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The fractional Neumann and Robin type boundary conditions for the regional fractional p-Laplacian

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Abstract. Let $p \in (1, \infty)$, $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^N$ a bounded open set with boundary $\partial\Omega$ of class $C^{1,1}$. In the first part of the article we prove an integration by parts formula for the fractional p-Laplace operator $(-\Delta)_p^s$ defined on $\Omega \subset \mathbb{R}^N$ and acting on functions that do not necessarily vanish at the boundary $\partial\Omega$. In the second part of the article we use the above mentioned integration by parts formula to clarify the fractional Neumann and Robin boundary conditions associated with the fractional p-Laplacian on open sets.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary $\partial\Omega$, $1 < p < \infty$ and let $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ be the p-Laplace operator. Using the following well known integration by parts formula,

$$-\int_\Omega v\Delta_p u\,dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} v|\nabla u|^{p-2}\nabla u \cdot \nu\,d\sigma, \quad (1.1)$$

valid for $u, v \in W^{1,p}(\Omega)$ such that $\Delta_p u \in L^{\frac{p}{p-1}}(\Omega)$ and $|\nabla u|^{p-2}\nabla u \cdot \nu = |\nabla u|^{p-2}\partial u/\partial\nu \in L^{\frac{p}{p-1}}(\partial\Omega, \sigma)$, where σ denotes the usual Lebesgue surface measure on $\partial\Omega$, it is nowadays classical to define the Neumann boundary conditions associated with the operator Δ_p, given by $|\nabla u|^{p-2}\nabla u \cdot \nu = g$ on $\partial\Omega$, and the Robin boundary conditions, $|\nabla u|^{p-2}\nabla u \cdot \nu + \gamma|u|^{p-2}u = g$ on $\partial\Omega$, where g is a given function on $\partial\Omega$. Here ν denotes the outer normal vector at $\partial\Omega$. We call $|\nabla u|^{p-2}\nabla u\cdot\nu$ the p-normal derivative of u so that the 2-normal derivative, $\nabla u\cdot\nu$, coincides with the classical normal derivative of u. A realization $\Delta_p^{\mathcal{N}}$ and Δ_p^R of the p-Laplace operator with zero ($g=0$) Neumann and Robin boundary conditions, respectively, can then be defined directly, if Ω has a Lipschitz continuous boundary, or by using the method of proper, convex, lower semi-continuous functional if Ω is not smooth enough (see e.g. [2,26]).

The main concerns in the present article are the following: let $p\in(1,\infty)$, $0<s<1$ and $\Omega\subset\mathbb{R}^N$ a bounded open set with boundary $\partial\Omega$ of class $C^{1,1}$.

- Find a suitable definition of an (s,p)-normal derivative of a function u defined only on Ω, so that as $s\uparrow 1$, it converges to $|\nabla u|^{p-2}\nabla u\cdot\nu$ (the p-normal derivative of u mentioned above).
- Find an integration by parts formula for the fractional p-Laplace operator defined on Ω and acting on functions defined only on Ω and do not necessarily vanish at the boundary $\partial\Omega$, that is, a formula comparable to (1.1) for the fractional p-Laplace operator.
- Define a realization in $L^2(\Omega)$ of the fractional p-Laplace operator with fractional Neumann and Robin type boundary conditions.

In the above items, we have insisted that the functions are defined only on Ω and do not necessarily vanish on $\partial\Omega$. In fact, this makes the situation difficult to define the recently studied fractional Laplace operator.

For the convenience of the reader and in order to make the paper as self-contained as possible, we start by introducing the fractional p-Laplace operator. Let $0<s<1$, $p\in(1,\infty)$ and set

$$\mathcal{L}^{p-1}(\Omega):=\left\{u:\ \Omega\to\mathbb{R}\ \text{ measurable},\ \int_\Omega\frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx<\infty\right\}.$$

For $u\in\mathcal{L}^{p-1}(\mathbb{R}^N)$, $x\in\mathbb{R}^N$ and $\varepsilon>0$, we write

$$(-\Delta)^s_{p,\varepsilon}u(x)=C_{N,p,s}\int_{\{y\in\mathbb{R}^N,|y-x|>\varepsilon\}}|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy,$$

with the normalized constant $C_{N,p,s}$ given by

$$C_{N,p,s}=\frac{s2^{2s}\Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)},\tag{1.2}$$

where Γ is the usual Gamma function (see e.g. [3,5,7,14,17,18] for the linear case $p=2$).

The fractional p-Laplacian $(-\Delta)^s_p u$ of the function u is defined by the formula

$$\begin{aligned}(-\Delta)^s_p u(x)&=\text{P.V. }C_{N,p,s}\int_{\mathbb{R}^N}|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy\\&=\lim_{\varepsilon\downarrow 0}(-\Delta)^s_{p,\varepsilon}u(x),\quad x\in\mathbb{R}^N,\end{aligned}\tag{1.3}$$

provided that the limit exists. We refer to Sect. 4 below for the class of functions for which the limit exists. We notice that if $0<s<\frac{p-1}{p}$ and u is smooth, for example, $u \in L^{\infty}(\mathbb{R}^N) \cap C^{0,1}(\mathbb{R}^N)$, then the integral in (1.3) is in fact not singular near x.

We mention that even in the case $\Omega = \mathbb{R}^N$, $\mathcal{L}^{p-1}(\mathbb{R}^N)$ is different from the space $L^{p-1}_{sp}(\mathbb{R}^N)$ introduced in [10,11,22]. In fact $\mathcal{L}^{p-1}(\mathbb{R}^N)$ is the right space on which for every $\varepsilon > 0$, $(-\Delta)^s_{p,\varepsilon} u$ is well defined on $\mathbb{R}^N$ and is continuous where the function u is continuous.

The linear fractional Laplacians being the generator of s-stable processes (Levy flights in some of the physical literature) are widely used to model systems of stochastic dynamics with applications in operation research, queuing theory, mathematical finance, risk estimate and others. Nonlocal models differ from the classical partial differential equation models in the fact that in the latter case interactions between two domains occur only due to contact, whereas in the former case interactions can occur at a distance. For more applications and details on these facts we refer to [3,6,9,27–29] and the references therein.

As we have mentioned above, of concern in this article is to define a realization of the fractional p-Laplace operator with fractional Neumann and Robin type boundary conditions on open subsets of $\mathbb{R}^N$. Since the operator $(-\Delta)^s_p$ is nonlocal, it cannot be used on domain automatically. We proceed as follows. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. For $u \in \mathcal{L}^{p-1}(\Omega)$, $x \in \Omega$ and $\varepsilon > 0$, we let

$$(-\Delta)^s_{\Omega,p,\varepsilon} u(x) = C_{N,p,s} \int_{\{y \in \Omega, |y-x|>\varepsilon\}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy,$$

and we define the operator

$$(-\Delta)^s_{\Omega,p} u(x) = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_{\Omega,p,\varepsilon} u(x), \quad x \in \Omega, \tag{1.4}$$

provided that the limit exists. In [17,18] the linear operator $(-\Delta)^s_{\Omega,2}$ has been called the *regional fractional Laplacian*. We shall use this terminology to call $(-\Delta)^s_{\Omega,p}$ the *regional fractional p-Laplace operator*.

We have the following. Let $u \in \mathcal{D}(\Omega)$. Since $u = 0$ on $\mathbb{R}^N \backslash \Omega$, a simple calculation gives

$$\begin{aligned}
(-\Delta)^s_{\Omega,p} u &= C_{N,p,s}\text{P.V.} \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy \\
&\quad - C_{N,p,s} \int_{\mathbb{R}^N \backslash \Omega} \frac{|u(x)|^{p-2} u(x)}{|x-y|^{N+ps}}\, dy \\
&= (-\Delta)^s_p u(x) - V_{\Omega,p}(x)|u(x)|^{p-2} u(x),
\end{aligned} \tag{1.5}$$

where the potential $V_{\Omega,p}$ is given by

$$V_{\Omega,p}(x) := C_{N,p,s} \int_{\mathbb{R}^N \backslash \Omega} \frac{1}{|x-y|^{N+ps}}\, dy. \tag{1.6}$$

The fractional p-Laplace operator $(-\Delta)^s_p$ (for all $p \in (1,\infty)$) with the Dirichlet boundary condition has been investigated in [16,21,29] and the references therein. More precisely in [16] some spectral properties of a realization in $L^2(\Omega)$ of this operator with Dirichlet boundary condition has been studied, in [21] the authors have shown some global Hölder continuity results of the associated quasi-linear elliptic problem with the Dirichlet boundary condition and finally in [29] some well-posedness, fine estimates of solutions and existence results to parabolic and non-linear elliptic problems associated to $(-\Delta)_p^s$ with Dirichlet boundary condition have been obtained. Similar elliptic problems associated to general non-local operators defined on $\mathbb{R}^N$ where $|x-y|^{-N-sp}$ is replaced by a general symmetric kernel $K(x,y)$ have been recently studied in [10,11,22]. We notice that in [16,21], $(-\Delta)_p^s$ has been introduced without a normalized constant, but this is not restrictive in the framework considered there. Here we have introduced an explicit constant $C_{N,p,s}$ that coincides with the well-known constant $C_{N,s}$ in the linear case $p=2$ contained in the above mentioned references. The justification of the constant $C_{N,p,s}$ is given in Remark 4.2 below. In fact the constant is needed to approach the classical p-Laplace operator as s goes to 1 (see e.g. [4,14]).

At our knowledge there is almost no reference (except the Ph.D Dissertation of Tang [25] where parabolic problems associated with the operator $(-\Delta)_{\Omega,p}^s$ have been mentioned without going into details) where the regional fractional p-Laplacian $(-\Delta)_{\Omega,p}^s$ has been deeply studied. We think that one of the reason is that there is no appropriate Green type formula associated with this operator when it acts on functions that are defined only on Ω and do not vanish on $\partial\Omega$. An integration by parts formula is crucial and the main tool in the study of parabolic, hyperbolic and elliptic partial differential equations. We also mention that for a general open set Ω, we are not sure that an identity as the one in (1.5) can be established for functions defined only on Ω and do not vanish on $\partial\Omega$. Indeed, letting $\tilde{u}$ be an extension of such a function u (if it is possible) to all $\mathbb{R}^N$ such that $(-\Delta)_p^s\tilde{u}$ exists, then a relation between $(-\Delta)_p^s\tilde{u}$ and $(-\Delta)_{\Omega,p}^s u$ can be established but the relation may not be independent of the choice of the extension except in the case where there is only one possible extension. Parabolic problems associated with the linear operator $(-\Delta)_{\Omega,2}^s$ have been intensively studied in [5,8,18] and the references therein, by using some probabilistic approach and in [27,28] by a direct method. The method of proper, convex and lower semi-continuous functional will be used to define a realization in $L^2(\Omega)$ of $(-\Delta)_{\Omega,p}^s$ with suitable fractional Neumann and Robin type boundary conditions. To characterize the Neumann or/and the Robin type boundary conditions for $(-\Delta)_{\Omega,p}^s$, one needs to introduce first the object in the regional fractional p-Laplacian playing the role that the p-normal derivative does in the case of the p-Laplace operator. Second, one needs a Green type formula for the regional fractional p-Laplace operator. The case $p=2$ has been recently investigated in [12,13,17,18,27]. More precisely, for the case $p=2$, Guan and Ma [18] (for the one-dimenional case) and Guan [17] (for the N-dimensional case) have shown that if $\frac{1}{2}<s<1$ and $u(x)=f_1(x)\rho(x)^{2s-1}+g_1(x)$, $v(x)=f_2(x)\rho(x)^{2s-1}+g_2(x)$ for some $f_1,f_2,g_1,g_2\in C^2(\overline{\Omega})$ where $\rho(x)=\mathrm{dist}(x,\partial\Omega)$, $x\in\Omega$, then one has the following integration by parts formula:

$$\begin{aligned}\int_\Omega v(-\Delta)^s_{\Omega,2}u\,dx =&\frac{C_{N,2,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy\\ &-B_{N,s}\int_{\partial\Omega} v\mathcal{N}_2^{2-2s}u\,d\sigma,\end{aligned} \tag{1.7}$$

where $B_{N,s}$ is an explicit constant and the boundary operator $\mathcal{N}_2^{2-2s}u$ is defined for every $z\in\partial\Omega$ by

$$\mathcal{N}_2^{2-2s}u(z)=\lim_{t\downarrow 0}\frac{du(z+t\vec{n}(z))}{dt}t^{2-2s},$$

and $\vec{n}(z)$ denotes the inner normal vector to $\partial\Omega$ at the point $z\in\partial\Omega$. We call the function $\mathcal{N}_2^{2-2s}u$, the $(s,2)$-normal derivative of u. In fact, by [27], (1.7) holds for every $v\in W^{s,2}(\Omega)$ and by [28] the constant $B_{N,s}=B_s$ depends only in s (see also Remark 3.10 below).

In the present article we take inspiration from Guan and Ma's approach for the linear case $p=2$ (see also [27] for a weak formulation) to obtain a general integration by parts formula for the quasi-linear operator $(-\Delta)^s_{\Omega,p}$ acting on functions that do not necessarily vanish on $\partial\Omega$. With the help of the obtained integration by parts formula, we will be able to characterize completely the fractional Neumann and Robin type boundary conditions for the operator $(-\Delta)^s_{\Omega,p}$ which are also consistent with the linear case $p=2$. It turns out that if $p\in(1,\infty)$, $\max\{\frac{p-1}{p},\frac{1}{p}\}<s<1$, $\beta=\frac{ps-1}{p-1}+1$ and $u(x)=f(x)\rho(x)^{\beta-1}+g(x)=u_0(x)+g(x)$ for some $f,g\in C^2(\overline{\Omega})$, then the fractional Neumann boundary conditions for $(-\Delta)^s_{\Omega,p}u$ correspond to

$$C_{p,s}\mathcal{N}_p^{2-\beta}u:=C_{p,s}|\beta-1|^{p-1}\left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2}\frac{u_0}{\rho^{\beta-1}}=g\quad\text{on}\qquad\partial\Omega, \tag{1.8}$$

where $C_{p,s}$ is an explicit constant (see formula (3.20) below). If $g\in C(\partial\Omega)$, then (1.8) means that for every $z\in\partial\Omega$,

$$C_{p,s}\mathcal{N}_p^{2-\beta}u(z):=C_{p,s}|\beta-1|^{p-1}\lim_{\Omega\ni x\to z}\left(\left|\frac{u_0(x)}{\rho(x)^{\beta-1}}\right|^{p-2}\frac{u_0(x)}{\rho(x)^{\beta-1}}\right)=g(z).$$

If $g\in L^{\frac{p}{p-1}}(\partial\Omega)$, then (1.8) means that the function $\left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2}\frac{u_0}{\rho^{\beta-1}}$ has a trace on $\partial\Omega$ and for every $v\in L^p(\partial\Omega)$ we have that

$$C_{p,s}|\beta-1|^{p-1}\int_{\partial\Omega}\left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2}\frac{u_0}{\rho^{\beta-1}}v\,d\sigma=\int_{\partial\Omega}gv\,d\sigma.$$

The function $\left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2}\frac{u_0}{\rho^{\beta-1}}$ is the object in the Green formula for the regional fractional p-Laplacian playing the role that $|\nabla u|^{p-2}\nabla u\cdot\nu$ does in the classical Green formula for the p-Laplace operator. If $0<s\le\frac{1}{p}$, since $W_0^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ coincide (see e.g. [27, Example 4.11] and [5] for the case $p=2$), we have that the Dirichlet and the fractional Neumann boundary conditions for the operator $(-\Delta)^s_{\Omega,p}$ coincide.

The rest of the paper is organized as follows. In Sect. 2 we give some notation, introduce the function spaces and recall some known results as they are used to obtain our main results. In Sect. 3 we state the main results of the paper. In particular, we introduce and give some properties of the (s,p)-normal derivative of a function u. Section 4 contains some existence results and some estimates for $(-\Delta)^s_{\Omega,p}u$ that will be used in the proofs of the main results. In Sect. 5 we prove the main results stated in Sect. 3. In the final section 6, we use the results obtained in Sect. 3 to define the realization in $L^2(\Omega)$ of the regional fractional p-Laplacian with fractional Neumann and Robin type boundary conditions.

2. Functional setup and notation

Let $\Omega\subset\mathbb{R}^N$ be an arbitrary open set with boundary $\partial\Omega$. We denote by $\mathcal{D}(\Omega)$ the space of test functions on Ω. For $k\in\mathbb{N}\cup\{0\}$ and $0<\beta\le 1$, $C^{k,\beta}(\Omega)$ is the space of all functions in $C^k(\Omega)$ whose partial derivatives of order k are locally Hölder continuous in Ω with exponent β. By $C^{k,\beta}(\overline{\Omega})$ we mean the space of all functions in $C^k(\overline{\Omega})$ whose partial derivatives of order k are uniformly Hölder continuous in Ω with exponent β. By definition,

$$C^{k,\beta}(\overline{\Omega})=\{u|_\Omega,\ u\in C^{k,\beta}(\mathcal{U})\ \text{ for some open set }\ \mathcal{U}\supset\overline{\Omega}\}.$$

For $p\in[1,\infty)$ and $s\in(0,1)$, we denote by

$$W^{s,p}(\Omega):=\left\{u\in L^p(\Omega):\ \int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy<\infty\right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)}:=\left(\int_\Omega|u|^p\,dx+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{1/p}.$$

We let

$$W_0^{s,p}(\Omega)=\overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}.$$

For more information on fractional order Sobolev spaces we refer to [1, 14, 20, 27] and their references.

Let $\Omega\subset\mathbb{R}^N$ be an open set and $\delta>0$ a real number. We shall use the notation:

$$\begin{aligned}
\rho(x)&=\operatorname{dist}(x,\partial\Omega)=\inf\{|y-x|:\ y\in\partial\Omega\},\quad x\in\Omega,\\
d_\Omega&=\text{diameter of }\ \Omega=\sup\{|x-y|:\ x,y\in\Omega\},\\
\Omega_\delta^\star&=\{x\in\Omega:\ 0<\rho(x)<\delta\},\\
\Omega^\delta&=\{x\in\Omega:\ \rho(x)>\delta\},\\
\vec{n}(z)&=\ \text{the inner normal vector of }\ \partial\Omega\ \text{ at the point }\ z\in\partial\Omega,\\
\nu(z)&=-\vec{n}(z)\ \text{the outer normal vector of }\ \partial\Omega\ \text{ at the point }\ z\in\partial\Omega,
\end{aligned}$$

$$\chi_A = \text{the characteritic function of the set } \ A \subset \mathbb{R}^N,$$

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{N} x_i y_i \ \text{ for } \ x = (x_1, \dots, x_N), \ y = (y_1, \dots, y_N) \in \mathbb{R}^N.$$

An open set $\Omega \subset \mathbb{R}^N$ is said to be of class $C^{1,1}$, if there exists a constant $r_0 > 0$ such that for every $z \in \partial\Omega$, we can find a $C^{1,1}$-function $\Gamma_z : \mathbb{R}^{N-1} \to \mathbb{R}$ and an orthonormal coordinate system such that

$$\Omega \cap B(z, r_0) = \{y = (y_1, \dots, y_N) : \ y_N > \Gamma_z(y_1, \dots, y_{N-1})\} \cap B(z, r_0). \quad (2.1)$$

It follows from (2.1) that for every $z \in \partial\Omega$ and $y \in \Omega \cap B(z, \frac{r_0}{2})$,

$$\rho(y) = \inf\{|y - \xi| : \ \xi = (\xi_1, \dots, \xi_N) \in B(z, r_0), \ \xi_N = \Gamma_z(z_1, \dots, z_{N-1})\}. \quad (2.2)$$

The following result is taken from [19, Proof of Lemma 14.16] (see also [23, Lemmas 2.1 and 2.2]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$ of class $C^{1,1}$. Then for every $\delta > 0$ small enough and $x \in \Omega_\delta^\star$, there exists a unique point $z = z(x) \in \partial\Omega$ such that*

$$|x - z| = \rho(x) \ \text{ and } \ x - z = -\nu(z)\rho(x) = \vec{n}(z)\rho(x). \quad (2.3)$$

Moreover, $z \in C^1(\overline{\Omega}^\star_{\delta_0})$ for some $\delta_0 \in (0, \frac{r_0}{2})$ and

$$\nabla\rho(x) = \frac{x - z(x)}{|x - z(x)|}, \qquad \forall\, x \in \Omega^\star_{\delta_0}.$$

In addition, we have that $\rho \in C^2(\overline{\Omega}^\star_{\delta_0})$. Here, $r_0 > 0$ is the constant specified in (2.1).

For $r, t > 0$ we denote by $\mathbb{B}(r, t)$ the usual beta function defined by

$$\mathbb{B}(r, t) = \int_0^1 \tau^{r-1}(1 - \tau)^{t-1}\, d\tau.$$

The following inequalities will be useful. Let $a > 0, b > 0$ and $1 < \alpha < 2$. Then

$$|b^{\alpha-1} - a^{\alpha-1}| \le b^{\alpha-2}|b - a|. \quad (2.4)$$

If $a, b \in \mathbb{R}$ and $q \ge 1$ (see e.g. [21]), then

$$||a|^{q-1}a - |b|^{q-1}b| \le q(|a|^{q-1} + |b|^{q-1})|a - b|. \quad (2.5)$$

3. Main results

Before, we state the main results of the paper, we need first to introduce the (s, p)-normal derivative of a function u and give its properties.

3.1. The (s,p)-normal derivative

Throughout the rest of this subsection, without any mention, $\Omega \subset \mathbb{R}^N$ denotes a bounded open set of class $C^{1,1}$ with boundary $\partial\Omega$.

We define the following boundary operator.

Definition 3.1. *For $0 \le \alpha < 2$, $p \in (1,\infty)$, $u \in C^1(\Omega)$ and $z \in \partial\Omega$, we define the operator $\mathcal{N}_p^\alpha$ on $\partial\Omega$ by*

$$\begin{aligned}\mathcal{N}_p^\alpha u(z) &:= -\lim_{t\downarrow 0}\left|\frac{du(z+t\vec{n}(z))}{dt}\right|^{p-2}\frac{du(z+t\vec{n}(z))}{dt}t^{\alpha(p-1)}\\ &= -\lim_{t\downarrow 0}\left|\frac{du(z+t\vec{n}(z))}{dt}t^\alpha\right|^{p-2}\frac{du(z+t\vec{n}(z))}{dt}t^\alpha, \qquad (3.1)\end{aligned}$$

provided that the limit exists.

The operator $\mathcal{N}_p^\alpha u$ is linear in u if and only if $p=2$ and $\mathcal{N}_2^\alpha$ in one dimension has been introduced by Guan and Ma [18, Definition 7.1]. The extension to the N-dimensional case has been given by Guan in [17, Definition 2.1]. Let $0 < s := 1-\frac{\alpha}{2} \le 1$ so that $\alpha = 2-2s$. We call the function $\mathcal{N}_p^\alpha u$ *the (s,p)-normal derivative* of u.

Remark 3.2. If $\alpha = 0$, that is, if $s=1$, then it follows from (3.1) that for every $p \in (1,\infty)$, $u \in C^1(\overline{\Omega})$ and $z \in \partial\Omega$,

$$\mathcal{N}_p^0 u(z) = -|\nabla u|^{p-2}\nabla u \cdot \vec{n}(z) = |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}(z), \qquad (3.2)$$

where $\frac{\partial u}{\partial\nu}(z)$ is the normal derivative of u at the point z. We have shown that the $(1,p)$-normal derivative of u coincides with the p-normal derivative mentioned in the introduction. Moreover, it follows from (3.1) that if $u \in C^1(\overline{\Omega})$, then $\mathcal{N}_p^\alpha u(z) = 0$ for every $0<\alpha<2$, $z \in \partial\Omega$ and $p \in (1,\infty)$.

Let $\beta > 0$ be a real number. It follows from Lemma 2.1 (see also [17]) that, there exist a constant $\delta > 0$ (depending on Ω) and a function $h_\beta \in C^2(\Omega)$ (depending on Ω and β) such that

$$h_\beta(x) = \begin{cases}\rho(x)^{\beta-1} & \forall\, x \in \Omega_\delta^\star, \quad \beta \in (0,1)\cup(1,\infty),\\ \ln(\rho(x)) & \forall\, x\in\Omega_\delta^\star, \quad \beta = 1.\end{cases} \qquad (3.3)$$

Next, for $k=1$ or $k=2$, we define the space

$$C_\beta^k(\overline{\Omega}) := \{u:\ u(x) = f(x)h_\beta(x) + g(x),\ \forall\, x \in \Omega,\ \text{ for some } f,g \in C^k(\overline{\Omega})\},$$

where $h_\beta \in C^2(\Omega)$ has been given in (3.3). When $\beta \in (1,\infty)$, we assume that the function $u \in C_\beta^k(\overline{\Omega})$ is defined on $\overline{\Omega}$ by continuous extension. Since Ω is smooth, we have that $C_\beta^1(\overline{\Omega}) = C^1(\overline{\Omega})$ when $\beta \ge 2$.

We have the following explicit representation of $\mathcal{N}_p^\alpha$ for functions in $C_\beta^1(\overline{\Omega})$.

Lemma 3.3. *Let $0<\beta<2$, $p\in(1,\infty)$ and $u \in C_\beta^1(\overline{\Omega})$. Then for every $z \in \partial\Omega$,*

$$\mathcal{N}_p^{2-\beta}u(z) = -(\beta-1)|\beta-1|^{p-2}|f(z)|^{p-2}f(z), \quad \beta\in(0,1)\cup(1,2); \qquad (3.4)$$

and

$$\mathcal{N}_p^1 u(z) = -|f(z)|^{p-2} f(z). \tag{3.5}$$

Proof. Let $0 < \beta < 2$, $u := fh_\beta + g \in C^1_\beta(\overline{\Omega})$ and $\delta > 0$ small enough. Let $z \in \partial\Omega$, $0 < t < \delta$ and $x = z + t\vec{n}(z)$. Then $x \in \Omega^\star_\delta$ and $x \to z$ if and only if $t \downarrow 0$.

First, let $\beta \in (0,1) \cup (1,2)$. By definition, $u(x) = f(x)\rho(x)^{\beta-1} + g(x)$ for some $f, g \in C^1(\overline{\Omega})$. Then

$$\begin{aligned} u(x) = u(z + t\vec{n}(z)) &= f(z + t\vec{n}(z))|z + t\vec{n}(z) - z|^{\beta-1} + g(z + t\vec{n}(z)) \\ &= f(z + t\vec{n}(z))t^{\beta-1} + g(z + t\vec{n}(z)). \end{aligned}$$

A simple calculation gives

$$\frac{du(z + t\vec{n}(z))}{dt} = \frac{df(z + t\vec{n}(z))}{dt} t^{\beta-1} + (\beta - 1)f(z + t\vec{n}(z))t^{\beta-2} + \frac{dg(z + t\vec{n}(z))}{dt}. \tag{3.6}$$

Note that

$$\begin{aligned} &\left|\frac{du(z + t\vec{n}(z))}{dt}\right|^{p-2} \frac{du(z + t\vec{n}(z))}{dt} t^{(2-\beta)(p-1)} \\ &\quad = \left|\frac{du(z + t\vec{n}(z))}{dt} t^{2-\beta}\right|^{p-2} \frac{du(z + t\vec{n}(z))}{dt} t^{(2-\beta)}. \end{aligned} \tag{3.7}$$

It follows from (3.6) that

$$\frac{du(z + t\vec{n}(z))}{dt} t^{2-\beta} = \frac{df(z + t\vec{n}(z))}{dt} t + (\beta - 1)f(z + t\vec{n}(z)) + \frac{dg(z + t\vec{n}(z))}{dt} t^{2-\beta}.$$

Since $f, g \in C^1(\overline{\Omega})$, we have that

$$\begin{aligned} -\lim_{t\downarrow 0} \frac{du(z + t\vec{n}(z))}{dt} t^{2-\beta} &= -\lim_{t\downarrow 0} \left[\frac{df(z + t\vec{n}(z))}{dt} t + (\beta - 1)f(z + t\vec{n}(z)) \right. \\ &\qquad \left. + \frac{dg(z + t\vec{n}(z))}{dt} t^{2-\beta}\right] \qquad (3.8) \\ &= -\lim_{t\downarrow 0} (\beta - 1)f(z + t\vec{n}(z)) = -(\beta - 1)f(z). \qquad (3.9) \end{aligned}$$

It follows from (3.7) and (3.8) that

$$\mathcal{N}_p^{2-\beta} u(z) = (1 - \beta)|\beta - 1|^{p-2}|f(z)|^{p-2} f(z),$$

and we have shown (3.4).

Now, let $\beta = 1$ and $u \in C^1_1(\overline{\Omega})$. Then

$$\begin{aligned} u(x) = u(z + t\vec{n}(z)) &= f(z + t\vec{n}(z)) \ln(|z + t\vec{n}(z) - z|) + g(z + t\vec{n}(z)) \\ &= f(z + t\vec{n}(z)) \ln(t) + g(z + t\vec{n}(z)). \end{aligned}$$

Proceeding as above we get the identity (3.5) and the proof is finished. $\square$

The following result gives a second characterization of $\mathcal{N}_p^{2-\beta}$.

Lemma 3.4. *Let* $0 < \beta < 2$, $p \in (1, \infty)$ *and* $u := fh_\beta + g \in C^1_\beta(\overline{\Omega})$. *Let* $u_0 := fh_\beta$ *so that* $u = u_0 + g$. *Then for every* $z \in \partial\Omega$,

$$\mathcal{N}_p^{2-\beta} u(z) = -(\beta - 1)|\beta - 1|^{p-2} \lim_{\Omega \ni x \to z} \left| \frac{u_0(x)}{\rho(x)^{\beta-1}} \right|^{p-2} \frac{u_0(x)}{\rho(x)^{\beta-1}}, \quad \beta \in (0, 1) \cup (1, 2), \tag{3.10}$$

and if $\beta = 1$, *then for every* $z \in \partial\Omega$,

$$\mathcal{N}_p^1 u(z) = -\lim_{\Omega \ni x \to z} \left| \frac{u_0(x)}{\ln \rho(x)} \right|^{p-2} \frac{u_0(x)}{\ln \rho(x)}. \tag{3.11}$$

Proof. Let $0 < \beta < 2$, $p \in (1, \infty)$ and $u := fh_\beta + g = u_0 + g \in C^1_\beta(\overline{\Omega})$. Since $g \in C^1(\overline{\Omega})$, it follows from Remark 3.2 that

$$\lim_{t \downarrow 0} \frac{dg(z + t\vec{n}(z))}{dt} t^{2-\beta} = 0.$$

Therefore,

$$\begin{aligned} \lim_{t \downarrow 0} \frac{du(z + t\vec{n}(z))}{dt} t^{2-\beta} &= \lim_{t \downarrow 0} \frac{du_0(z + t\vec{n}(z))}{dt} t^{2-\beta} + \lim_{t \downarrow 0} \frac{dg(z + t\vec{n}(z))}{dt} t^{2-\beta} \\ &= \lim_{t \downarrow 0} \frac{du_0(z + t\vec{n}(z))}{dt} t^{2-\beta}. \end{aligned}$$

Let $\delta > 0$ be small enough, $z \in \partial\Omega$, $0 < t < \delta$ and let $x = z + t\vec{n}(z)$. Then $x \in \Omega_\delta^\star$ and $t \downarrow 0$ if and only if $x \to z$. Using Lemma 3.3, we get that if $\beta \in (0, 1) \cup (1, 2)$, then

$$\begin{aligned} \lim_{t \downarrow 0} \frac{du_0(z + t\vec{n}(z))}{dt} t^{2-\beta} &= (\beta - 1) \lim_{t \downarrow 0} f(z + t\vec{n}(z)) = (\beta - 1) \lim_{t \downarrow 0} \frac{u_0(z + t\vec{n}(z))}{\rho(z + t\vec{n}(z))^{\beta-1}} \\ &= (\beta - 1) \lim_{\Omega \ni x \to z} \frac{u_0(x)}{\rho(x)^{\beta-1}}. \end{aligned}$$

Similarly we get that

$$\lim_{t \downarrow 0} \frac{du_0(z + t\vec{n}(z))}{dt} t = \lim_{\Omega \ni x \to z} \frac{u_0(x)}{\ln \rho(x)}.$$

We have shown (3.10) and (3.11). The proof of lemma is finished. □

Next, we give a third characterization of $\mathcal{N}_p^{2-\beta}$.

Lemma 3.5. *Let* $1 < \beta \leq 2$, $p \in (1, \infty)$ *and* $u := fh_\beta + g \in C^1_\beta(\overline{\Omega})$. *Then for every* $z \in \partial\Omega$,

$$\mathcal{N}_p^{2-\beta} u(z) = -|\beta - 1|^{p-1} \lim_{\Omega \ni x \to z} \left(\left| \frac{u(x) - u(z)}{\rho(x)^{\beta-1}} \right|^{p-2} \frac{u(x) - u(z)}{\rho(x)^{\beta-1}} \right). \tag{3.12}$$

Proof. Let $p \in (1,\infty)$. First, let $\beta \in (1,2)$ and $u := fh_\beta + g \in C^1_\beta(\overline{\Omega})$. Let $\delta > 0$ be small enough, $z \in \partial\Omega$, $0 < t < \delta$ and $x = z + t\vec{n}(z)$. By (3.3), $u(x) = f(x)\rho(x)^{\beta-1} + g(x)$. Using (3.4), we get that

$$\begin{aligned}\mathcal{N}_p^{2-\beta}u(z) =& -|\beta-1|^{p-1}|f(z)|^{p-2}f(z)\\ =& -|\beta-1|^{p-1}\lim_{\Omega\ni x\to z}\left(\left|\frac{u(x)-g(x)}{\rho(x)^{\beta-1}}\right|^{p-2}\frac{u(x)-g(x)}{\rho(x)^{\beta-1}}\right).\end{aligned}$$

Since $g \in C^1(\overline{\Omega})$, $u \in C(\overline{\Omega})$, $u(z) = g(z)$ and $g(x) - g(z) = \nabla g(\eta)(x-z)$ for some η in the line segment from x to z (by using the generalized mean value theorem), we have that

$$\begin{aligned}\frac{u(x)-g(x)}{\rho(x)^{\beta-1}} =& \frac{u(x)-u(z)}{\rho(x)^{\beta-1}} - \frac{g(x)-u(z)}{\rho(x)^{\beta-1}} = \frac{u(x)-u(z)}{\rho(x)^{\beta-1}} - \frac{g(x)-g(z)}{\rho(x)^{\beta-1}}\\ =& \frac{u(x)-u(z)}{\rho(x)^{\beta-1}} - \frac{\nabla g(\eta)\cdot(x-z)}{\rho(x)^{\beta-1}}.\end{aligned} \tag{3.13}$$

Since $g \in C^1(\overline{\Omega})$ and $2-\beta > 0$, we have that

$$\lim_{\Omega\ni x\to z}\left|\frac{\nabla g(\eta)\cdot(x-z)}{\rho(x)^{\beta-1}}\right| = \lim_{x\to z}|\nabla g(\eta)||x-z|^{2-\beta} = 0. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\lim_{\Omega\ni x\to z}\frac{u(x)-g(x)}{\rho(x)^{\beta-1}} = \lim_{\Omega\ni x\to z}\frac{u(x)-u(z)}{\rho(x)^{\beta-1}},$$

and we have shown (3.12) for $1 < \beta < 2$.

Now, let $\beta = 2$ and $u \in C^1(\overline{\Omega})$. By (3.2),

$$\mathcal{N}_p^0 u = |\nabla u|^{p-2}\nabla u\cdot\nu = |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}.$$

We have to prove that

$$-\lim_{x\to z}\frac{u(x)-u(z)}{\rho(x)} = \nabla u\cdot\nu(z).$$

Let $z \in \partial\Omega$ and $x \in \Omega$. Note that $u(x) - u(z) = \nabla u(\xi)\cdot(x-z)$ for some point ξ in the line segment from x to z. Hence,

$$-\lim_{\Omega\ni x\to z}\frac{u(x)-u(z)}{\rho(x)} = -\lim_{x\to z}\nabla u(\xi)\cdot\frac{x-z}{|x-z|} = -\nabla u\cdot\vec{n}(z) = \nabla u\cdot\nu(z).$$

We have shown (3.12) for $\beta = 2$ and this completes the proof of the lemma. □

Remark 3.6. It follows also from (3.12) that, if $1 < \beta < 2$ and $u \in C^1(\overline{\Omega})$, then $\mathcal{N}_p^{2-\beta}u(z) = 0$ for every $p \in (1,\infty)$ and $z \in \partial\Omega$. Indeed,

$$\lim_{\Omega\ni x\to z}\frac{u(x)-u(z)}{\rho(x)^{\beta-1}} = \lim_{t\downarrow 0}\frac{u(z+t\vec{n})-u(z)}{t}t^{2-\beta} = 0.$$

We conclude this subsection with the following useful result.

Lemma 3.7. *Let* $1 < \beta < 2$, $p \in (1,\infty)$, $u := fh_\beta + g \in C^1_\beta(\overline{\Omega})$ *and* $z \in \partial\Omega$. *Then for any unit vector* $\vec{\theta}$ *such that* $\langle \vec{n}(z), \vec{\theta}\rangle > 0$, *we have*

$$\lim_{t\downarrow 0} \left|\frac{du(z+t\vec{\theta})}{dt} t^{2-\beta}\right|^{p-2} \frac{du(z+t\vec{\theta})}{dt} t^{2-\beta}$$
$$= (\beta-1)^{p-1}|f(z)|^{p-2}f(z)\langle \vec{n}(z), \vec{\theta}\rangle^{(p-1)(\beta-1)}. \tag{3.15}$$

Proof. By [17, Lemma 2.7],

$$\lim_{t\downarrow 0} \frac{du(z+t\vec{\theta})}{dt} t^{2-\beta} = (\beta-1)f(z)\langle \vec{n}(z), \vec{\theta}\rangle^{\beta-1}. \tag{3.16}$$

Now (3.15) follows from (3.16) by using (3.7). □

3.2. The integration by parts formula

Throughout this subsection without any mention, $\Omega \subset \mathbb{R}^N$ denotes a bounded open set of class $C^{1,1}$ with boundary $\partial\Omega$. We have the first integration by parts formula which is valid for smooth functions.

Theorem 3.8. *Let* $p \in (1,\infty)$ *and* $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$. *Then for every* $u \in C^2(\overline{\Omega})$ *and* $v \in W^{s,p}(\Omega)$,

$$\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx$$
$$= \frac{C_{N,p,s}}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy. \tag{3.17}$$

The fractional Green formula associated with the operator $(-\Delta)^s_{\Omega,p}$ is given in the following theorem.

Theorem 3.9. *Let* $p \in (1,\infty)$, $\max\{\frac{1}{p}, \frac{p-1}{p}\} < s < 1$ *and* $\beta := \frac{ps-1}{p-1} + 1$. *Then, for every* $u \in C^2_\beta(\overline{\Omega})$ *and* $v \in W^{s,p}(\Omega)$,

$$\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx$$
$$= \frac{C_{N,p,s}}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dx\, dy$$
$$- B_{N,p,s}\int_{\partial\Omega} v\mathcal{N}^{2-\beta}_p u\, d\sigma, \tag{3.18}$$

where $\mathcal{N}^{2-\beta}_p$ *is the boundary operator defined in* (3.1), *the constant* $B_{N,p,s}$ *is such that*

$$\frac{C_{1,p,s}}{C_{N,p,s}} B_{N,p,s} := C_{p,s}\int_{\{|x|=1, x_N>0, x\in\mathbb{R}^N\}} x_N^{\beta(p-1)}\, d\sigma, \tag{3.19}$$

and the constant $C_{p,s}$ *is given by*

$$C_{p,s} = \frac{(p-1)C_{1,p,s}}{(ps-(p-2))(ps-(p-2)-1)}\int_0^\infty \frac{(1\vee s)^{p-ps-1} - |1-s|^{(p+2)+1-ps}}{s^{p-ps}} ds. \tag{3.20}$$

Remark 3.10. If $N = 1$, then

$$\int_{\{|x|=1, x_N>0, x\in\mathbb{R}^N\}} x_N^{\beta(p-1)}\, d\sigma = 1.$$

If $N \geq 2$, then using polar coordinates and a change of variable, we get that

$$\begin{aligned}
\int_{\{|x|=1, x_N>0, x\in\mathbb{R}^N\}} x_N^{\beta(p-1)}\, d\sigma &= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^{\frac{\pi}{2}} \cos^{\beta(p-1)}(\theta) \sin^{N-2}(\theta)\, d\theta \\
&= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \frac{1}{2} \int_0^1 t^{\frac{\beta(p-1)+1}{2}-1}(1-t)^{\frac{N-1}{2}-1}\, dt \\
&= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \mathbb{B}\left(\frac{\beta(p-1)+1}{2}, \frac{N-1}{2}\right) \\
&= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \frac{\Gamma\left(\frac{\beta(p-1)+1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{\beta(p-1)+N}{2}\right)} \\
&= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \frac{\Gamma\left(\frac{ps+p-1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{ps+p+N-2}{2}\right)}. \qquad (3.21)
\end{aligned}$$

Replacing (3.21) into (3.19) and using the expressions of $C_{1,p,s}$ and $C_{N,p,s}$ given in (1.2), we get that $B_{N,p,s} = C_{p,s}$ given in (3.20) and hence, it does not depend on N. We think that a careful calculation of $C_{p,s}$ will give a constant that does not depend on p and depends only on s. Since this is not the main concern of the present paper, we will not go into details. We also mention that we have chosen our constant $C_{N,p,s}$ in (1.2) in a way such that

$$\frac{C_{N,p,s}}{C_{1,p,s}} \int_{\{|x|=1, x_N>0, x\in\mathbb{R}^N\}} x_N^{\beta(p-1)}\, d\sigma = 1,$$

and $C_{N,2,s}$ coincides with the known constant in the linear case $p = 2$ included in any reference on this topic.

Corollary 3.11. *Let p, s and β be as in the statement of Theorem 3.9. Then, for every $u := fh_\beta + g = u_0 + g \in C^2_\beta(\overline{\Omega})$ and $v \in W^{s,p}(\Omega)$,*

$$\begin{aligned}
\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx = \frac{C_{N,p,s}}{2} \int_\Omega \int_\Omega |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dx\, dy \\
+ C_{p,s}|\beta-1|^{p-1} \int_{\partial\Omega} v \left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2} \left(\frac{u_0}{\rho^{\beta-1}}\right) d\sigma, \qquad (3.22)
\end{aligned}$$

where for every $z \in \partial\Omega$,

$$\left|\frac{u_0}{\rho^{\beta-1}}\right|^{p-2} \left(\frac{u_0}{\rho^{\beta-1}}\right)(z) := \lim_{\Omega\ni x\to z} \left(\left|\frac{u_0(x)}{\rho(x)^{\beta-1}}\right|^{p-2} \frac{u_0(x)}{\rho(x)^{\beta-1}}\right).$$

Corollary 3.12. *Let p, s and β be as in the statement of Theorem 3.9. Then, for every $u \in C^2_\beta(\overline{\Omega})$ and $v \in W^{s,p}(\Omega)$,*

$$\int_\Omega v(-\Delta)^s_{\Omega,p}u\,dx = \frac{C_{N,p,s}}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy$$
$$+ C_{p,s}|\beta-1|^{p-1}$$
$$\int_{\partial\Omega} v(z)\lim_{\Omega\ni x\to z}\left(\left|\frac{u(x)-u(z)}{\rho(x)^{\beta-1}}\right|^{p-2}\frac{u(x)-u(z)}{\rho(x)^{\beta-1}}\right)\,d\sigma_z. \quad (3.23)$$

Remark 3.13. We make some comments about the Green formula given in Theorem 3.9.

(a) The identity (3.18) for the case $p = 2$ has been proved in [17, Theorem 3.3] under the assumption that v also belongs to $C^2_\beta(\overline{\Omega})$. The case $v \in W^{s,2}(\Omega)$ is obtained by a simple density argument (see e.g. [27]). The identities (3.22) and (3.23) in the case $p = 2$ are contained in [27]. We notice that the left-hand side integral in (3.18) makes sense since $v \in W^{s,p}(\Omega)$, and $(-\Delta)^s_{\Omega,p}u \in L^q(\Omega)$ for every $q \in [1,\infty)$ (by Proposition 4.1 below), and the right hand-side integrals also make sense since $u, v \in W^{s,p}(\Omega)$ (by Proposition 5.2 below) and $\mathcal{N}_p^{2-\beta}u(z) = |\beta-1|^{p-1}|f(z)|^{p-2}f(z)$, where $f \in C^2(\overline{\Omega})$ is the function from the definition of $C^2_\beta(\overline{\Omega})$.

(b) We have assumed that $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$. This is not a restriction since by [27, Corollary 4.9], if $0 < s \le \frac{1}{p}$ then $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$, and hence, there will be no boundary integral in the right hand side of (3.18). Moreover, in that case, it is easy to show that for every $u \in W^{s,p}_0(\Omega)$ such that $(-\Delta)^s_{\Omega,p}u \in L^{\frac{p}{p-1}}(\Omega)$ and for every $v \in W^{s,p}_0(\Omega)$, we have

$$\int_\Omega v(-\Delta)^s_{\Omega,p}u\,dx$$
$$= \frac{C_{N,p,s}}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

(c) Since $\mathcal{N}_p^{2-\beta}u = 0$ for every $u \in C^1(\overline{\Omega})$ and $p \in (1,\infty)$, we have that in (3.18) there will be no boundary term when the function u is smooth. This is surprising if one compares with the classical Green formula for the p-Laplace operator given in (1.1) where the boundary terms exist and are well defined for smooth functions. But there is an explanation. Since the operator $(-\Delta)^s_{\Omega,p}$ is non-local, then the values of $(-\Delta)^s_{\Omega,p}u(x)$, no matter how far $\mathrm{dist}(x,\partial\Omega)$ is, is always effected by the values of u and its first order derivatives near the boundary. Consequently if u and its first order derivatives are uniformly bounded on Ω (which is the case for functions in $C^1(\overline{\Omega})$), then the integrals $\int_\Omega v(-\Delta)^s_{\Omega,p}u\,dx$ and $\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{Np2s}}\,dxdy$ have accumulated enough the effect of the values near the boundary and hence, there is no extra boundary term appearing in (3.18). However, if the first order derivatives of u are not uniformly bounded, then either $(-\Delta)^s_{\Omega,p}u$ does not exist, or in the integration by parts formula we must add a term reflecting the singularity of the first order derivatives near the boundary.

(d) Let $1 < p < \infty$, $0 < s < 1$, $\Phi :\ \mathbb{R} \to \mathbb{R}$ continuous, satisfying $\Phi(0) = 0$ and the monotonicity property

$C_1|t|^p \leq \Phi(t)t \leq C_2|t|^p,\ \forall\, t \in \mathbb{R}$ and some constants $0 < C_1 \leq C_2$.

Let $K :\ \Omega \times \Omega \to \mathbb{R}$ be measurable, symmetric and there are two constants $0 < \lambda \leq \Lambda$ such that

$$\lambda \leq K(x,y)|x-y|^{N+sp} \leq \Lambda,\ \ \forall\, x,y \in \Omega,\ x \neq y.$$

Let the operator $\mathcal{L}_{\Phi,\Omega}$ be formally given by

$$\mathcal{L}_{\Phi,\Omega}u(x) = \text{P.V.} \int_\Omega K(x,y)\Phi(u(x)-u(y))\,dy,\ \ x \in \Omega.$$

We think that our method can be used to find the corresponding result in Theorem 3.9 for the operator $\mathcal{L}_{\Phi,\Omega}$. Such a result cannot be deduced directly from our theorem and needs a careful study. This will be done in a forthcoming investigation. Finally we mention that elliptic problems associated with $\mathcal{L}_{\Phi,\mathbb{R}^N}$ and satisfying the Dirichlet boundary condition on $\mathbb{R}^N\backslash\Omega$ have been recently studied in [10, 11, 22].

4. Existence and some estimates of $(-\Delta)^s_{\Omega,p}u$

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set, $p \in (1,\infty)$ and $0 < s < 1$. Recall that for $u \in \mathcal{L}^{p-1}(\Omega)$ and $x \in \Omega$,

$$\begin{aligned}(-\Delta)^s_{\Omega,p}u(x) &= \lim_{\varepsilon\downarrow 0}(-\Delta)^s_{\Omega,p,\varepsilon}u(x)\\ &= C_{N,p,s}\lim_{\varepsilon\downarrow 0}\int_{\{y\in\Omega,\ |x-y|>\varepsilon\}} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy,\end{aligned}$$

provided that the limit exists. The main concern here is to prove the following theorem which is the main result of this section.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u \in C^2_\beta(\overline{\Omega})$. Then $(-\Delta)^s_{p,\Omega}u \in L^q(\Omega)$ for every $q \in [1,\infty)$.*

To prove the theorem, we need some preparation.

Remark 4.2. Let $\Omega \subset \mathbb{R}^N$ be an open set and $K \subset \Omega$ a compact set. It is well-known that there exists a constant $R_K > 0$ such that

$$\Omega_K := \{x \in \mathbb{R}^N :\ \text{dist}(x,K) \leq R_K\} \subset \Omega.$$

In that case for $x \in K$, $B(x,R_K)$ is the so-called Lebesgue ball.

Next, we give some existence results and some estimates.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in (1,\infty)$, $\alpha \in [0,1]$, $u \in C^{1,\alpha}(\Omega)$ and $K \subset \Omega$ a compact set. Then the following assertions hold.*

(a) *If $p \geq 2$, then there exist two constants $C = C(K,u) > 0$ and $R_K > 0$ such that for every $x \in K$ and $z \in B(x, R_K)$, we have*

$$\Big||u(x)-u(x+z)|^{p-2}(u(x)-u(x+z))+(|u(x)-u(x-z)|^{p-2}(u(x)-u(x-z))\Big| \\ \leq C|z|^{\alpha+p-1}. \tag{4.1}$$

(b) *If $1 < p < 2$, then there exist two constants $C = C(K,u) > 0$ and $R_K > 0$ such that for every $x \in K$ and $z \in B(x, R_K)$, we have*

$$||u(x) - u(x+z)|^{p-2}(u(x) - u(x+z)) + (|u(x) - u(x-z)|^{p-2}(u(x) \\ - u(x-z))| \leq C|z|^{(\alpha+1)(p-1)}. \tag{4.2}$$

Proof. Let Ω_K and $R_K > 0$ be given by Remark 4.2, $\alpha \in [0,1]$ and $u \in C^{1,\alpha}(\Omega)$.

(a) Let $p \geq 2$. Since $u \in C^{1,\alpha}(\Omega_K)$, using the mean value theorem, we have that for $x \in K$, $z \in B(x, R_K)$, there exist $\xi \in \Omega_K$ in the line segment from x to $x+z$ and $\eta \in \Omega_K$ in the line segment from x to $x-z$ such that

$$u(x) - u(x+z) = \nabla u(\xi) \cdot z \qquad \text{and} \qquad u(x) - u(x-z) = -\nabla u(\eta) \cdot z.$$

Using these identities and (2.5) we get that

$$\begin{aligned} &||u(x)-u(x+z)|^{p-2}(u(x)-u(x+z))+(|u(x)-u(x-z)|^{p-2}(u(x) - u(x-z))| \\ &\quad = ||\nabla u(\xi) \cdot z|^{p-2}\nabla u(\xi) \cdot z - |\nabla u(\eta) \cdot z|^{p-2}\nabla u(\eta) \cdot z| \\ &\quad \leq (p-1) \sup_{B(x,R_K)} |\nabla u|^{p-2}|z|^{p-2}|(\nabla u(\xi) - \nabla u(\eta)) \cdot z| \\ &\quad \leq C\|\nabla u\|^{p-1}_{C^{0,\alpha}(\Omega_K)}|z|^{\alpha+p-1}, \end{aligned}$$

and we have shown (4.1).

(b) Let $1 < p < 2$. Since the function $t \mapsto |t|^{p-2}t$ is globally $(p-1)$-Hölder continuous, then with the same notation as in part (a), we get that

$$\begin{aligned} &||u(x)-u(x+z)|^{p-2}(u(x)-u(x+z))+(|u(x)-u(x-z)|^{p-2}(u(x)-u(x-z))| \\ &\quad \leq C|(\nabla u(\xi) - \nabla u(\eta)) \cdot z| \\ &\quad \leq C\|\nabla u\|^{p-1}_{C^{0,\alpha}(\Omega_K)}|z|^{\alpha(p-1)}|z|^{p-1}. \end{aligned}$$

We have shown (4.2) and the proof of lemma is finished.

□

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in (1,\infty)$, $s \in (0,1)$ and $u \in \mathcal{L}^{p-1}(\Omega)$. Then the following assertions hold.*

(a) *Let $0 < s < \frac{p-1}{p}$. If $u \in C^{0,\alpha}(\Omega)$ for some α satisfying $\alpha(p-1) > ps$, then $(-\Delta)^s_{\Omega,p}u(x)$ exists for every $x \in \Omega$. In addition, if $u \in C^{0,\alpha}(\overline{\Omega})$, then $(-\Delta)^s_{\Omega,p}u$ is continuous on Ω and $(-\Delta)^s_{\Omega,p,\varepsilon}u$ converges to $(-\Delta)^s_{\Omega,p}u$ locally uniformly as $\varepsilon \downarrow 0$.*

(b) *Let $p \geq 2$ and $\frac{p-1}{p} \leq s < 1$. If $u \in C^{1,\alpha}(\Omega)$ for some α satisfying $\alpha > ps-(p-1)$, then $(-\Delta)^s_{\Omega,p}u(x)$ exists for every $x \in \Omega$. In addition, if $u \in C^{1,\alpha}(\overline{\Omega})$, then $(-\Delta)^s_{\Omega,p}u$ is continuous on Ω and $(-\Delta)^s_{\Omega,p,\varepsilon}u$ converges to $(-\Delta)^s_{\Omega,p}u$ locally uniformly as $\varepsilon \downarrow 0$.*

(c) *Let $1 < p < 2$ and $\frac{1}{p} \leq s < 1$. If $u \in C^{1,\alpha}(\Omega)$ for some α satisfying $\alpha > \frac{ps-(p-1)}{p-1}$, then $(-\Delta)^s_{\Omega,p}u(x)$ exists for every $x \in \Omega$. In addition, if $u \in C^{1,\alpha}(\overline{\Omega})$, then $(-\Delta)^s_{\Omega,p}u$ is continuous on Ω and $(-\Delta)^s_{\Omega,p,\varepsilon}u$ converges to $(-\Delta)^s_{\Omega,p}u$ locally uniformly as $\varepsilon \downarrow 0$.*

Proof. By definition, if $u \in \mathcal{L}^{p-1}(\Omega)$ and $\varepsilon > 0$, then $(-\Delta)^s_{\Omega,p,\varepsilon}u$ is well defined on Ω and $(-\Delta)^s_{\Omega,p,\varepsilon}u$ is continuous where u is continuous.

(a) Let $0 < s < \frac{p-1}{p}$ and assume that $u \in C^{0,\alpha}(\Omega)$ for some α satisfying $\alpha(p-1) > ps$. Then for $x \in \Omega$, we can take $t > 0$ such that

$$M_\alpha := \sup_{y\in\Omega,|x-y|<t} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty. \tag{4.3}$$

Let $0 < \varepsilon < \delta < t$ and $x \in \Omega$. Using polar coordinates, we get that

$$\begin{aligned}
&\left|(-\Delta)^s_{\Omega,p,\delta}u(x) - (-\Delta)^s_{\Omega,p,\varepsilon}u(x)\right| \\
&\quad= C_{N,p,s}\left|\int_{\{y\in\Omega:\ \varepsilon<|x-y|\leq\delta\}} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy\right| \\
&\quad\leq C_{N,p,s}\int_{\{y\in\Omega:\ \varepsilon<|x-y|\leq\delta\}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\quad\leq C_{N,p,s}M_\alpha^{p-1}\int_{\{\varepsilon<|x-y|\leq\delta\}} \frac{1}{|x-y|^{N+ps-\alpha(p-1)}}\,dy \\
&\quad= C_{N,p,s}(2\pi)^N M_\alpha^{p-1}\int_\varepsilon^\delta r^{\alpha(p-1)-ps-1}\,dr \\
&\quad= \frac{C_{N,p,s}M_\alpha^{p-1}(2\pi)^N}{\alpha(p-1)-ps}(\delta^{\alpha(p-1)-ps} - \varepsilon^{\alpha(p-1)-ps}).
\end{aligned} \tag{4.4}$$

We have shown that $(-\Delta)^s_{\Omega,p}u(x)$ exists. Now, if $u \in C^{0,\alpha}(\overline{\Omega})$, then for every compact set $K \subset \Omega$, there exist $t > 0$ and $M_\alpha > 0$ such that (4.3) holds for all $x \in K$. In this case, it follows from (4.4) that $(-\Delta)^s_{\Omega,p,\varepsilon}u$ converges to $(-\Delta)^s_{\Omega,p}u$ uniformly on K as $\varepsilon \downarrow 0$. The proof of part (a) is finished.

(b) Let $p \geq 2$, $\frac{p-1}{p} \leq s < 1$ and $u \in C^{1,\alpha}(\Omega)$ for some α satisfying $\alpha > ps-(p-1)$. Since the first derivatives of u are locally α-Hölder continuous, then for $x \in \Omega^t$ we can find $0 < \tau < t$ such that

$$G_\alpha := \sup_{y\in\Omega,|x-y|<\tau} \frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^\alpha} < \infty. \tag{4.5}$$

If $u \in C^{1,\alpha}(\overline{\Omega})$, then for every compact set $K \subset \Omega^t$, we can find $0 < \tau < t$ and $G_\alpha > 0$ such that (4.5) holds for all $x \in K$. Next let $0 < \varepsilon < \delta < \tau$

and let $K \subset \Omega$ be an arbitrary compact set. Let Ω_K and $R_K > 0$ be given by Remark 4.2. For $x \in K$ and $z \in B(x, R_K)$ we set

$$F(x,z,u(x),u(z)) := \frac{|u(x)-u(x+z)|^{p-2}(u(x)-u(x+z)) + |u(x)-u(x-z)|^{p-2}(u(x)-u(x-z))}{|x-y|^{N+sp}}. \tag{4.6}$$

Using a change of variable, (4.1) and polar coordinates, we get that for $x \in K$,

$$\begin{aligned}
&\left|(-\Delta)^s_{\Omega,p,\delta}u(x) - (-\Delta)^s_{\Omega,p,\varepsilon}u(x)\right| \\
&= C_{N,p,s}\left|\int_{\{y\in\Omega:\ \varepsilon<|x-y|\le\delta\}} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy\right| \\
&\le \frac{C_{N,p,s}}{2}\int_{\{z\in B(x,R_K),\ \varepsilon<|z|\le\delta\}}\left|\frac{F(x,z,u(x),u(z))}{|z|^{N+ps}}\right|\,dz \\
&\le \frac{C_{N,p,s}}{2}G_\alpha^{p-1}\int_{\{z\in B(x,R_K),\ \varepsilon<|z|\le\delta\}}\frac{1}{|z|^{N-1+ps-\alpha-(p-2)}}\,dz \\
&\le C_{N,p,s}G_\alpha^{p-1}\int_\varepsilon^\delta r^{\alpha-ps+(p-2)}\,dr \\
&= \frac{C_{N,p,s}G_\alpha^{p-1}}{\alpha+(p-1)-ps}\left(\delta^{\alpha+(p-1)-ps} - \varepsilon^{\alpha+(p-1)-ps}\right).
\end{aligned}$$

Since the compact K was arbitrary we get the same conclusion as in the assertion (a).

(c) The proof of this part follows the lines of the proof of part (b) by using (4.2) in Lemma 4.3. The proof of proposition is finished. □

Next, we give some estimates for $(-\Delta)^s_{\Omega,p}u$.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $p \in (1,\infty)$ and $0 < s < 1$. Then the following assertions hold.*

(a) *If $p \ge 2$, $\frac{p-1}{p} < s < 1$ and $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > sp+1-p$, then for every $x \in \Omega$,*

$$\begin{aligned}
|(-\Delta)^s_{\Omega,p}u(x)| \le& C_{N,p,s}(2\pi)^N\frac{M_1^{p-1}}{sp-p+1}\left(\rho(x)^{p-sp-1} - d_\Omega^{p-sp-1}\right) \\
&+ C_{N,p,s}(2\pi)^N\frac{G_\alpha^{p-1}}{\alpha+p-sp-1}\rho(x)^{\alpha+p-sp-1}.
\end{aligned} \tag{4.7}$$

Moreover, $(-\Delta)^s_{\Omega,p}u$ is bounded on $\Omega^\star_\delta$ for every $\delta > 0$.

(b) *If $1 < p < 2$, $\frac{1}{p} < s < 1$ and $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{sp+1-p}{p-1}$, then for every $x \in \Omega$,*

$$
\begin{aligned}
|(-\Delta)^s_{\Omega,p}u(x)| \leq & C_{N,p,s}(2\pi)^N \frac{M_1^{p-1}}{sp-p+1}\Big(\rho(x)^{p-sp-1} - d_\Omega^{p-sp-1}\Big) \\
& + C_{N,p,s}(2\pi)^N \frac{G_\alpha^{p-1}}{\alpha+p-sp-1}\rho(x)^{(\alpha+1)(p-1)-sp}. \qquad (4.8)
\end{aligned}
$$

Moreover, $(-\Delta)^s_{\Omega,p}$ is bounded in $\Omega^\star_\delta$ for every $\delta > 0$.

In (4.7) and (4.8), we have set

$$M_1 := \sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|} \qquad \text{and} \qquad G_\alpha := \sup_{x,y\in\Omega}\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^\alpha}.$$

Proof. Let $\alpha \in [0,1]$. Since $\overline{\Omega}$ is compact, it follows from the definition of $C^{1,\alpha}(\overline{\Omega})$ and Remark 4.2 that there exists an open set $\mathcal{U}$ such that $\overline{\Omega} \subset \mathcal{U}$, and for every $x \in \overline{\Omega}$ there exists a radius $R_{\overline{\Omega}} > 0$ such that $\mathcal{U}_\Omega := \{x \in \mathbb{R}^N : \mathrm{dist}(x,\overline{\Omega}) \leq R_{\overline{\Omega}}\} \subset \mathcal{U}$. Next, let $p \in (1,\infty)$ and $0 < s < 1$.

(a) Let $p \geq 2$ and $\frac{p-1}{p} < s < 1$. Let $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > sp+1-p$. Let $x \in \Omega$, $R_{\overline{\Omega}}$, $\mathcal{U}$ and $\mathcal{U}_\Omega$ be as above. For $x \in \Omega$, $z \in B(x,R_{\overline{\Omega}})$ and $u \in C^{1,\alpha}(\overline{\Omega})$, let $F(x,z,u(x),u(z))$ be as in (4.6). Using a change of variable, (4.1) in Lemma 4.3 and the fact that $|u(x)-u(y)|^{p-1} \leq M_1^{p-1}|x-y|^{p-1}$, we get that

$$
\begin{aligned}
&|(-\Delta)^s_{\Omega,p}u(x)| \\
&\leq \lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{\Omega\cap B(x,\varepsilon)^c}\left|\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\right| dy \\
&\leq \frac{1}{2}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))^c}\left|\frac{F(x,z,u(x),u(z))}{|z|^{N+sp}}\right| dz \\
&\quad + \frac{1}{2}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))\cap B(x,\varepsilon)^c}\left|\frac{F(x,z,u(x),u(z))}{|z|^{N+sp}}\right| dz \\
&\leq \frac{1}{2}M_1^{p-1}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))^c}\frac{|z|^{p-1}}{|z|^{N+sp}}\, dz \\
&\quad + \frac{1}{2}G_\alpha^{p-1}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))\cap B(x,\varepsilon)^c}\frac{|z|^{\alpha+p-1}}{|z|^{N+sp}}\, dz \\
&\leq \lim_{\varepsilon\downarrow 0} C_{N,p,s}\left[\int_{\rho(x)}^{R_{\overline{\Omega}}}\frac{M_1^{p-1}}{r^{sp-p+2}}\, dr + \int_\varepsilon^{\rho(x)}\frac{G_\alpha^{p-1}}{r^{sp-p+2-\alpha}}\, dr\right] \\
&= C_{N,p,s}(2\pi)^N\left[\frac{M_1^{p-1}}{sp-p+1}\Big(\rho(x)^{p-sp-1} - R_{\overline{\Omega}}^{p-sp-1}\Big)\right. \\
&\qquad \left. + \frac{G_\alpha^{p-1}}{\alpha+p-sp-1}\rho(x)^{\alpha+p-sp-1}\right].
\end{aligned}
$$

We have shown (4.7). It follows from (4.7) that $(-\Delta)^s_{\Omega,p}u$ is bounded on $\Omega^\star_\delta$ for every $\delta > 0$.

(b) Let $1 < p < 2$, $\frac{1}{p} < s < 1$ and $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{sp+1-p}{p-1}$. Proceeding exactly as in part (a) by using here (4.2) in Lemma 4.3, we get that

$$
\begin{aligned}
&|(-\Delta)^s_{\Omega,p}u(x)| \\
&= \lim_{\varepsilon\downarrow 0} C_{N,p,s}\left|\int_{\Omega\cap B(x,\varepsilon)^c}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy\right| \\
&\le \frac{1}{2}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))^c}\left|\frac{F(x,z,u(x),u(z))}{|z|^{N+sp}}\right|\,dz \\
&\quad+\frac{1}{2}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))\cap B(x,\varepsilon)^c}\left|\frac{F(x,z,u(x),u(z))}{|z|^{N+sp}}\right|\,dz \\
&\le \frac{1}{2}M_1^{p-1}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))^c}\frac{|z|^{p-1}}{|z|^{N+sp}}\,dz \\
&\quad+\frac{1}{2}G_\alpha^{p-1}\lim_{\varepsilon\downarrow 0} C_{N,p,s}\int_{B(x,R_{\overline{\Omega}})\cap B(x,\rho(x))\cap B(x,\varepsilon)^c}\frac{|z|^{(\alpha+1)(p-1)}}{|z|^{N+sp}}\,dz \\
&\le \lim_{\varepsilon\downarrow 0} C_{N,p,s}\left[\int_{\rho(x)}^{R}\frac{M_1^{p-1}}{r^{sp-p+2}}\,dr+\int_{\varepsilon}^{\rho(x)}\frac{G_\alpha^{p-1}}{r^{sp+1-(\alpha+1)(p-1)}}\,dr\right] \\
&= C_{N,p,s}(2\pi)^N\left[\frac{M_1^{p-1}}{sp-p+1}\left(\rho(x)^{p-sp-1}-R_\Omega^{p-sp-1}\right)\right. \\
&\quad\left.+\frac{G_\alpha^{p-1}}{(\alpha+1)(p-1)-sp}\rho(x)^{(\alpha+1)(p-1)-sp}\right],
\end{aligned}
$$

and we have shown (4.8). It follows also from (4.8) that $(-\Delta)^s_{\Omega,p}u$ is bounded on $\Omega^\star_\delta$ for every $\delta>0$. The proof of lemma is finished. □

Lemma 4.6. *Let* $\mathbb{R}^N_+ := \{(x_1,\dots,x_N)\in\mathbb{R}^N,\ x_N>0\}$, $p\in(1,\infty)$, $\max\{\frac{p-1}{p},\frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1}+1$ *and* $w_\beta(x) := \rho(x)^{\beta-1}$ *where we recall that* $\rho(x) = dist(x,\partial\mathbb{R}^N_+)$, $x\in\mathbb{R}^N_+$. *Then*

$$(-\Delta)^s_{\mathbb{R}^N_+,p}w_\beta(x)=0,\ \ \forall\, x\in\mathbb{R}^N_+. \tag{4.9}$$

Proof. For $x=(x_1,x_2,\dots,x_N)\in\mathbb{R}^N$ we write $x=(\tilde{x},x_N)$, where $\tilde{x}=(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}$. Let $x\in\mathbb{R}^N_+$ and $w_\beta(x):=\rho(x)^{\beta-1}$. Then

$$w_\beta(x)=\begin{cases} x_N^{\beta-1} & x_N>0\\ 0 & x_N\le 0,\end{cases}$$

and

$$(-\Delta)^s_{\mathbb{R}^N_+,p}w_\beta(x)=C_{N,p,s}\lim_{\varepsilon\downarrow 0^+}\int_{\mathbb{R}^N_+\setminus B(x,\varepsilon)}\frac{|x_N^{\beta-1}-y_N^{\beta-1}|^{p-2}(x_N^{\beta-1}-y_N^{\beta-1})}{|x-y|^{N+sp}}\,dy.$$

Since w_β is of class C^2 in a neighborhood of x and $w_\beta\in\mathcal{L}^{p-1}(\mathbb{R}^N)$, it follows from Proposition 4.4 that the limit exists. Let $e_N=(0,\dots,0,1)\in\mathbb{R}^N$. Using the change of variable $x_N z = y-(\tilde{x},0)$ we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N_+\setminus B(x,\varepsilon)} \frac{|x_N^{\beta-1}-y_N^{\beta-1}|^{p-2}(x_N^{\beta-1}-y_N^{\beta-1})}{|x-y|^{N+sp}}\,dy\\
&= x_N^{(\beta-1)(p-1)-ps}\int_{\mathbb{R}^N_+\setminus B(x,\varepsilon)} \frac{|1-z_N^{\beta-1}|^{p-2}(1-z_N^{\beta-1})}{|z-e_N|^{N+sp}}\,dz.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(-\Delta)^s_{\mathbb{R}^N_+,p}u(x) = C_{N,p,s}x_N^{(\beta-1)(p-1)-ps}W \quad \text{with}\\
&W = P.V.\int_{\mathbb{R}^N_+} \frac{|1-z_N^{\beta-1}|^{p-2}(1-z_N^{\beta-1})}{|z-e_N|^{N+sp}}\,dz.
\end{aligned}$$

Using a changing of variable, we get that

$$\begin{aligned}
W &:= P.V.\int_{\mathbb{R}^N_+} \frac{|1-z_N^{\beta-1}|^{p-2}(1-z_N^{\beta-1})}{|z-e_N|^{N+sp}}\,dz\\
&= P.V.\int_{\mathbb{R}^{N-1}}\int_0^\infty \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{||\xi|^2+(t-1)^2|^{\frac{N+sp}{2}}}\,dtd\xi\\
&= P.V.\int_{\mathbb{R}^{N-1}}(|\xi|^2+1)^{-\frac{N+sp}{2}}\int_0^\infty \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dtd\xi.
\end{aligned}$$

Thus

$$W = \int_{\mathbb{R}^{N-1}}(|\xi|^2+1)^{-\frac{N+sp}{2}}\,d\xi\ \text{P.V.}\int_0^\infty \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dt.$$

Using polar coordinates and the change of variable $\tau = r^2/(r^2+1)$ we get that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}}(|\xi|^2+1)^{-\frac{N+sp}{2}}\,d\xi &= \omega_{N-1}\int_0^\infty r^{N-2}(r^2+1)^{-\frac{N+sp}{2}}\,dr\\
&= \frac{\omega_{N-1}}{2}\int_0^1 \tau^{\frac{N-1}{2}-1}(1-\tau)^{\frac{sp+1}{2}-1}\,d\tau\\
&= \frac{\omega_{N-1}}{2}\mathbb{B}\left(\frac{N-1}{2},\frac{ps+1}{2}\right),
\end{aligned}$$

where we recall that $\mathbb{B}$ denotes the usual Beta function and ω_{N-1} is the $(N-2)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^{N-1}$.

Let $\varepsilon \in (0,1)$. Calculating and using a changing a variable we get that

$$\begin{aligned}
&\int_{(0,\infty)\setminus(1-\varepsilon,1+\varepsilon)} \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dt\\
&= \int_0^{1-\varepsilon} \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dt + \int_{1+\varepsilon}^\infty \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dt\\
&= \int_0^{1-\varepsilon} \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\,dt + \int_0^{1/(1+\varepsilon)} \frac{|1-t^{1-\beta}|^{p-2}(1-t^{1-\beta})t^{-2}}{|1/t-1|^{sp+1}}\,dt\\
&= \int_0^{1-\varepsilon} \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\left[1-t^{sp-(\beta-1)(p-1)-1}\right]\,dt + R_\varepsilon
\end{aligned}$$

with

$$R_\varepsilon := \int_{1-\varepsilon}^{1/(1+\varepsilon)} |t^{\beta-1}-1|^{p-2}(1-t^{\beta-1})(1-t)^{-sp-1}t^{sp-(\beta-1)(p-1)-1}\, dt.$$

Since $\beta = \frac{ps-1}{p-1}+1$, we have that $sp-(\beta-1)(p-1)-1 = sp-sp+1-1=0$. Therefore,

$$R_\varepsilon = \int_{1-\varepsilon}^{1/(1+\varepsilon)} |t^{\beta-1}-1|^{p-2}(1-t^{\beta-1})(1-t)^{-sp-1}\, dt.$$

Since $1-\varepsilon \le \dfrac{1}{1+\varepsilon} < 1-\varepsilon+\varepsilon^2 < 1$, it follows that

$$\begin{aligned}
|R_\varepsilon| &\le \int_{1-\varepsilon}^{1/(1+\varepsilon)} |1-t^{\beta-1}|^{p-1}(1-t)^{-sp-1}\, dt\\
&\le \varepsilon^2 \frac{|1-(1-\varepsilon)^{\beta-1}|^{p-1}}{(1-(1-\varepsilon+\varepsilon^2))^{sp+1}} \le \varepsilon^2 \frac{|1-(1-\varepsilon)^{\beta-1}|^{p-1}}{\varepsilon^{sp+1}}(1+\varepsilon)^{sp+1}\\
&\le \varepsilon^{2-\frac{sp}{p-1}} \frac{|1-(1-\varepsilon)^{\beta-1}|^{p-1}}{\varepsilon^{sp+1-\frac{sp}{p1}}} \le \varepsilon^{2-\frac{sp}{p-1}} \left|\frac{1-(1-\varepsilon)^{\beta-1}}{\varepsilon}\right|^{p-1} \varepsilon^{p+\frac{sp}{p-1}-sp-2}\\
&\le \varepsilon^{p-sp}\left|\frac{1-(1-\varepsilon)^{\beta-1}}{\varepsilon}\right|^{p-1}.
\end{aligned} \tag{4.10}$$

Since $p-sp>0$ and

$$\lim_{\varepsilon\to 0}\left|\frac{1-(1-\varepsilon)^{\beta-1}}{\varepsilon}\right| = \beta-1,$$

then taking the limit of both sides of (4.10) as $\varepsilon\to 0$, we get that

$$\lim_{\varepsilon\to 0}|R_\varepsilon| \le (\beta-1)^{p-1}\lim_{\varepsilon\to 0}\varepsilon^{p-sp} = 0.$$

This implies that $\lim\limits_{\varepsilon\to 0}|R_\varepsilon| = 0$ and hence,

$$\begin{aligned}
P.V.\int_0^\infty & \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\, dt\\
&= \int_0^1 \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\left[1-t^{sp-(\beta-1)(p-1)-1}\right] dt.
\end{aligned}$$

Since $sp-(\beta-1)(p-1)-1=0$, the preceding identity implies that

$$P.V.\int_0^\infty \frac{|1-t^{\beta-1}|^{p-2}(1-t^{\beta-1})}{|t-1|^{sp+1}}\, dt = 0.$$

We have shown that $(-\Delta)^s_{\mathbb{R}^N_+,p} w_\beta(x) = 0$ for every $x\in\mathbb{R}^N_+$, that is, (4.9) holds. The proof of lemma is finished. □

We mention that Lemma 4.6 in the case $p=2$ has been proved in [5, Lemma 5.1].

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$. Let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u_\beta(x) = \rho(x)^{\beta-1}$. Let $\delta_0 > 0$ be the constant given in Lemma 2.1. Then there exist two constants $C_1, C_2 \geq 0$ (depending only on s, p, N and Ω) such that*

$$|(-\Delta)^s_{\Omega,p} u_\beta(x)| \leq C_1 |\ln(\rho(x))| + C_2, \ \ \forall\, x \in \Omega^\star_{\delta_0}. \tag{4.11}$$

Proof. Let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u_\beta(x) = \rho(x)^{\beta-1}$. Let $\delta_0 > 0$ be the constant given in Lemma 2.1. Since $(-\Delta)^s_{\Omega,p} u_\beta$ is continuous on $\Omega^\star_{\delta_0}$, it suffices to consider the case when $x = (x_1, \ldots, x_N) \in \Omega^\star_{\frac{\delta_0}{2} \wedge \delta_0^2}$. By Lemma 2.1 there is a unique point $z_0 \in \partial\Omega$ such that $|x - z_0| = \rho(x)$. By rotating the coordinate system, we can assume that $z_0 = 0$ and $x = (0, \ldots, 0, \rho(x))$. Set

$$U_{z_0} = \{y = (y_1, \ldots, y_N) : \ y \in \Omega^\star_{\delta_0}, \ |(y_1, \ldots, y_{N-1})| < \delta_0\}. \tag{4.12}$$

Let Γ_{z_0} be the function defined on $\partial\Omega$ near the point z_0 given in (2.1). Since $\partial\Omega$ is of class $C^{1,1}$, it follows that there exists a constant $k_1 > 0$ such that

$$|\Gamma_{z_0}(y_1, \ldots, y_{N-1})| < k_1 |(y_1, \ldots, y_{N-1})|^2 \quad \text{and} \quad |y_n| < k_1 \delta_0, \ \ \forall\, y \in U_{z_0}. \tag{4.13}$$

By Lemma 2.1, $\partial\Omega^\delta$ is tangent to the plane $\{x = (x_1, \ldots, x_N), \ x_N = \delta\}$ at the point $(0, \ldots, 0, \delta)$ for $0 < \delta < \delta_0$. So there exists a constant $k_2 > 0$ such that

$$|y_N - \rho(y)| < k_2 |(y_1, \ldots, y_{N-1})|^2, \ \ \forall\, y \in U_{z_0}. \tag{4.14}$$

Let $\varepsilon > 0$. Since $\rho(x) = x_N$, we have the following estimates:

$$\begin{aligned}
&\left| \int_{\{y \in U_{z_0}, |y-x| > \varepsilon\}} |\rho(x)^{\beta-1} - \rho(y)^{\beta-1}|^{p-2} \frac{\rho(x)^{\beta-1} - \rho(y)^{\beta-1}}{|x-y|^{N+ps}}\, dy \right| \\
&\quad \leq \int_{\{y \in U_{z_0}, |y-x| > \varepsilon\}} \frac{|\rho(x)^{\beta-1} - \rho(y)^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\, dy \\
&\quad = \int_{\{y \in U_{z_0}, |y-x| > \varepsilon\}} \frac{|x_N^{\beta-1} - \rho(y)^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\, dy \\
&\quad \leq 2^{p-1} \int_{\{y \in U_{z_0}, |y-x| > \varepsilon\}} \frac{|x_N^{\beta-1} - |y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\, dy \\
&\qquad + 2^{p-1} \int_{\{y \in U_{z_0}, |y-x| > \varepsilon\}} \frac{||y_N|^{\beta-1} - \rho(y)^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\, dy.
\end{aligned} \tag{4.15}$$

Let

$$\begin{aligned}
A := \{y : \ &\Gamma_{z_0}(y_1, \ldots, y_{N-1}) < y_N < 0 \ \text{ or } \ 0 < y_N \\
&< \Gamma_{z_0}(y_1, \ldots, y_{N-1}), \ |(y_1, \ldots, y_{N-1})| < \delta_0\}.
\end{aligned}$$

As $x \in \Omega^\star_{\frac{\delta_0}{2} \wedge \delta_0^2}$, then using (4.9) in Lemma 4.6, the first inequality in (4.13) and the fact that the function $t \mapsto |t|^{\beta-1}$ is globally $(\beta-1)$-Hölder continuous, we get that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\{y\in U_{z_0},|y-x|>\varepsilon\}}\frac{|x_N^{\beta-1}-|y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,dy\\
&\quad\le\int_A\frac{|x_N^{\beta-1}-|y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,dy+\int_{\{|y-x|>\frac{\delta_0}{2}\}}\frac{|x_N^{\beta-1}-|y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,dy\\
&\quad\le\int_0^{\delta_0}dr\int_{\{|(y_1,\dots,y_{N-1}|=1\}}\chi_A(y)\frac{|x_N^{\beta-1}-|y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,d\sigma_y\\
&\qquad+\int_{\{|y-x|>\frac{\delta_0}{2}\}}\frac{1}{|x-y|^{N+ps-(\beta-1)(p-1)}}\,dy\\
&\quad=\int_0^{\delta_0}dr\int_{\{|(y_1,\dots,y_{N-1}|=1\}}\chi_A(y)\frac{|x_N^{\beta-1}-|y_N|^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,d\sigma_y\\
&\qquad+\int_{\{|y-x|>\frac{\delta_0}{2}\}}\frac{1}{|x-y|^{N+1}}\,dy\\
&\quad\le(2\pi)^N k_1(1+k_1^{ps-1})\left[\int_0^{\sqrt{\rho(x)}}\frac{\rho(x)^{(\beta-1)(p-1)}}{(\frac{r+\rho(x)}{2})^{ps}}dr+\int_{\sqrt{\rho(x)}}^{\delta_0}\frac{r^{2(ps-1)}}{(\frac{r+\rho(x)}{2})^{ps}}dr\right]\\
&\qquad+(2\pi)^N\frac{2}{\delta_0}\\
&\quad=(2\pi)^N k_1(1+k_1^{ps-1})\frac{2^{ps}}{ps-1}+(2\pi)^N k_1(1+k_1^{ps-1})\frac{2^{ps}\delta_0^{ps-1}}{ps-1}+(2\pi)^N\frac{2}{\delta_0},
\end{aligned}\tag{4.16}
$$

where we have also used that $|x-y|\ge\dfrac{\rho(x)+r}{2}$, $y\in A$ which follows from the fact that $|x-y|\ge r$ and $|x-y|\ge\rho(x)$ for $y\in A$. Now we estimate the second term in the right hand-side of (4.15). Using (4.14), (2.4) and the second inequality in (4.13), we get that

$$
\begin{aligned}
&\int_{\{y\in U_{z_0},|y-x|>\varepsilon\}}\frac{||y_N|^{\beta-1}-\rho(y)^{\beta-1}|^{p-1}}{|x-y|^{N+ps}}\,dy\\
&\quad\le\int_{\{y\in U_{z_0},|y-x|>\varepsilon\}}\frac{k_2^{p-1}|y_N|^{(\beta-2)(p-1)}}{|x-y|^{N+ps-2}}\,dy\\
&\quad\le\int_{-k_1\delta_0}^{k_1\delta_0}dr\int_{\{y_N=r\}}\chi_{\{y\in U_{z_0},|y-x|>\varepsilon\}}\frac{k_2^{p-1}r^{ps-p}}{|x-y|^{N+ps-2}}\,d\sigma_y\\
&\quad\le 2\int_0^{k_1\delta_0}dr\int_{\{y_N=r,|(y_1,\dots,y_{N-1}|<\delta_0\}}\frac{k_2^{p-1}r^{ps-1}}{|x-y|^{N+ps-2}}\,d\sigma_y\\
&\quad\le\frac{2^{ps+1}k_2^{p-1}(2\pi)^N}{ps-1}\int_0^{k_1\delta_0}\frac{r^{ps-p}}{|r-\rho(x)|^{ps-1}}\,dr
\end{aligned}
$$

$$= \frac{2^{ps+1}k_2^{p-1}(2\pi)^N}{ps-1}\left(\int_0^{2\rho(x)} \frac{1}{r^{-ps+p}|r-\rho(x)|^{ps-1}}\,dr + \int_{2\rho(x)}^{k_1\delta_0} \frac{1}{r-\rho(x)}\,dr\right)$$
$$\leq \frac{2^{ps+1}k_2^{p-1}(2\pi)^N}{ps-1}\left(\int_0^{2} \frac{r^{ps-p}}{|r-1|^{ps-1}}\,dr + \ln(k_1\delta_0) - \ln(\rho(x))\right). \qquad (4.17)$$

Combining (4.15), (4.16) and (4.17) we get the estimate (4.11) and the proof is finished. □

We have the following result.

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u_\beta(x) = f(x)\rho(x)^{\beta-1}$ for some $f \in C^2(\overline{\Omega})$. Then u_β satisfies the estimate (4.11).*

Proof. We first mention the following fact. Let $p \in (1,\infty)$, $0 < s < 1$ and let u, v be two functions such $(-\Delta)^s_{\Omega,p} u$ and $(-\Delta)^s_{\Omega,p} v$ exist. Then

$$\left|\int_\Omega |u(x)v(x)-u(y)v(y)|^{p-2}\frac{u(x)v(x)-u(y)v(y)}{|x-y|^{N+ps}}\,dy\right|$$
$$\leq \int_\Omega \frac{|u(x)v(x)-u(y)v(y)|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$\leq 2^{p-1}|u(x)|^{p-1}\int_\Omega \frac{|v(x)-v(y)|^{p-1}}{|x-y|^{N+ps}}\,dy + 2^{p-1}\int_\Omega \frac{|v(y)(u(x)-u(y))|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$\leq 2^{p-1}|u(x)|^{p-1}\int_\Omega \frac{|v(x)-v(y)|^{p-1}}{|x-y|^{N+ps}}\,dy + 2^{p-1}|v(x)|^{p-1}\int_\Omega \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$+ 2^{p-1}\int_\Omega \frac{|(v(y)-v(x))(u(x)-u(y))|^{p-1}}{|x-y|^{N+ps}}\,dy. \qquad (4.18)$$

Using (4.18) we obtain that

$$\left|(-\Delta)^s_{\Omega,p}(u(x)v(x))\right| \leq 2^{p-1}|u(x)|^{p-1}\lim_{\varepsilon\downarrow 0}\int_{\{y\in\Omega,|x-y|>\varepsilon\}} \frac{|v(x)-v(y)|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$+ 2^{p-1}|v(x)|^{p-1}\lim_{\varepsilon\downarrow 0}\int_{\{y\in\Omega,|x-y|>\varepsilon\}} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$+ 2^{p-1}\int_\Omega \frac{|(v(y)-v(x))(u(x)-u(y))|^{p-1}}{|x-y|^{N+ps}}\,dy. \qquad (4.19)$$

Next, let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u_\beta(x) = f(x)\rho(x)^{\beta-1}$ for some $f \in C^2(\overline{\Omega})$. Since $(-\Delta)^s_{\Omega,p} f$ and $(-\Delta)^s_{\Omega,p}\rho^{\beta-1}$ exist and satisfy the estimate (4.11), then by (4.19), it suffices to show that the function $\displaystyle\int_\Omega \frac{|(f(y)-f(x))(\rho(x)^{\beta-1}-\rho(y)^{\beta-1})|^{p-1}}{|x-y|^{N+ps}}\,dy$ satisfies the estimate (4.11). We omit the proof since it follows the lines of the proof of Lemma 4.7. □

Now, we are ready to give the proof of the main result of this section.

Proof of Theorem 4.1. Let $t \geq 0$ be small enough. By [18, Lemma 3.1], there exist two constants $a_1, a_2 > 0$ such that

$$\int_{\Omega^\tau \setminus \Omega^t} dy \leq a_1(t-\tau), \quad 0 < \tau < t < a_2. \tag{4.20}$$

Let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and $u = fh_\beta + g \in C^2_\beta(\overline{\Omega})$. We have to show that $(-\Delta)^s_{\Omega,p} u \in L^q(\Omega)$ for every $q \in [1,\infty)$. Let $\delta > 0$ be small enough. Since $g \in C^2(\overline{\Omega})$ and $fh_\beta \in C^2(\overline{\Omega}^\star_\delta)$ (hence $u = fh_\beta + g \in C^2(\overline{\Omega}^\star_\delta)$), it follows from (4.7) that $|(-\Delta)^s_{\Omega,p} u|$ is bounded on $\Omega^\star_\delta$. It also follows from Lemma 4.8 that $(-\Delta)^s_{\Omega,p} u$ satisfies the estimate (4.11). Let a_1, a_2 be as in (4.20). Let $m \in \mathbb{N}$ be such that $\frac{1}{m} \leq a_2$ and set

$$B_k := \left\{ x \in \Omega : \ \frac{1}{k+1} < \rho(x) \leq \frac{1}{k} \right\}, \quad k \geq m, \quad k \in \mathbb{N}.$$

Using (4.20) and (4.11) we get that

$$\begin{aligned}
\int_{\Omega^\star_{\frac{1}{m}}} |(-\Delta)^s_{\Omega,p} u|^q \, dx &= \int_{\Omega \setminus \Omega^{\frac{1}{m}}} |(-\Delta)^s_{\Omega,p} u|^q \, dx \\
&\leq \sum_{k \geq m} \int_{B_k} |(-\Delta)^s_{\Omega,p} u|^q \, dx \\
&\leq \sum_{k \geq m} \frac{a_1}{k(k+1)} \left(C_1 \ln(k+1) + C_2\right)^q < \infty. \qquad (4.21)
\end{aligned}$$

Now (4.21) together with the fact that $|(-\Delta)^s_{\Omega,p} u|$ is bounded on $\Omega^{\frac{1}{m}}$ imply that $(-\Delta)^s_{\Omega,p} u \in L^q(\Omega)$ for every $q \in [1,\infty)$. The proof is finished. □

In Theorem 4.1 we notice that in general $(-\Delta)^s_{\Omega,p} u$ does not belong to $L^\infty(\Omega)$.

5. Proofs of the main results

In this section we give the proofs of the main results stated in Sect. 3.

Proof of Theorem 3.8. Let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$ and $u \in C^2(\overline{\Omega})$. It follows from Theorem 4.1 that $(-\Delta)^s_{\Omega,p} u \in L^{\frac{p}{p-1}}(\Omega)$. Let $v \in W^{s,p}(\Omega)$ and $\varepsilon > 0$. Since

$$\begin{aligned}
&\int_\Omega v (-\Delta)^s_{\Omega,p,\varepsilon} u \, dx \\
&\quad = C_{N,p,s} \int_\Omega v(x) \int_{\{y \in \Omega: \ |x-y| > \varepsilon\}} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}} \, dydx \\
&\quad = \frac{C_{N,p,s}}{2} \int_\Omega \int_\Omega \chi_{\{y \in \Omega: \ |x-y| > \varepsilon\}} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y)}{|x-y|^{N+ps}} \, dydx
\end{aligned}$$

and $(-\Delta)^s_{\Omega,p,\varepsilon} u$ satisfies the estimates (4.7)–(4.8) in Lemma 4.5, we get (3.17) by applying the Lebesgue Dominated Convergence Theorem. □

To prove Theorem 3.9, we need a preliminary lemma and a density result.

Lemma 5.1. *Let* $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ *and* $T > 0$. *Let* $u \in C[0,T] \cap C^1(0,T]$ *and* $v \in C[0,T]$. *If* $\mathcal{N}_p^{2-\beta} u(0)$ *exists, then*

$$\lim_{\delta \downarrow 0} C_{1,p,s} \int_0^\delta \int_\delta^T |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))v(x)}{|x-y|^{1+ps}}\, dxdy = C_{p,s} v(0) \mathcal{N}_p^{2-\beta} u(0), \tag{5.1}$$

where the constant $C_{p,s}$ *is given by* (3.9).

Proof. To prove the lemma, we follow the ideas of the proof of the linear version ($p=2$) contained in [17, Lemma 3.2] (see also [18, Theorem 7.5]). Let $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$ and $\beta := \frac{ps-1}{p-1} + 1$. Since $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$, we have that

$$1 < ps - (p-2) < 2, \quad 0 < ps - (p-2) - 1 < 1 \text{ and } (2-\beta)(p-1) = p - ps.$$

Let $T > 0$, $u \in C^1(0,T]$, $v \in C[0,T]$ and assume that $\mathcal{N}_p^{2-\beta} u(0)$ exists. Without any restriction we may assume that $T = 1$ and $\mathcal{N}_p^{2-\beta} u(0) = -1$. Let $w \in C[0,1] \cap C^1(0,1]$ be such that $w(x) - w(y) = |u(x)-u(y)|^{p-2}(u(x)-u(y))$ for all $x, y \in [0,1]$. Let $y \in [0,1]$ be fixed. We would like to have an expression of the derivative $w'(x)$ for $x \in (0,1]$. Calculating we get that for every $x \in (0,1]$,

$$\begin{aligned} w'(x) &= (p-2)|u(x)-u(y)|^{p-3} u'(x) \mathrm{sgn}(u(x)-u(y))(u(x)-u(y)) \\ &\quad + |u(x)-u(u)|^{p-2} u'(x) \\ &= (p-2)|u(x)-u(y)|^{p-2} u'(x) + |u(x)-u(u)|^{p-2} u'(x) \\ &= (p-1)|u(x)-u(y)|^{p-2} u'(x). \end{aligned} \tag{5.2}$$

Using the function w and (5.2) we get that

$$\begin{aligned} |u(x)-u(y)|^{p-2}(u(x)-u(y)) &= w(x) - w(y) = \int_y^x w'(s)\, ds \\ &= (p-1) \int_y^x |u(s)-u(y)|^{p-2} u'(s)\, ds. \end{aligned}$$

Using Fubini's Theorem, we get that,

$$\begin{aligned} &\int_0^\delta \int_\delta^1 |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\, dxdy \\ &= \int_0^\delta \int_\delta^1 \frac{(w(x)-w(y))v(y)}{|x-y|^{1+ps}}\, dxdy \\ &= \int_0^\delta \int_\delta^1 \frac{v(y)}{|x-y|^{1+ps}} \int_y^x w'(s)\, ds\, dxdy \\ &= (p-1) \int_0^\delta dy \int_\delta^1 \frac{v(y)}{|x-y|^{1+ps}} dx \int_y^x |u(s)-u(x)|^{p-2} u'(s) ds \\ &= (p-1) \int_0^\delta dy \int_y^\delta ds \int_\delta^1 \frac{v(y)}{|x-y|^{1+ps}} |u(s)-u(x)|^{p-2} u'(s) dx \end{aligned}$$

$$
\begin{aligned}
&+ (p-1)\int_0^\delta dy\int_\delta^1 ds\int_s^1 \frac{v(y)}{|x-y|^{1+ps}}|u(s)-u(x)|^{p-2}u'(s)dx\\
&= (p-1)\int_0^\delta dy\int_y^\delta ds\int_\delta^1 \frac{v(y)}{|x-y|^{1+ps-(p-2)}}|u'(s)|^{p-2}u'(s)dx\\
&\quad + (p-1)\int_0^\delta dy\int_\delta^1 ds\int_s^1 \frac{v(y)}{|x-y|^{1+ps-(p-2)}}|u'(s)|^{p-2}u'(s)dx + J\\
&= \frac{p-1}{ps-(p-2)}\int_0^\delta dy\int_y^\delta \left[\frac{v(y)|u'(s)|^{p-2}u'(s)}{|\delta-y|^{ps-(p-2)}} - \frac{v(y)|u'(s)|^{p-2}u'(s)}{|1-y|^{ps-(p-2)}}\right] ds\\
&\quad + \frac{p-1}{ps-(p-2)}\int_0^\delta dy\int_\delta^1 \left[\frac{v(y)|u'(s)|^{p-2}u'(s)}{|s-y|^{ps-(p-2)}} - \frac{v(y)|u'(s)|^{p-2}u'(s)}{|1-y|^{ps-(p-2)}}\right]\\
&\quad \times ds + J, \qquad (5.3)
\end{aligned}
$$

where

$$
\begin{aligned}
J &= (p-1)\int_0^\delta dy\int_y^\delta ds\int_\delta^1\\
&\quad \times \frac{v(y)|u(s)-u(x)|^{p-2}u'(s) - v(y)|u'(s)|^{p-2}|x-y|^{p-2}u'(s)}{|x-y|^{1+ps}}dx\\
&\quad + (p-1)\int_0^\delta dy\int_\delta^1 ds\int_s^1\\
&\quad \times \frac{v(y)|u(s)-u(x)|^{p-2}u'(s) - v(y)|u'(s)|^{p-2}|x-y|^{p-2}u'(s)}{|x-y|^{1+ps}}dx. \qquad (5.4)
\end{aligned}
$$

It follows from (5.3) that

$$
\begin{aligned}
&\int_0^\delta\int_\delta^1 |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\, dxdy\\
&\quad = \frac{p-1}{ps-(p-2)}\int_0^\delta ds\int_0^s \frac{v(y)|u'(s)|^{p-2}u'(s)}{|\delta-y|^{ps-(p-2)}}dy\\
&\qquad + \frac{p-1}{ps-(p-2)}\int_\delta^1 ds\int_0^\delta \frac{v(y)|u'(s)|^{p-2}u'(s)}{|s-y|^{ps-(p-2)}}dy + J + I_1 \qquad (5.5)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 = -\frac{p-1}{ps-(p-2)}\Bigg[&\int_0^\delta dy\int_y^\delta \frac{v(y)|u'(s)|^{p-2}u'(s)}{|1-y|^{ps-(p-2)}}ds\\
&+ \int_0^\delta dy\int_\delta^1 \frac{v(y)|u'(s)|^{p-2}u'(s)}{|1-y|^{ps-(p-2)}}ds\Bigg]. \qquad (5.6)
\end{aligned}
$$

Let

$$
C_p := \frac{p-1}{(ps-(p-2))(ps-(p-2)-1)} = \frac{p-1}{(ps-p+2)(ps-p+1)}.
$$

By (5.5) we have that

$$\begin{aligned}
&\int_0^\delta \int_\delta^1 |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\,dxdy\\
&\quad = \frac{p-1}{ps-(p-2)}\int_0^\delta ds\int_0^s \frac{(v(y)-v(0)+v(0))|u'(s)|^{p-2}u'(s)}{|\delta-y|^{ps-(p-2)}}dy\\
&\qquad + \frac{p-1}{ps-(p-2)}\int_\delta^1 ds\int_0^\delta \frac{(v(y)-v(0)+v(0))|u'(s)|^{p-2}u'(s)}{|s-y|^{ps-(p-2)}}dy + J + I_1\\
&\quad = \frac{p-1}{ps-(p-2)}\int_0^\delta ds\int_0^s \frac{v(0)|u'(s)|^{p-2}u'(s)}{|\delta-y|^{ps-(p-2)}}dy\\
&\qquad + \frac{p-1}{ps-(p-2)}\int_\delta^1 ds\int_0^\delta \frac{v(0)|u'(s)|^{p-2}u'(s)}{|s-y|^{ps-(p-2)}}dy + J + I_1 + I_2\\
&\quad = C_p\int_0^\delta \left[\frac{v(0)|u'(s)|^{p-2}u'(s)}{\delta^{ps-(p-2)-1}} - \frac{v(0)|u'(s)|^{p-2}u'(s)}{|\delta-s|^{ps-(p-2)-1}}\right]ds\\
&\qquad + C_p\int_\delta^1 \left[\frac{v(0)|u'(s)|^{p-2}u'(s)}{s^{ps-(p-2)-1}} - \frac{v(0)|u'(s)|^{p-2}u'(s)}{|s-\delta|^{ps-(p-2)-1}}\right]ds + J + I_1 + I_2,
\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}
I_2 = &\frac{p-1}{ps-(p-2)}\left[\int_0^\delta ds\int_0^s \frac{(v(y)-v(0))|u'(s)|^{p-2}u'(s)}{|\delta-y|^{ps-(p-2)}}dy\right.\\
&\left. + \int_\delta^1 ds\int_0^\delta \frac{(v(y)-v(0))|u'(s)|^{p-2}u'(s)}{|s-y|^{ps-(p-2)}}dy\right].
\end{aligned} \tag{5.8}$$

It follows from (5.7) that

$$\begin{aligned}
&\int_0^\delta \int_\delta^1 |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\,dxdy\\
&\quad = C_pv(0)\int_0^\delta \left[\frac{1}{s^{(2-\beta)(p-1)}\delta^{ps-(p-2)-1}} - \frac{1}{s^{(2-\beta)(p-1)}|\delta-s|^{ps-(p-2)-1}}\right]ds\\
&\qquad + C_pv(0)\int_\delta^1 \left[\frac{1}{s^{(\beta-2)(p-1)}s^{ps-(p-2)-1}} - \frac{1}{s^{(2-\beta)(p-1)}|s-\delta|^{ps-(p-2)-1}}\right]ds\\
&\qquad + J + I_1 + I_2 + I_3\\
&\quad = C_pv(0)\int_0^\delta \left[\frac{1}{s^{(2-\beta)(p-1)}\delta^{ps-(p-2)-1}} - \frac{1}{s^{(2-\beta)(p-1)}|\delta-s|^{ps-(p-2)-1}}\right]ds\\
&\qquad + C_pv(0)\int_\delta^1 \left[\frac{1}{s} - \frac{1}{s^{(2-\beta)(p-1)}|s-\delta|^{ps-(p-2)-1}}\right]ds\\
&\qquad + J + I_1 + I_2 + I_3,
\end{aligned} \tag{5.9}$$

where

$$I_3 = C_p \int_0^\delta \left[\frac{v(0)}{\delta^{ps-(p-2)-1}} - \frac{v(0)}{|\delta - s|^{ps-(p-2)-1}}\right]\left[|u'(s)|^{p-2}u'(s) - \frac{1}{s^{(2-\beta)(p-1)}}\right] ds$$
$$+ C_p \int_\delta^1 \left[\frac{v(0)}{s^{ps-(p-2)-1}} - \frac{v(0)}{|s-\delta|^{ps-(p-2)-1}}\right]\left[|u'(s)|^{p-2}u'(s) - \frac{1}{s^{(2-\beta)(p-1)}}\right] ds. \tag{5.10}$$

Using (5.9) we obtain that

$$\int_0^\delta \int_\delta^1 |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\,dxdy$$
$$= C_p v(0)\left[\int_0^1 \frac{1-|1-s|^{(p+2)+1-ps}}{s^{p-ps}} ds + \int_1^{\frac{1}{\delta}} \frac{s^{p-ps-1} - |1-s|^{(p+2)+1-ps}}{s^{p-ps}} ds\right]$$
$$+ J + I_1 + I_2 + I_3$$
$$= C_p v(0) \int_0^{\frac{1}{\delta}} \frac{(1\vee s)^{p-ps-1} - |1-s|^{(p+2)+1-ps}}{s^{p-ps}} ds + J + I_1 + I_2 + I_3. \tag{5.11}$$

Next, let J be given by (5.4). Since $u(s) - u(x) = u'(\xi)(s-x)$, $(s,x) \in [y,\delta]\times[\delta,1]$, or $(s,x) \in [\delta,1]\times[s,1]$ for some ξ on the line segment from s to x, and since $2 < 1+ps-(p-2) < 3$, we have that the two integrals exist. Using the Dominated Convergence Theorem, we obtain that $\lim_{\delta\downarrow 0} J = 0$. It is also an easy task to show that I_1 given by (5.6) exists and $\lim_{\delta\downarrow 0} I_1 = 0$ by the Dominated Convergence Theorem. Now, since v is continuous on $[0,1]$ we have that $\lim_{y\downarrow 0}(v(y)-v(0)) = 0$. The existence of (5.8) is also easy to verify and applying the Dominated Convergence Theorem, we get that $\lim_{\delta\downarrow 0} I_2 = 0$. Finally, since $\lim_{t\downarrow 0} |u'(t)|^{p-2}u'(t)t^{(2-\beta)(p-1)} = \lim_{t\downarrow 0} |u'(t)t^{2-\beta}|^{p-2}u'(t)t^{2-\beta} = -1$, we have that $|u'(t)|^{p-2}u'(t) = o\left(\frac{1}{t^{(2-\beta)(p-1)}}\right)$. Hence, we can verify that I_3 exists and $\lim_{\delta\downarrow 0} I_3 = 0$. We have shown that J, I_1, I_2 and I_3 are finite and that

$$\lim_{\delta\downarrow 0} J = \lim_{\delta\downarrow 0} I_1 = \lim_{\delta\downarrow 0} I_2 = \lim_{\delta\downarrow 0} I_3 = 0. \tag{5.12}$$

Taking the limit of (5.11) as $\delta \downarrow 0$ and using (5.12) we get that

$$\lim_{\delta\downarrow 0} C_{1,p,s} \int_0^\delta \int_\delta^1 |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))v(y)}{|x-y|^{1+ps}}\,dxdy$$
$$= C_{1,p,s}C_p v(0) \int_0^\infty \frac{(1\vee s)^{p-ps-1} - |1-s|^{(p+2)+1-ps}}{s^{p-ps}} ds$$
$$= -C_{1,p,s}C_p v(0) \int_0^\infty \frac{|1-s|^{(p+2)+1-ps} - (1\vee s)^{p-ps-1}}{s^{p-ps}} ds$$
$$= C_{p,s} v(0) \mathcal{N}_p^{2-\beta} u(0),$$

where we recall that $C_{p,s}$ is given in (3.20). We have shown (5.1) and the proof is finished. □

Next, we give a density result.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$, $p \in (1,\infty)$, $\max\{\frac{p-1}{p}, \frac{1}{p}\} < s < 1$ and $\beta := \frac{ps-1}{p-1} + 1$. Then $C^1_\beta(\overline{\Omega})$ is a dense subspace of $W^{s,p}(\Omega)$.*

Proof. We show first that $C^1_\beta(\overline{\Omega})$ is a subspace of $W^{s,p}(\Omega)$. Let $u \in C^1_\beta(\overline{\Omega})$. It is clear that $u \in L^p(\Omega)$. We have to prove that

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy < \infty. \tag{5.13}$$

Since $C^1(\overline{\Omega}) \subset W^{s,p}(\Omega)$ we have that (5.13) holds for every $u \in C^1(\overline{\Omega})$. Therefore we have to prove (5.13) for functions u of the form $u = fh_\beta$ with $f \in C^1(\overline{\Omega})$. Let then $u = fh_\beta$ for some $f \in C^1(\overline{\Omega})$. Then

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|f((x)h_\beta(x) - f(y)h_\beta(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&\quad = \int_\Omega \int_\Omega \frac{|f(x)h_\beta(x) - f(x)h_\beta(y) + f(x)h_\beta(y) - f(y)h_\beta(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&\quad \le 2^{p-1}\|f\|^p_{L^\infty(\Omega)} \int_\Omega \int_\Omega \frac{|h_\beta(x)-h_\beta(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&\qquad + 2^{p-1}\|h_\beta\|^p_{L^\infty(\Omega)} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+ps}}\, dxdy.
\end{aligned}$$

Since

$$\|h_\beta\|^p_{L^\infty(\Omega)} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+ps}}\, dxdy < \infty,$$

it suffices to show that

$$\int_\Omega \int_\Omega \frac{|h_\beta(x)-h_\beta(y)|^p}{|x-y|^{N+ps}}\, dxdy < \infty.$$

It follows from (3.17) in Theorem 3.8 that

$$\begin{aligned}
&\frac{C_{N,p,s}}{2} \int_{\Omega^\delta} \int_{\Omega^\delta} \frac{|h_\beta(x)-h_\beta(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&\quad = \int_{\Omega^\delta} h_\beta (-\Delta)^s_{p,\Omega^\delta} h_\beta\, dx \\
&\quad = \int_{\Omega^\delta} \int_{\Omega^\star_\delta} \frac{|h_\beta(x)-h_\beta(y)|^{p-2}(h_\beta(x)-h_\beta(y))h_\beta(x)}{|x-y|^{N+ps}}\, dxdy \\
&\qquad + \int_{\Omega^\delta} h_\beta (-\Delta)^s_{p,\Omega} h_\beta\, dx.
\end{aligned} \tag{5.14}$$

It follows from (2.4) and the fact that ρ is Lipschitz continuous, that there is a constant $C > 0$ such that

$$|h_\beta(x) - h_\beta(y)| \le \rho(x)^{\beta-2}|\rho(x)-\rho(y)| \le C\rho(x)^{\beta-2}|x-y|, \quad \forall\, x \in \Omega^\star_\delta,\ y \in \Omega^\delta.$$

Letting $M := \|h_\beta\|_{L^\infty(\Omega)}$ and using a changing of variable, we get that,

$$\begin{aligned}
\int_{\Omega^\delta}\int_{\Omega_\delta^\star} \frac{|h_\beta(x)-h_\beta(y)|^{p-1}|h_\beta(x)|}{|x-y|^{N+ps}}\,dxdy &\le M\int_{\Omega^\delta}\int_{\Omega_\delta^\star} \frac{|h_\beta(x)-h_\beta(y)|^{p-1}}{|x-y|^{N+ps}}\,dxdy\\
&\le C\int_{\Omega_\delta^\star}\rho(x)^{(\beta-2)(p-1)}\\
&\quad \int_{\Omega^\delta}\frac{1}{|x-y|^{N-1+p(s-1)+2}}\,dydx\\
&\le C\int_{\Omega_\delta^\star}\rho(x)^{ps-p}\int_{\delta-\rho(x)}^{d_\Omega}\frac{1}{r^{p(s-1)+2}}\,drdx\\
&\le C\int_{\Omega_\delta^\star}\rho(x)^{ps-p}(\delta-\rho(x))^{p(1-s)-1}\,dx\\
&\le C\int_0^\delta r^{ps-p}(\delta-r)^{p(1-s)-1}\,dr\\
&= C\int_0^1 r^{ps-p}(1-r)^{p(1-s)-1}\,dr\\
&= C\mathbb{B}\left(ps-p+1,p(1-s)\right)<\infty. \qquad (5.15)
\end{aligned}$$

Combining (5.14), (5.15) and noticing that $(-\Delta)_{\Omega,p}^s h_\beta \in L^1(\Omega)$ (by Theorem 4.1), we obtain the estimate (5.13). Hence, $C_\beta^1(\overline{\Omega}) \subset W^{s,p}(\Omega)$. Finally, since $C^1(\overline{\Omega}) \subset C_\beta^1(\overline{\Omega})$ and is dense in $W^{s,p}(\Omega)$, we also have that $C_\beta^1(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ and the proof is finished. □

Some density results for some spaces similar to $W^{s,p}(\Omega)$ (but different) have been obtained in [15]. We give a simple version of their spaces that are comparable to the one defined in the present paper. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. For $1 < p < \infty$ and $0 < s < 1$, let

$$\widetilde{W}_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N):\ u = 0 \ \text{ a.e. on } \ \mathbb{R}^N\setminus\Omega\}.$$

By definition, $\widetilde{W}_0^{s,p}(\Omega)$ is always a subspace of $W^{s,p}(\Omega)$ but in general there is no obvious inclusion between $\widetilde{W}_0^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$. It has been shown in [15, Theorem 6] that if Ω is an open set with continuous boundary, then $\mathcal{D}(\Omega)$ is dense in $\widetilde{W}_0^{s,p}(\Omega)$. This has been also previously obtained in [20, Theorem 1.4.2.2]. But we mention that in [15] the defined spaces are more general than $\widetilde{W}_0^{s,p}(\Omega)$. In fact we have the following. If Ω is a bounded open set with Lipschitz continuous boundary, then when $s \neq \frac{1}{p}$, we have $\widetilde{W}_0^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ and furthermore, when $0 < s < \frac{1}{p}$, then $\widetilde{W}_0^{s,p}(\Omega) = W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$. We refer to [20, Chapter 1] for a further details on this topic.

Before we give the proof of Theorem 3.9 , we introduce some notation and give some useful properties of open sets of class $C^{1,1}$.

Let $\delta > 0$ be small enough. Let $(\theta_i)_{i=1}^{N-1} \subset \mathbb{R}$ be such that $0 \le \theta_i \le \pi$, $i = 1, 2, \ldots, N-2$, and $0 < \theta_{N-1} < 2\pi$. We shall write

$$\Phi(\theta_1,\ldots,\theta_{N-2}) := \sin^{N-2}(\theta_1)\sin^{N-3}(\theta_2)\cdots\sin(\theta_{N-2}),$$

and set

$$\vec{\theta} := \Big(\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-1}), \ldots, \sin(\theta_1)\cos(\theta_2), \cos(\theta_1) \Big).$$

Since $\Omega_\delta^\star$ is an open set, for any direction $\vec{\theta}$ and $x \in \overline{\Omega}$, the set

$$K_x^{\delta,\vec{\theta}} := \{y : \; y = x + t\vec{\theta} \in \Omega_\delta^\star \;\text{ for some }\; t \in \mathbb{R}\}$$

is a one dimensional open set. Moreover $K_x^{\delta,\vec{\theta}}$ is the union of finite one dimensional open intervals. For each open interval U of $K_x^{\delta,\vec{\theta}}$, denote $a_1 := x + t_1\vec{\theta}$, $a_2 := x + t_2\vec{\theta}$, where $t_1 := \inf\{t : \;\; y = x + t\vec{\theta} \in U\}$ and $t_2 := \sup\{t : \; y = x + t\vec{\theta} \in U\}$. We call a_1 and a_2 the left end point and the right hand point of the interval U, respectively. Then, $a_1, a_2 \in \partial\Omega \cup \partial\Omega^\delta$. So we have four types of open intervals:

$$\begin{aligned} &(1)\; a_1 \in \partial\Omega, a_2 \in \partial\Omega^\delta; \quad (2)\; a_1 \in \partial\Omega, a_2 \in \partial\Omega; \\ &(3)\; a_1 \in \partial\Omega^\delta, a_2 \in \partial\Omega; \quad (4)\; a_1 \in \partial\Omega^\delta, a_2 \in \partial\Omega^\delta. \end{aligned}$$

According to these four types of intervals, $\Omega_\delta^\star$ can be divided into four parts.

$$\Omega_i^{\delta,\vec{\theta}} := \bigcup_{x\in\Omega} \{(a_1, a_2) : \; (a_1, a_2) \;\text{ is a type } i \text{ open interval of }\; K_x^{\delta,\vec{\theta}}\}, \; i = 1, 2, 3, 4.$$

Define

$$\begin{cases} \Gamma_1^{\delta,\vec{\theta}} := \{z \in \partial\Omega : \; z \;\text{ is the left end point of a type 1 open interval of }\; K_z^{\delta,\vec{\theta}}\} \\ \Gamma_2^{\delta,\vec{\theta}} := \{z \in \partial\Omega : \; z \;\text{ is the left end point of a type 4 open interval of }\; K_z^{\delta,\vec{\theta}}\}. \end{cases} \tag{5.16}$$

For $z \in \Gamma_1^{\delta,\vec{\theta}} \cup \Gamma_2^{\delta,\vec{\theta}}$, we denote

$$L_z^{\delta,\vec{\theta}} := \{(a_1, a_2) : \; (a_1, a_2) \;\text{ is the open interval of } K_z^{\delta,\vec{\theta}} \;\text{ taking } z \text{ as the left end point}\}.$$

For $z \in \Gamma_1^{\delta,\vec{\theta}}$, we denote

$$\begin{aligned} H_z^{\delta,\vec{\theta}} := \bigcup_{t>0} \{\xi \in K_z^{\delta,\vec{\theta}} : \; \xi = z - t\vec{\theta}, \; \xi \notin G_4^{\delta,\vec{\theta}} \\ \text{and } \; \{y : \; y = z - s\vec{\theta}, \; s \in (0, t)\} \cap \Gamma_1^{\delta,\vec{\theta}} = \emptyset\}. \end{aligned}$$

The proof of the following result is contained in [17, Lemma 3.4].

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$. Then the following assertions hold.*

(a) *For all $z \in \Gamma_1^{\delta,\vec{\theta}}$ we have that $\langle \vec{n}(z), \vec{\theta} \rangle \ge 0$.*
(b) *There exists a constant $k_1 > 0$ such that for every unit vector $\vec{\theta}$,*

$$\rho(z + t\vec{\theta}) \ge \frac{t}{4} \langle \vec{n}(z), \vec{\theta} \rangle, \;\; \forall\, z \in \partial\Omega, \; 0 < t < k_1 \langle \vec{n}(z), \vec{\theta} \rangle.$$

(c) *Let $\rho_\delta(y) = dist(y, \partial\Omega_\delta^\star)$. Then there exists a constant $k_2 > 0$ such that for every unit vector $\vec{\theta}$,*

$$\rho_\delta(z - t\vec{\theta}) \ge \frac{t}{4} \langle \vec{n}_\delta(z), \vec{\theta} \rangle, \;\; \forall\, z \in \partial\Omega_\delta^\star, \; 0 < t < k_2 \langle \vec{n}_\delta(z), \vec{\theta} \rangle.$$

Here, $\vec{n}_\delta(z)$ is the inner normal vector to $\partial\Omega^\delta$ at the point z.

The properties of open sets of class $C^{1,1}$ given here have been clarified in [17,24]. We have included it here for the convenience of the reader and to make the paper as self-contained as possible.

Proof of Theorem 3.9 Let $p \in (1,\infty)$, $\max\{\frac{1}{p}, \frac{p-1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1}+1$ and $(-\Delta)^s_{\Omega,p}$ the (quasi-linear) nonlocal operator defined in (1.4). Throughout the proof, we use the notation of sets $\Omega_i^{\delta,\vec{\theta}}$, $\Gamma_i^{\delta,\vec{\theta}}$, $L_x^{\delta,\vec{\theta}}$, $H_x^{\delta,\vec{\theta}}$, function Φ, and vectors $\vec{\theta}$, $\vec{n}(x)$ introduced above. Let $u \in C^2_\beta(\overline{\Omega})$. First we assume that $v \in C^1_\beta(\overline{\Omega})$. Using Theorem 3.8 and Lemma 4.8 we get that

$$
\begin{aligned}
&\frac{C_{N,p,s}}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&=\lim_{\delta\downarrow 0}\int_{\Omega^\delta} v(-\Delta)^s_{\Omega^\delta,p}u\,dx\\
&=\lim_{\delta\downarrow 0} C_{N,p,s}\int_{\Omega^\delta}dx\int_{\Omega_\delta^\star} v(x)|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy\\
&\quad+\lim_{\delta\downarrow 0}\int_{\Omega^\delta} v(-\Delta)^s_{\Omega,p}u\,dx\\
&=\lim_{\delta\downarrow 0} C_{N,p,s}\int_{\Omega^\delta}dx\int_{\Omega_\delta^\star} v(y)|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy\\
&\quad+\lim_{\delta\downarrow 0} C_{N,p,s}\int_{\Omega^\delta}\int_{\Omega_\delta^\star} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad+\lim_{\delta\downarrow 0}\int_{\Omega} v(-\Delta)^s_{\Omega^\delta,p}u\,dx\\
&=\lim_{\delta\downarrow 0} C_{N,p,s}\int_{\Omega_\delta^\star}\int_{\Omega^\delta} v(x)|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad+\lim_{\delta\downarrow 0}\int_{\Omega} v(-\Delta)^s_{\Omega^\delta,p}u\,dx. \qquad (5.17)
\end{aligned}
$$

Calculating the first term in the right hand-side of (5.17) by using polar coordinates we get that

$$
\begin{aligned}
&C_{N,p,s}\int_{\Omega_\delta^\star}\int_{\Omega^\delta} v(x)|u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dx\,dy\\
&=C_{N,p,s}\int_{\Omega_\delta^\star} v(x)\,dx\int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})d\theta_{N-1}\\
&\quad\times\int_{\{x+r\vec{\theta}\in\Omega^\delta,\, r>0\}} |u(x)-u(x+r\vec{\theta})|^{p-2}\frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}}\,dr\\
&=C_{N,p,s}\int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})d\theta_{N-1}\\
&\quad\times\sum_{i=1}^4\int_{\Omega_i^{\delta,\vec{\theta}}} v(x)\,dx\int_{\{x+r\vec{\theta}\in\Omega^\delta,\, r>0\}} |u(x)-u(x+r\vec{\theta})|^{p-2}\frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}}\,dr.
\end{aligned}
\qquad (5.18)
$$

Calculating the integral in the right-hand side of (5.18) along the vector $\vec{\theta}$ we get that

$$\begin{aligned}
&\int_{\Omega_1^{\delta,\vec{\theta}}} v(x)\,dx \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}} |u(x)-u(x+r\vec{\theta})|^{p-2}\frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}}\,dr\\
&\quad= \int_{\Gamma_1^{\delta,\vec{\theta}}} \langle \vec{n}(x),\vec{\theta}\rangle\,d\sigma_x \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}} v(x+t\vec{\theta})\,dt\\
&\qquad\times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} |u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})|^{p-2}\frac{u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})}{r^{1+ps}}\,dr.
\end{aligned} \tag{5.19}$$

If $x\in\Omega_i^{\delta,\vec{\theta}}$ $(i=2,3)$ and $\{z:\ z=x+t\vec{\theta}\in\Omega^\delta\ \text{ for some }\ t>0\}\neq\emptyset$, then

$$\{z:\ z=x+t\vec{\theta}\ \text{ for some }\ t>0\}\cap\Gamma_1^{\delta,\vec{\theta}}\neq\emptyset.$$

So, as in (5.19), we have

$$\begin{aligned}
&\sum_{i=2}^3\int_{\Omega_i^{\delta,\vec{\theta}}} v(x)\,dx \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}} |u(x)-u(x+r\vec{\theta})|^{p-2}\frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}}\,dr\\
&\quad= \int_{\Gamma_1^{\delta,\vec{\theta}}} \langle \vec{n}(x),\vec{\theta}\rangle\,d\sigma_x \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}} v(x+t\vec{\theta})\,dt\\
&\qquad\times \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}} |u(x-t\vec{\theta})-u(x+r\vec{\theta})|^{p-2}\frac{u(x-t\vec{\theta})-u(x+r\vec{\theta})}{(t+r)^{1+ps}}\,dr.
\end{aligned} \tag{5.20}$$

For the integral over the set $\Omega_4^{\delta,\vec{\theta}}$ we have

$$\begin{aligned}
&\int_{\Omega_4^{\delta,\vec{\theta}}} v(x)\,dx \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}} |u(x)-u(x+r\vec{\theta})|^{p-2}\frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}}\,dr\\
&\quad= \int_{\Gamma_2^{\delta,\vec{\theta}}} v(x)\langle \vec{n}_\delta(x),-\vec{\theta}\rangle\,d\sigma_x \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}} v(x+t\vec{\theta})\,dt\\
&\qquad\times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} |u(x+t\vec{\theta})\\
&\qquad - u(x+(r+t)\vec{\theta})|^{p-2}\frac{u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})}{r^{1+ps}}\,dr.
\end{aligned} \tag{5.21}$$

In (5.21), $\vec{n}_\delta(x)$ is the inner normal vector of $\partial\Omega^\delta$ at the point x.

Next we claim that

$$\begin{aligned}
&\langle \vec{n}(x),\vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\,dt\\
&\quad\times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} |u(x+t\vec{\theta})\\
&\quad - u(x+(r+t)\vec{\theta})|^{p-2}\frac{u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})}{r^{1+ps}}\,dr,\quad x\in\Gamma_1^{\delta,\vec{\theta}},
\end{aligned} \tag{5.22}$$

and

$$\langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}}\,dt$$
$$\times \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}}|u(x-t\vec{\theta})-u(x+r\vec{\theta})|^{p-2}$$
$$\frac{u(x-t\vec{\theta})-u(x+r\vec{\theta})}{(t+r)^{1+ps}}\,dr,\ x\in\Gamma_1^{\delta,\vec{\theta}}, \tag{5.23}$$

and

$$\langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\,dt$$
$$\times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}}|u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})|^{p-2}$$
$$\frac{u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})}{r^{1+ps}}\,dr, \quad x\in\Gamma_2^{\delta,\vec{\theta}}, \tag{5.24}$$

are all uniformly bounded for $\delta>0$, $\vec{\theta}$ and the corresponding x. The $\vec{n}(x)$ in (5.22) and (5.23) and the $\vec{n}_\delta(x)$ in (5.24) are the inner normal vector to $\partial\Omega$ and $\partial\Omega^\delta$ at the point x, respectively. Indeed, let

$$T := \sup\{t:\ x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}, \tag{5.25}$$

and assume first that $u\in C^2(\overline{\Omega})$. Consider the expression in (5.22). Using the mean value theorem and a change of variable we get that there exists a constant $C>0$ such that for $x\in\Gamma_1^{\delta,\vec{\theta}}$,

$$\begin{aligned}
&\langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\,dt\\
&\qquad \times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}}\frac{|u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})|^{p-1}}{r^{1+ps}}\,dr\\
&\quad\le C\|\nabla u\|_{L^\infty(\Omega)}^{p-1}\int_0^T dt\int_0^\infty \frac{1}{r^{1+ps-(p-1)}}\,dr\\
&\quad= C\|\nabla u\|_\infty^{p-1}\int_0^T dt\int_{T-t}^\infty \frac{1}{r^{1+ps-(p-1)}}\,dr\\
&\quad\le C_1\|\nabla u\|_{L^\infty(\Omega)}^{p-1}T^{p-ps},
\end{aligned}$$

which is uniformly bounded for $\delta>0$, $\vec{\theta}$, the corresponding x and we have shown (5.22) for $u\in C^2(\overline{\Omega})$. Similarly, it is easy to verify that (5.23) and (5.24) hold for a function $u\in C^2(\overline{\Omega})$. Hence, we can assume that $u=fh_\beta$. Written

$$f(y)h_\beta(y)-f(x)h_\beta(x)=(f(y)-f(x))h_\beta(y)+(h_\beta(y)-h_\beta(x))f(x)$$

and applying (5.22), (5.23) and (5.24), the proof of the claims (5.22), (5.23) and (5.24) can be also reduced to the case where $u=h_\beta=\rho(x)^{\beta-1}$. Therefore, to prove the claims we assume that $u(x)=\rho(x)^{\beta-1}$.

Next, we prove the claim (5.22). Recall that by Lemma 5.3, $\langle \vec{n}(x), \vec{\theta}\rangle \geq 0$ and there exists a constant $k_1 > 0$ such that $\rho(x + t\vec{\theta}) \geq \frac{t}{4}$ for all $x \in \partial\Omega$ and $0 < t < k_1\langle \vec{n}(x), \vec{\theta}\rangle$.

Assume first that $\langle \vec{n}(x), \vec{\theta}\rangle \geq \frac{3\sqrt{\delta}}{\sqrt{k_1}}$ and let T be given by (5.25). It follows from Lemma 5.3(b) that the length of $L_x^{\delta,\vec{\theta}}$ is less than or equal to $3\sqrt{\delta k_1}$ and hence, $\rho(x + t\vec{\theta}) \geq \frac{t}{4}\langle \vec{n}(x), \vec{\theta}\rangle$ for $t \in (0, T)$. Therefore, using an addition a change of variable, we get that for every $x \in \Gamma_1^{\delta,\vec{\theta}}$,

$$
\begin{aligned}
&\langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\,dt \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} \\
&\quad \frac{|u(x+t\vec{\theta}) - u(x+(r+t)\vec{\theta})|^{p-1}}{r^{1+ps}}\,dr \\
&\quad \leq (\beta-1)^{p-1}\langle \vec{n}(x), \vec{\theta}\rangle \int_0^T dt \int_0^\infty \chi_{x+(t+r)\vec{\theta}\in\Omega^\delta} \\
&\quad \frac{\left(\frac{t}{4}\right)^{(\beta-2)(p-1)}\langle \vec{n}(x), \vec{\theta}\rangle^{(\beta-2)(p-1)}}{r^{2+ps-p}}\,dr \\
&\quad \leq 4\langle \vec{n}(x), \vec{\theta}\rangle^{(\beta-2)(p-1)+1} \int_0^T t^{(\beta-2)(p-1)}dt \int_0^\infty \frac{1}{(T-t+r)^{2-ps-p}}\,dr \\
&\quad = 4\langle \vec{n}(x), \vec{\theta}\rangle^{(\beta-2)(p-1)+1} \int_0^T t^{(\beta-2)(p-1)}(T-t)^{p-ps-1}\,dt \\
&\quad \leq 4\int_0^1 t^{(\beta-2)(p-1)}(1-t)^{p-ps-1}\,dt \\
&\qquad = 4\mathbb{B}\left((\beta-2)(p-1)+1, p-ps\right) = 4\mathbb{B}\left(ps-p+1, p-ps\right).
\end{aligned}
$$

Next, assume that $0 < \langle \vec{n}(x), \vec{\theta}\rangle < \frac{3\sqrt{\delta}}{\sqrt{k_1}}$. Using a change of variable, we get that for every $x \in \Gamma_1^{\delta,\vec{\theta}}$,

$$
\begin{aligned}
&\langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\,dt \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta}) - u(x+(r+t)\vec{\theta})|^{p-1}}{r^{1+ps}}\,dr \\
&\quad \leq \langle \vec{n}(x), \vec{\theta}\rangle \int_{T-\delta}^T dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\,dr \\
&\qquad + \langle \vec{n}(x), \vec{\theta}\rangle \int_0^{T-\delta} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\,dr.
\end{aligned}
\tag{5.26}
$$

If $T - \delta < t < T$, then $\rho(x + t\vec{\theta}) \geq \delta - (T - t)$. Recall that $u(x) = \rho(x)^{\beta-1}$. Using the inequality (2.4) and the fact that $(\beta-2)(p-1) < 0$, we get that

$$
\begin{aligned}
|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})| &= |\rho(x+t\vec{\theta})^{\beta-1} - \rho(x+(r+T)\vec{\theta})^{\beta-1}| \\
&\leq \rho(x+t\vec{\theta})^{\beta-2}|r+T-t| \\
&\leq [\delta-(T-t)]^{(\beta-2)(p-1)}|r+T-t|.
\end{aligned}
\tag{5.27}
$$

Using (5.27) we get that there exists a constant $C > 0$ (depending only on N, s and p) such that

$$
\begin{aligned}
&\leq \langle \vec{n}(x), \vec{\theta} \rangle \int_{T-\delta}^{T} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\, dr \\
&\leq C \int_{T-\delta}^{T} dt \int_0^\infty \frac{(\delta-(T-t))^{(\beta-2)(p-1)}}{(T-t+r)^{ps-p+2}}\, dr \\
&= \frac{C}{p-ps+1} \int_{T-\delta}^{T} (\delta-(T-t))^{(\beta-2)(p-1)} (T-t)^{-ps+p-1}\, dt \\
&\leq \frac{C}{p-ps+1} \int_0^1 t^{(\beta-2)(p-1)} (1-t)^{p-ps-1}\, dt \\
&= \frac{C}{p-ps+1} \mathbb{B}\left((\beta-2)(p-1)+1, p-ps\right) \\
&= \frac{C}{p-ps+1} \mathbb{B}\left(ps-p+1, p-ps\right).
\end{aligned} \tag{5.28}
$$

For the second term in the right-hand side of (5.26), noticing that $\rho(x+(r+T)\vec{\theta}) \geq T-t+r$ and proceeding as in (5.27), we get that

$$
\begin{aligned}
|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})| &= |\rho(x+t\vec{\theta})^{\beta-1} - \rho(x+(r+T)\vec{\theta})^{\beta-1}| \\
&\leq \rho(x+(r+T)\vec{\theta})^{\beta-2} |r+T-t| \leq |r+T-t|^{\beta-1}.
\end{aligned} \tag{5.29}
$$

Using (5.29), we get that

$$
\begin{aligned}
&\langle \vec{n}(x), \vec{\theta} \rangle \int_0^{T-\delta} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega_\delta\}} \frac{|u(x+t\vec{\theta}) - u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\, dr \\
&\leq \frac{3\sqrt{\delta}}{\sqrt{k_1}} \int_0^{T-\delta} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{1}{(T-t+r)^{1+ps-(\beta-1)(p-1)}}\, dr \\
&= \frac{3\sqrt{\delta}}{\sqrt{k_1}} \int_0^{T-\delta} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{1}{(T-t+r)^2}\, dr \\
&\leq \frac{3\sqrt{\delta}}{\sqrt{k_1}} \int_0^{T-\delta} \frac{1}{T-t}\, dt = \frac{3\sqrt{\delta}}{\sqrt{k_1}} \left(\ln(T) - \ln(\delta)\right) \leq \frac{3\sqrt{\delta}}{\sqrt{k_1}} \left(\ln(d_\Omega) - \ln(\delta)\right).
\end{aligned} \tag{5.30}
$$

Combining (5.26), (5.28) and (5.30) we obtain the claim (5.22).

Next, we prove (5.23). Since $\partial\Omega$ is of class $C^{1,1}$, we have that there is a constant $k_3 > 0$ such that

$$
|y-x| \geq k_3 \langle \vec{n}(x), \vec{\theta} \rangle, \;\; y \in H_x^{\delta,\vec{\theta}}, \; x \in \Gamma_1^{\delta,\vec{\theta}}.
$$

Proceeding as in (5.29) we get that

$$
|u(x-t\vec{\theta}) - u(x+r\vec{\theta})| \leq |r+t|^{\beta-1}. \tag{5.31}
$$

Using (5.31) we get that

$$\begin{aligned}
&\langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}}\, dt \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega^\delta\}} \frac{|u(x-t\vec{\theta})-u(x+r\vec{\theta})|^{p-1}}{(t+r)^{1+ps}}\, dr\\
&\le \langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}}\, dt \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega_\delta\}} \frac{1}{(t+r)^{1+ps-(\beta-1)(p-1)}}\, dr\\
&= \langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}}\, dt \int_0^\infty \chi_{\{x+r\vec{\theta}\in\Omega_\delta\}} \frac{1}{(t+r)^{2}}\, dr\\
&= \langle \vec{n}(x), \vec{\theta}\rangle \int_0^\infty \chi_{\{x-t\vec{\theta}\in H_x^{\delta,\vec{\theta}}\}} \frac{1}{t}\, dt\\
&\le \langle \vec{n}(x), \vec{\theta}\rangle \int_{k_3\langle \vec{n}(x),\vec{\theta}\rangle}^{d_\Omega} \frac{dt}{t} = \langle \vec{n}(x), \vec{\theta}\rangle \Big(\ln(d_\Omega) - \ln(k_3\langle \vec{n}(x), \vec{\theta}\rangle)\Big),
\end{aligned}$$

which is bounded for all $\langle \vec{n}(x), \vec{\theta}\rangle$ and we have proved the claim (5.23).

Next, we show (5.24). It follows from Lemma 5.3(c) that there is a constant $k_2 > 0$ such that

$$0 < \langle \vec{n}_\delta(x), -\vec{\theta}\rangle \le \frac{3\sqrt{\delta}}{\sqrt{k_2}}, \quad x \in \Gamma_2^{\delta,\vec{\theta}}, \tag{5.32}$$

where $\vec{n}_\delta(x)$ is the inner normal vector of $\partial\Omega^\delta$ at the point x. Proceeding as in (5.26) we have that for every $x \in \Gamma_2^{\delta,\vec{\theta}}$,

$$\begin{aligned}
&\langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_0^\infty \chi_{\{x+t\vec{\theta}\in L_x^{\delta,\vec{\theta}}\}}\, dt\\
&\quad \times \int_0^\infty \chi_{\{x+(t+r)\vec{\theta}\in\Omega^\delta\}} |u(x+t\vec{\theta}) - u(x+(r+t)\vec{\theta})|^{p-2}\\
&\quad \frac{u(x+t\vec{\theta})-u(x+(r+t)\vec{\theta})}{r^{1+ps}}\, dr\\
&\le \langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_{T-\delta}^{T} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta})-u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\, dr\\
&\quad + \langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_0^{T-\delta} dt\\
&\quad \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta})-u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\, dr.
\end{aligned} \tag{5.33}$$

Using (5.32) and (5.27) and proceeding as in (5.28) we obtain that there exists a constant $C > 0$ (depending only on N, s, p, δ and k_2) such that

$$\begin{aligned}
&\langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_{T-\delta}^{T} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta})-u(x+(r+T)\vec{\theta})|^{p-}}{(T-t+r)^{1+ps}}\, dr\\
&\quad \le C\mathbb{B}(ps-p+1, p-ps).
\end{aligned}$$

For the second term in (5.33), using (5.32) and (5.29) and proceeding as in (5.30), we get that there exists a constant $C > 0$ (depending only on N, s, p, δ

and k_2) such that

$$\langle \vec{n}_\delta(x), -\vec{\theta}\rangle \int_0^{T-\delta} dt \int_0^\infty \chi_{\{x+(T+r)\vec{\theta}\in\Omega^\delta\}} \frac{|u(x+t\vec{\theta})-u(x+(r+T)\vec{\theta})|^{p-1}}{(T-t+r)^{1+ps}}\, dr \\ \leq C\left(\ln(d_\Omega)-\ln(\delta)\right).$$

The proof of the claims (5.22), (5.23) and (5.24) is finished.

Now, in the rest of the proof, the function $u = fh_\beta + g$ is in his general form. It is easy to check that the term in (5.23) goes to zero as $\delta \downarrow 0$. Hence, using (5.20) and the Dominated Convergence Theorem, we get that

$$\lim_{\delta\downarrow 0} \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{N-2} \int_0^{2\pi} \Phi(\theta_1,\ldots,\theta_{N-2}) d\theta_{N-1} \\ \times \sum_{i=2}^{3} \int_{\Omega_i^{\delta,\vec{\theta}}} v(x)\, dx \int_{\{x+r\vec{\theta}\in\Omega_\delta^\star, r>0\}} |u(x)-u(x+r\vec{\theta})|^{p-2} \frac{u(x)-u(x+r\vec{\theta})}{r^{1+ps}} \\ dr = 0.$$

Let M_1 be the bound of (5.24) and $M := \|v\|_{\infty,\Omega}$. It follows from Lemma 2.1 that

$$M_2 := \sup_{0<\delta<\delta_1} \int_{\partial\Omega_\delta^\star} d\sigma < \infty. \tag{5.34}$$

Using (5.34), (5.21) and (5.32), we get that

$$\lim_{\delta\downarrow 0} \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{N-2} \int_0^{2\pi} \Phi(\theta_1,\ldots,\theta_{N-2}) d\theta_{N-1} \\ \times \int_{\Omega_4^{\delta,\vec{\theta}}} |v(x)|\, dx \int_{\{x+r\vec{\theta}\in\Omega_\delta^\star, r>0\}} \frac{|u(x)-u(x+r\vec{\theta})|^{p-1}}{r^{1+ps}}\, dr \\ \leq M_1 M \lim_{\delta\downarrow 0} \int_{\Gamma_4^{\delta,\vec{\theta}}} d\sigma \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{N-2} \int_0^{2\pi} \Phi(\theta_1,\ldots,\theta_{N-2}) d\theta_{N-1} \\ \leq M_1 M \lim_{\delta\downarrow 0} \int_{\partial\Omega_\delta^\star} d\sigma \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{N-2} \int_0^{2\pi} \Phi(\theta_1,\ldots,\theta_{N-2}) \\ \chi_{\{}0 < \langle \vec{n}(x), -\vec{\theta}\rangle \leq \frac{3\sqrt{\delta}}{\sqrt{k_1}}\} d\theta_{N-1} \\ \leq M_1 M_2 M (2\pi)^N \lim_{\delta\downarrow 0} \frac{3\sqrt{\delta}}{\sqrt{k_1}} = 0. \tag{5.35}$$

We note that for every $x \in \partial\Omega$,

$$\{\vec{\theta}:\ \langle \vec{n}(x), \vec{\theta}\rangle > 0\} \subseteq \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty \{\vec{\theta}:\ x \in \Gamma_1^{\frac{1}{m},\vec{\theta}}\} \subseteq \{\vec{\theta}:\ \langle \vec{n}(x), \vec{\theta}\rangle \geq 0\}.$$

Hence, using Lemma 3.7 and (5.19), we get that

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\int_0^{\pi} d\theta_1\cdots\int_0^{\pi} d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})d\theta_{N-1}\\
&\quad\times\int_{\Omega_1^{\delta,\vec\theta}} v(x)\,dx\int_{\{x+r\vec\theta\in\Omega_\delta,r>0\}}|u(x)-u(x+r\vec\theta)|^{p-2}\frac{u(x)-u(x+r\vec\theta)}{r^{1+ps}}\,dr\\
&\quad=\lim_{\delta\downarrow 0}\int_{\Gamma_1^{\delta,\vec\theta}} d\sigma\int_0^{\pi} d\theta_1\cdots\int_0^{\pi} d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})d\theta_{N-1}\\
&\quad\times\langle\vec n(x),\vec\theta\rangle\int_0^{\infty}\chi_{\{x+t\vec\theta\in L_x^{\delta,\vec\theta}\}}v(x+t\vec\theta)\,dt\\
&\quad\times\int_0^{\infty}\chi_{\{x+(t+r)\vec\theta\in\Omega^\delta\}}|u(x+t\vec\theta)\\
&\quad-u(x+(t+r)\vec\theta)|^{p-2}\frac{u(x+t\vec\theta)-u(x+(t+r)\vec\theta)}{r^{1+ps}}\,dr.
\end{aligned}\tag{5.36}$$

Let $\tilde u(t)=u(x+t\vec\theta)$ and $\tilde v(t)=v(x+t\vec\theta)$. Then $\tilde u\in C[0,T]\cap C^1(0,T]$, $\tilde v\in C[0,T]$, where we recall that T is given in (5.25). Since $\tilde u(t)=f(x+t\vec\theta)\rho(x+t\vec\theta)^{\beta-1}+g(x+t\vec\theta)$ for some $f,g\in C^2(\overline\Omega)$ we have that $\mathcal N_p^{2-\beta}\tilde u(0)$ exists and is given by $\mathcal N_p^{2-\beta}\tilde u(0)=-|\beta-1|^{p-1}|f(x)|^{p-2}f(x)$. Applying Lemma 5.1 to the functions $\tilde u$ and $\tilde v$ we get from (5.36) that

$$\begin{aligned}
&C_{N,p,s}\lim_{\delta\downarrow 0}\int_{\Gamma_1^{\delta,\vec\theta}} d\sigma\int_0^{\pi} d\theta_1\cdots\int_0^{\pi} d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})d\theta_{N-1}\\
&\quad\times\langle\vec n(x),\vec\theta\rangle\int_0^{\infty}\chi_{\{x+t\vec\theta\in L_x^{\delta,\vec\theta}\}}v(x+t\vec\theta)\,dt\\
&\quad\times\int_0^{\infty}\chi_{\{x+(t+r)\vec\theta\in\Omega^\delta\}}|u(x+t\vec\theta)\\
&\quad-u(x+(t+r)\vec\theta)|^{p-2}\frac{u(x+t\vec\theta)-u(x+(t+r)\vec\theta)}{r^{1+ps}}\,dr\\
&\quad=\frac{(\beta-1)^{p-1}C_{N,p,s}C_{p,s}}{C_{1,p,s}}\int_{\partial\Omega}v(x)|f(x)|^{p-2}f(x)d\sigma\\
&\quad\times\int_0^{\frac{\pi}{2}} d\theta_1\cdots\int_0^{\pi} d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})\langle\vec n(x),\vec\theta\rangle^{\beta(p-1)}\\
&\quad\chi_{\{\langle\vec n(x),\vec\theta\rangle>0\}}d\theta_{N-1}\\
&\quad=\frac{(\beta-1)^{p-1}C_{N,p,s}C_{p,s}}{C_{1,p,s}}\int_{\partial\Omega}v(x)|f(x)|^{p-2}f(x)d\sigma\\
&\quad\times\int_0^{\frac{\pi}{2}} d\theta_1\cdots\int_0^{\pi} d\theta_{N-2}\int_0^{2\pi}\Phi(\theta_1,\ldots,\theta_{N-2})\cos(\theta_1)^{\beta(p-1)}d\theta_{N-1}\\
&\quad=\frac{C_{N,p,s}C_{p,s}}{C_{1,p,s}}\int_{\{|x|=1,x_N>0,x\in\mathbb R^N\}}x_N^{\beta(p-1)}d\sigma\int_{\partial\Omega}v(\beta-1)^{p-1}|f|^{p-2}f\,d\sigma\\
&\quad=-C_{p,s}\int_{\partial\Omega}v\mathcal N_p^{2-\beta}u\,d\sigma.
\end{aligned}\tag{5.37}$$

We have shown that the first term in (5.17) is given by (5.37) for every $u \in C^2_\beta(\overline{\Omega})$ and $v \in C^1_\beta(\overline{\Omega})$. It is easy to see that the second term in (5.17) is given by $\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx$ for every $u \in C^2_\beta(\overline{\Omega})$ and $v \in C^1_\beta(\overline{\Omega})$. Combining these two results we get (3.18) for $u \in C^2_\beta(\overline{\Omega})$ and $v \in C^1_\beta(\overline{\Omega})$. Finally, since $C^1_\beta(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ (by Proposition 5.2), we obtain (3.18) for $u \in C^2_\beta(\overline{\Omega})$ and $v \in W^{s,p}(\Omega)$ by density. The proof of theorem is finished.

Proof of Corollaries 3.11 and 3.12 The identity (3.22) follows directly from (3.18) and the expression (3.10) of $\mathcal{N}^{2-\beta}_p u$ given in Lemma 3.4. The identity (3.23) also follows from (3.18) and the expression (3.12) of $\mathcal{N}^{2-\beta}_p u$ given in Lemma 3.5.

6. The fractional Neumann and Robin boundary conditions

In this section we use the results obtained in Sect. 3 to define a realization of the regional fractional p-Laplacian $(-\Delta)^s_{\Omega,p}$ with fractional Neumann and Robin type boundary conditions. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary $\partial\Omega$. Let $p \in (1,\infty)$, $0 < s < 1$ and γ a non-negative measurable function in $L^\infty(\partial\Omega)$. Let Φ_γ be the functional with effective domain $D(\Phi_\gamma) = W^{s,p}(\Omega) \cap L^2(\Omega)$ and defined on $L^2(\Omega)$ by

$$\Phi_\gamma(u) := \begin{cases} \dfrac{C_{N,p,s}}{2p} \displaystyle\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy + \frac{1}{p}\int_{\partial\Omega} \gamma |u|^p\, d\sigma, & u \in W^{s,p}(\Omega), \\ \infty, & u \in L^2(\Omega)\backslash W^{s,p}(\Omega). \end{cases} \tag{6.1}$$

We have the following result.

Theorem 6.1. *Let $p \in (1,\infty)$ and $0 < s < 1$. Let Φ_γ be the functional defined in (6.1). Then Φ_γ is proper, convex and lower semi-continuous. Let $f \in L^2(\Omega)$, $u \in W^{s,p}(\Omega) \cap L^2(\Omega)$ and $\partial\Phi_\gamma$ the single-valued subgradient of Φ_γ. Then the following assertions hold.*

(a) *If $\frac{1}{p} < s < 1$, then $f = \partial\Phi_\gamma(u)$ if and only if for every $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$,*

$$\frac{C_{N,p,s}}{2} \int_\Omega \int_\Omega |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy + \int_{\partial\Omega} \gamma |u|^{p-2} u(x)v(x)\, d\sigma = \int_\Omega fv\, dx. \tag{6.2}$$

(b) *If $0 < s \le \frac{1}{p}$, then $u \in W^{s,p}_0(\Omega) \cap L^2(\Omega)$ and $f = \partial\Phi_\gamma(u)$ if and only if for every $v \in W^{s,p}_0(\Omega) \cap L^2(\Omega)$,*

$$\frac{C_{N,p,s}}{2} \int_\Omega \int_\Omega |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy = \int_\Omega fv\, dx. \tag{6.3}$$

Moreover, if Ω *is of class* $C^{1,1}$*,* $p \in (1,\infty)$*,* $\max\{\frac{1}{p}, \frac{p-1}{p}\} < s < 1$ *and* $\beta := \frac{ps-1}{p-1} + 1$*, then*

$$\begin{cases} D(\partial\Phi_\gamma) \cap C^2_\beta(\overline{\Omega}) = \{u \in C^2_\beta(\overline{\Omega}),\ \ (-\Delta)^s_{\Omega,p} u \in L^2(\Omega),\ C_{p,s}\mathcal{N}^{2-\beta}_p u \\ \quad +\gamma|u|^{p-2}u = 0 \ \ on\ \partial\Omega\} \\ \partial\Phi_\gamma(u) = (-\Delta)^s_{\Omega,p} u. \end{cases} \tag{6.4}$$

Proof. Let $p \in (1,\infty)$, $0 < s < 1$ and Φ_γ the functional defined in (6.1). It is clear that Φ_γ is proper and convex. It is also easy to show that Φ_γ is lower semi-continuous. Let $\partial\Phi_\gamma$ be the subgradient of Φ_γ. It is clear that $\partial\Phi_\gamma$ is single valued. Let $f \in L^2(\Omega)$, $\frac{1}{p} < s < 1$ and $u \in W^{s,p}(\Omega) \cap L^2(\Omega)$. It is straightforward (this follows as the case of the classical p-Laplacian) to show that $f = \partial\Phi_\gamma(u)$ if and only if (6.2) holds for every $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$. If $0 < s \leq \frac{1}{p}$, then the identity (6.3) is obtained similarly by observing that in this case $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$. It remains to show (6.4). Let $\max\{\frac{1}{p}, \frac{p-1}{p}\} < s < 1$, $\beta := \frac{ps-1}{p-1} + 1$ and assume that Ω is of class $C^{1,1}$. Set

$$\mathcal{W}_\gamma := \{u \in C^2_\beta(\overline{\Omega}),\ \ (-\Delta)^s_{\Omega,p} u \in L^2(\Omega),\ C_{p,s}\mathcal{N}^{2-\beta}_p u + \gamma|u|^{p-2}u = 0 \ \text{ on } \partial\Omega\}.$$

Let $u \in D(\partial\Phi_\gamma) \cap C^2_\beta(\overline{\Omega})$ and $f := \partial\Phi_\gamma(u)$. Then by definition, $f \in L^2(\Omega)$ and (6.2) holds for every $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$. Using the integration by parts formula (3.18), we get from (6.2) that, for every $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$,

$$\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx + \int_{\partial\Omega} v\left[C_{p,s}\mathcal{N}^{2-\beta}_p u + \gamma|u|^{p-2}u\right]\, d\sigma = \int_\Omega fv\, dx. \tag{6.5}$$

In particular, it follows from (6.5) that for every $v \in \mathcal{D}(\Omega)$,

$$\int_\Omega v(-\Delta)^s_{\Omega,p} u\, dx = \int_\Omega fv\, dx.$$

Hence, $\partial\Phi_\gamma(u) = (-\Delta)^s_{\Omega,p} u = f$. Since $f \in L^2(\Omega)$, we have that $\partial\Phi_\gamma(u) = (-\Delta)^s_{\Omega,p} u \in L^2(\Omega)$. From this equality and (6.5) we also get that $C_{p,s}\mathcal{N}^{2-\beta}_p u + \gamma|u|^{p-2}u = 0$ on $\partial\Omega$. We have shown that $u \in \mathcal{W}_\gamma$ and $\partial\Phi_\gamma(u) = (-\Delta)^s_{\Omega,p} u$. To prove the converse inclusion, let $u \in \mathcal{W}_\gamma$. Then it follows from Theorem 3.9 again that (6.2) holds for every $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$ with $f = (-\Delta)^s_{\Omega,p} u = \partial\Phi_\gamma(u)$. Hence, $u \in D(\partial\Phi_\gamma) \cap C^2_\beta(\overline{\Omega})$ and this completes the proof of the theorem. □

In fact, using Theorem 4.1, we have that

$$D(\partial\Phi_\gamma) \cap C^2_\beta(\overline{\Omega}) = \{u \in C^2_\beta(\overline{\Omega}),\ \ C_{p,s}\mathcal{N}^{2-\beta}_p u + \gamma|u|^{p-2}u = 0 \ \text{ on } \partial\Omega\}.$$

We conclude the paper with the following remark.

Remark 6.2. If $\gamma \equiv 0$ and $\frac{1}{p} < s < 1$, then $\partial\Phi_0$ is a realization in $L^2(\Omega)$ of $(-\Delta)^s_{\Omega,p}$ with fractional Neumann boundary conditions $\mathcal{N}^{2-\beta}_p u = 0$ on $\partial\Omega$, if $\gamma \not\equiv 0$ and $\frac{1}{p} < s < 1$, then $\partial\Phi_\gamma$ is a realization in $L^2(\Omega)$ of $(-\Delta)^s_{\Omega,p}$ with Robin boundary conditions $C_{p,s}\mathcal{N}^{2-\beta}_p u + \gamma|u|^{p-2}u = 0$ on $\partial\Omega$. The case $0 < s \leq \frac{1}{p}$ corresponds to a realization in $L^2(\Omega)$ of $(-\Delta)^s_{\Omega,p}$ with Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

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