and $a_r \in B(y_{j_r}, \eta)$ with $j_i \neq j_r$. Let

$$u_1 = \sum_{i \in D} \alpha_i P \delta_i.$$

u_1 has to satisfy one of the two cases above, that is, $u_1 \in V_i(\text{card}(D), \varepsilon, \eta, \sigma)$ for $i = 1$ or 2 . Thus, we can apply the associated vector field which we will denote Z'_3 and we have the estimate

$$\begin{aligned} (-\partial J(u), Z'_3) &\geq c \sum_{i \in D} \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i} + \sum_{i, j \in D} \varepsilon_{ij} \right) \\ &\quad + O \left(\sum_{k \in D, r \notin D} \varepsilon_{kr} \right) + o \left(\sum_{i \notin D} \frac{1}{\lambda_i} \right). \end{aligned}$$

Observe that for $k \in D$ and $r \notin D$, we have either $r \in B^q := \{i/\bar{\chi}(\lambda_i) \neq 0\} \cap (\cup_{j=1}^q B_j)$ or $r \in (B^q)^c$. If $r \in B^q$, we have ε_{kr} in the lower bound of (4.5). If $r \in (B^q)^c$, in this case since $r \notin D$ we have $\lambda_{i_0}/\lambda_r < \gamma'$. Furthermore, we can prove that a_k and a_r are not in the same set $B(y, \eta)$ and therefore $|a_k - a_r| > c$. Thus

$$\varepsilon_{kr} \leq \frac{c}{\sqrt{\lambda_k \lambda_r}} \leq \frac{c\gamma'}{\sqrt{\lambda_k \lambda_{i_0}}} = o(\varepsilon_{ki_0})$$

(γ' small). Since $i_0 \in D$ (i_0 is defined by (4.6)), then from $1/\lambda_{i_0}$, we can appear on the lower bound all the $1/\lambda_i$ and ε_{ir} for $i, r \in (B^q)^c$ (since for those indices we have $|a_i - a_r| > c$). Thus, we derive that

$$(-\partial J(u), Z'_3) \geq c \left(\sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i} + \sum_{i, j \in (B^q)^c} \varepsilon_{ij} \right) + O \left(\sum_{k \in D, r \in B^q} \varepsilon_{kr} \right). \quad (4.7)$$

Now, we define $X_3 = CZ_3 + Z'_3 + mZ_a$ where C is a large constant and m is a small constant. We obtain

$$(-\partial J(u), X_3) \geq c \left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right). \quad (4.8)$$

The vector field W_1 will be as a convex combination of all above cases.

4.2. Construction of W_2 on $V_2(p, \varepsilon, \eta)$

Let us define

$$B' = \{i/a_i \notin \cup B(y_k, 2\mu)\},$$

$$B = B' \cup \{j \notin B' : \exists(i_1, \dots, i_r) \text{ such that } i_r \in B', i_1 = j, |a_{i_{k-1}} - a_{i_k}| < \frac{\mu}{p}, \forall k \leq r\}.$$

We observe that

- (1) For each $j \notin B$ we have $a_j \in \cup B(y_k, 2\mu)$.
- (2) The advantage of B is that if $i \in B$ and $j \notin B$, then $|a_i - a_j| \geq \mu/p$.

We define the vector field X_1 as follows:

$$X_1 = \sum_{i \in B} \frac{\alpha_i}{\lambda_i} \frac{\partial P \delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}.$$

Using Proposition 3.4, we get

$$\begin{aligned} (-\partial J(u), X_1) &\geq c \sum_{i \in B} \frac{1}{\lambda_i} + O \left(\sum_{i, j \in B, |a_i - a_j| < \mu/p} \varepsilon_{ij} \right) \\ &\quad + O \left(\sum_{i \in B, j \notin B} \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right) + o \left(\sum_{j \neq i} \varepsilon_{ij} \right) + o \left(\sum_{j=1}^p \frac{1}{\lambda_j} \right). \end{aligned} \quad (4.9)$$

Observe that, for $i \in B$ and $j \notin B$, we have $\lambda_i |a_i - a_j| \rightarrow +\infty$. Thus, we have

$$\left| \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \leq \frac{c}{\lambda_i |a_i - a_j|} \varepsilon_{ij} = o(\varepsilon_{ij}). \quad (4.10)$$

Combining (4.9) and (4.10), we get

$$\begin{aligned} (-\partial J(u), X_1) &\geq c \sum_{i \in B} \frac{1}{\lambda_i} + O \left(\sum_{i, j \in B, |a_i - a_j| < \mu/p} \varepsilon_{ij} \right) \\ &\quad + o \left(\sum_{j \neq i} \varepsilon_{ij} \right) + o \left(\sum_{j=1}^p \frac{1}{\lambda_j} \right). \end{aligned} \quad (4.11)$$

Now, for $i \in B$, we introduce the set

$$T_i = \left\{ j \in B : \exists(i_1, \dots, i_r) \text{ such that } i_r = i, i_1 = j, |a_{i_{k-1}} - a_{i_k}| < \frac{\mu}{p}, \forall k \leq r \right\}.$$

We observe that

- (1) For each $i \in T_j$ we have $i \in T_k \ \forall k \in T_i$.
- (2) If $k \notin T_i$, then $\forall j \in T_i \ |a_j - a_k| \geq \mu/p$.
- (3) If $j \in T_i$, then $T_i = T_j$.

In T_i , we order the λ_i 's: $\lambda_{i_0} \leq \lambda_{i_1} \leq \lambda_{i_2} \leq \dots \leq \lambda_{i_s}$. We have the following lemma.

Lemma 4.2. *If $T_i \neq \{i\}$, then there exists a vector field Y_i such that*

$$(-\partial J(u), Y_i) \geq c \sum_{1 \leq k \leq s, j \neq i_k} \varepsilon_{ikj} + o\left(\sum_{k \neq r} \varepsilon_{kr}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j}\right).$$

Proof. We define the following vector field Y_i

$$Y_i = - \sum_{k=1}^s 2^{k-1} \alpha_{i_k} \lambda_{i_k} \frac{\partial P \delta_{i_k}}{\partial \lambda_{i_k}}.$$

Using Proposition 3.3, we get

$$(-\partial J(u), Y_i) \geq c \sum_{1 \leq k \leq s, j \neq i_k} \varepsilon_{ikj} + O\left(\sum_{k=1}^s \frac{1}{\lambda_{i_k}}\right) + o\left(\sum_{k \neq r} \varepsilon_{kr}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j}\right).$$

Observe that, $\forall 1 \leq k \leq s$, we have $\frac{1}{\lambda_{i_k}} = o(\varepsilon_{i_k i_0})$. Indeed, since $|a_{i_k} - a_{i_0}| < \mu/p$, we have

$$\frac{1}{\lambda_{i_k}^2 \varepsilon_{i_k i_0}^2} = \frac{\lambda_{i_0}}{\lambda_{i_k}^3} + \frac{1}{\lambda_{i_k} \lambda_{i_0}} + \frac{\lambda_{i_0}}{\lambda_{i_k}} |a_{i_k} - a_{i_0}|^2 = o(1).$$

The Lemma follows. $\square$

We define the vector field $X_2 = \sum_i Y_i$. Following Lemma 4.2, we have

$$(-\partial J(u), X_2) \geq \sum_{i, j \in B, |a_i - a_j| < \mu/p} \varepsilon_{ij} + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j}\right). \quad (4.12)$$

Now, we define the vector $X_3 = M_1 X_2 + X_1$, where M_1 is a large positive constant. By (4.11) and (4.12), we obtain

$$(-\partial J(u), X_3) \geq c \sum_{i \in B} \frac{1}{\lambda_i} + \sum_{i, j \in B, |a_i - a_j| < \mu/p} \varepsilon_{ij} + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j}\right). \quad (4.13)$$

In B , we order the λ_i 's : $\lambda_{r_1} \leq \lambda_{r_2} \leq \dots \leq \lambda_{r_h}$. We define the following vector field

$$X_4 = - \sum_{k=1}^h 2^{k-1} \alpha_{r_k} \lambda_{r_k} \frac{\partial P \delta_{r_k}}{\partial \lambda_{r_k}}.$$

Using Proposition 3.4, we get

$$(-\partial J(u), X_4) \geq c \sum_{i \in B, j \neq i} \varepsilon_{ij} + O\left(\sum_{i \in B} \frac{1}{\lambda_i}\right) + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j}\right). \quad (4.14)$$

We define the vector $X_5 = M_2 X_3 + X_4$, where M_2 is a large positive constant. By (4.13) and (4.14), we obtain

$$(-\partial J(u), X_5) \geq c \sum_{i \in B, j \neq i} \varepsilon_{ij} + \sum_{i \in B} \frac{1}{\lambda_i} + o \left(\sum_{j \neq i} \varepsilon_{ij} \right) + o \left(\sum_{j=1}^p \frac{1}{\lambda_j} \right). \quad (4.15)$$

It remains to make appear all the term $\frac{1}{\lambda_i}$ for $i \notin B$ and ε_{ij} for $i, j \notin B$. Let

$$u_1 = \sum_{i \notin B} \alpha_i P \delta_i.$$

Observe that $u_1 \in V_1(p - \text{card}(B), \varepsilon, \eta)$. We then use the previous constructions in section 4.3 to u_1 , which means the previous constructions to the subpack of functions $u = \sum_{i=1}^{p - \text{card}(B)} \alpha_i P \delta_i$. We define $X_6 = W_1(u_1)$. We have

$$(-\partial J(u), X_6) \geq c \sum_{i \notin B} \left(\frac{1}{\lambda_i} + \sum_{j \notin B, j \neq i} \varepsilon_{ij} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + O \left(\sum_{k \notin B, r \in B} \varepsilon_{kr} \right). \quad (4.16)$$

Thus, we can define $W_2 = M_3 X_5 + X_6$, where M_3 is a large positive constant.

Finally, we use a convex combination of W_1 and W_2 to build our global vector field W in $V(p, \varepsilon, \eta)$ and the estimate (i) follows.

We can obtain the estimate (ii), arguing as in Appendix B of [8], using the estimates (i), $\|\partial J(u + \bar{v})\|$ and $\|\bar{v}\|$. Hence our theorem follows. $\square$

Proposition 4.3. *Under assumptions (H_1) , (H_2) and (H_3) , the only critical points at infinity in $V(p, \varepsilon, \eta)$ correspond to $\sum_{i=1}^p K(y_{j_i})^{-1/4} P \delta_{(y_{j_i}, \infty)}$, where $p \in \mathbb{N}^*$ and the points y_{j_i} are critical points of K satisfying $M(y_{j_1}, \dots, y_{j_p})$ is positive definite. Such a critical point at infinity has a morse index equal to $4p - 1 - \sum_{i=1}^p k_{j_i}$, where $k_{j_i} = \text{index}(K, y_{j_i})$.*

Proof. From Theorem 1.1, we derive that we just need to see what happens in $V(p, \varepsilon, \eta)$ for $p \geq 1$. From Theorem 4.1, we deduce that the only region where the λ_i 's are not bounded is when each a_j is near critical point y_{j_i} with $j \neq k$ and the matrix $M(\tau_p)$ is positive definite. In this region, arguing as in [3, 8], under assumptions (H_1) and (H_2) , let $\varepsilon > 0$ sufficiently small be given; for each $u = \sum_{i=1}^p \alpha_i P \delta_{(a_i, \lambda_i)} + v$ in $V(p, \varepsilon, \eta)$, after a change of variable $v \rightarrow V$, where V belongs to a neighborhood of zero in a fixed Hilbert space, one can write

$$J \left(\sum_{i=1}^p \alpha_i P \delta_{(a_i, \lambda_i)} + \bar{v} \right) = J \left(\sum_{i=1}^p \alpha_i P \delta_{(a_i, \lambda_i)} \right) + \bar{v} + |V|^2.$$

Furthermore, if each a_i belongs to a neighborhood of a critical point y_{j_i} such that $j_i \neq j_k$ for $i \neq k$ and $M(Y)$ is positive definite, where $Y = (y_{j_1}, \dots, y_{j_p})$, then we can find another change of variables $(a_i, \lambda_i) \rightarrow (\tilde{a}_i, \tilde{\lambda}_i)$ such that, for β a fixed small constant

$$J\left(\sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + \bar{v}\right) = \frac{\sum \alpha_i^2 S^{2/3}}{(\sum \alpha_i^6 K(\bar{a}_i))^{1/3}} (1 + (c - \beta)\Lambda^t M(\tau_p)\Lambda),$$

where ${}^T\Lambda = (1/\sqrt{\tilde{\lambda}_1}, \dots, 1/\sqrt{\tilde{\lambda}_p})$. The index of such a critical point at infinity is the same of the critical point of

$$g(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p) := \frac{\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^6 K(a_i))^{1/3}}.$$

In the variable α_i 's we have a nondegenerated critical point $(\bar{\alpha}_1, \dots, \bar{\alpha}_p)$ which satisfies

$$\frac{\bar{\alpha}_i^4 K(a_i)}{\bar{\alpha}_j^4 K(a_j)} = 1.$$

This critical point has a index equal to $p-1$ (since the function g is homogenous in the variables α_i and the critical point corresponds to a maximum), and we have

$$g(\bar{\alpha}_1, \dots, \bar{\alpha}_p, a_1, \dots, a_p) = \left(\sum_{i=1}^p \frac{1}{K(a_i)}\right)^{1/3} (1 - |X|^2),$$

where X belongs to $\mathbb{R}^{p-1}$. Thus the index is equal to

$$p-1 + \sum_{i=1}^p (3 - \text{index}(K, y_{j_i})) = 4p-1 - \sum_{i=1}^p k_{j_i}.$$

□

5. Proof of Theorem 1.4

We argue by contradiction. We assume that (P) has no solution. First, we will define a vector field X in Σ by a convex combination of $-\partial J$ (defined outside of $\cup_p V(p, \varepsilon/2, \eta)$) and the vector field W (defined on each $V(p, \varepsilon, \eta)$ by Theorem 4.1). Observe that the set Σ^+ is not invariant under the flow generated by X . For this effect, we introduce, for $\varepsilon_0 > 0$ small enough, the set $V_{\varepsilon_0}(\Sigma^+) = \{u \in \Sigma/J(u)^4 | u^-|_{L^6}^4 < \varepsilon_0\}$ where $u^- = \max(0, -u)$. Using Lemma 4.1 of [8], we have $V_{\varepsilon_0}(\Sigma^+)$ is invariant under the flow generated by X . For $\tau_s = (i_1, \dots, i_s)$ such that $M(\tau_s)$ is positive definite, using Theorem 4.1, we have $\sum_{i=1}^p K(y_{j_i})^{-1/4} P\delta_{(y_{j_i}, \infty)}$ is a critical point at infinity of index $4s-1 - \sum_{j=1}^s k_{i_j}$, where $k_{i_j} = \text{index}(K, y_{i_j})$. Let $c(\tau_s)$ be the critical value associated to this critical point. (For simplicity, we will assume that all the critical values are different). Let χ be the Euler–Poincaré characteristic. Thus for $\theta > 0$ small enough (we choose θ so that the only critical value between $c(\tau_s) + \theta$ and $c(\tau_s) - \theta$ is $c(\tau_s)$), we have

$$\chi(J_{c(\tau_s)+\theta}) = \chi(J_{c(\tau_s)-\theta}) + (-1)^{4s-1-\sum_{j=1}^s k_{i_j}}.$$

Since, $V_{\varepsilon_0}(\Sigma^+)$ is contractible, $\chi(V_{\varepsilon_0}(\Sigma^+)) = 1$. Thus

$$\chi(V_{\varepsilon_0}(\Sigma^+)) = \sum_{s=1}^{m_1} \sum_{\tau_s=(i_1, \dots, i_s)/M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^s k_{i_j}}$$

which is a contradiction with the assumption of the theorem. Therefore, (P) has a solution $u_0 \in V_{\varepsilon_0}(\Sigma^+)$.

Proposition 5.1. *The solution u_0 found by the Theorem 1.4 is positive.*

Proof. We can write $u_0 = u_0^+ - u_0^-$. By definition of $V_{\varepsilon_0}(\Sigma^+)$ we have $J(u)^4 |u^-|_{L^6}^4 < \varepsilon_0$. Multiplying the equation by the test function u_0^- and integrating, we obtain

$$|\nabla u_0^-|_{L^2} \leq c |u_0^-|_{L^2}^3 \leq c' |\nabla u_0^-|_{L^2}^3.$$

Hence, either $u_0^- = 0$, or $|\nabla u_0^-|_{L^2} \geq c_0$ for some constant $c_0 > 0$, a contradiction with ε_0 is small enough. Thus, $u_0^- = 0$. Therefore, $u_0 > 0$. $\square$

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Rabeh Ghouli
Faculté des Sciences
Université de Gabès
Cité El Riadh
Gabès, Tunisia
e-mail: ghouli.rabeh@yahoo.fr

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