Intersection properties of radial solutions and global bifurcation diagrams for supercritical quasilinear elliptic equations

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Abstract. We study the positive solution $u(r, \rho)$ of the quasilinear elliptic equation

$$\begin{cases} r^{-(\gamma-1)}(r^\alpha |u'|^{\beta-1} u')' + |u|^{p-1} u = 0, & 0 < r < \infty, \\ u(0) = \rho > 0, \quad u'(0) = 0. \end{cases}$$

This class of differential operators includes the usual Laplace, m -Laplace, and k -Hessian operators in the space of radial functions. The equation has a singular positive solution $u^*(r)$ under certain conditions on α , β , γ , and p . A generalized Joseph–Lundgren exponent, which we denote by p_{JL}^* , is obtained. We study the intersection numbers between $u(r, \rho)$ and $u^*(r)$ and between $u(r, \rho_0)$ and $u(r, \rho_1)$, and see that p_{JL}^* plays an important role. We also determine the bifurcation diagram of the problem

$$\begin{cases} r^{-(\gamma-1)}(r^\alpha |u'|^{\beta-1} u')' + \lambda(u+1)^p = 0, & 0 < r < 1, \\ u(r) > 0, & 0 \leq r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

The main technique used in the proofs is a phase plane analysis.

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1. Introduction and main results

Let $\mathbb{R}_+ := \{r \in \mathbb{R}; r \geq 0\}$, and let $u_0(r), u_1(r) \in C^0(\mathbb{R}_+)$. We denote the number of the zeros of the function $u_0(r)$ defined on $\mathbb{R}_+$ by $\mathcal{Z}[u_0(\cdot)] := \#\{r \in$

$\mathbb{R}_+$; $u_0(r) = 0\}$. We define the intersection number of the two functions $u_0(r)$ and $u_1(r)$ on $\mathbb{R}_+$ by

$$\mathcal{Z}[u_0(\cdot) - u_1(\cdot)] = \#\{r \in \mathbb{R}_+; u_0(r) - u_1(r) = 0\}.$$

We recall intersection properties of the case of the semilinear elliptic equation

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \quad (1.1)$$

There are two important exponents for the structure of solutions of (1.1):

$$p_S := \begin{cases} \frac{N+2}{N-2}, & \text{if } N \geq 3, \\ \infty, & \text{if } N = 1, 2, \end{cases} \quad (1.2)$$

$$p_{JL} := \begin{cases} 1 + \frac{4}{N-4-2\sqrt{N-1}}, & \text{if } N \geq 11, \\ \infty, & \text{if } 2 \leq N \leq 10. \end{cases} \quad (1.3)$$

The exponent p_{JL} appears only when $N \geq 11$. We see that $p_{JL} > p_S$ if $N \geq 11$. As we see in Proposition 1.1 below, p_{JL} plays an important role in our study. We consider the critical and supercritical cases, i.e., $p \geq p_S$. The Eq. (1.1) has a one-parameter family of radial solutions $u(r, \rho)$, where u satisfies

$$\begin{cases} r^{-(N-1)}(r^{N-1}u')' + |u|^{p-1}u = 0, & 0 < r < \infty, \\ u(0) = \rho > 0, \quad u'(\infty) = 0. \end{cases} \quad (1.4)$$

It is known that $u(r, \rho) > 0$ for $r \geq 0$ and $u(r, \rho) \rightarrow 0$ as $r \rightarrow \infty$. The Eq. (1.1) also has the singular radial solution

$$u^*(r) := A_0 r^{-\frac{2}{p-1}}, \quad \text{where} \quad A_0^{p-1} := \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \quad (1.5)$$

if $p > N/(N-2)$. We are interested in the intersection numbers of two solutions, i.e.,

$$\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] \quad (\rho_0 \neq \rho_1) \quad \text{and} \quad \mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)].$$

Proposition 1.1. *Assume that $N \geq 3$. Let $\rho_0 > 0$, $\rho_1 > 0$, ($\rho_0 \neq \rho_1$). Let $u(r, \rho_j)$, $j = 0, 1$, be solutions of (1.4), and let $u^*(r)$ be the singular solution defined by (1.5).*

- (i) *If $p \geq p_{JL}$, then $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = 0$ and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 0$.*
- (ii) *If $p_S < p < p_{JL}$, then $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = \infty$ and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = \infty$.*
- (iii) *If $p = p_S$, then $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = 1$ and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 2$.*

Note that if $p \geq p_{JL}$, then (i) indicates that if $\rho_0 < \rho_1$, then $u(r, \rho_0) < u(r, \rho_1)$ for $r \geq 0$, and hence the intersection number can be seen as a generalization of the separation property. In Joseph–Lundgren [22] they mainly studied the asymptotic behavior of $u(\cdot, \rho_0)$, using a phase plane analysis. The exponent (1.3) firstly appeared in [22]. The phase plane diagram in [22] shows that $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 0$ ($p_{JL} \leq p$), $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = \infty$ ($p_S < p < p_{JL}$), and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 2$ ($p = p_S$). However, in [22] the statement was

not written and the rigorous proof was not given. According to [18, Proposition 2.16], Proposition 1.1 was firstly stated and proved by Wang [34]. In [34, Propositions 3.5 and 3.7] he studied intersection numbers not only between regular and singular solutions but also between two regular solutions.

Intersection properties, which includes the separation property, play a key role in studies of (1.1) and the parabolic problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.6)$$

in the critical and supercritical cases $p \geq p_S$. These properties were firstly used for the study of the asymptotic behavior of the solution of (1.6) and of the blow-up criteria in [34]. Gui et al. [18] studied the stability and instability of the stationary solution of (1.6), using these properties. These properties were used for the study of various behaviors of solutions of (1.6) in [11–13, 19, 31, 32]. Later, blow-up phenomena of solutions of (1.6) were studied in [24–26, 28–30]. Let $\varepsilon > 0$, and let $g(t) \in C(\mathbb{R})$ be a function such that $g(t) = O(t^{p-\varepsilon})$ as $t \rightarrow \infty$. If $\tilde{u}(s, \rho)$ is the radial solution of

$$\begin{cases} \Delta \tilde{u} + \tilde{u}^p + g(\tilde{u}) = 0 & \text{in } \mathbb{R}^N, \\ \tilde{u}(0) = \rho, \tilde{u}'(0) = 0, \end{cases}$$

then $\bar{u}(r) := \rho^{-1}\tilde{u}(s)$ ($s := \rho^{-\frac{p-1}{2}}r$) satisfies $\Delta \bar{u} + \bar{u}^p + \rho^{-p}g(\rho\bar{u}) = 0$ and $\bar{u}(0) = 1$. Because of the assumption on g , $\rho^{-p}g(\rho\bar{u}) \rightarrow 0$ pointwisely converges to 0 as $\rho \rightarrow \infty$, and hence $\bar{u}(r)$ converges to the solution $u(r, 1)$ of (1.4) in an appropriate sense. This indicates that intersection properties can be used for large solutions of elliptic equations with rather general nonlinearities. These properties were used for the study of the structure of the positive solutions of supercritical semilinear elliptic equations in [20, 22, 27].

In this paper we study intersection numbers of two solutions of the quasilinear elliptic equation

$$\begin{cases} r^{-(\gamma-1)}(r^\alpha|u'|^{\beta-1}u')' + |u|^{p-1}u = 0, & 0 < r < \infty, \\ u(0) = \rho > 0, u'(0) = 0, \end{cases} \quad (1.7)$$

under certain conditions on α , β , γ , and p . We will see in Sect. 2 that (1.7) has the unique solution $u(r, \rho)$ and that $u(r, \rho) > 0$ for $r \geq 0$. The first term of the equation in (1.7) includes the usual Laplace, m -Laplace, and k -Hessian operators as Table 1 shows. Therefore, this differential operator enables us to treat these operators in a unified way.

We define q by

$$q := \frac{-\alpha + \beta + \gamma}{p - \beta}. \quad (1.8)$$

If

$$\gamma > \alpha > \beta > 0 \quad (1.9)$$

TABLE 1. Relations between operators and parameters

Operator	α	β	γ
Laplacian	$N - 1$	1	N
m -Laplacian ($m > 1$)	$N - 1$	$m - 1$	N
k -Hessian ($k \in \{1, 2, \dots, N\}$)	$N - k$	k	N

This table is taken from Jacobsen–Schmitt [21, p. 287]

and

$$p > \beta + \frac{\beta(-\alpha + \beta + \gamma)}{\alpha - \beta}, \quad (1.10)$$

then $-\alpha + \beta + \gamma > 0$, $p - \beta > 0$, and $\alpha - \beta(q + 1) > 0$. Since $q > 0$ and $\alpha - \beta(q + 1) > 0$, the equation in (1.7) has the singular solution

$$u^*(r) := Ar^{-q}, \quad \text{where} \quad A^{p-\beta} := q^\beta \{\alpha - \beta(q + 1)\}. \quad (1.11)$$

Hereafter we always assume (1.9) and (1.10). Therefore, the singular solution (1.11) exists. In order to state the main results we need two exponents

$$p^*(\alpha, \beta, \gamma) := \beta + \frac{2\beta(-\alpha + \beta + \gamma)}{\alpha - \beta} \quad (1.12)$$

and

$$p_{JL}^*(\alpha, \beta, \gamma) := \begin{cases} \beta + \frac{(\beta+1)^2(-\alpha+\beta+\gamma)}{(\beta+3)(\alpha-\beta)-2\gamma-2\sqrt{\frac{(-\alpha+\beta+\gamma)(\alpha+\beta\gamma-\beta)}{\beta}}}, & \text{if } (\alpha - \beta)(\beta + 4) > 4\gamma, \\ \infty, & \text{if } (\alpha - \beta)(\beta + 4) \leq 4\gamma. \end{cases} \quad (1.13)$$

The exponent $p_{JL}^*(\alpha, \beta, \gamma)$ is a generalization of p_{JL} . Indeed, $p_{JL}^*(N-1, 1, N) = p_{JL}$. On the other hand, p^* is not corresponding to p_S in spite that $p^*(N-1, 1, N) = p_S$. See Remarks 1.2, 1.3, 1.6, and 1.8 below.

If α , β , and γ satisfy (1.9) and $p \geq p^*$, then (1.10) holds, and hence the singular solution (1.11) exists. The first main result is the following:

Theorem A. *Let p^* and p_{JL}^* be numbers defined by (1.12) and (1.13), respectively. Assume that (1.9) holds and that*

$$p \geq p^*, \beta \geq 1, \quad \text{and} \quad (p, \beta) \neq (p^*, 1). \quad (1.14)$$

Let $\rho_0 > 0$, $\rho_1 > 0$, ($\rho_0 \neq \rho_1$). Let $u(r, \rho_j)$, $j = 0, 1$, be solutions of (1.7) with $\rho = \rho_j$. Then, $u(r, \rho_0) > 0$ for $r \geq 0$ and the followings hold:

- (i) *If $p \geq p_{JL}^*$, then $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = 0$ and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 0$.*
- (ii) *If $p^* \leq p < p_{JL}^*$, then $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = \infty$ and $\mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = \infty$.*

The inequality $p^* < p_{JL}^*$ is not necessarily valid. If $p^* \geq p_{JL}^*$, then the assumption of (ii) is not satisfied. When $(\alpha, \beta, \gamma) = (N-1, 1, N)$, Theorem A recovers Proposition 1.1 (i) and (ii). When $(\alpha, \beta, \gamma) = (N-1, 1, N+\delta)$ ($\delta > 0$),

(1.7) gives the radial solution of the Hénon equation $\Delta u + r^\delta u = 0$. In this case Theorem A recovers [34, Proposition 3.7 (iii)(iv)].

Remark 1.2. In [7, (1.1)] the critical exponent

$$p_S^*(\alpha, \beta, \gamma) := \beta + \frac{(\beta + 1)(-\alpha + \beta + \gamma)}{\alpha - \beta} \quad (1.15)$$

was defined provided that (1.9) holds. Note that $p_S^* < p_{JL}^*$. If $p \leq p_S^*$, then

$$\left(\int_0^1 r^{\gamma-1} |u(r)|^{p+1} dr \right)^{\frac{1}{p+1}} < C \left(\int_0^1 r^\alpha |u'(r)|^{\beta+1} dr \right)^{\frac{1}{\beta+1}} \quad (1.16)$$

holds for a certain class of functions such that $u(1) = 0$. If $p < p_S^*$, then the embedding associated to (1.16) is compact, and hence p_S^* is corresponding to p_S . When $p < p_S^*$, using this compactness, one can show that the solution $u(r, \rho)$ of (1.7) has the first positive zero, i.e., there is $r_*(\rho) > 0$ such that $u(r, \rho) > 0$ for $0 < r < r_*(\rho)$ and $u(r_*(\rho), \rho) = 0$. (See [7, Remark 4.1] for details.) If $\rho_1 > \rho_0 > 0$, then $r_*(\rho_0) > r_*(\rho_1) > 0$ and a simple zero number argument shows that

$$\#\{r \in (0, r_*(\rho_1)); u(r, \rho_0) = u(r, \rho_1)\} = 1.$$

Remark 1.3. If $\beta > 1$ and (1.9) holds, then

$$p^* - p_S^* = \frac{(\beta - 1)(-\alpha + \beta + \gamma)}{\alpha - \beta} > 0,$$

and hence there is a gap $p_S^* < p < p^*$ in the supercritical region $p > p_S^*$ where intersection properties are not clarified by Theorem A. See Remark 2.3 for the reason of the assumption $p \geq p^*$.

Remark 1.4. When $p = p_S^*$, the solution $u(r, \rho)$ of (1.7) can be written explicitly as

$$u(r, \rho) = \rho \left(1 + c(\rho^{\frac{\beta+1}{\alpha-\beta}} r)^{-\frac{\alpha+\beta+\gamma}{\beta}} \right)^{-\frac{\alpha-\beta}{-\alpha+\beta+\gamma}} \quad \text{where} \quad c := \gamma^{-\frac{1}{\beta}} \frac{\beta}{\alpha - \beta}. \quad (1.17)$$

See [7, Section 1] for details. If $\beta = 1$, then $p^* = p_S^*$. Therefore, the case $(p, \beta) = (p^*, 1)$, which is an exception in (1.14), is included in the case $p = p_S^*$. If $p = p_S^*$ and (1.9) holds, then by direct calculation we see that

$$\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = 1 \quad \text{and} \quad \mathcal{Z}[u(\cdot, \rho_0) - u^*(\cdot)] = 2. \quad (1.18)$$

These equalities hold for $\beta > 0$, and the assumption $\beta \geq 1$, which is assumed in Theorem A, is not needed. If $(\alpha, \beta, \gamma) = (N-1, 1, N)$, then (1.18) is Proposition 1.1 (iii).

See Table 2 for the summary of intersection numbers of two solutions of the equation in (1.7).

We consider the case of the m -Laplacian. The m -Laplace operator, which is denoted by Δ_m , is defined by $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2} \nabla u)$. When u is radial, $\Delta_m u = r^{-(N-1)}(r^{N-1}|u'|^{m-2}u')'$ and the radial solution of the problem

$$\Delta_m u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N$$

TABLE 2. Intersection numbers described in Theorem A and Remark 1.4

Intersection number	$p = p_S^*, \beta > 0$	$p^* < p < p_{JL}^*, \beta \geq 1$ or $p = p^* < p_{JL}^*, \beta > 1$	$p_{JL}^* \leq p^* < p, \beta \geq 1,$ $p_{JL}^* \leq p^* = p, \beta > 1,$ or $p^* < p_{JL}^* \leq p, \beta \geq 1$
$\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)]$	1	∞	0
$\mathcal{Z}[u(\cdot, \rho) - u^*(\cdot)]$	2	∞	0

satisfies the equation in (1.7) with $(\alpha, \beta, \gamma) = (N - 1, m - 1, N)$. Note that if $m = 2$, then Δ_2 becomes the usual Laplacian Δ . Because of (1.9), $1 < m < N$. The condition (1.10) is equivalent to the condition $p > (m - 1)N/(N - m)$. In this case the singular solution (1.11) becomes

$$u^*(r) = Ar^{-q}, \quad \text{where } q = \frac{m}{p - m + 1} \quad \text{and} \\ A^{p-m+1} = q^{m-1} \{N - 1 - (q + 1)(m - 1)\}.$$

Two exponents become

$$p^* = m - 1 + \frac{2m(m - 1)}{N - m}, \quad (1.19)$$

$$p_{JL}^* = \begin{cases} m - 1 + \frac{m^2}{N - m - 2 - 2\sqrt{\frac{N-1}{m-1}}}, & \text{if } N > \frac{m(m+3)}{m-1}, \\ \infty, & \text{if } N \leq \frac{m(m+3)}{m-1}. \end{cases} \quad (1.20)$$

We see by direct calculation that if $1 < m \neq 2$, then

$$\begin{cases} p_{JL}^* \leq p^*, & \text{if } N \geq m \frac{m^2 + 4m - 4}{(m-2)^2} \quad \text{and if } N > \frac{m(m+3)}{m-1}, \\ p_{JL}^* > p^*, & \text{if } N < m \frac{m^2 + 4m - 4}{(m-2)^2} \quad \text{or if } N \leq \frac{m(m+3)}{m-1}. \end{cases}$$

Note that

$$\begin{cases} \frac{m(m^2 + 4m - 4)}{(m-2)^2} > \frac{m(m+3)}{m-1}, & \text{if } m > \sqrt{2}, \\ \frac{m(m^2 + 4m - 4)}{(m-2)^2} \leq \frac{m(m+3)}{m-1}, & \text{if } 1 < m \leq \sqrt{2}. \end{cases} \quad (1.21)$$

Theorem A becomes as follows:

Corollary 1.5. (m -Laplacian) *Let p^* and p_{JL}^* be numbers defined by (1.19) and (1.20), respectively. Assume that $N \geq 3$ and $2 < m < N$.*

- (i) *If $N \geq m(m^2 + 4m - 4)/(m - 2)^2$, then $p_{JL}^* \leq p^*$ and the following holds:
If $p \geq p^*$, then the conclusions of (i) of Theorem A hold.*
- (ii) *If $N < m(m^2 + 4m - 4)/(m - 2)^2$, then $p_{JL}^* > p^*$ and the following holds:
If $p \geq p_{JL}^*$ (resp. $p^* \leq p < p_{JL}^*$), then the conclusions of (i) (resp. (ii)) of Theorem A hold.*

When $m = 2$, the m -Laplace operator is the Laplacian. We define $m(m^2 + 4m - 4)/(m - 2)^2 = \infty$ if $m = 2$. Then Corollary 1.5 (ii) corresponds to Proposition 1.1 (i) and (ii). The exponent (1.20) was found in the study of an L^∞ bound of a stable solution in Cabré-Sanchón [6].

Remark 1.6. Since $p_S^*(N - 1, m - 1, N) = (Nm - N + m)/(N - m)$, p_S^* is the critical exponent of the embedding $W^{1,m}(\mathbb{R}^N) \hookrightarrow L^{p_S^*+1}(\mathbb{R}^N)$.

We consider the case of the k -Hessian operator. Let $\{\lambda_j\}_{j=1}^N$ be the set of the eigenvalues of the Hessian matrix D^2u . Let $k \in \{1, 2, \dots, N\}$. The k -Hessian operator is defined by

$$S_k(D^2u) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k}.$$

For instance $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$, the Monge-Ampère operator. When u is radial, $\{u'', \overbrace{u'/r, \dots, u'/r}^{N-1}\}$ is the set of the eigenvalues of D^2u . Therefore,

$$\begin{aligned} S_k(D^2u) &= \binom{N-1}{k-1} u'' \left(\frac{u'}{r}\right)^{k-1} + \binom{N-1}{k} \left(\frac{u'}{r}\right)^k \\ &= \frac{1}{k} \binom{N-1}{k-1} r^{-(N-1)} (r^{N-k} (u')^k)'. \end{aligned}$$

We consider the case where $u'(r) < 0$ for $r > 0$. Then, $|u'| = -u'$. Therefore,

$$\begin{aligned} r^{-(N-1)} (r^{N-k} |u'|^{k-1} u')' &= c_{N,k} S_k(D^2u), \quad \text{where} \\ c_{N,k} &:= (-1)^{k-1} k \binom{N-1}{k-1}^{-1}. \end{aligned}$$

The radial solution of the problem

$$c_{N,k} S_k(D^2u) + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^N$$

satisfies the equation in (1.7) with $(\alpha, \beta, \gamma) = (N-k, k, N)$. Because of (1.9), $0 < 2k < N$. The condition (1.10) is equivalent to the condition $p > Nk/(N-2k)$. In this case the singular solution (1.11) becomes

$$u^*(r) = Ar^{-q}, \quad \text{where } q = \frac{2k}{p-k} \quad \text{and} \quad A^{p-k} = q^k \{N-k-(q+1)k\}.$$

Two exponents become

$$p^* = k + \frac{4k^2}{N-2k}, \tag{1.22}$$

$$p_{JL}^* = \begin{cases} k + \frac{2k(k+1)^2}{(k+1)N-2k(k+3)-2\sqrt{2\{(k+1)N-2k\}}}, & \text{if } N > 2k+8, \\ \infty, & \text{if } N \leq 2k+8. \end{cases} \tag{1.23}$$

We see by direct calculation that if $2 < 2k < N$, then

$$\begin{cases} p_{JL}^* \leq p^*, & \text{if } N \geq 2k \frac{k^2+6k+1}{(k-1)^2}, \\ p_{JL}^* > p^*, & \text{if } N < 2k \frac{k^2+6k+1}{(k-1)^2}. \end{cases}$$

Note that $2k(k^2+6k+1)/(k-1)^2 > 2k+8$ if $k > 1$. Theorem A becomes as follows:

Corollary 1.7. (k -Hessian) *Let p^* and p_{JL}^* be numbers defined by (1.22) and (1.23), respectively. Assume that $N \geq 5$, $2 < 2k < N$, and $k \in \mathbb{N}$.*

- (i) If $N \geq 2k(k^2 + 6k + 1)/(k - 1)^2$, then $p_{JL}^* \leq p^*$ and the following holds:
 If $p \geq p^*$, then the conclusions of (i) of Theorem A hold.
- (ii) If $N < 2k(k^2 + 6k + 1)/(k - 1)^2$, then $p_{JL}^* > p^*$ and the following holds:
 If $p \geq p_{JL}^*$ (resp. $p^* \leq p < p_{JL}^*$), then the conclusions of (i) (resp. (ii)) of Theorem A hold.

When $k = 1$, the k -Hessian operator is the Laplacian. We define $2k(k^2 + 6k + 1)/(k - 1)^2 = \infty$ if $k = 1$. Then Corollary 1.7 (ii) corresponds to Proposition 1.1 (i) and (ii). The exponent p_{JL}^* defined by (1.23) seems new.

Remark 1.8. Since $p_S^*(N - k, k, N) = k(N + 2)/(N - 2k)$, p_S^* is the critical exponent obtained in the study of the k -Hessian operator in Tso [33, Theorem].

Using the same technique used in the proof of Theorem A, we can determine the global bifurcation diagram of the quasilinear elliptic Dirichlet problem

$$\begin{cases} R^{-(\gamma-1)}(R^\alpha|U'|^{\beta-1}U')' + \lambda(U+1)^p = 0, & 0 < R < 1, \\ U > 0, & 0 \leq R < 1, \\ U(1) = 0, \end{cases} \quad (1.24)$$

where $U = U(R)$. Let B be a unit ball in $\mathbb{R}^N$. The bifurcation diagram of (1.24) gives that of the radial solutions of

$$\begin{cases} \Delta_m U + \lambda R^\delta(U+1)^p = 0 & \text{in } B, \\ U > 0 & \text{in } B, \\ U = 0 & \text{on } \partial B, \end{cases} \quad (1.25)$$

if $(\alpha, \beta, \gamma) = (N - 1, m - 1, N + \delta)$, and

$$\begin{cases} c_{N,k} S_k(D^2 U) + \lambda R^\delta(U+1)^p = 0 & \text{in } B, \\ U > 0 & \text{in } B, \\ U = 0 & \text{on } \partial B, \end{cases} \quad (1.26)$$

if $(\alpha, \beta, \gamma) = (N - k, k, N + \delta)$. Let $U(R, \tau)$ denote a solution of (1.24) such that $U(0, \tau) = \tau > 0$ and $U'(0) = 0$. We later see that $U'(R, \tau) < 0$ for $0 < R \leq 1$. Therefore, $\|U(\cdot, \tau)\|_\infty = \tau$. It is easy to see that the solution set of (1.24) can be described as a graph of τ , i.e., $\lambda = \lambda(\tau)$. Therefore, the shape of the graph $\lambda = \lambda(\tau)$ becomes important for the structure of the solutions. The second main result is the following:

Theorem B. Let $U(R, \tau)$ be the solution of (1.24) such that $U(0, \tau) = \tau$ and $U'(0, \tau) = 0$. Let p^* , p_{JL}^* , and p_S^* be numbers defined by (1.12), (1.13), and (1.15), respectively. Assume that (1.9) holds. Let

$$\lambda^* := A^{p-\beta} = q^\beta \{\alpha - \beta(q+1)\}.$$

Here, q and A are defined by (1.11) and (1.8), respectively. Then, $\lim_{\tau \downarrow 0} \lambda(\tau) = 0$ and the followings hold:

- (i) If $p \geq p_{JL}^*$ and (1.14) holds, then $\lambda(\tau)$ is monotonically increasing and $\lambda(\tau) \rightarrow \lambda^*$ ($\tau \rightarrow \infty$).

- (ii) If $p^* \leq p < p_{jL}^*$ and (1.14) holds, then $\lambda(\tau) > 0$ for $\tau > 0$, $\lambda(\tau) \rightarrow \lambda^*$ ($\tau \rightarrow \infty$), and $\lambda(\tau)$ oscillates around λ^* . Moreover, $\lambda(\tau_2) < \lambda(\tau_4) < \dots < \lambda^* < \dots < \lambda(\tau_3) < \lambda(\tau_1)$ and $\lim_{n \rightarrow \infty} \lambda(\tau_n) = \lambda^*$, where $\{\tau_j\}_{j=1}^\infty$, $\tau_1 < \tau_2 < \dots \rightarrow \infty$, is the set of the critical points of $\lambda(\tau)$. In particular, $\tau_1, \tau_3, \dots$ are the local maximum points and $\tau_2, \tau_4, \dots$ are the local minimum points.
- (iii) If $p = p_S^*$, then $\lambda(\tau) > 0$ for $\tau > 0$, $\lambda(\tau) \rightarrow 0$ ($\tau \rightarrow \infty$), and $\lambda(\tau)$ has exactly one critical point which is the maximum point. In particular, if $(p, \beta) = (p^*, 1)$, then $p = p^* = p_S^*$, and hence the assertions hold.

If Theorem B (i) occurs, then (1.24) with $U'(0) = 0$ has a unique solution for each $\lambda \in (0, \lambda^*)$. If Theorem B (ii) occurs, then (1.24) with $U'(0) = 0$ has infinitely many solutions for $\lambda = \lambda^*$. If Theorem B (iii) occurs, then (1.24) with $U'(0) = 0$ has exactly two solutions for each $\lambda \in (0, \bar{\lambda})$ and a unique solution for $\lambda = \bar{\lambda}$, where $\bar{\lambda} := (\alpha - \beta)^\beta (\beta + 1)^{-\beta-1} \gamma$. When $(\alpha, \beta, \gamma) = (N - 1, 1, N)$, Theorem B recovers a part of [22, Theorem 1 (iii) (iv) (v)]. In the case of the m -Laplacian Theorem B (i) essentially follows from [6, Proposition 1.3]. The subcritical case of (1.25) with $\delta = 0$ is studied in Ferrero [10].

Let us explain technical details. The operator $r^{-(\gamma-1)}(r^\alpha |u'|^{\beta-1} u')'$, which allows us to use a unified approach, was used in [7, 8, 21]. In [7] the Brezis–Nirenberg type problem for quasilinear elliptic equations including this operator was studied. In [21] Jacobsen and Schmitt determined the bifurcation diagram of the problem

$$\begin{cases} r^{-(\gamma-1)}(r^\alpha |u'|^{\beta-1} u')' + \lambda e^u = 0, & 0 < r < 1, \\ u(r) > 0, & 0 \leq r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases}$$

The proofs of Theorems A and B are essentially the same as Joseph–Lundgren [22]. Most calculations are elementary. However, they are somewhat complicated. The Eq. (1.7) can be transformed into the first order autonomous system (2.12) in Sect. 2 by the change of variables (2.5) and (2.6) which is a quasilinear version of that used in [9, 22]. This change of variables for the semilinear case appeared in Emden [9, p. 48] which was published in 1907, and it is called Emden’s transformation. In the quasilinear case more suitable change of variables was introduced by Bidaut–Véron in [5] and independently by Franca in [14] and in a more general context in [15]. We show that this system has a Lyapunov function and a heteroclinic orbit between the two equilibria $(0, 0)$ and $(1, 0)$. The equilibrium $(1, 0)$ is a stable node or a stable spiral. Using these properties, we prove Theorem A and Theorem B (i) and (ii). Under the assumptions of Theorem B (iii) the solution can be written explicitly. Then, $\lambda = \lambda(\tau)$ has an exact expression which leads to the conclusions of Theorem B (iii).

Intersection properties for the inhomogeneous semilinear elliptic equation $\Delta u + K(x)u^p = 0$ were studied in [1–4, 23] under various conditions on K . We refer the papers [16, 17] in which phase plane analysis are used for studies of

quasilinear elliptic equations. This work is motivated by Clément et al. [7], Gui et al. [18], Jacobsen–Schmitt [21], and Joseph–Lundgren [22].

This paper consists of three sections. In Sect. 2 we recall the existence and uniqueness of the solution of (1.7) and prove Theorem A. In Sect. 3 we prove Theorem B.

2. Intersection properties

In this section we always assume (1.9). Then (1.7) has the singular solution (1.11) provided that (1.10) holds.

We consider the initial value problem (1.7). First, we consider the existence and uniqueness of the solution near $r = 0$.

Proposition 2.1. *Assume that (1.9) holds and $p > 1$. Then the initial value problem (1.7) has the unique solution $u(r, \rho)$ near $r = 0$ and $u(\cdot, \rho) \in C^2((0, r_0)) \cap C^1([0, r_0])$ for small $r_0 > 0$.*

Proposition 2.1 is more or less known. However, we present a proof for the self-containedness of the paper.

Proof. We define

$$\mathcal{B} := \{u(r) \in C^0([0, r_0]); u(0) = \rho > 0, |u(r) - \rho| \leq m\},$$

$$\mathcal{F}(u) := \rho - \int_0^r \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u(s)|^{p-1} u(s) ds \right)^{\frac{1}{\beta}} dt, \quad (2.1)$$

where $r_0 > 0$ is defined later and $m := \rho/2$. If $u \in \mathcal{B}$ and if $u = \mathcal{F}(u)$, then $u(0) = \rho$. Moreover, $\mathcal{F}(u) \in C^2((0, r_0])$, and hence $u \in C^2((0, r_0])$. Differentiating $u = \mathcal{F}(u)$ with respect to r , we have $u'(r) = -\left(\frac{1}{r^\alpha} \int_0^r s^{\gamma-1} |u|^{p-1} u ds\right)^{\frac{1}{\beta}}$, and hence

$$|u'(r)|^\beta \leq \frac{1}{r^\alpha} \int_0^r s^{\gamma-1} (\rho + m)^p ds =: c_0 r^{-\alpha+\gamma}. \quad (2.2)$$

Because of (1.9), $\beta > 0$ and $-\alpha + \gamma > 0$. Hence, $\lim_{r \downarrow 0} u'(r) = 0$. Because of (2.1),

$$\begin{aligned} |\mathcal{F}(u) - \rho| &\leq \left| \int_0^r \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u(s)|^p ds \right)^{\frac{1}{\beta}} dt \right| \\ &\leq \left| \int_0^r \left\{ \frac{(\rho + m)^p}{\gamma} t^{-\alpha+\gamma} \right\}^{\frac{1}{\beta}} dt \right| \\ &= \left(\frac{(\rho + m)^p}{\gamma} \right)^{\frac{1}{\beta}} \frac{\beta}{-\alpha + \beta + \gamma} r^{-\frac{\alpha+\gamma}{\beta}+1} \\ &=: c_1 r^{-\frac{\alpha+\gamma}{\beta}+1}. \end{aligned} \quad (2.3)$$

Hence, $\lim_{r \downarrow 0} u(r) = \rho$. Thus, $u \in C^1([0, r_0])$. The function u is a solution of (1.7). We easily see that the converse is also true, i.e., if u is a solution of (1.7), then $u = \mathcal{F}(u)$ and $u \in \mathcal{B}$ for some small $r_0 > 0$.

We show that $\mathcal{F}$ has a unique fixed point in $\mathcal{B}$ in order to prove the existence and uniqueness of the solution of (1.7). Since $0 \leq u(r) \leq \rho + m$ ($0 \leq r \leq r_0$),

$$\rho - \int_0^r \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} (\rho + m)^p ds \right)^{\frac{1}{\beta}} dt \leq \mathcal{F}(u) \leq \rho.$$

Because of (2.3), we choose small $r_1 > 0$ such that $|\mathcal{F}(u) - \rho| \leq c_1 r_1^{\frac{-\alpha+\gamma}{\beta}+1} \leq m$. We see that $\mathcal{F}(\mathcal{B}) \subset \mathcal{B}$ provided that $r_0 \leq r_1$. Let $u_1, u_2 \in \mathcal{B}$. We show that there is $L \in (0, 1)$ such that $\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\| \leq L\|u_1 - u_2\|$. Here $\|\cdot\| := \|\cdot\|_{C^0([0, r_1])}$. Because of (1.9), $\beta > 0$. We divide the proof into two cases.

First, we consider the case $\beta \geq 0$. Using the inequalities

$$|\xi^{\frac{1}{\beta}} - \eta^{\frac{1}{\beta}}| \leq \frac{2}{\xi^{\frac{\beta-1}{\beta}} + \eta^{\frac{\beta-1}{\beta}}} |\xi - \eta| \quad \text{for } \xi, \eta > 0$$

and

$$\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u_j|^{p-1} u_j ds \geq \frac{1}{t^\alpha} \int_0^t s^{\gamma-1} (\rho - m)^p ds = c_2 t^{-\alpha+\gamma} \quad \text{for } j = 1, 2,$$

we have

$$\begin{aligned} & \left| \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u_1|^{p-1} u_1 ds \right)^{\frac{1}{\beta}} - \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u_2|^{p-1} u_2 ds \right)^{\frac{1}{\beta}} \right| \\ & \leq \frac{\frac{2}{t^\alpha} \int_0^t s^{\gamma-1} \left| |u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right| ds}{\left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u_1|^{p-1} u_1 ds \right)^{\frac{\beta-1}{\beta}} + \left(\frac{1}{t^\alpha} \int_0^t s^{\gamma-1} |u_2|^{p-1} u_2 ds \right)^{\frac{\beta-1}{\beta}}} \\ & \leq c_3 t^{(-\alpha+\gamma)\frac{-\beta+1}{\beta}} \frac{1}{t^\alpha} \int_0^t s^{\gamma-1} \left| |u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right| ds \\ & \leq c_4 t^{(-\alpha+\gamma)\frac{-\beta+1}{\beta}} \frac{1}{t^\alpha} \int_0^t s^{\gamma-1} ds \|u_1 - u_2\| \\ & \leq c_5 t^{\frac{-\alpha+\gamma}{\beta}} \|u_1 - u_2\|. \end{aligned} \tag{2.4}$$

Here we use $\left| |u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right| \leq c|u_1 - u_2|$ for $u_1, u_2 \in \mathcal{B}$, since $p > 1$. Using (2.4), we have

$$\begin{aligned} |\mathcal{F}(u_1) - \mathcal{F}(u_2)| & \leq \int_0^r c_5 t^{\frac{-\alpha+\gamma}{\beta}} \|u_1 - u_2\| dt \\ & \leq c_6 r^{\frac{-\alpha+\gamma}{\beta}+1} \|u_1 - u_2\| \quad \text{for } 0 \leq r \leq r_1. \end{aligned}$$

We choose small $r_2 > 0$ such that $c_6 r_2^{\frac{-\alpha+\gamma}{\beta}+1} < 1$. We define $r_0 := \min\{r_1, r_2\}$. Then $\mathcal{F}$ is a contraction mapping on $\mathcal{B}$. It follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point in $\mathcal{B}$. It is clear that all solutions of (1.7) near $r = 0$ are in $\mathcal{B}$ provided that $r_0 > 0$ is small. The assertion of the lemma holds in the case $\beta \geq 0$.

Second, we consider the case $0 < \beta < 1$. Using the inequality

$$|\xi^{\frac{1}{\beta}} - \eta^{\frac{1}{\beta}}| \leq \frac{1}{\beta} (\max\{\xi, \eta\})^{-\frac{\beta-1}{\beta}} |\xi - \eta| \quad \text{for } \xi, \eta > 0,$$

we can obtain

$$|\mathcal{F}(u_1) - \mathcal{F}(u_2)| \leq c_7 r^{-\frac{\alpha+\gamma}{\beta}+1} \|u_1 - u_2\|.$$

The same argument as above leads to the local existence and uniqueness of the solution of (1.7). The proof is complete. $\square$

We have constructed the solution for $\beta > 0$ in Proposition 2.1. However, we consider the case $\beta \geq 1$ hereafter.

The solution $u'(r)$ is actually Hölder continuous on $[0, r_0)$. Therefore, $u(r) \in C^2((0, r_0)) \cap C^{1,\mu}([0, r_0))$ for some $\mu \in (0, 1)$. See [7, Lemma 2.3].

Let $u(r)$ denote the solution obtained by Proposition 2.1. Let $u^*(r)$ be the singular solution defined by (1.11). We introduce the variable

$$v(r) := \frac{u(r)}{u^*(r)}. \quad (2.5)$$

This change of variables was also used in Joseph–Lundgren [22] and Emden [9]. The Eq. in (1.7) becomes

$$\begin{aligned} \left| \frac{rv'}{q} - v \right|^{\beta-1} [q^{\beta-1}\beta r^2 v'' + q^{\beta-1}(\alpha - 2\beta q)rv' \\ - q^\beta \{\alpha - \beta(q+1)\}v] + A^{p-\beta}|v|^{p-1}v = 0. \end{aligned}$$

We change the variables

$$x(t) := v(r) (= \frac{u(r)}{u^*(r)}) \quad \text{and} \quad t := \frac{1}{a} \log r, \quad (2.6)$$

where the constant a is defined later. Then,

$$\left| \frac{x'}{aq} - x \right|^{\beta-1} \left\{ x'' + \frac{\alpha - \beta - 2\beta q}{\beta} ax' - \frac{a^2 A^{p-\beta}}{q^{\beta-1}\beta} x \right\} + \frac{a^2 A^{p-\beta}}{q^{\beta-1}\beta} |x|^{p-1} x = 0. \quad (2.7)$$

We define

$$a := \sqrt{\frac{\beta}{q(\alpha - \beta - \beta q)}}. \quad (2.8)$$

Because of (1.10), $\alpha - \beta - \beta q > 0$, and hence $a \in \mathbb{R}$. Then, $a^2 A^{p-\beta} / (q^{\beta-1}\beta) = 1$, and (2.7) becomes

$$|\kappa x' - x|^{\beta-1} (x'' + \eta x' - x) + |x|^{p-1} x = 0, \quad (2.9)$$

where we define

$$\kappa := \frac{1}{aq} \quad \text{and} \quad \eta := \frac{\alpha - \beta - 2\beta q}{\beta} a. \quad (2.10)$$

The change of variables (2.5), (2.6), and (2.8) transforms the inhomogeneous Eq. in (1.7) into the homogeneous Eq. (2.9).

Hereafter, we assume that $\kappa x' - x < 0$ which is shown later. Let

$$y(t) := x'(t). \quad (2.11)$$

Then, by (2.9) we obtain the following autonomous system

$$\begin{cases} x' = y, \\ y' = x - \eta y - \frac{|x|^{p-1}x}{|\kappa y - x|^{\beta-1}}. \end{cases} \quad (2.12)$$

We define the “energy” of (2.12) by

$$I(x, y) := \frac{y^2}{2} - \frac{x^2}{2} + \frac{|x|^{p-\beta+2}}{p-\beta+2}. \quad (2.13)$$

The function I is similar to the Pohozaev function. In the proof of Lemma 2.2 below we see that if (1.14) holds, then $\frac{d}{dt}I(x(t), y(t)) < 0$ for the solution $(x(t), y(t))$ of (2.12), and hence $(x(t), y(t)) \in \{(x, y); I(x, y) < 0\}$.

Lemma 2.2. *Suppose that (1.9) and (1.14) hold. Let $u(r)$ be the solution given by Proposition 2.1, which is defined near $r = 0$. Then $u(r)$ is defined for all $r \geq 0$, $u(r) > 0$ for $r \geq 0$, and $u'(r) < 0$ for $r > 0$.*

In the m -Laplacian case the existence of the positive solution was already proved in [14] for $\beta < 1$ and in [16] for $\beta > 1$. However, we present a proof for readers' convenience.

Proof. Since $p \geq p^*$, $\beta \geq 1$, and $(p, \beta) \neq (p^*, 1)$, we see that $p > 1$. The assumptions of Proposition 2.1 are satisfied, and hence the solution $u(r)$ can actually be defined near $r = 0$.

Let

$$r^* := \sup\{r^* > 0; u(r) \text{ can be defined in } 0 < r < r^*\}.$$

We prove the lemma by contradiction. Suppose the contrary, i.e., $r^* < \infty$. We show by contradiction that $u(r) > 0$ for $0 < r < r^*$. We assume that there is $r_0 \in (0, r^*)$ such that $u(r) > 0$ for $0 < r < r_0$ and $u(r_0) = 0$. Let $x(t)$ and $y(t)$ be defined by (2.6) and (2.11), respectively. Let $I(x, y)$ be defined as (2.13). Since $(u(0), u'(0)) = (\gamma, 0)$,

$$\begin{aligned} \lim_{t \rightarrow -\infty} x(t) &= \lim_{r \downarrow 0} \frac{u(r)}{u^*(r)} = 0, \\ \lim_{t \rightarrow -\infty} y(t) &= \lim_{r \downarrow 0} ar \left(\frac{u'(r)}{u^*(r)} - \frac{u(r)(u^*(r))'}{(u^*(r))^2} \right) = 0. \end{aligned}$$

Thus, $(x(-\infty), y(-\infty)) = (0, 0)$. We have

$$\frac{d}{dt}I(x(t), y(t)) = -\eta y^2(t) - |x(t)|^{p-1}x(t)y(t) \left(\frac{1}{|\kappa y(t) - x(t)|^{\beta-1}} - \frac{1}{|x(t)|^{\beta-1}} \right). \quad (2.14)$$

Since $p \geq p^*$, $\eta \geq 0$. Since

$$r^\alpha |u'(r)|^{\beta-1} u'(r) = - \int_0^r s^{\gamma-1} |u(s)|^{p-1} u(s) ds < 0, \quad (2.15)$$

we have

$$\operatorname{sgn}(-u'(r)) = \operatorname{sgn}(x(t) - \kappa y(t)) > 0 \quad (2.16)$$

for $0 < r < r_0$, and hence

$$|\kappa y(t) - x(t)|^{\beta-1} = (x(t) - \kappa y(t))^{\beta-1}. \quad (2.17)$$

Because of (2.17), if $\beta > 1$, then

$$-|x(t)|^{p-1}x(t)y(t) \left(\frac{1}{|\kappa y(t) - x(t)|^{\beta-1}} - \frac{1}{|x(t)|^{\beta-1}} \right) \begin{cases} = 0, & \text{if } y(t) = 0, \\ < 0, & \text{if } y(t) \neq 0. \end{cases}$$

Therefore, if $p \geq p^*$ and $\beta \geq 1$, then $\frac{d}{dt}I(x(t), y(t)) \leq 0$ and $I(x(t), y(t)) \leq I(x(-\infty), y(-\infty)) = 0$. Let $t_0 := \frac{1}{a} \log r_0$. Since $(p, \beta) \neq (p^*, 1)$, $I(x(t_0), y(t_0)) < 0$, otherwise $\frac{d}{dt}I(x(t), y(t)) = 0$ and $y(t) = 0$ for all $-\infty < t < t_0$, which is a contradiction. Note that $\{(x, y); I(x, y) < 0, x \geq 0\} \cap \{x = 0\} = \emptyset$. Since $I(x(t_0), y(t_0)) < 0$, $x(t_0) (= u(r_0)/u^*(r_0)) > 0$. We obtain a contradiction, because we assumed that $u(r_0) (= x(t_0)u^*(r_0)) = 0$. Thus, $u'(r) < 0$ for $0 < r < r^*$. We see by (2.15) that $u'(r) < 0$ for $0 < r < r^*$. Since $u(r) > 0$ for $0 < r < r^*$ and $u'(r) < 0$ for $0 < r < r^*$, the limit $\lim_{r \uparrow r^*} u(r)$ exists. We define $u(r^*) := \lim_{r \uparrow r^*} u(r)$. Then, by (2.15) we see that $u'(r^*) < 0$. Because $u'(r^*) \neq 0$, there is $\varepsilon > 0$ such that we can define $u(r)$ for $0 < r < r^* + \varepsilon$ as a solution of (1.7). This contradicts the definition of r^* . Thus, $r^* = \infty$. The other assertions follow from the above arguments. The proof is complete. $\square$

Remark 2.3. If $p_S^* < p < p^*$, then $\eta < 0$. We cannot conclude $\frac{d}{dt}I(x(t), y(t)) \leq 0$ from (2.14).

Corollary 2.4. Suppose that (1.9) and (1.14) hold. Let $u(r)$ be the solution of (1.7). Then,

$$u(r) < \left(\frac{p - \beta + 2}{2} \right)^{\frac{1}{p-\beta}} u^*(r) \quad \text{for } r > 0.$$

Proof. In the proof of Lemma 2.2 we show that $(x(t), y(t))$ satisfies $I(x(t), y(t)) < 0$. By direct calculation we see that $\{I(x, y) < 0\} \subset \{x < \left(\frac{p-\beta+2}{2} \right)^{\frac{1}{p-\beta}}\}$. $\square$

Lemma 2.5. Suppose that (1.9) and (1.14) hold. Let $u(r)$ be the solution of (1.7). Let $x(t)$ and $y(t)$ be as defined by (2.6) and (2.11), respectively. Then the orbit $(x(t), y(t))$ is defined for all $t \in \mathbb{R}$ and it connects $(0, 0)$ and $(1, 0)$. In particular, $\lim_{t \rightarrow \infty} (x(t), y(t)) = (1, 0)$. Moreover, the following hold:

- (i) If $p \geq p_{JL}^*$, then $(1, 0)$ is a stable node.
- (ii) If $p < p_{JL}^*$, then $(1, 0)$ is a stable spiral.

Proof. It follows from Lemma 2.2 that $u(r)$ is defined for $r > 0$. Because of (2.6) and (2.11), $(x(t), y(t))$ is defined for all $t \in \mathbb{R}$. In the proof of Lemma 2.2 we already saw that $\lim_{t \rightarrow -\infty} (x(t), y(t)) = (0, 0)$, that $(x(t), y(t))$ is in a tear-shaped region $\{I(x, y) < 0, x > 0\}$, and that

$$\frac{d}{dt}I(x(t), y(t)) \begin{cases} = 0, & \text{if } y(t) = 0, \\ < 0, & \text{if } y(t) \neq 0. \end{cases}$$

Therefore, there is no periodic orbit in $\{I(x, y) < 0, x > 0\}$. By Poincaré–Bendixson theorem we see that $(x(t), y(t))$ converges to $(1, 0)$, which is a unique equilibrium point in $\{I(x, y) < 0, x > 0\}$. We have shown that the orbit $(x(t), y(t))$ is a heteroclinic orbit between $(0, 0)$ and $(1, 0)$.

We consider the linearization at the equilibrium $(1, 0)$ of (2.12). Then the Jacobian matrix is given by

$$J := \begin{pmatrix} 0 & 1 \\ -(p - \beta) & -\{\eta + (\beta - 1)\kappa\} \end{pmatrix}, \quad \text{and hence } \operatorname{tr} J = -\{\eta + (\beta - 1)\kappa\} \\ \text{and } \det J = p - \beta,$$

where η and κ are defined by (2.10). The eigenvalues of J are

$$\lambda_{\pm} := \frac{1}{2}\operatorname{tr} J \pm \frac{1}{2}\sqrt{D}, \quad \text{where } D := (\operatorname{tr} J)^2 - 4\det J. \quad (2.18)$$

Because of (1.14), $\eta + (\beta - 1)\kappa > 0$, and hence $\operatorname{tr} J < 0$. Since $p - \beta > 0$, $\det J > 0$. Because $\operatorname{tr} J < 0$ and $\det J > 0$,

$$\begin{cases} \lambda_{\pm} \in \mathbb{R}, \lambda_{-} \leq \lambda_{+} < 0, & \text{if } D \geq 0, \\ \lambda_{\pm} \in \mathbb{C} \setminus \mathbb{R}, \operatorname{Re}(\lambda_{+}) < 0, \operatorname{Re}(\lambda_{-}) < 0, & \text{if } D < 0. \end{cases} \quad (2.19)$$

The inequalities (2.19) indicate that the equilibrium $(1, 0)$ is a stable node or a stable spiral. We study the sign of D .

First, we consider the case $D < 0$ and prove (ii). Since

$$D = \{\eta + (\beta - 1)\kappa\}^2 - 4(p - \beta), \quad \eta + (\beta - 1)\kappa = \sqrt{\frac{\beta}{q} \frac{\alpha - \beta - (\beta + 1)q}{\alpha - \beta - \beta q}}, \\ \text{and } p - \beta = \frac{-\alpha + \beta + \gamma}{q},$$

$D < 0$ if and only if

$$\frac{\beta\{\alpha - \beta - (\beta + 1)q\}^2}{(\alpha - \beta - \beta q)q} < 4 \frac{-\alpha + \beta + \gamma}{q}.$$

Since $p \geq p^*$,

$$\alpha - \beta - \beta q > 0. \quad (2.20)$$

Using (2.20) and $q > 0$, we have the quadratic inequality in q

$$(\beta + 1)^2 q^2 - 2\xi q + (\alpha - \beta)^2 - \frac{4(\alpha - \beta)(-\alpha + \beta + \gamma)}{\beta} < 0, \quad (2.21)$$

where $\xi := (\beta + 3)(\alpha - \beta) - 2\gamma$. By E we denote the discriminant of (2.21), i.e.,

$$E := \xi^2 - (\beta + 1)^2 \left\{ (\alpha - \beta)^2 - \frac{4(\alpha - \beta)(-\alpha + \beta + \gamma)}{\beta} \right\}.$$

Then, by direct calculation we have

$$E = \frac{(-\alpha + \beta + \gamma)(\alpha + \beta\gamma - \beta)}{\beta} > 0.$$

The inequality (2.21) is equivalent to

$$\frac{\xi - 2\sqrt{E}}{(\beta + 1)^2} < q \quad (2.22)$$

and

$$q < \frac{\xi + 2\sqrt{E}}{(\beta + 1)^2}. \quad (2.23)$$

We consider (2.23). Since $\xi = (\beta + 1)(\alpha - \beta) - 2(-\alpha + \beta + \gamma)$ and

$$E = (-\alpha + \beta + \gamma)^2 + \frac{\beta + 1}{\beta}(\alpha - \beta)(-\alpha + \beta + \gamma) > (-\alpha + \beta + \gamma)^2,$$

we have

$$\xi + 2\sqrt{E} = (\beta + 1)(\alpha - \beta) + 2\left\{\sqrt{E} - (-\alpha + \beta + \gamma)\right\} > 0. \quad (2.24)$$

By the definition of q in (1.8) we see that (2.23) is equivalent to

$$p > \beta + \frac{(\beta + 1)^2(-\alpha + \beta + \gamma)}{\xi + 2\sqrt{E}} =: \tilde{p}_{JL}^*. \quad (2.25)$$

Since

$$\begin{aligned} p^* - \tilde{p}_{JL}^* &= \frac{(-\alpha + \beta + \gamma)}{(\alpha - \beta)(\xi + 2\sqrt{E})} \left\{ 2\beta\xi + 4\beta\sqrt{E} - (\beta + 1)^2(\alpha - \beta) \right\} \\ &= \frac{(-\alpha + \beta + \gamma)}{(\alpha - \beta)(\xi + 2\sqrt{E})} \left[(\beta - 1)(\beta + 1)(\alpha - \beta) + 4\beta \left\{ \sqrt{E} - (-\alpha + \beta + \gamma) \right\} \right] \\ &> 0 \end{aligned} \quad (2.26)$$

and $p \geq p^*$, we see that $p \geq p^* > \tilde{p}_{JL}^*$, and hence (2.25) holds. The inequality (2.23) always holds. We consider (2.22). If $\xi - 2\sqrt{E} > 0$, then this inequality is equivalent to

$$(\beta + 4)(\alpha - \beta) > 4\gamma. \quad (2.27)$$

Under this condition, by (2.22), we have

$$p < \beta + \frac{(\beta + 1)^2(-\alpha + \beta + \gamma)}{\xi - 2\sqrt{E}} = p_{JL}^*.$$

Thus, if $p < p_{JL}^*$, then both (2.22) and (2.23) hold, and hence $D < 0$. The equilibrium $(1, 0)$ is a stable spiral and the assertion (ii) holds. If $\xi - 2\sqrt{E} \leq 0$, then (2.22) clearly holds. The inequality $\xi - 2\sqrt{E} \leq 0$ is equivalent to

$$(\beta + 4)(\alpha - \beta) \leq 4\gamma.$$

In this case both (2.22) and (2.23) hold and (ii) holds. The proof of (ii) is complete.

Second, we consider the case $D \geq 0$ and prove (i). By almost the same calculation we see that $D \geq 0$ if and only if

$$q \leq \frac{\xi - 2\sqrt{E}}{(\beta + 1)^2} \quad (2.28)$$

or

$$\frac{\xi + 2\sqrt{E}}{(\beta + 1)^2} \leq q. \quad (2.29)$$

We consider (2.29). Since $\xi + 2\sqrt{E} > 0$ (see (2.24)), by (2.29) we have

$$p \leq \beta + \frac{(\beta + 1)^2(-\alpha + \beta + \gamma)}{\xi + 2\sqrt{E}} = \tilde{p}_{JL}^* < p^*,$$

where it follows from (2.26) that $\tilde{p}_{JL}^* < p^*$. This inequality contradicts $p \geq p^*$, and hence (2.29) does not hold. We consider (2.28). If $\xi - 2\sqrt{E} \leq 0$, then $p < \beta < p^*$ and this case does not occur. If $\xi - 2\sqrt{E} > 0$, then (2.28) is equivalent to

$$p \geq \beta + \frac{(\beta + 1)^2(-\alpha + \beta + \gamma)}{\xi - 2\sqrt{E}} = p_{JL}^*.$$

Thus if $p \geq p_{JL}^*$ which is equivalent to (2.27), then (2.28) holds, and hence $D \geq 0$. The equilibrium $(1, 0)$ is a stable node and the assertion (i) holds. The proof of (i) is complete.

The proof of the lemma is complete. $\square$

Lemma 2.6. *Suppose that (1.9) and (1.14) hold. Then the assertion (i) of Theorem A holds.*

Proof. Let $u(r, \rho)$ denote the solution of (1.7). Let $x(t, \rho)$ and $y(t, \rho)$ be as defined by (2.6) and (2.11), respectively. Since we consider the case $p \geq p_{JL}^*$, in the proof of Lemma 2.5 we saw that $\lambda_{\pm} \in \mathbb{R}$ and $\lambda_- \leq \lambda_+ < 0$, where $\lambda_{\pm}$ are given by (2.18). We analyze the phase plane of (2.12). We study the direction of the vector of the vector field

$$F \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} y \\ x - \eta y - \frac{|x|^{p-1}x}{|\kappa y - x|^{\beta-1}} \end{pmatrix}$$

on the line segment

$$\ell_0 := \{(1-t, -\lambda_+ t) \in \mathbb{R}^2; 0 \leq t < t_*\},$$

where $t_* := 1/(1 - \lambda_+ \kappa)$ and κ is defined by (2.10). We define

$$f(t) := \frac{(1-t)^p}{\{(\lambda_+ \kappa - 1)t + 1\}^{\beta-1}} - 1 + \{p + \lambda_+(\beta - 1)\kappa - (\beta - 1)\}t.$$

Then,

$$\begin{aligned} F \begin{pmatrix} 1-t \\ -\lambda_+ t \end{pmatrix} &= \begin{pmatrix} 0 \\ -(p - \beta) - \{\eta + (\beta - 1)\kappa\} \end{pmatrix} \begin{pmatrix} -t \\ -\lambda_+ t \end{pmatrix} + \begin{pmatrix} 0 \\ -f(t) \end{pmatrix} \\ &= (-\lambda_+ t) \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + \begin{pmatrix} 0 \\ -f(t) \end{pmatrix}. \end{aligned} \quad (2.30)$$

By direct calculation we have

$$\begin{aligned} f'(t) &= -\frac{p(1-t)^{p-1}}{\{(\lambda_+ \kappa - 1)t + 1\}^{\beta-1}} - \frac{(\beta - 1)(\lambda_+ \kappa - 1)(1-t)^p}{\{(\lambda_+ \kappa - 1)t + 1\}^{\beta}} \\ &\quad + p + \lambda_+(\beta - 1)\kappa - (\beta - 1), \end{aligned}$$

$$f''(t) = \frac{(\lambda_+ \kappa - 1)^2 (1-t)^p}{\{(\lambda_+ \kappa - 1)t + 1\}^{\beta+1}} \left\{ p(p-1) \left(\tilde{t} + \frac{\beta-1}{p-1} \right)^2 + \frac{(\beta-1)(p-\beta)}{p-1} \right\} > 0,$$

where $\tilde{t} := \frac{(\lambda_+ \kappa - 1)t + 1}{(\lambda_+ \kappa - 1)(1-t)}$. Therefore, $f'(0) = 0$ and $f''(t) > 0$ for $0 \leq t < t_*$. We see that $f'(t) > 0$ for $0 < t < t_*$. Since $f(0) = 0$, $f(t) > 0$ for $0 < t < t_*$. It follows from (2.30) that the vector F on ℓ_0 points to the region $\{(x, y); y < \lambda_+(x-1)\}$. This indicates that the orbit $(x(t, \rho), y(t, \rho))$ cannot go across the line segment ℓ_0 and that $(x(t, \rho), y(t, \rho)) \subset \{(x, y); y < \lambda_+(x-1)\}$. Let $\ell_1 := \{(t, 0); 0 < t < 1\}$. Since

$$F \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ t - t^{p-\beta+1} \end{pmatrix}$$

and $t - t^{p-\beta+1} > 0$ for $0 < t < 1$, the orbit cannot go across the line segment ℓ_1 . Here we use the inequality $p - \beta > 1$. Therefore, the orbit stays in $\{(x, y); 0 < y < \lambda_+(x-1)\}$. In particular, $x(t, \rho) < 1$ for $t \in \mathbb{R}$. Because of (2.6), $u(r) < u^*(r)$ and $\mathcal{Z}[u(\cdot, \rho) - u^*(\cdot)] = 0$.

Let $\tilde{u}(r, \rho) := \rho u(\rho^{\frac{1}{q}} r, 1)$. Then $\tilde{u}(r, \rho)$ satisfies (1.7). By the uniqueness of the solution of (1.7) we see that $\rho u(\rho^{\frac{1}{q}} r, 1) = u(r, \rho)$. Let $t(\rho) := \frac{1}{aq} \log \rho$. Using $\rho u^*(\rho^{\frac{1}{q}} r) = u^*(r)$, we have

$$x(t, \rho) = \frac{u(r, \rho)}{u^*(r)} = \frac{\rho u(\rho^{\frac{1}{q}} r, 1)}{\rho u^*(\rho^{\frac{1}{q}} r)} = x(t + t(\rho), 1).$$

If $\rho_0 > \rho_1 > 0$, then $x(t + t(\rho_0), 1) > x(t + t(\rho_1), 1)$, because $x'(t, 1) = y(t, 1) > 0$. Since $x(t, \rho_0) > x(t, \rho_1)$, $u(r, \rho_0) > u(r, \rho_1)$ for $r > 0$ and $\mathcal{Z}[u(\cdot, \rho_0) - u(\cdot, \rho_1)] = 0$. $\square$

Lemma 2.7. *Suppose that (1.9) and (1.14) hold. Then the assertion (ii) of Theorem A holds.*

Proof. Let $u(r, \rho)$ denote the solution of (1.7). Let $x(t, \rho)$ and $y(t, \rho)$ be as defined by (2.6) and (2.11), respectively. We consider the case where $p^* \leq p < p_{JL}^*$ and $(p, \beta) \neq (p^*, 1)$. By Lemma 2.5 we see that $(1, 0)$ is a stable spiral. The orbit $(x(t, \rho), y(t, \rho))$ starts from the origin at $t = -\infty$ and rotates around the point $(1, 0)$ clockwise. Because of (2.12), there is $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$ such that $t_1 < t_2 < \dots$, $y(t_n, \rho) = 0$ for all n , and $x(t_2, \rho) < x(t_4, \rho) < \dots < 1 < \dots < x(t_3, \rho) < x(t_1, \rho)$. Let $r_n := e^{at_n}$. Then,

$$u(t_n, \rho) \begin{cases} > u^*(r_n, \rho), & \text{if } n \in \{1, 3, \dots\}, \\ < u^*(r_n, \rho), & \text{if } n \in \{2, 4, \dots\}, \end{cases}$$

and hence, $\mathcal{Z}[u(\cdot, \rho) - u^*(\cdot)] = \infty$.

Because of (2.12), we see that if $n \in \{1, 3, \dots\}$, then

$$x(t, \rho) < x(t_n, \rho) \quad \text{for } t > t_n, \quad (2.31)$$

and if $n \in \{2, 4, \dots\}$, then

$$x(t, \rho) > x(t_n, \rho) \quad \text{for } t > t_n. \quad (2.32)$$

Let ρ_1 be positive real numbers such that $\rho_1 > \rho > 0$, and let $t(\rho) = \frac{1}{aq} \log \rho$. Then $t(\rho_1) > t(\rho)$. Using (2.31) and (2.32), we see that

$$x(t_n + t(\rho_1) - t(\rho), \rho) \begin{cases} < x(t_n, \rho), & \text{if } n \in \{1, 3, \dots\}, \\ > x(t_n, \rho), & \text{if } n \in \{2, 4, \dots\}. \end{cases} \quad (2.33)$$

Since $x(t_n, \rho_1) = x(t_n + t(\rho_1), 1) = x(t_n + t(\rho_1) - t(\rho), \rho)$, (2.33) indicates that

$$u(r_n, \rho_1) \begin{cases} < u(r_n, \rho), & \text{if } n \in \{1, 3, \dots\}, \\ > u(r_n, \rho), & \text{if } n \in \{2, 4, \dots\}, \end{cases}$$

where $r_n := e^{at_n}$. Thus, $\mathcal{Z}[u(\cdot, \rho_1) - u(\cdot, \rho)] = \infty$. $\square$

Proof of Theorem A. Lemma 2.2 says that $u(r, \rho) > 0$ for $r > 0$. The assertions (i) and (ii) of Theorem A follows from Lemmas 2.6 and 2.7, respectively. $\square$

Proof of Corollary 1.5. When $N > \frac{m(m+3)}{m-1}$,

$$p_{JL}^* = m - 1 + \frac{m^2}{N - m - 2 - 2\sqrt{\frac{N-1}{m-1}}}.$$

By direct calculation we see that $p_{JL}^* \leq p^*$ if $N \geq \frac{m(m^2+4m-4)}{(m-2)^2}$. Thus (i) of Corollary 1.5 follows from Theorem A.

We prove (ii). If $N < m(m^2+4m-4)/(m-2)^2$, then by direct calculation we see that $p_{JL}^* > p^*$. Here p_{JL}^* may be ∞ . Thus (ii) of Corollary 1.5 follows from Theorem A. $\square$

Proof of Corollary 1.7. The proof is almost the same as the proof of Corollary 1.5. We omit the proof. $\square$

We consider the case $(p, \beta) = (p^*, 1)$. Let η be defined by (2.10). Since $p = p^*$, $\eta = 0$. By (2.14) we see that the orbit $(x(t, \rho), y(t, \rho))$ is a homoclinic orbit to the origin. Hence, by the phase plane we easily see that there are exactly two constants t_1 and t_2 such that $x(t_j, \rho) = 1$ for $j = 1, 2$. Hence, $\mathcal{Z}[u(\cdot, \rho) - u^*(\cdot)] = 2$. We also see that there is exactly one constant t_0 such that $x(t_0 + t(\rho_1) - t(\rho), \rho) = x(t_0, \rho)$. Hence, $\mathcal{Z}[u(\cdot, \rho_1) - u(\cdot, \rho)] = 1$. We obtain (1.18) in Remark 1.4 in the case $(p, \beta) = (p^*, 1)$.

3. Global bifurcation diagrams

Proof of Theorem B. Let $U(R, \tau)$ be the solution of the Cauchy problem

$$\begin{cases} R^{-(\gamma-1)}(R^\alpha |U'|^{\beta-1} U')' + \lambda(U+1)^p = 0, & R > 0, \\ U(0) = \tau > 0, U'(0) = 0, \end{cases} \quad (3.1)$$

where λ is a positive parameter. Let $u(r, \rho) := U(R, \tau) + 1$, $\rho := \tau + 1$, and $r := \lambda^{\frac{1}{-\alpha+\beta+\gamma}} R$. Then, $u(r, \rho)$ satisfies (1.7). Since (1.7) has a unique solution, (3.1) also has a unique solution, which is denoted by $U(R, \tau)$. Since $u'(r, \rho) < 0$ for $r > 0$, $U'(R, \tau) < 0$ for $R > 0$. The function $U(R, \tau)$ is the solution of (1.24) if and only if $U(1, \tau) = 0$. We define $g(\lambda, \tau) := U(1, \tau) = u(\lambda^{\frac{1}{-\alpha+\beta+\gamma}}, \tau + 1) - 1$.

Let $x(t, \rho)$ be defined by (2.6), and let $u^*(r)$ be defined by (1.11). Let $y(t, \rho) := x_t(t, \rho)$. Then we follow the orbit $\{(x(t, \rho), y(t, \rho))\}$ which is invariant for translation in ρ . In particular, $\{(x(t, \rho), y(t, \rho)); -\infty < t \leq t_0\}$ corresponds to the solution of (3.1), where $t_0 := \frac{1}{a} \log \lambda^{-\frac{1}{-\alpha+\beta+\gamma}}$. All the desired properties of $\lambda(\tau)$ follows from the properties of the end point $(x(t_0, \rho), y(t_0, \rho))$.

We study the zero level set of $g(\lambda, \tau)$. We have

$$\begin{aligned} g(\lambda, \tau) &= x\left(\frac{1}{a} \log \lambda^{-\frac{1}{-\alpha+\beta+\gamma}}, \tau + 1\right) u^*\left(\lambda^{-\frac{1}{-\alpha+\beta+\gamma}}\right) - 1 \\ &= x(z(\lambda, \tau), 1) A \lambda^{-\frac{1}{p-\beta}} - 1, \end{aligned} \quad (3.2)$$

where

$$z(\lambda, \tau) := \frac{1}{aq(p-\beta)} \log \lambda + \frac{1}{aq} \log(\tau + 1). \quad (3.3)$$

Since $u'(r, \rho) < 0$ for $r > 0$, by (2.16) we see that $\kappa x'(z(\lambda, \tau), 1) - x(z(\lambda, \tau), 1) < 0$, and hence

$$\frac{\partial g}{\partial \lambda}(\lambda, \tau) = \frac{A \lambda^{-\frac{1}{p-\beta}-1}}{p-\beta} (\kappa x'(z(\lambda, \tau), 1) - x(z(\lambda, \tau), 1)) < 0, \quad (3.4)$$

where κ is defined by (2.10). We have

$$\frac{\partial g}{\partial \tau}(\lambda, \tau) = \frac{1}{aq(\tau+1)} x'(z(\lambda, \tau), 1) A \lambda^{-\frac{1}{p-\beta}}. \quad (3.5)$$

It follows from (3.4) and the implicit function theorem that λ is locally a graph of τ , i.e., $\lambda = \lambda(\tau)$ and $g(\lambda(\tau), \tau) = 0$. Using (3.4) and (3.5), we have

$$\begin{aligned} \lambda'(\tau) &= -\frac{\frac{\partial g}{\partial \tau}(\tau(\lambda), \lambda)}{\frac{\partial g}{\partial \lambda}(\tau(\lambda), \lambda)} \\ &= -\frac{(p-\beta)\lambda x'(z(\lambda, \tau), 1)}{aq(\tau+1)(\kappa x'(z(\lambda, \tau), 1) - x(z(\lambda, \tau), 1))}. \end{aligned} \quad (3.6)$$

Since $u'(r, \rho) < 0$ for $r > 0$ and $u(r, \rho) \rightarrow 0$ ($r \rightarrow \infty$), for each $\tau > 0$ there is exactly one $\lambda > 0$ such that $u(\lambda^{-\frac{1}{-\alpha+\beta+\gamma}}, \tau + 1) - 1 = 0$, which indicates that $\lambda = \lambda(\tau)$ can be defined for all $\tau > 0$. We define $\tilde{z}(\tau)$ by

$$\tilde{z}(\tau) := z(\lambda(\tau), \tau).$$

Using (3.3) and (3.6), we have

$$\begin{aligned} \tilde{z}'(\tau) &= \frac{\partial z}{\partial \lambda}(\lambda(\tau), \tau) \lambda'(\tau) + \frac{\partial z}{\partial \tau}(\lambda(\tau), \tau) \\ &= \frac{-x(z(\lambda(\tau), \tau), 1)}{aq(\tau+1)(\kappa x'(z(\lambda(\tau), \tau), 1) - x(z(\lambda(\tau), \tau), 1))} > 0, \end{aligned}$$

where we use $\kappa x'(\tilde{z}(\tau), 1) - x(\tilde{z}(\tau), 1) < 0$. Since $\tilde{z}(\tau)$ is a strictly increasing function in τ , there is a constant $z^* \in \mathbb{R}$ such that $\tilde{z}(\tau) \rightarrow z^*$ ($\tau \rightarrow \infty$) or $\tilde{z}(\tau) \rightarrow \infty$ ($\tau \rightarrow \infty$). If $\tilde{z}(\tau) \rightarrow z^*$ ($\tau \rightarrow \infty$), then by (3.2) we see that

$$\lambda(\tau) \rightarrow (Ax(z^*, 1))^{p-\beta} \quad (\tau \rightarrow \infty). \quad (3.7)$$

Because of (3.3),

$$\lambda(\tau) = \left(\frac{e^{aq\tilde{z}(\tau)}}{\tau + 1} \right)^{p-\beta} \rightarrow 0 \quad (\tau \rightarrow \infty).$$

This contradicts (3.7), because $x(z^*, 1) > 0$. Thus, $\tilde{z}(\tau) \rightarrow \infty$ ($\tau \rightarrow \infty$). We define $\tilde{x}(\tau)$ by

$$\tilde{x}(\tau) := x(\tilde{z}(\tau), 1).$$

Because of (3.2), we have

$$\lambda(\tau) = (A\tilde{x}(\tau))^{p-\beta}, \quad (3.8)$$

$$\lambda'(\tau) = (p - \beta)A^{p-\beta}(\tilde{x}(\tau))^{p-\beta-1}\tilde{x}'(\tau). \quad (3.9)$$

We show that $\lim_{\tau \downarrow 0} \lambda(\tau) = 0$. Since $\lambda' > 0$ and $\lambda > 0$, $\lim_{\tau \downarrow 0} \lambda(\tau)$ exists. Let $\lambda_0 := \lim_{\tau \downarrow 0} \lambda(\tau)$. Then, $\lambda_0 \geq 0$. We assume that $\lambda_0 > 0$. Then $z(\lambda(\tau), \tau) \rightarrow -\infty$ ($\tau \downarrow 0$), and hence $\tilde{x}(\tau) \rightarrow 0$ ($\tau \downarrow 0$). By (3.8) we see that $\lambda(\tau) = (A\tilde{x}(\tau))^{p-\beta} \rightarrow 0$ ($\tau \downarrow 0$). This contradicts that $\lambda(\tau) \rightarrow \lambda_0 > 0$ ($\tau \downarrow 0$). Thus, $\lim_{\tau \downarrow 0} \lambda(\tau) = \lambda_0 = 0$.

First, we consider the case where (1.14) holds. Then the assumptions of Theorem B (i) or (ii) are satisfied. In this case, $\tilde{x}(\tau) \rightarrow 1$ as $\tau \rightarrow \infty$. By (3.8) we see that

$$\lambda(\tau) \rightarrow A^{p-\beta} \quad \text{as } \tau \rightarrow \infty.$$

If the assumptions of Theorem B (i) hold, then $\tilde{x}'(\tau) (= y(\tilde{z}(\tau), 1)) > 0$ (see Lemma 2.6), and hence by (3.9) we see that $\lambda' > 0$ for $\tau > 0$. We have proved the assertion (i).

If the assumptions of Theorem B (ii) hold, then $(1, 0)$ is a spiral point (see Lemma 2.5), and hence $\tilde{x}(\tau)$ oscillates around 1 as τ increases. Specifically, there is a sequence $\{\tau_n\}_{n=1}^{\infty}$ ($-\infty = \tau_0 < \tau_1 < \tau_2 < \dots \rightarrow \infty$) such that

$$\tilde{x}'(\tau_n) = 0 \quad \text{for } n \geq 1,$$

$$\tilde{x}'(\tau) = \begin{cases} > 0, & \text{if } \tau \in (\tau_0, \tau_1) \cup (\tau_2, \tau_3) \cup \dots, \\ < 0, & \text{if } \tau \in (\tau_1, \tau_2) \cup (\tau_3, \tau_4) \cup \dots, \end{cases}$$

$$\tilde{x}(\tau_2) < \tilde{x}(\tau_4) < \dots < 1 < \dots < \tilde{x}(\tau_3) < \tilde{x}(\tau_1), \quad \text{and } \lim_{n \rightarrow \infty} \tilde{x}(\tau_n) = 1.$$

Because of (3.8) and (3.9),

$$\lambda'(\tau_n) = 0 \quad \text{for } n \geq 1,$$

$$\lambda'(\tau) = \begin{cases} > 0, & \text{if } \tau \in (\tau_0, \tau_1) \cup (\tau_2, \tau_3) \cup \dots, \\ < 0, & \text{if } \tau \in (\tau_1, \tau_2) \cup (\tau_3, \tau_4) \cup \dots, \end{cases}$$

$$\lambda(\tau_2) < \lambda(\tau_4) < \dots < A^{p-\beta} < \dots < \lambda(\tau_3) < \lambda(\tau_1), \quad \text{and } \lim_{n \rightarrow \infty} \lambda(\tau_n) = A^{p-\beta}.$$

We have proved the assertion (ii).

We consider the case (iii). Since $p = p_S^*$, (1.17) is the exact solution of (1.7). Then,

$$\begin{aligned} g(\lambda, \tau) &= u\left(\lambda^{-\frac{1}{-\alpha+\beta+\gamma}}, \tau+1\right) - 1 \\ &= (\tau+1)u\left((\tau+1)^{\frac{\beta+1}{\alpha-\beta}}\lambda^{-\frac{1}{-\alpha+\beta+\gamma}}, 1\right) - 1 \\ &= (\tau+1)\left(1 + c(\tau+1)^{\frac{(\beta+1)(-\alpha+\beta+\gamma)}{\beta(\alpha-\beta)}}\lambda^{\frac{1}{\beta}}\right)^{-\frac{\alpha-\beta}{-\alpha+\beta+\gamma}} - 1. \end{aligned}$$

We can explicitly solve $g(\lambda, \tau) = 0$ in λ , i.e.,

$$\lambda = c^{-\beta} \left((\tau+1)^{-\frac{-\alpha+\beta+\gamma}{\beta(\alpha-\beta)}} - (\tau+1)^{-\frac{(\beta+1)(-\alpha+\beta+\gamma)}{\beta(\alpha-\beta)}} \right)^{\beta}.$$

By elementary calculation we can check the conclusions of (iii). The proof is complete. $\square$

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References

- [1] Bae, S.: Separation structure of positive radial solutions of a semilinear elliptic equation in $\mathbf{R}^n$. J. Differ. Equ. **194**, 460–499 (2003)
- [2] Bae, S.: Infinitely multiplicity and separation structure of positive solutions for a semilinear elliptic equation in $\mathbf{R}^n$. J. Differ. Equ. **200**, 274–311 (2004)
- [3] Bae, S.: Positive entire solutions of semilinear elliptic equations with quadratically vanishing coefficient. J. Differ. Equ. **237**, 159–197 (2007)
- [4] Bae, S., Chang, T.-K.: On a class of semilinear elliptic equations in $\mathbf{R}^n$. J. Differ. Equ. **185**, 225–250 (2002)
- [5] Bidaut-Véron, M.-F.: Local and global behavior of solutions of quasilinear equations of Emden–Fowler type. Arch. Ration. Mech. Anal. **107**, 293–324 (1989)
- [6] Cabré, X., Sanchón, M.: Semi-stable and extremal solutions of reaction equations involving the p -Laplacian. Commun. Pure Appl. Anal. **6**, 43–67 (2007)
- [7] Clément, P., Figueiredo, D., de Mitidieri, E.: Quasilinear elliptic equations with critical exponents. Topol. Methods Nonlinear Anal. **7**, 133–170 (1996)
- [8] Egnell, H.: Elliptic boundary value problems with singular coefficients and critical nonlinearities. Indiana Univ. Math. J. **38**, 235–251 (1989)

- [9] Emden, R.: Gaskugeln: Anwendungen der mechanischen Wärmetheorie auf kosmologische und meteorologische Probleme (German) In: Teubner, B.G. (ed.) Leipzig; Berlin (1907). https://en.wikipedia.org/wiki/Springer_Vieweg_Verlag
- [10] Ferrero, A.: On the solutions of quasilinear elliptic equations with a polynomial-type reaction term. *Adv. Differ. Equ.* **9**, 1201–1234 (2004)
- [11] Fila, M., King, J., Winkler, M., Yanagida, E.: Optimal lower bound of the growth rate for a supercritical parabolic equation. *J. Differ. Equ.* **228**, 339–356 (2006)
- [12] Fila, M., Winkler, M., Yanagida, E.: Convergence rate for a parabolic equation with supercritical nonlinearity. *J. Dyn. Differ. Equ.* **17**, 249–269 (2005)
- [13] Fila, M., Winkler, M., Yanagida, E.: Slow convergence to zero for a parabolic equation with a supercritical nonlinearity. *Math. Ann.* **340**, 477–496 (2008)
- [14] Franca, M.: Classification of positive solutions of p -Laplace equation with a growth term. *Arch. Math. (Brno)* **40**, 415–434 (2005)
- [15] Franca, M.: Some results on the m -Laplace equations with two growth terms. *J. Dyn. Differ. Equ.* **17**, 391–425 (2005)
- [16] Franca, M.: Fowler transformation and radial solutions for quasilinear elliptic equations. I. The subcritical and the supercritical case. *Can. Appl. Math. Q.* **16**, 123–159 (2008)
- [17] Franca, M.: Fowler transformation and radial solutions for quasilinear elliptic equations. II. Nonlinearities of mixed type. *Ann. Mat. Pura Appl.* **189**, 67–94 (2010)
- [18] Gui, C., Ni, W.-M., Wang, X.: On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$. *Comm. Pure Appl. Math.* **45**, 1153–1181 (1992)
- [19] Gui, C., Ni, W.-M., Wang, X.: Further study on a nonlinear heat equation. *J. Differ. Equ.* **169**, 588–613 (2001)
- [20] Guo, Z., Wei, J.: Global solution branch and Morse index estimates of a semilinear elliptic equation with super-critical exponent. *Trans. Am. Math. Soc.* **363**, 4777–4799 (2011)
- [21] Jacobsen, J., Schmitt, K.: The Liouville–Bratu–Gelfand problem for radial operators. *J. Differ. Equ.* **184**, 283–298 (2002)
- [22] Joseph, Y., Lundgren, T.: Quasilinear Dirichlet problems driven by positive sources. *Arch. Ration. Mech. Anal.* **49**, 241–269 (1972/73)
- [23] Liu, Y., Li, Y., Deng, Y.: Separation property of solutions for a semilinear elliptic equation. *J. Differ. Equ.* **163**, 381–406 (2000)
- [24] Matano, H., Merle, F.: On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Comm. Pure Appl. Math.* **57**, 1494–1541 (2004)
- [25] Matano, H., Merle, F.: Classification of type I and type II behaviors for a supercritical nonlinear heat equation. *J. Funct. Anal.* **256**, 992–1064 (2009)

- [26] Matano, H., Merle, F.: Threshold and generic type I behaviors for a supercritical nonlinear heat equation. *J. Funct. Anal.* **261**, 716–748 (2011)
- [27] Miyamoto, Y.: Structure of the positive solutions for supercritical elliptic equations in a ball. *J. Math. Pures Appl.* **102**, 672–701 (2014)
- [28] Mizoguchi, N., Yanagida, E.: Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation. *Math. Ann.* **307**, 663–675 (1997)
- [29] Mizoguchi, N.: Multiple blowup of solutions for a semilinear heat equation. *Math. Ann.* **331**, 461–473 (2005)
- [30] Mizoguchi, N.: Rate of type II blowup for a semilinear heat equation. *Math. Ann.* **339**, 839–877 (2007)
- [31] Poláčik, P., Yanagida, E.: On bounded and unbounded global solutions of a supercritical semilinear heat equation. *Math. Ann.* **327**, 745–771 (2003)
- [32] Poláčik, P., Yanagida, E.: Convergence of anisotropically decaying solutions of a supercritical semilinear heat equation. *J. Dyn. Differ. Equ.* **21**, 329–341 (2009)
- [33] Tso, K.: Remarks on critical exponents for Hessian operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7**, 113–122 (1990)
- [34] Wang, X.-F.: On the Cauchy problem for reaction-diffusion equations. *Trans. Am. Math. Soc.* **337**, 549–590 (1993)

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