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Uniqueness of viscosity solutions for a class of integro-differential equations

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Abstract. We prove comparison theorems and uniqueness of viscosity solutions for a class of nonlocal equations. This class of equations includes Bellman–Isaacs equations containing operators of Lévy type with measures depending on x and control parameters, as well as elliptic nonlocal equations that are not strictly monotone in the u variable. The proofs use the knowledge about regularity of viscosity solutions of such equations.

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1. Introduction

In this paper, we study comparison principles and uniqueness of viscosity solutions for nonlocal equations of the type

$$G(x, u, I[x, u]) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $I[x, u]$ is an integro-differential operator. The function u is real-valued. The nonlinearity $G\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is degenerate elliptic, i.e., for any $x \in \Omega$, $r \in \mathbb{R}$, $l_1, l_2 \in \mathbb{R}$

$$G(x, r, l_1) \le G(x, r, l_2) \quad \text{if } l_1 \ge l_2, \tag{1.2}$$

and coercive, i.e. there is a non-negative constant γ such that, for any $x \in \Omega$, $r \ge s$, $l \in \mathbb{R}$

$$\gamma(r - s) \le G(x, r, l) - G(x, s, l). \tag{1.3}$$

The nonlocal operator I has the form

$$I[x, u] := \int_{\mathbb{R}^n} [u(x + z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \mu_x(dz), \tag{1.4}$$

where $\mathbb{1}_{B_1(0)}$ denotes the indicator function of the unit ball $B_1(0)$ and $\{\mu_x : x \in \Omega\}$ is a family of Lévy measures, i.e. non-negative, Borel measures on $\mathbb{R}^n \setminus \{0\}$ such that

$$\int_{\mathbb{R}^n} \min\{|z|^2, 1\}\mu_x(dz) < +\infty \quad \text{for all } x \in \Omega. \tag{1.5}$$

We extend μ_x to measures on $\mathbb{R}^n$ by setting $\mu_x(\{0\}) = 0$. The operator $I[x, u]$ is thus well defined at least for functions $u \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$ for some $\delta > 0$. We point out that the solution u has to be given in the whole space $\mathbb{R}^n$ even if (1.1) is satisfied only in Ω. We will also be interested in equations of Bellman–Isaacs type

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha,\beta}(x)\} = 0, \quad \text{in } \Omega, \tag{1.6}$$

where each $I_{\alpha\beta}[x, u]$ is of the form (1.4).

Comparison principles and uniqueness results are well known for Eq. (1.1) and classical Bellman–Isaacs equations when $\gamma > 0$ and the nonlocal operators I and $I_{\alpha\beta}$ have the form

$$I_{\alpha\beta}[x, u] = \int_{\mathbb{R}^n} [u(x+\gamma_{\alpha\beta}(x, z)) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot \gamma_{\alpha\beta}(x, z)]\mu(dz). \tag{1.7}$$

In this case the Lévy measure is fixed which, in the stochastic control/ differential game interpretation of the Bellman–Isaacs equations, means that we can only control the state through the diffusion coefficients $\gamma_{\alpha\beta}$ of a stochastic differential equation driven by a fixed Lévy process or a fixed random measure. The first comparison and uniqueness results for such equations were obtained in [24,26,27] and many other results can be found in the literature, including results for equations with second order PDE terms, see [1–4,6–10,16,18,19].

The case when we have a family of measures μ_x depending on x is much more difficult. Some comparison results for time dependent equation like (1.1) were obtained in [2] however with restrictive assumptions. In particular the measures $\mu_{t,x}$, which depend on t and x there, are bounded. We prove several comparison theorems for Eqs. (1.1) and (1.6). In Sect. 3 we first look at the case when equations are strictly monotone in the u variable, i.e. when $\gamma > 0$ in (1.3) and in (1.6). Since standard comparison proofs do not work for these equations, the idea is to try to prove comparison assuming that either a viscosity subsolution or a supersolution is more regular. Of particular interest is the case when one of them is in $C^r_{\text{loc}}(\Omega)$ for some $r > 1$. We adapt to the nonlocal case the technique from [11, Section 5.6] (see also [20]). There are many recent $C^r_{\text{loc}}(\Omega)$ regularity results [6,8,12–14,21,23,25] for Eqs. (1.1) and (1.6) and we show in Sect. 7 that comparison theorems obtained in this paper can be applied to various classes of problems.

Another largely open problem considered in this manuscript is comparison results for Eqs. (1.1) and (1.6) when they are not strictly monotone in the u variable, i.e. when $\gamma = 0$. The only result in this direction in [12, Section 5], is for equations corresponding to the case when the measures μ_x are independent of x. There is also a remark made in [15, Theorem 9.2], about comparison for a class of equations being a consequence of an Aleksandrov–Bakelman–Pucci estimate for nonlocal equations, however it is not supported by any proof and it is probably false without additional assumptions about the nonlocal operator. Our small contribution here in Sect. 4 is in showing how comparison results of Sect. 3 can be extended to the case $\gamma = 0$ when equations are elliptic with respect to a good enough class of linear nonlocal operators. We follow a typical strategy of perturbing viscosity sub/supersolutions to strict viscosity sub/supersolutions (see [11,17]). The reader can consult [5,11,17,20] for comparison results for fully nonlinear elliptic PDE which are not strictly monotone in the u variable.

In Sect. 5 we show how viscosity sub/supersolutions of Eqs. (1.1) and (1.6) can be regularized by special sup- and inf-convolutions that depend on a family of smooth functions. We also show how to use these special sup/inf-convolutions to prove that the difference of a viscosity subsolution and a viscosity supersolution of the same elliptic equation is a viscosity subsolution of a nonlocal Pucci extremal equation. Knowing this one can use an Aleksandrov–Bakelman–Pucci estimate of [15] to prove a comparison principle but this part appears to be missing in [15].

2. Definitions and assumptions

We will write $B_\delta(x)$ for the open ball centered at x with radius $\delta > 0$ and $BUC(\mathbb{R}^n)$ for the space of bounded and uniformly continuous functions in $\mathbb{R}^n$. If Ω' is an open set, $r = k + \alpha, k = 1, 2, \ldots, 0 < \alpha < 1$, we will write $C^r(\Omega')$ to denote the standard Hölder space $C^{k,\alpha}(\Omega')$ equipped with the usual norm which we will denote by $\|\cdot\|_{C^r(\Omega')}$. We will use this notation instead of the standard notation $C^{k,\alpha}(\Omega')$ or $C^{k+\alpha}(\Omega')$ to simplify the statements. The space of Lipschitz continuous functions will be denoted by $C^{0,1}(\Omega')$. We will denote by $C^r_{\text{loc}}(\Omega')$ [respectively, $C^{0,1}_{\text{loc}}(\Omega')$] the space of all functions which are in $C^r(\Omega'')$ [respectively, $C^{0,1}(\Omega'')$] for every open set $\Omega'' \subset\subset \Omega'$. If $0 < \alpha < 1$ we will use the standard notation

$$[u]_{C^\alpha(\Omega')} := \sup_{x,y\in\Omega'} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Suppose that G is continuous and (1.2), (1.3), and (1.5) hold. We recall two equivalent definitions of a viscosity solution of (1.1). In order to do it, we introduce two associated operators $I^{1,\delta}$ and $I^{2,\delta}$,

$$I^{1,\delta}[x, p, u] = \int_{|z|<\delta} [u(x + z) - u(x) - \mathbb{1}_{B_1(0)}(z)p \cdot z]\mu_x(dz),$$

$$I^{2,\delta}[x, p, u] = \int_{|z|\geq\delta} [u(x + z) - u(x) - \mathbb{1}_{B_1(0)}(z)p \cdot z]\mu_x(dz).$$

Definition 2.1. A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (1.1) if whenever $u - \varphi$ has a maximum over $\mathbb{R}^n$ at $x \in \Omega$ for some test function

$\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$, then

$$G(x, u(x), I[x, \varphi]) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (1.1) if whenever $u - \varphi$ has a minimum over $\mathbb{R}^n$ at $x \in \Omega$ for a test function $\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$, then

$$G(x, u(x), I[x, \varphi]) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1.1) if it is both a viscosity subsolution and viscosity supersolution of (1.1).

It is easy to see that Definition 2.1 is equivalent to the definition in which the requirement that $\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ is replaced by the requirement that $\varphi \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$ for some $\delta > 0$. The equivalence of Definitions 2.1 and 2.2 is also standard.

Definition 2.2. A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (1.1) if whenever $u - \varphi$ has a maximum over $B_\delta(x)$ at $x \in \Omega$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then

$$G\left(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]\right) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (1.1) if whenever $u - \varphi$ has a minimum over $B_\delta(x)$ at $x \in \Omega$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then

$$G\left(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]\right) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1.1) if it is both a viscosity subsolution and viscosity supersolution of (1.1).

We make the following assumptions on the nonlinearity G and the family of Lévy measures $\{\mu_x\}$.

(H1) For each $\Omega' \subset\subset \Omega$, there is a nondecreasing continuous function $w_{\Omega'}$ satisfying $w_{\Omega'}(0) = 0$ and a non-negative constant $\Lambda_{\Omega'}$ such that

$$G(y, r, l_2) - G(x, r, l_1) \leq \Lambda_{\Omega'}(l_1 - l_2) + w_{\Omega'}(|x - y|)$$

for any $x, y \in \Omega'$ and $r, l_1, l_2 \in \mathbb{R}$.

(H2) For every $x \in \Omega$ the measure μ_x is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, i.e. $\mu_x(dz) = a(x, z)dz$, where $a(x, \cdot) \geq 0$ is measurable, and there exist two constants $0 < \theta \leq 1$, $0 < \sigma < 2$ and a positive constant C such that, for any $x, y \in \Omega$, we have

$$|a(x, z) - a(y, z)| \leq C\frac{|x - y|^\theta}{|z|^{n+\sigma}} \quad \text{in } B_1(0),$$

$$a(x, z) \leq \frac{C}{|z|^{n+\sigma}} \quad \text{in } B_1(0),$$

$$\int_{\mathbb{R}^n \setminus B_1(0)} |a(x, z) - a(y, z)| dz \leq C|x - y|^\theta,$$

$$\int_{\mathbb{R}^n \setminus B_1(0)} \mu_x(dz) \leq C.$$

3. Uniqueness of viscosity solutions of (1.1) for $\gamma > 0$

In this section we prove the main comparison theorem which will be a basis for other comparison results.

Theorem 3.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (1.1) is continuous and satisfies (1.3) with $\gamma > 0$ and (H1). Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, u is a viscosity subsolution of (1.1), v is a viscosity supersolution of (1.1), $u \leq v$ in Ω^c, and either u or v is in $C^r_{\text{loc}}(\Omega)$, we have $u \leq v$ in $\mathbb{R}^n$.*

Proof. Without loss of generality we assume that $u \in C^r_{\text{loc}}(\Omega)$. The proof is divided into two cases.

Case 1 $0 < \sigma \leq 1$.

Without loss of generality we can assume in this case that $0 < r < 1$. Suppose that $\max_\Omega(u - v) = \nu > 0$. Let $K \subset \Omega$ be a compact neighborhood of the set of maximum points of $u - v$ in Ω. Then (see Proposition 3.7 of [11]), for ϵ sufficiently small, there are $\hat{x}$, $\hat{y} \in K$ such that

$$u(\hat{x}) - v(\hat{y}) - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 = \sup_{x,y}\left\{u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2\right\} \geq \nu.$$

Moreover, we can assume that there is $0 < c < 1$ such that $B_{2c}(\hat{x}) \cup B_{2c}(\hat{y}) \subset \Omega$. Since

$$u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2 \leq u(\hat{x}) - v(\hat{y}) - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2,$$

for any x, $y \in \mathbb{R}^n$, putting $x = y = \hat{y}$, we thus have

$$\frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 \leq u(\hat{x}) - u(\hat{y}) \leq C|\hat{x} - \hat{y}|^r$$

for some $C > 0$ independent of ϵ, which gives us

$$\frac{|\hat{x} - \hat{y}|^{2-r}}{2\epsilon} \leq C. \tag{3.1}$$

By the definition of viscosity subsoltions and supersolutions, we have for $0 < \delta < c$,

$$G\left(\hat{x}, u(\hat{x}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, \frac{|\cdot - \hat{y}|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, u(\cdot)\right]\right) \leq 0,$$
$$G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, -\frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, v(\cdot)\right]\right) \geq 0.$$

Therefore, by (1.3) and assumption (H1), we have

$$\begin{aligned}
&\gamma(u(\hat{x}) - v(\hat{y}))\\
&\quad \leq G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, -\frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, v(\cdot)\right]\right)\\
&\qquad -G\left(\hat{x}, v(\hat{y}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, \frac{|\cdot - \hat{y}|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, u(\cdot)\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \Lambda_K \left\{ I^{1,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon}, \frac{|\cdot-\hat{y}|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon}, u(\cdot)\right] \right. \\
&\quad \left. - \left(I^{1,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon}, -\frac{|\hat{x}-\cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon}, v(\cdot)\right]\right)\right\} + w_K(|\hat{x}-\hat{y}|) \\
&\le \Lambda_K \left\{ \int_{|z|<\delta} \left[\frac{1}{2\epsilon}|\hat{x}-\hat{y}+z|^2 - \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 - \frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right]\mu_{\hat{x}}(dz) \right. \\
&\quad + \int_{|z|<\delta} \left[\frac{1}{2\epsilon}|\hat{x}-\hat{y}-z|^2 - \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 + \frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right]\mu_{\hat{y}}(dz) \\
&\quad + \int_{|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right]\mu_{\hat{x}}(dz) \\
&\quad \left. - \int_{|z|\ge\delta} \left[v(\hat{y}+z) - v(\hat{y}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right]\mu_{\hat{y}}(dz) \right\} + w_K(|\hat{x}-\hat{y}|) \\
&= \Lambda_K \left\{ \int_{|z|<\delta} \frac{|z|^2}{2\epsilon}\mu_{\hat{x}}(dz) + \int_{|z|<\delta} \frac{|z|^2}{2\epsilon}\mu_{\hat{y}}(dz) \right. \\
&\quad + \int_{|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad \left. + \int_{|z|\ge\delta} [u(\hat{x}+z) - u(\hat{x}) - v(\hat{y}+z) + v(\hat{y})]\mu_{\hat{y}}(dz) \right\} + w_K(|\hat{x}-\hat{y}|).
\end{aligned}
$$

Since $u(x) - v(y) - \frac{1}{2\epsilon}|x-y|^2$ attains a global maximum at $(\hat{x},\hat{y})$, we have

$$
u(\hat{x}+z) - u(\hat{x}) \le v(\hat{y}+z) - v(\hat{y}), \quad \text{for any } z \in \mathbb{R}^n.
$$

Moreover, by assumption (H2) and the boundedness of u, we have

$$
\begin{aligned}
&\int_{|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad \le \int_{|z|\ge 1} [u(\hat{x}+z) - u(\hat{x})](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\qquad + \int_{1>|z|\ge c} \left[u(\hat{x}+z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\qquad + \int_{c>|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad \le \int_{c>|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\qquad + C|\hat{x}-\hat{y}|^\theta + C\frac{|\hat{x}-\hat{y}|^{1+\theta}}{\epsilon}.
\end{aligned}
$$

Therefore,

$$\gamma\big(u(\hat{x})-v(\hat{y})\big) \le \Lambda_K\Bigg\{\int_{|z|<\delta}\frac{|z|^2}{2\epsilon}\mu_{\hat{x}}(dz)+\int_{|z|<\delta}\frac{|z|^2}{2\epsilon}\mu_{\hat{y}}(dz) + \int_{c>|z|\ge\delta}\left[u(\hat{x}+z)-u(\hat{x})-\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz)-\mu_{\hat{y}}(dz)) + C|\hat{x}-\hat{y}|^\theta + C\frac{|\hat{x}-\hat{y}|^{1+\theta}}{\epsilon}\Bigg\} + w_K(|\hat{x}-\hat{y}|). \tag{3.2}$$

Now by assumption (H2), we have for some $C>0$

$$\int_{|z|<\delta}\frac{|z|^2}{2\epsilon}\mu_{\hat{x}}(dz)+\int_{|z|<\delta}\frac{|z|^2}{2\epsilon}\mu_{\hat{y}}(dz) \le C\frac{\delta^{2-\sigma}}{\epsilon}, \tag{3.3}$$

$$\begin{aligned}
&\int_{c>|z|\ge\delta}\left[u(\hat{x}+z)-u(\hat{x})-\frac{1}{\epsilon}(\hat{x}-\hat{y})\cdot z\right](\mu_{\hat{x}}(dz)-\mu_{\hat{y}}(dz))\\
&\le \int_{c>|z|\ge\delta}\frac{C\left(|z|^r|\hat{x}-\hat{y}|^\theta+\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta}|z|\right)}{|z|^{n+\sigma}}dz\\
&\le \begin{cases}
C|\hat{x}-\hat{y}|^\theta\delta^{r-1}-C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta}\ln\delta & \text{if } r<\sigma=1,\\
C|\hat{x}-\hat{y}|^\theta\delta^{r-\sigma}+C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta} & \text{if } r<\sigma<1,\\
-C|\hat{x}-\hat{y}|^\theta\ln\delta+C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta} & \text{if } r=\sigma<1,\\
C|\hat{x}-\hat{y}|^\theta+C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta} & \text{if } \sigma<r<1.
\end{cases}
\end{aligned}$$

In the rest of the proof we will only consider the case $r<\sigma$. The case $\sigma\le r<1$ is easier and can be handled similarly. Let $\delta=n^{-\alpha}$ and $\epsilon=n^{-\beta}$. By (3.1), we have

$$|\hat{x}-\hat{y}|\le Cn^{-\frac{\beta}{2-r}}.$$

If $r<\sigma<1$, we have

$$\begin{aligned}
C\frac{\delta^{2-\sigma}}{\epsilon} &= Cn^{\alpha(\sigma-2)+\beta},\\
C|\hat{x}-\hat{y}|^\theta\delta^{r-\sigma} &\le Cn^{-\frac{\theta\beta}{2-r}+\alpha(\sigma-r)},\\
C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta} &\le Cn^{\frac{\beta}{2-r}(1-r-\theta)}.
\end{aligned}$$

Thus, if

$$\beta<(2-\sigma)\alpha, \tag{3.4}$$

$$\alpha(\sigma-r)<\frac{\theta\beta}{2-r}, \tag{3.5}$$

$$\theta>1-r, \tag{3.6}$$

it follows

$$C\frac{\delta^{2-\sigma}}{\epsilon}\to 0, \tag{3.7}$$

$$C|\hat{x}-\hat{y}|^\theta\delta^{r-\sigma}\to 0, \tag{3.8}$$

$$C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta} \to 0. \tag{3.9}$$

It remains to find proper $\alpha > 0$, $\beta > 0$, and $0 < r_0 < \sigma$ so that (3.4) and (3.5) hold. We set $\beta = 1$ and $\alpha > 1/(2-\sigma)$ so that (3.4) is satisfied. Then obviously there exists a positive constant $r_0 < \sigma$ such that (3.5) is satisfied if $r_0 < r < \sigma$.

If $r < \sigma = 1$, we have

$$C\frac{\delta^{2-\sigma}}{\epsilon} = Cn^{\beta-\alpha},$$
$$C|\hat{x}-\hat{y}|^{\theta}\delta^{r-\sigma} \le Cn^{-\frac{\theta\beta}{2-r}+\alpha(1-r)},$$
$$C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta}(-\ln\delta) \le Cn^{\frac{\beta}{2-r}(1-r-\theta)}\ln(n).$$

Thus, if

$$\beta < \alpha, \tag{3.10}$$
$$\alpha(1-r) < \frac{\theta\beta}{2-r}, \tag{3.11}$$
$$\theta > 1-r, \tag{3.12}$$

we have

$$C\frac{\delta^{2-\sigma}}{\epsilon} \to 0, \tag{3.13}$$
$$C|\hat{x}-\hat{y}|^{\theta}\delta^{r-\sigma} \to 0, \tag{3.14}$$
$$C\frac{1}{\epsilon}|\hat{x}-\hat{y}|^{1+\theta}(-\ln\delta) \to 0. \tag{3.15}$$

Using the same strategy as before, for any $\theta > 0$, we set $\beta = 1, \alpha > 1$, and then choose $0 < r_0 < \sigma$ such that (3.11) is satisfied if $r_0 < r < 1$.

Therefore, using (3.7)–(3.9) and (3.13)–(3.15) in (3.2), we conclude

$$\gamma\nu \le \limsup_{n\to+\infty} \gamma\big(u(\hat{x}) - v(\hat{y})\big) \le 0$$

if $r_0 < r < \sigma$. This contradiction thus implies that we must have $u \le v$ in $\mathbb{R}^n$.

Case 2 $1 < \sigma < 2$.

We assume that $r > 1$. Suppose that $\max_{\Omega}(u-v) = \nu > 0$. Let $K \subset \Omega$ be a compact neighborhood of the set of maximum points of $u-v$ in Ω. There is a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\psi_n\}_n$ such that

$$u - \psi_n \to 0 \quad \text{as } n \to +\infty \quad \text{uniformly on } \mathbb{R}^n, \tag{3.16}$$

and

$$\begin{cases} |Du - D\psi_n| \le Cn^{1-r} & \text{on } K, \\ |D^2\psi_n| \le Cn^{2-r} & \text{on } K, \\ |D^2\psi_n(x) - D^2\psi_n(y)| \le Cn^{3-r}|x-y| & \text{on } K, \end{cases} \tag{3.17}$$

where C is a positive constant (see [11]). Let ρ be a modulus of continuity of u and v.

Let $(\hat{x},\hat{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ be a maximum point of

$$(u(x) - \psi_n(x)) - (v(y) - \psi_n(y)) - \frac{1}{2\epsilon}|x-y|^2$$

over $\mathbb{R}^n \times \mathbb{R}^n$. Again it is standard to notice (see Proposition 3.7 of [11]) that

$$\lim_{\epsilon\to 0}(u(\hat{x}) - v(\hat{y})) = \max_{\Omega}(u - v), \tag{3.18}$$

and there must exist $0 < c < 1$ such that $B_c(\hat{x}) \cup B_c(\hat{y}) \subset K$ if ϵ is sufficiently small. Moreover, since $u(\cdot) - \psi_n(\cdot) - \big(v(\hat{y}) - \psi_n(\hat{y})\big) - \frac{1}{2\epsilon}|\cdot - \hat{y}|^2$ has a global maximum at $\hat{x}$, we have

$$Du(\hat{x}) - D\psi_n(\hat{x}) = \frac{\hat{x} - \hat{y}}{\epsilon}.$$

Thus, we get

$$\frac{|\hat{x} - \hat{y}|}{\epsilon} \le Cn^{1-r}. \tag{3.19}$$

By the definition of viscosity subsolutions and supersolutions, we have, for any $0 < \delta < c$,

$$\begin{aligned} &G\left(\hat{x}, u(\hat{x}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right]\right. \\ &\qquad \left. + I^{2,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right]\right) \le 0, \\ &G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right]\right. \\ &\qquad \left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right]\right) \ge 0. \end{aligned}$$

Therefore, by (1.3) and assumption (H1), we have

$$\begin{aligned} &\gamma(u(\hat{x}) - v(\hat{y})) \\ &\quad \le G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right]\right. \\ &\qquad \left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right]\right) \\ &\qquad - G\left(\hat{x}, v(\hat{y}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right]\right. \\ &\qquad \left. + I^{2,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right]\right) \\ &\quad \le \Lambda_K \left\{ I^{1,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right]\right. \\ &\qquad + I^{2,\delta}\left[\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right] - \left(I^{1,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right]\right. \\ &\qquad \left.\left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{x}-\hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right]\right)\right\} + w_K(|\hat{x} - \hat{y}|) \\ &\quad \le \Lambda_K \left\{ \int_{|z|<\delta} \left[\psi_n(\hat{x} + z) + \frac{1}{2\epsilon}|\hat{x} - \hat{y} + z|^2 - \left(\psi_n(\hat{x}) + \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2\right)\right.\right. \\ &\qquad \left. - \left(\frac{1}{\epsilon}(\hat{x} - \hat{y}) + D\psi_n(\hat{x})\right) \cdot z\right] \mu_{\hat{x}}(dz) \end{aligned}$$

$$
\begin{aligned}
&-\int_{|z|<\delta}\left[\psi_n(\hat y+z)-\frac{1}{2\epsilon}|\hat x-\hat y-z|^2-\left(\psi_n(\hat y)-\frac{1}{2\epsilon}|\hat x-\hat y|^2\right)\right.\\
&\left.-\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat y)\right)\cdot z\right]\mu_{\hat y}(dz)\\
&+\int_{|z|\ge\delta}\left[u(\hat x+z)-u(\hat x)-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat x)\right)\cdot z\right]\mu_{\hat x}(dz)\\
&\left.-\int_{|z|\ge\delta}\left[v(\hat y+z)-v(\hat y)-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat y)\right)\cdot z\right]\mu_{\hat y}(dz)\right\}\\
&+w_K(|\hat x-\hat y|)\\
\le\ &\Lambda_K\left\{\int_{|z|<\delta}\left[\frac{1}{2\epsilon}|z|^2+\psi_n(\hat x+z)-\psi_n(\hat x)-D\psi_n(\hat x)\cdot z\right]\mu_{\hat x}(dz)\right.\\
&-\int_{|z|<\delta}\left[-\frac{1}{2\epsilon}|z|^2+\psi_n(\hat y+z)-\psi_n(\hat y)-D\psi_n(\hat y)\cdot z\right]\mu_{\hat y}(dz)\\
&+\int_{|z|\ge\delta}\left[u(\hat x+z)-u(\hat x)-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat x)\right)\cdot z\right]\\
&\times(\mu_{\hat x}(dz)-\mu_{\hat y}(dz))+\int_{|z|\ge\delta}[u(\hat x+z)-u(\hat x)-v(\hat y+z)+v(\hat y)\\
&\left.-\mathbb{1}_{B_1(0)}(z)\left(D\psi_n(\hat x)-D\psi_n(\hat y)\right)\cdot z]\mu_{\hat y}(dz)\right\}+w_K(|\hat x-\hat y|).
\end{aligned}
$$

Since $(\hat x,\hat y)$ is a global maximum point of $(u(x)-\psi_n(x))-(v(y)-\psi_n(y))-\frac{1}{2\epsilon}|x-y|^2$, we have

$$
\begin{aligned}
&u(\hat x+z)-u(\hat x)-v(\hat y+z)+v(\hat y)\le\psi_n(\hat x+z)-\psi_n(\hat x)-\psi_n(\hat y+z)+\psi_n(\hat y),\\
&\text{for all } z\in\mathbb{R}^n.
\end{aligned}
$$

Thus, by (3.17) and the uniform continuity of u,v, we have

$$
\begin{aligned}
&\int_{|z|\ge\delta}\left[u(\hat x+z)-u(\hat x)-v(\hat y+z)+v(\hat y)-\mathbb{1}_{B_1(0)}(z)(D\psi_n(\hat x)-D\psi_n(\hat y))\cdot z\right]\mu_{\hat y}(dz)\\
&\quad\le\int_{c\ge|z|\ge\delta}[(\psi_n(\hat x+z)-\psi_n(\hat x)-D\psi_n(\hat x)\cdot z)\\
&\qquad-(\psi_n(\hat y+z)-\psi_n(\hat y)-D\psi_n(\hat y)\cdot z)]\,\mu_{\hat y}(dz)+C|\hat x-\hat y|^{r-1}+C\rho(|\hat x-\hat y|).
\end{aligned}
$$

Moreover, by assumption (H2), the boundedness of u and $Du(\hat x)=\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat x)$ (in n and ϵ), we have

$$
\begin{aligned}
&\int_{|z|\ge\delta}\left[u(\hat x+z)-u(\hat x)-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat x)\right)\cdot z\right](\mu_{\hat x}(dz)-\mu_{\hat y}(dz))\\
&\quad\le\int_{c\ge|z|\ge\delta}+\int_{|z|\ge c}\le\int_{c\ge|z|\ge\delta}\left[u(\hat x+z)-u(\hat x)-\left(\frac{1}{\epsilon}(\hat x-\hat y)+D\psi_n(\hat x)\right)\cdot z\right]\\
&\qquad\times(\mu_{\hat x}(dz)-\mu_{\hat y}(dz))+C|\hat x-\hat y|^{\theta}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&\gamma\big(u(\hat{x}) - v(\hat{y})\big) \\
&\quad \le \Lambda_K \bigg\{ \int_{|z|<\delta} \left[\frac{1}{2\epsilon}|z|^2 + \psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x})\cdot z\right] \mu_{\hat{x}}(dz) \\
&\qquad - \int_{|z|<\delta} \left[-\frac{1}{2\epsilon}|z|^2 + \psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y})\cdot z\right] \mu_{\hat{y}}(dz) \\
&\qquad + \int_{c\ge|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \left(\frac{1}{\epsilon}(\hat{x}-\hat{y}) + D\psi_n(\hat{x})\right)\cdot z\right] \\
&\qquad \times\big(\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)\big) \\
&\qquad + \int_{c\ge|z|\ge\delta} \big[\big(\psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x})\cdot z\big) \\
&\qquad -\big(\psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y})\cdot z\big)\big]\,\mu_{\hat{y}}(dz)\bigg\} \\
&\qquad + C\rho(|\hat{x}-\hat{y}|) + C|\hat{x}-\hat{y}|^{r-1} + C|\hat{x}-\hat{y}|^{\theta} + w_K(|\hat{x}-\hat{y}|). \qquad (3.20)
\end{aligned}$$

Estimate (3.3) holds. Moreover, by (H2) and (3.17), we have

$$\left|\int_{|z|<\delta}[\psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x})\cdot z]\mu_{\hat{x}}(dz)\right| \le Cn^{2-r}\delta^{2-\sigma},$$

$$\left|\int_{|z|<\delta}[\psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y})\cdot z]\mu_{\hat{y}}(dz)\right| \le Cn^{2-r}\delta^{2-\sigma},$$

$$\begin{aligned}
&\int_{c\ge|z|\ge\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \left(\frac{1}{\epsilon}(\hat{x}-\hat{y}) + D\psi_n(\hat{x})\right)\cdot z\right]\big(\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)\big) \\
&\le C\int_{c\ge|z|\ge\delta} \frac{|z|^r|\hat{x}-\hat{y}|^\theta}{|z|^{n+\sigma}}dz \le \begin{cases} C\delta^{r-\sigma}|\hat{x}-\hat{y}|^\theta & \text{if } r<\sigma, \\ -C|\hat{x}-\hat{y}|^\theta\ln\delta & \text{if } r=\sigma, \\ C|\hat{x}-\hat{y}|^\theta & \text{if } \sigma<r<2. \end{cases}
\end{aligned}$$

We recall a simple identity. If $f\in C^2(\mathbb{R}^n)$ then for every $x, z\in\mathbb{R}^n$

$$f(x+z) = f(x) + Df(x)\cdot z + \int_0^1\int_0^1 D^2 f(x+stz)z\cdot z\, tdsdt.$$

Using it, (H2), and recalling that $B_c(\hat{x})\cup B_c(\hat{y})\subset K$, we obtain

$$\begin{aligned}
&\int_{c\ge|z|\ge\delta}[(\psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x})\cdot z) \\
&\qquad -(\psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y})\cdot z)]\mu_{\hat{y}}(dz) \\
&\quad = \int_{c\ge|z|\ge\delta}\int_0^1\int_0^1 [D^2\psi_n(\hat{x}+stz) - D^2\psi_n(\hat{y}+stz)]z\cdot z\, tdsdt\mu_{\hat{y}}(dz) \\
&\quad \le C\int_{c\ge|z|\ge\delta} n^{3-r}|\hat{x}-\hat{y}|\frac{|z|^2}{|z|^{n+\sigma}}dz \le Cn^{3-r}|\hat{x}-\hat{y}|.
\end{aligned}$$

In the remainder of the proof we will only consider the case $r < \sigma$. The case $\sigma \leq r < 2$ is easier and can be done similarly (see also Remark 3.2). Assume then that $1 < r < \sigma$. Let again $\delta = n^{-\alpha}$ and $\epsilon = n^{-\beta}$. By (3.19), we have

$$C\frac{\delta^{2-\sigma}}{\epsilon} = Cn^{\alpha(\sigma-2)+\beta},$$
$$Cn^{2-r}\delta^{2-\sigma} = Cn^{2-r+\alpha(\sigma-2)},$$
$$C|\hat{x}-\hat{y}|^{\theta}\delta^{r-\sigma} \leq Cn^{-\theta[(r-1)+\beta]-\alpha(r-\sigma)},$$
$$Cn^{3-r}|\hat{x}-\hat{y}| \leq Cn^{-(r-1)-\beta+(3-r)}.$$

Thus, if

$$\beta < (2-\sigma)\alpha, \tag{3.21}$$
$$2-r < \alpha(2-\sigma), \tag{3.22}$$
$$\alpha(\sigma-r) < \theta(r-1+\beta), \tag{3.23}$$
$$(4-2r) < \beta, \tag{3.24}$$

we have

$$C\frac{\delta^{2-\sigma}}{\epsilon} \to 0, \tag{3.25}$$
$$Cn^{2-r}\delta^{2-\sigma} \to 0, \tag{3.26}$$
$$C|\hat{x}-\hat{y}|^{\theta}\delta^{r-\sigma} \to 0, \tag{3.27}$$
$$Cn^{3-r}|\hat{x}-\hat{y}| \to 0. \tag{3.28}$$

We need to find $\alpha > 0$, $\beta > 0$, and $1 \leq r_0 < \sigma$ so that (3.21)–(3.24) are satisfied if $r_0 < r < \sigma$. First fix β such that (3.24) is satisfied. Then, fix α such that (3.21) and (3.22) are satisfied. It is then clear that there exists a positive constant $1 \leq r_0 < \sigma$ such that (3.23) is satisfied if $r_0 < r < \sigma$.

Thus, letting $n \to +\infty$ in (3.20) and using (3.18) and (3.25)–(3.28), we obtain $\gamma\nu \leq 0$ which is a contradiction. Therefore, $u \leq v$ in $\mathbb{R}^n$. □

Remark 3.1. It follows from the proof of Theorem 3.1 that if the kernel functions $a(x,\cdot)$ are symmetric, the requirement $\theta > \max\{0, 1-r\}$ can be replaced by a weaker requirement $\theta > 0$. The same remark applies to Theorems 3.2, 4.1, 4.2, 5.1, Lemmas 5.1, 5.2, and Corollaries 3.1, 3.2, 4.1, 4.2.

Corollary 3.1. *Let the assumptions of Theorem 3.1 be satisfied,* $0 < \sigma < 2, \theta > \max\{0, 1-r\}, 0 < r < 2$. *If u is a viscosity subsolution, v is a viscosity supersolution of (1.1), $u \leq v$ in Ω^c, and either u or v is in $C^r_{\mathrm{loc}}(\Omega)$, then:*

(i) *For $0 < \sigma \leq 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \leq v$ in $\mathbb{R}^n$.*

(ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, we have $u \leq v$ in $\mathbb{R}^n$.*

Proof. (i) Let $\beta = 1$ and $\alpha = 1/(2-\sigma) + \eta$, where $\eta > 0$. Then (3.4) and (3.6) hold and (3.5) will be satisfied if

$$\left(\frac{1}{2-\sigma} + \eta\right)(\sigma - r) < \frac{\theta}{2-r}.$$

An easy calculation shows that the above will be true for some $\eta > 0$ if

$$\sigma < \frac{\theta(2-r)}{2-r+\theta} + r.$$

(ii) Set

$$\beta = 4 - 2r + \eta_1, \quad \alpha = \frac{4-2r+\eta_1}{2-\sigma} + \eta_2,$$

where $\eta_1, \eta_2 > 0$. Then (3.21), (3.22) and (3.24) are satisfied, and (3.23) will be satisfied if

$$\left(\frac{4-2r+\eta_1}{2-\sigma} + \eta_2\right)(\sigma - r) < \theta\left(r - 1 + 4 - 2r + \eta_1\right)$$

for some $\eta_1, \eta_2 > 0$. Again a simple calculation yields that this inequality will be satisfied for some $\eta_1, \eta_2 > 0$ if

$$\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r) + (4-2r)}.$$

□

Let us consider another important fully nonlinear integro-PDE appearing in the study of stochastic optimal control and stochastic differential games for processes with jumps, namely the Bellman–Isaacs equation (1.6)

$$\gamma u + \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \{-I_{\alpha\beta}[x,u] + f_{\alpha,\beta}(x)\} = 0, \quad \text{in } \Omega,$$

where $I_{\alpha\beta}[x,u] = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(z)\cdot z] \mu_x^{\alpha\beta}(dz)$ and $\{\mu_x^{\alpha\beta}\}$ is a family of Lévy measures with indices α and β ranging in some sets $\mathcal{A}$ and $\mathcal{B}$. Equation (1.6) is not of the same form as (1.1), which means that the following theorem and corollary are not corollaries of Theorem 3.1 and Corollary 3.1, however the proofs follow the same arguments. Similar results would be true if we included other typical purely local first and second order terms in (1.6).

Theorem 3.2. *Let Ω be a bounded domain. Suppose that $\gamma > 0$, the family of Lévy measures $\{\mu_x^{\alpha\beta}\}$ satisfies assumption (H2) uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and $f_{\alpha,\beta}$ are uniformly bounded in Ω and uniformly continuous in every compact subset $K \subset \Omega$, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Then, for any $0 < \sigma < 2$, there exists a constant $0 \le r_0 < \sigma$ ($r_0 \ge 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, u is a viscosity subsolution of (1.6), v is a viscosity supersolution of (1.6), $u \le v$ in Ω^c, and either u or v is in $C^r_{\mathrm{loc}}(\Omega)$, we have $u \le v$ in $\mathbb{R}^n$.*

Corollary 3.2. *Let the assumptions of Theorem 3.2 be satisfied, $0 < \sigma < 2, \theta > \max\{0, 1-r\}, 0 < r < 2$. If u is a viscosity subsolution of (1.6), v is a viscosity supersolution of (1.6), $u \le v$ in Ω^c, and either u or v is in $C^r_{\mathrm{loc}}(\Omega)$, then:*

(i) *For $0 < \sigma \le 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \le v$ in $\mathbb{R}^n$.*

(ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, we have $u \le v$ in $\mathbb{R}^n$.*

Remark 3.2. Suppose that the kernel function $a(x,z)$ satisfies the second condition of (H2). If $r > \max(\sigma, 1)$, or if $r > \sigma$ and the kernels $a(x,\cdot)$ are symmetric, then a viscosity subsolution/supersolution of (1.1) which is in $C^r_{\rm loc}(\Omega)$ can be considered to be a classical subsolution/supersolution of (1.1). In such a case comparison theorem is standard and we do not need the full assumptions of Theorem 3.1. The same remark applies to Theorem 3.2, and Theorems 4.1 and 4.2 if condition (H3) is satisfied.

4. Uniqueness of viscosity solutions of (1.1) for $\gamma = 0$

In this section we investigate uniqueness of viscosity solutions of (1.1) when $\gamma = 0$ in (1.3). As always we assume that G is continuous and (1.2), (1.3), and (1.5) hold. To compensate for the fact that $\gamma = 0$, we will assume that the nonlinearity G is uniformly elliptic with respect to a class of linear nonlocal operators $\mathcal{L}$. A class $\mathcal{L}$ is a set of linear nonlocal operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x)\cdot z]\mu^L(dz),$$

where the Lévy measures μ^L are symmetric and satisfy $\sup_{L\in\mathcal{L}} \int_{\mathbb{R}^n} \min\{1,|z|^2\}$ $\mu^L(dz) < +\infty$. We say that the nonlinearity G in (1.1) is uniformly elliptic with respect to $\mathcal{L}$ if for every $\varphi,\psi \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n), x \in \Omega, r \in \mathbb{R}, \delta > 0$,

$$M^-_{\mathcal{L}}(\varphi-\psi)(x) \le G\left(x,r,I[x,\varphi]\right) - G\left(x,r,I[x,\psi]\right) \le M^+_{\mathcal{L}}(\varphi-\psi)(x),$$

where

$$M^+_{\mathcal{L}}\varphi(x) = \sup_{L\in\mathcal{L}} -L\varphi(x),$$

$$M^-_{\mathcal{L}}\varphi(x) = \inf_{L\in\mathcal{L}} -L\varphi(x).$$

In order to have a comparison principle for the case $\gamma = 0$, we need to impose an additional minimal ellipticity condition on the class $\mathcal{L}$. We will assume that the following condition holds.

(H3) There exist a non-negative function $\varphi \in C^2(\Omega) \cap BUC(\mathbb{R}^n)$ and $\delta_0 > 0$, such that $L\varphi > \delta_0$ in Ω for every $L \in \mathcal{L}$.

Theorem 4.1. *Let Ω be a bounded domain and let a class $\mathcal{L}$ satisfy (H3). Suppose that the nonlinearity G in* (1.1) *is continuous and uniformly elliptic with respect to $\mathcal{L}$, and satisfies* (1.3) *with $\gamma = 0$ and* (*H1*)*. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption* (*H2*)*. Then, for any $0 < \sigma < 2$, there exists a constant $0 \le r_0 < \sigma$ ($r_0 \ge 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, u is a viscosity subsolution of* (1.1)*, v is a viscosity supersolution of* (1.1)*, $u \le v$ in Ω^c, and either u or v is in $C^r_{\rm loc}(\Omega)$, we have $u \le v$ in $\mathbb{R}^n$.*

Proof. By (H3), there is a positive constant $M > 0$ such that $\varphi \le M$ in $\mathbb{R}^n$. For any $\epsilon > 0$, let $\varphi_\epsilon = \epsilon(1 - \frac{1}{M}\varphi)$ in $\mathbb{R}^n$. Obviously, we have $0 \le \varphi_\epsilon \le \epsilon$ in $\mathbb{R}^n$ and $M^-_{\mathcal{L}}(\varphi_\epsilon) \ge \frac{\epsilon\delta_0}{M}$ in Ω.

We claim that $v+\varphi_\epsilon$ is a viscosity supersolution of $G=\frac{\epsilon\delta_0}{M}$ in Ω. Suppose that $x\in\Omega,\delta>0$ and $\psi\in C^2(B_\delta(x))\cap BUC(\mathbb{R}^n)$ are such that $v+\varphi_\epsilon-\psi$ has a minimum over $\mathbb{R}^n$ at x. Since v is a viscosity supersolution of (1.1), we have $G(x,v(x),I[x,\psi-\varphi_\epsilon])\geq 0$. By (1.3) and the uniform ellipticity, we get

$$\begin{aligned} &G(x,v(x)+\varphi_\epsilon(x),I[x,\psi])\\ &\quad\geq G(x,v(x)+\varphi_\epsilon(x),I[x,\psi])-G(x,v(x),I[x,\psi-\varphi_\epsilon])\\ &\quad\geq G(x,v(x),I[x,\psi])-G(x,v(x),I[x,\psi-\varphi_\epsilon])\\ &\quad\geq M^-_{\mathcal{L}}(\varphi_\epsilon)\geq\frac{\epsilon\delta_0}{M}. \end{aligned}$$

Therefore, the proof of the claim is complete.

We notice that $u\leq v+\varphi_\epsilon$ in Ω^c. We can now repeat the proof of Theorem 3.1 to obtain $u\leq v+\varphi_\epsilon\leq v+\epsilon$ in $\mathbb{R}^n$. (Instead of the contradiction $\gamma\nu\leq 0$ we will now get a contradiction $\frac{\epsilon\delta_0}{M}\leq 0$.) Letting $\epsilon\to 0^+$, we thus conclude that $u\leq v$ in $\mathbb{R}^n$. $\square$

Combining the proofs of Corollary 3.1 and Theorem 4.1, we have the following corollary.

Corollary 4.1. *Let the assumptions of Theorem 4.1 be satisfied,* $0<\sigma<2,\theta>\max\{0,1-r\},0<r<2$. *If* u *is a viscosity subsolution,* v *is a viscosity supersolution of* (1.1), $u\leq v$ *in* Ω^c, *and either* u *or* v *is in* $C^r_{\mathrm{loc}}(\Omega)$, *then:*

- (i) *For* $0<\sigma\leq 1$, *if* $\sigma<\frac{\theta(2-r)}{2-r+\theta}+r$, *we have* $u\leq v$ *in* $\mathbb{R}^n$.
- (ii) *For* $1<\sigma<2$ *and* $r>1$, *if* $\sigma<2-2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, *we have* $u\leq v$ *in* $\mathbb{R}^n$.

The same techniques also produce the following two results for Eq. (1.6).

Theorem 4.2. *Let* Ω *be a bounded domain. Suppose that* $\gamma=0$, *the family of Lévy measures* $\{\mu_x^{\alpha\beta}\}$ *satisfies assumption* (*H2*) *uniformly in* $\alpha\in\mathcal{A},\beta\in\mathcal{B}$, *and* $f_{\alpha,\beta}$ *are uniformly bounded in* Ω *and uniformly continuous in every compact subset* $K\subset\Omega$, *uniformly in* $\alpha\in\mathcal{A},\beta\in\mathcal{B}$, *and the class* $\{I_{\alpha\beta}\}$ *satisfies* (*H3*). *Then, for any* $0<\sigma<2$, *there exists a constant* $0\leq r_0<\sigma$ ($r_0\geq 1$ *if* $\sigma>1$) *such that if* $r_0<r<2$, $\theta>\max\{0,1-r\}$, u *is a viscosity subsolution of* (1.6), v *is a viscosity supersolution of* (1.6), $u\leq v$ *in* Ω^c, *and either* u *or* v *is in* $C^r_{\mathrm{loc}}(\Omega)$, *we have* $u\leq v$ *in* $\mathbb{R}^n$.

Corollary 4.2. *Let the assumptions of Theorem 4.2 be true,* $0<\sigma<2$, $\theta>\max\{0,1-r\}$ *and* $0<r<2$. *If* u *is a viscosity subsolution of* (1.6), v *is a viscosity supersolution of* (1.6), $u\leq v$ *in* Ω^c, *and either* u *or* v *is in* $C^r_{\mathrm{loc}}(\Omega)$, *then:*

- (i) *For* $0<\sigma\leq 1$, *if* $\sigma<\frac{\theta(2-r)}{2-r+\theta}+r$, *we have* $u\leq v$ *in* $\mathbb{R}^n$.
- (ii) *For* $1<\sigma<2$ *and* $r>1$, *if* $\sigma<2-2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, *we have* $u\leq v$ *in* $\mathbb{R}^n$.

5. Regularization by sup/inf-convolutions

In this section we show how techniques of Sect. 3 can be adapted to regularize viscosity sub/supersolutions by sup/inf-convolutions. It is a generally expected principle in the theory of viscosity solutions of PDE that whenever one is able to prove a comparison principle then one should be able to prove that a sup-convolution of a viscosity subsolution (respectively, inf-convolution of a viscosity supersolution) is a viscosity subsolution (respectively, supersolution) of a slightly perturbed equation. The same principle also seems to work for viscosity sub/supersolutions of integro-PDE under standard assumptions, see e.g. [22] for a proof for a standard Bellman–Isaacs equation. Here the situation is a bit more complicated. Since in our case the proof of comparison principle uses auxiliary functions ψ_n, we have to introduce a notion of sup/inf-convolution that depends on a parameter $\epsilon > 0$ and on a function ψ. Such sup/inf convolutions have been used in [20]. We will also show that if G is uniformly elliptic with respect to a class $\mathcal{L}$ of linear nonlocal operators, u is a viscosity subsolution of (1.1) and v is a viscosity supersolution of (1.1), then $u - v$ satisfies $M^-_{\mathcal{L}}(u-v) \leq 0$ in the viscosity sense. Similar results can also be proved for Eq. (1.6).

We will always assume that G is continuous and satisfies (1.2), (1.3), and (1.5). We first give yet another equivalent definition of viscosity solutions of (1.1).

Definition 5.1. A function φ is said to be $C^{1,1}$ at the point x, and we write $u \in C^{1,1}(x)$, if there are a vector $p \in \mathbb{R}^n$, a constant $M > 0$ and a neighborhood N_x of x such that

$$|\varphi(y) - \varphi(x) - p \cdot (y - x)| \leq M|y - x|^2 \quad \text{for } y \in N_x.$$

The definition implies that $D\varphi(x) = p$.

Definition 5.2. A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (1.1) if for any test function $\varphi(x) \in C^{1,1}(x) \cap BUC(B_\delta(x))$ such that $u - \varphi$ has a maximum over $B_\delta(x)$ at $x \in \Omega$,

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (1.1) if for any test function $\varphi \in C^{1,1}(x) \cap BUC(B_\delta(x))$ such that $u - \varphi$ has a minimum over $B_\delta(x)$ at $x \in \Omega$,

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1.1) if it is both a viscosity subsolution and viscosity supersolution of (1.1).

Proposition 5.1. *Let G be continuous and* (1.2), (1.3), *and* (1.5) *hold. Then Definition* 2.2 *is equivalent to Definition* 5.2.

Proof. Obviously if u is a viscosity sub/supersolution in the sense of Definition 5.2, it is a viscosity sub/supersolution in the sense of Definition 2.2. Assume now that u is a viscosity subsolution in the sense of Definition 2.2. Let $\varphi \in$

$C^{1,1}(x) \cap BUC(B_\delta(x))$ and $u - \varphi$ have a maximum over $B_\delta(x)$ at x. Then $I^{1,\delta}[x, D\varphi(x), \varphi], I^{2,\delta}[x, D\varphi(x), u]$) are well defined. Also because φ is $C^{1,1}(x)$, there exist a sequence of $C^2(B_\delta(x))$ functions $\{\varphi_n\}_n$ and a positive constant C such that $\varphi - \varphi_n$ has a maximum point at x over $B_\delta(x)$, $\varphi_n \geq \varphi, \varphi_n \to \varphi$ uniformly in $B_\delta(x)$ and $|\varphi_n(x+z) - \varphi_n(x) - D\varphi_n(x)\cdot z| \leq C|z|^2$. Thus $u - \varphi_n$ has a maximum at x over $B_\delta(x)$ and $D\varphi(x) = D\varphi_n(x)$. Therefore, by Definition 2.2,

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi_n] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

Letting $n \to +\infty$ and using the Lebesgue dominated convergence theorem we thus conclude

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

□

Definition 5.3. (See [20]) Given $u, \psi \in BUC(\mathbb{R}^n), \epsilon > 0$, the ψ-sup-convolution $u^{\psi,\epsilon}$ of u is defined by

$$u^{\psi,\epsilon}(x) := (u - \psi)^\epsilon(x) + \psi(x) = \sup_{y\in\mathbb{R}^n} \left\{ u(y) - \psi(y) - \frac{|x-y|^2}{2\epsilon} \right\} + \psi(x),$$

and the ψ-inf-convolution $u_{\psi,\epsilon}$ of u is defined by

$$u_{\psi,\epsilon}(x) := (u - \psi)_\epsilon(x) + \psi(x) = \inf_{y\in\mathbb{R}^n} \left\{ u(y) - \psi(y) + \frac{|x-y|^2}{2\epsilon} \right\} + \psi(x).$$

Remark 5.1. The functions $u^{0,\epsilon}$ and $u_{0,\epsilon}$ are the usual sup- and inf-convolutions of u respectively, and we will denote them by u^ϵ and u_ϵ (see [11]).

Remark 5.2. $u^{\psi_\alpha,\epsilon}(x), u_{\psi_\alpha,\epsilon}(x) \to u(x)$ uniformly for $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}$ as $\epsilon \to 0$ if the functions $\{\psi_\alpha\}_{\alpha\in\mathcal{A}} \subset BUC(\mathbb{R}^n)$ have a uniform modulus of continuity.

Lemma 5.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (1.1) is continuous and $G(x, \cdot, l)$ is uniformly continuous, uniformly for $x \in \Omega, l \in \mathbb{R}$. Assume moreover that G satisfies (1.3) with $\gamma = 0$ and (H1), and the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, $\Omega' \subset\subset \Omega$ is an open set, $u \in C^r_{\mathrm{loc}}(\Omega)$ is a viscosity subsolution of (1.1), then there are a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\psi_n\}_n$ with a uniform modulus of continuity, a sequence of positive numbers $\{\epsilon_n\}_n$ with $\epsilon_n \to 0$, and a modulus ρ such that u^{ψ_n,ϵ_n} is a viscosity subsolution of*

$$G(x, u^{\psi_n,\epsilon_n}, I[x, u^{\psi_n,\epsilon_n}]) = \rho\left(\frac{1}{n}\right) \quad \text{in } \Omega'. \tag{5.1}$$

Proof. **Case 1** $0 < \sigma \leq 1$.

As in the proof of Theorem 3.1, without loss of generality, we can assume that $0 < r < 1$. For any $\hat{x} \in \Omega'$ and $B_{\hat{\delta}}(\hat{x}) \subset \Omega'$, suppose that there is a test function $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ such that $u^\epsilon - \varphi$ has a maximum (equal 0) at $\hat{x}$ over $B_{\hat{\delta}}(\hat{x})$. Since $u \in BUC(\mathbb{R}^n)$, there exists a point $\hat{y} \in \Omega'$ such that

$u^\epsilon(\hat{x}) = u(\hat{y}) - \frac{|\hat{x}-\hat{y}|^2}{2\epsilon}$ if ϵ is sufficiently small. Thus $u(y) - \frac{1}{2\epsilon}|x-y|^2 - \varphi(x)$ has a maximum at $(\hat{y}, \hat{x})$ over $\mathbb{R}^n \times B_{\hat{\delta}}(\hat{x})$ and $u(\hat{y}) \geq u^\epsilon(\hat{x})$. Therefore, we have

$$\frac{|\hat{x}-\hat{y}|^{2-r}}{2\epsilon} \leq C \tag{5.2}$$

for some $C > 0$ independent of ϵ. Notice that u^ϵ is semi-convex, which implies that there is a paraboloid touching its graph from below at $\hat{x}$. Since $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ touches the graph of u^ϵ from above at $\hat{x}$, we get $u^\epsilon \in C^{1,1}(\hat{x}) \cap BUC(\mathbb{R}^n)$. For any $0 < \delta < \min\{\hat{\delta}, 1\}$ and small $\epsilon > 0$, we have by (1.3) and (H1),

$$\begin{aligned}
&G\left(\hat{x}, u^\epsilon(\hat{x}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]\right)\\
&\quad -G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, \frac{|\hat{x}-\cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, u\right]\right)\\
&\leq G\left(\hat{x}, u(\hat{y}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]\right)\\
&\quad -G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, \frac{|\hat{x}-\cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, u\right]\right)\\
&\leq \Lambda_{\Omega'}\Bigg\{I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, \frac{|\hat{x}-\cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, u\right]\\
&\quad -\left(I^{1,\delta}\left[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon\right] + I^{2,\delta}\left[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon\right]\right)\Bigg\} + w_{\Omega'}(|\hat{x}-\hat{y}|)\\
&\leq \Lambda_{\Omega'}\Bigg\{\int_{|z|<\delta}\left[\frac{1}{2\epsilon}|\hat{x}-\hat{y}-z|^2 - \frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 - \frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\mu_{\hat{y}}(dz)\\
&\quad -\int_{|z|<\delta}\left[u^\epsilon(\hat{x}+z) - u^\epsilon(\hat{x}) - \frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\mu_{\hat{x}}(dz)\\
&\quad +\int_{|z|\geq\delta}\left[u(\hat{y}+z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\mu_{\hat{y}}(dz)\\
&\quad -\int_{|z|\geq\delta}\left[u^\epsilon(\hat{x}+z) - u^\epsilon(\hat{x}) - \mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\mu_{\hat{x}}(dz)\Bigg\}\\
&\quad +w_{\Omega'}(|\hat{x}-\hat{y}|).
\end{aligned} \tag{5.3}$$

Since $\frac{\hat{y}-\hat{x}}{\epsilon} = Du^\epsilon(\hat{x})$ and $u^\epsilon(z) + \frac{|z|^2}{2\epsilon}$ is convex, we have

$$-\frac{|z|^2}{2\epsilon} \leq u^\epsilon(\hat{x}+z) - u^\epsilon(\hat{x}) - \frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z. \tag{5.4}$$

Thus, by (5.3) and (5.4),

$$\begin{aligned}
&G\left(\hat{x}, u^\epsilon(\hat{x}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]\right)\\
&\quad -G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, \frac{|\hat{x}-\cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon}, u\right]\right)
\end{aligned}$$

$$\leq \Lambda_{\Omega'} \Bigg\{ \int_{|z|<\delta} \frac{1}{2\epsilon}|z|^2\big(\mu_{\hat{y}}(dz)+\mu_{\hat{x}}(dz)\big) + \int_{|z|\geq\delta} \left[u(\hat{y}+z)-u(\hat{y})-\mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\big(\mu_{\hat{y}}(dz)-\mu_{\hat{x}}(dz)\big) + \int_{|z|\geq\delta} \left[u(\hat{y}+z)-u(\hat{y})-\big(u^\epsilon(\hat{x}+z)-u^\epsilon(\hat{x})\big)\right]\mu_{\hat{x}}(dz)\Bigg\} + w_{\Omega'}(|\hat{x}-\hat{y}|). \tag{5.5}$$

By the definition of u^ϵ, we have

$$u^\epsilon(\hat{x}+z) \geq u(\hat{y}+z) - \frac{|\hat{x}-\hat{y}|^2}{2\epsilon},$$

which implies

$$u^\epsilon(\hat{x}+z)-u^\epsilon(\hat{x}) \geq u(\hat{y}+z)-u(\hat{y}). \tag{5.6}$$

Thus, by (5.5) and (5.6), it follows

$$G\left(\hat{x},u^\epsilon(\hat{x}),I^{1,\delta}[\hat{x},Du^\epsilon(\hat{x}),u^\epsilon]+I^{2,\delta}[\hat{x},Du^\epsilon(\hat{x}),u^\epsilon]\right) - G\left(\hat{y},u(\hat{y}),I^{1,\delta}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon},\frac{|\hat{x}-\cdot|^2}{2\epsilon}\right]+I^{2,\delta}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon},u\right]\right)$$
$$\leq \Lambda_{\Omega'}\Bigg\{\int_{|z|<\delta}\frac{1}{2\epsilon}|z|^2\big(\mu_{\hat{y}}(dz)+\mu_{\hat{x}}(dz)\big) + \int_{|z|\geq\delta}\left[u(\hat{y}+z)-u(\hat{y})-\mathbb{1}_{B_1(0)}(z)\frac{1}{\epsilon}(\hat{y}-\hat{x})\cdot z\right]\big(\mu_{\hat{y}}(dz)-\mu_{\hat{x}}(dz)\big)\Bigg\} + w_{\Omega'}(|\hat{x}-\hat{y}|).$$

We now let $\delta_n = n^{-\alpha}$ and $\epsilon_n = n^{-\beta}$, and use the same estimates as in Case 1 of the proof of Theorem 3.1 to show that, we can find $\alpha > 0$, $\beta > 0$, and $0 < r_0 < \sigma$ such that, if $r_0 < r < 1$ and $\theta > 1-r$, then

$$G\left(\hat{x},u^{\epsilon_n}(\hat{x}),I^{1,\delta_n}[\hat{x},Du^{\epsilon_n}(\hat{x}),u^{\epsilon_n}]+I^{2,\delta_n}[\hat{x},Du^{\epsilon_n}(\hat{x}),u^{\epsilon_n}]\right) - G\left(\hat{y},u(\hat{y}),I^{1,\delta_n}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon_n},\frac{|\hat{x}-\cdot|^2}{2\epsilon_n}\right]+I^{2,\delta_n}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon_n},u\right]\right) \leq \rho\left(\frac{1}{n}\right)$$

for some modulus ρ. Since u is a viscosity subsolution of (1.1), this implies

$$G(\hat{x},u^{\epsilon_n}(\hat{x}),I[\hat{x},u^{\epsilon_n}]) = G\left(\hat{x},u^{\epsilon_n}(\hat{x}),I^{1,\delta_n}[\hat{x},Du^{\epsilon_n}(\hat{x}),u^{\epsilon_n}]+I^{2,\delta_n}[\hat{x},Du^{\epsilon_n}(\hat{x}),u^{\epsilon_n}]\right) \leq \rho\left(\frac{1}{n}\right).$$

Case 2 $1<\sigma<2$.

We take $r>1$. Let $\{\psi_n\}_n$ be a sequence of $C^2(\mathbb{R}^n)\cap BUC(\mathbb{R}^n)$ functions which are uniformly bounded and have a uniform (in n) modulus of continuity h, which satisfy (3.16) and (3.17) with K replaced by Ω'.

Let $\hat{x}\in\Omega'$, $B_{\hat{\delta}}(\hat{x})\subset\Omega'$, and suppose that there is a test function $\varphi\in C^2(B_{\hat{\delta}}(\hat{x}))$ such that $u^{\psi_n,\epsilon}-\varphi$ has a maximum (equal 0) at $\hat{x}$ over $B_{\hat{\delta}}(\hat{x})$. Since $u\in BUC(\mathbb{R}^n)$ and $\psi_n\in BUC(\mathbb{R}^n)$, there exists a point $\hat{y}\in\Omega'$ such that $u^{\psi_n,\epsilon}(\hat{x}) = u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}) - \frac{|\hat{x}-\hat{y}|^2}{2\epsilon}$ if ϵ is sufficiently small. Thus $u(y) - \psi_n(y) + \psi_n(x) - \frac{|x-y|^2}{2\epsilon} - \varphi(x)$ has a maximum at $(\hat{y},\hat{x})$ over $\mathbb{R}^n \times B_{\hat{\delta}}(\hat{x})$ and $u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}) \geq u^{\psi_n,\epsilon}(\hat{x})$. Since $u(\cdot) - \psi_n(\cdot) - \frac{|\hat{x}-\cdot|^2}{2\epsilon}$ has a maximum at $\hat{y}$ over $\mathbb{R}^n$, we have

$$Du(\hat{y}) - D\psi_n(\hat{y}) = \frac{\hat{y}-\hat{x}}{\epsilon}.$$

Thus, by (3.17),

$$\frac{|\hat{y}-\hat{x}|}{\epsilon} \leq Cn^{1-r}. \tag{5.7}$$

Since $u^{\psi_n,\epsilon}$ is semi-convex, there is a paraboloid touching its graph from below at $\hat{x}$. Since $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ touches the graph of $u^{\psi_n,\epsilon}$ from above at $\hat{x}$, we obtain that $u^{\psi_n,\epsilon} \in C^{1,1}(\hat{x}) \cap BUC(\mathbb{R}^n)$. Thus, for any $0 < \delta < \min\{\hat{\delta}, 1\}$ and small $\epsilon > 0$, we have, by (1.3), (H1), (3.17) and (5.7), uniform continuity of the ψ_n and the continuity properties of G,

$$\begin{aligned}
&G(\hat{x}, u^{\psi_n,\epsilon}(\hat{x}), I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]) \\
&\quad - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x}-\cdot|^2}{2\epsilon} + \psi_n\right]\right. \\
&\quad \left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), u\right]\right) \\
&\leq G(\hat{x}, u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}), I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] \\
&\quad + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]) \\
&\quad - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x}-\cdot|^2}{2\epsilon} + \psi_n\right]\right. \\
&\quad \left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), u\right]\right) \\
&\leq G\left(\hat{x}, u(\hat{y}), I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]\right) + \rho_1\left(\frac{1}{n}\right) \\
&\quad - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x}-\cdot|^2}{2\epsilon} + \psi_n\right]\right. \\
&\quad \left. + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), u\right]\right) \\
&\leq \Lambda_{\Omega'}\left\{I^{1,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x}-\cdot|^2}{2\epsilon} + \psi_n\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon} + D\psi_n(\hat{y}), u\right]\right. \\
&\quad \left. - \left(I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]\right)\right\} + \rho_1\left(\frac{1}{n}\right) \\
&\leq \Lambda_{\Omega'}\left\{\int_{|z|<\delta}\left[\left(\frac{1}{2\epsilon}|\hat{x}-\hat{y}-z|^2 + \psi_n(\hat{y}+z)\right) - \left(\frac{1}{2\epsilon}|\hat{x}-\hat{y}|^2 + \psi_n(\hat{y})\right)\right.\right. \\
&\quad \left. - \left(\frac{1}{\epsilon}(\hat{y}-\hat{x}) + D\psi_n(\hat{y})\right)\cdot z\right]\mu_{\hat{y}}(dz) \\
&\quad - \int_{|z|<\delta}\left[u^{\psi_n,\epsilon}(\hat{x}+z) - u^{\psi_n,\epsilon}(\hat{x}) - \left(\frac{1}{\epsilon}(\hat{y}-\hat{x}) + D\psi_n(\hat{x})\right)\cdot z\right]\mu_{\hat{x}}(dz)
\end{aligned}$$

$$+\int_{|z|\geq\delta}\left[u(\hat{y}+z)-u(\hat{y})-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat{y}-\hat{x})+D\psi_n(\hat{y})\right)\cdot z\right]\mu_{\hat{y}}(dz)$$
$$-\int_{|z|\geq\delta}\left[u^{\psi_n,\epsilon}(\hat{x}+z)-u^{\psi_n,\epsilon}(\hat{x})-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat{y}-\hat{x})+D\psi_n(\hat{x})\right)\cdot z\right]\mu_{\hat{x}}(dz)\Bigg\}$$
$$+\rho_1\left(\frac{1}{n}\right) \tag{5.8}$$

for some modulus ρ_1 independent of δ, ϵ.

Since $\frac{\hat{y}-\hat{x}}{\epsilon}+D\psi_n(\hat{x})=Du^{\psi_n,\epsilon}(\hat{x})$ and $u^{\psi_n,\epsilon}(z)+\frac{|z|^2}{2\epsilon}+(\sup_{\Omega'}|D^2\psi_n|)|z|^2$ is convex on $B_{\hat{\delta}}(\hat{x})$, we have for $|z|<\hat{\delta}$

$$-\frac{|z|^2}{2\epsilon}-\left(\sup_{\Omega'}|D^2\psi_n|\right)|z|^2\leq u^{\psi_n,\epsilon}(\hat{x}+z)-u^{\psi_n,\epsilon}(\hat{x})-\left(\frac{\hat{y}-\hat{x}}{\epsilon}+D\psi_n(\hat{x})\right)\cdot z. \tag{5.9}$$

Moreover, by the definition of $u^{\psi_n,\epsilon}$,

$$u^{\psi_n,\epsilon}(\hat{x}+z)\geq u(\hat{y}+z)-\psi_n(\hat{y}+z)+\psi_n(\hat{x}+z)-\frac{|\hat{x}-\hat{y}|^2}{2\epsilon},$$

which gives

$$u(\hat{y}+z)-u(\hat{y})-(u^{\psi_n,\epsilon}(\hat{x}+z)-u^{\psi_n,\epsilon}(\hat{x}))\leq\psi_n(\hat{y}+z)-\psi_n(\hat{y})-(\psi_n(\hat{x}+z)-\psi_n(\hat{x})). \tag{5.10}$$

Thus, by (5.8)–(5.10), we have

$$G\left(\hat{x},u^{\psi_n,\epsilon}(\hat{x}),I^{1,\delta}[\hat{x},Du^{\psi_n,\epsilon}(\hat{x}),u^{\psi_n,\epsilon}]+I^{2,\delta}[\hat{x},Du^{\psi_n,\epsilon}(\hat{x}),u^{\psi_n,\epsilon}]\right)$$
$$-G\left(\hat{y},u(\hat{y}),I^{1,\delta}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon}+D\psi_n(\hat{y}),\frac{|\hat{x}-\cdot|^2}{2\epsilon}+\psi_n\right]\right.$$
$$\left.+I^{2,\delta}\left[\hat{y},\frac{\hat{y}-\hat{x}}{\epsilon}+D\psi_n(\hat{y}),u\right]\right)$$
$$\leq\Lambda_{\Omega'}\Bigg\{\int_{|z|<\delta}\left[\frac{1}{2\epsilon}|z|^2+\left(\psi_n(\hat{y}+z)-\psi_n(\hat{y})-D\psi_n(\hat{y})\cdot z\right)\right]\mu_{\hat{y}}(dz)$$
$$+\int_{|z|<\delta}\left[\frac{1}{2\epsilon}|z|^2+\left(\sup_{\Omega'}|D^2\psi_n|\right)|z|^2\right]\mu_{\hat{x}}(dz)$$
$$+\int_{|z|\geq\delta}\left[u(\hat{y}+z)-u(\hat{y})-\mathbb{1}_{B_1(0)}(z)\left(\frac{1}{\epsilon}(\hat{y}-\hat{x})+D\psi_n(\hat{y})\right)\cdot z\right]$$
$$\times(\mu_{\hat{y}}(dz)-\mu_{\hat{x}}(dz))$$
$$+\int_{|z|\geq\delta}[\psi_n(\hat{y}+z)-\psi_n(\hat{y})-\mathbb{1}_{B_1(0)}(z)D\psi_n(\hat{y})\cdot z$$
$$-(\psi_n(\hat{x}+z)-\psi_n(\hat{x})-\mathbb{1}_{B_1(0)}(z)D\psi_n(\hat{x})\cdot z)]\mu_{\hat{x}}(dz)\Bigg\}+\rho_1\left(\frac{1}{n}\right).$$

We now again set $\delta_n=n^{-\alpha}$ and $\epsilon_n=n^{-\beta}$ and use the same estimates as these in Case 2 of the proof of Theorem 3.1, to obtain that for any $\theta>0$, we can find $\alpha>0$, $\beta>0$, and $1\leq r_0<\sigma$ such that, if $r_0<r<2$, then

$$\begin{aligned}&G(\hat{x}, u^{\psi_n,\epsilon_n}(\hat{x}), I^{1,\delta_n}[\hat{x}, Du^{\psi_n,\epsilon_n}(\hat{x}), u^{\psi_n,\epsilon_n}] + I^{2,\delta_n}[\hat{x}, Du^{\psi_n,\epsilon_n}(\hat{x}), u^{\psi_n,\epsilon_n}])\\&\quad -G\left(\hat{y}, u(\hat{y}), I^{1,\delta_n}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon_n} + D\psi_n(\hat{y}), \frac{|\hat{x}-\cdot|^2}{2\epsilon_n} + \psi_n\right]\right.\\&\quad \left. + I^{2,\delta_n}\left[\hat{y}, \frac{\hat{y}-\hat{x}}{\epsilon_n} + D\psi_n(\hat{y}), u\right]\right)\\&\le \rho\left(\frac{1}{n}\right)\end{aligned}$$

for some modulus ρ. Since u is a viscosity subsolution of (1.1), this implies

$$\begin{aligned}&G(\hat{x}, u^{\psi_n,\epsilon_n}(\hat{x}), I[\hat{x}, u^{\psi_n,\epsilon_n}])\\&\quad = G\left(\hat{x}, u^{\psi_n,\epsilon_n}(\hat{x}), I^{1,\delta_n}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon_n}] + I^{2,\delta_n}[\hat{x}, Du^{\psi_n,\epsilon_n}(\hat{x}), u^{\psi_n,\epsilon_n}]\right)\\&\quad \le \rho\left(\frac{1}{n}\right).\end{aligned}$$

□

The same proof gives the following result for viscosity supersolutions.

Lemma 5.2. *Suppose that the assumptions of Lemma 5.1 are true. Then, for any $0 < \sigma < 2$, there exists a constant $0 \le r_0 < \sigma$ ($r_0 \ge 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, $\Omega' \subset\subset \Omega$ is an open set, $u \in C^r_{\rm loc}(\Omega)$ is a viscosity supersolution of (1.1), then there are a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\tilde{\psi}_n\}_n$ with a uniform modulus of continuity, a sequence of positive numbers $\{\tilde{\epsilon}_n\}_n$ with $\tilde{\epsilon}_n \to 0$, and a modulus $\tilde{\rho}$ such that $u^{\tilde{\psi}_n,\tilde{\epsilon}_n}$ is a viscosity supersolution of*

$$G(x, u^{\tilde{\psi}_n,\tilde{\epsilon}_n}, I[x, u^{\tilde{\psi}_n,\tilde{\epsilon}_n}]) = -\tilde{\rho}\left(\frac{1}{n}\right) \quad in\ \Omega'. \tag{5.11}$$

We remark that it is clear from the proofs of Lemmas 5.1 and 5.2 that we can always have $\epsilon_n = \tilde{\epsilon}_n$.

The next lemma is standard and can be deduced from Lemmas 4.2 and 4.5 of [12].

Lemma 5.3. *Let $\{u_n\}_n$ be a sequence of bounded and uniformly continuous functions on $\mathbb{R}^n$ such that:*

(i) *u_n is a viscosity subsolution of $M^-_{\mathcal{L}}(u_n) = f_n$ in Ω.*
(ii) *The sequence $\{u_n\}$ converges to u uniformly in $\mathbb{R}^n$ for some $u \in BUC(\mathbb{R}^n)$.*
(iii) *The sequence $\{f_n\}$ converges to f uniformly in Ω for some $f \in C(\Omega)$.*

Then u is a viscosity subsolution of $M^-_{\mathcal{L}}(u) = f$ in Ω.

Theorem 5.1. *Let the assumptions of Lemma 5.1 be satisfied and let G be uniformly elliptic with respect to $\mathcal{L}$. Then, for any $0 < \sigma < 2$, there exists a constant $0 \le r_0 < \sigma$ ($r_0 \ge 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1-r\}$, $u \in C^r_{\rm loc}(\Omega)$ is a viscosity subsolution of (1.1) and $v \in C^r_{\rm loc}(\Omega)$ is a viscosity supersolution of (1.1), then $u - v$ is a viscosity subsolution of*

$$M^-_{\mathcal{L}}(u-v) = 0 \tag{5.12}$$

in $\Omega \cap \{u - v > 0\}$*. If* $G(x, r, l)$ *is independent of the second variable* r*, then* (5.12) *holds in* Ω*.*

Proof. For any $\Omega' \subset\subset \Omega$, let $x \in \Omega', u^{\psi_n,\epsilon_n}(x) > v_{\tilde{\psi}_n,\epsilon_n}(x)$, and let φ be a $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ test function whose graph is touching the graph of $u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}$ from above at x. Since u^{ψ_n,ϵ_n} and $-v_{\tilde{\psi}_n,\epsilon_n}$ are semi-convex in a neighborhood of x, each of them has a paraboloid touching its graph from below at x. Therefore, u^{ψ_n,ϵ_n} and $-v_{\tilde{\psi}_n,\epsilon_n}$ must be in $C^{1,1}(x) \cap BUC(\mathbb{R}^n)$. Thus, by Proposition 5.1 and Lemmas 5.1 and 5.2, we have

$$G(x, u^{\psi_n,\epsilon_n}(x), I[x, u^{\psi_n,\epsilon_n}]) \leq \rho\left(\frac{1}{n}\right)$$

and

$$G(x, v_{\tilde{\psi}_n,\epsilon_n}(x), I[x, v_{\tilde{\psi}_n,\epsilon_n}]) \geq -\rho\left(\frac{1}{n}\right)$$

for some modulus ρ. Thus, by (1.3) and the uniform ellipticity, we obtain

$$M_{\mathcal{L}}^{-}(u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n})(x) \leq 2\rho\left(\frac{1}{n}\right).$$

Thus, we have

$$M_{\mathcal{L}}^{-}\varphi(x) \leq 2\rho\left(\frac{1}{n}\right).$$

Therefore, we have proved that $u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}$ is a viscosity subsolution of

$$M_{\mathcal{L}}^{-}(u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}) = 2\rho\left(\frac{1}{n}\right)$$

in $\Omega' \cap \{u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n} > 0\}$.

By Remark 5.2, we have that $u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}$ converges uniformly to $u - v$ in $\mathbb{R}^n$. Thus, for any $\epsilon > 0$, there exists a sufficiently large n_ϵ such that $\Omega' \cap \{u - v > \epsilon\} \subset \Omega' \cap \{u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n} > 0\}$ if $n > n_\epsilon$. Therefore, $u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}$ is a viscosity subsolution of $M_{\mathcal{L}}^{-}(u^{\psi_n,\epsilon_n} - v_{\tilde{\psi}_n,\epsilon_n}) = 2\rho(\frac{1}{n})$ in $\Omega' \cap \{u - v > \epsilon\}$ if $n > n_\epsilon$, and hence, by Lemma 5.3, $u - v$ is a viscosity subsolution of $M_{\mathcal{L}}^{-}(u - v) = 0$ in $\Omega' \cap \{u - v > \epsilon\}$. Since $\Omega' \subset\subset \Omega$ and $\epsilon > 0$ are arbitrary, $u - v$ is a viscosity subsolution of $M_{\mathcal{L}}^{-}(u - v) = 0$ in $\Omega \cap \{u - v > 0\}$. □

Remark 5.3. Theorem 5.1, combined with an Aleksandrov–Bakelman–Pucci estimate of [15], can be used as an alternative way to prove comparison theorem when $\gamma = 0$, at least for some class of equations which are independent of the u variable.

6. Regularity

In this section we recall some regularity results for nonlocal equations. We first recall regularity results proved in [6,8]. Here, we only state their simplified versions applicable for our equations, which can be deduced from the results and techniques of [6,8]. The full theorems of [6,8] are much more general. An equivalent of Theorem 6.2 has not been stated in [6,8] but it can be deduced easily from the proofs there. We impose here an additional requirement $\theta > \max\{0, 1-\sigma\}$. It is possible that Theorems 6.1 and 6.2 are true without this assumption but it would require some more substantial changes in the proofs of [6,8].

Theorem 6.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (1.1) is continuous and satisfies (1.3) with $\gamma = 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2) with $\theta > \max\{0, 1-\sigma\}$ and, there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathcal{S}^{n-1}$, $\eta \in (0,1), \delta \in (0,1)$,*

$$\int_{\{z:|z|\le\delta,|d\cdot z|\ge(1-\eta)|z|\}} |z|^2 \mu_x(dz) \ge C\eta^{\frac{n-1}{2}}\delta^{2-\sigma}. \tag{6.1}$$

Then, we have:

(1) *If $0 < \sigma \le 1$, any viscosity solution u of (1.1) is $C^r_{\rm loc}(\Omega)$ for any $r < \sigma$.*
(2) *If $1 < \sigma$, any viscosity solution u of (1.1) is $C^{0,1}_{\rm loc}(\Omega)$.*

Theorem 6.2. *Let Ω be a bounded domain. Suppose that $\gamma \ge 0$ in (1.6), the family of Lévy measures $\{\mu_x^{\alpha\beta}\}$ satisfies assumption (H2) with $\theta > \max\{0, 1-\sigma\}$, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and $f_{\alpha,\beta}$ are uniformly continuous in Ω, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Suppose that there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathbb{S}^{n-1}$, $\eta \in (0,1), \delta \in (0,1)$, $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$,*

$$\int_{\{z:|z|\le\delta,|d\cdot z|\ge(1-\eta)|z|\}} |z|^2 \mu_x^{\alpha\beta}(dz) \ge C\eta^{\frac{n-1}{2}}\delta^{2-\sigma}.$$

Then, we have:

(1) *If $0 < \sigma \le 1$, any viscosity solution u of (1.6) is $C^r_{\rm loc}(\Omega)$ for any $r < \sigma$.*
(2) *If $1 < \sigma$, any viscosity solution u of (1.6) is $C^{0,1}_{\rm loc}(\Omega)$.*

Let us now introduce some definitions and regularity theorems from [13,23,25]. Consider the following nonlocal equations

$$\gamma u - I[x,u] = f(x) \quad \text{in } \Omega, \tag{6.2}$$

where $\gamma \ge 0$, Ω is a bounded domain, f is bounded and continuous in Ω, and $I[x,u]$ is a nonlocal operator of the form

$$\begin{aligned} I[x,u] &= \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}} I_{\alpha\beta}[x,u] \\ &:= \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}} \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x)\cdot z] K_{\alpha\beta}(x,z)dz. \end{aligned}$$

We will denote

$$I_{\alpha\beta,x_0}[x,u] := \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]K_{\alpha\beta}(x_0,z)dz.$$

Remark 6.1. It is easy to see that if $K_{\alpha\beta}(x,z) = \frac{a_{\alpha\beta}(x,z)}{|z|^{n+\sigma}}$, $\lambda \le a_{\alpha\beta}(x,z) \le \Lambda$ and $|a_{\alpha\beta}(x_1,z) - a_{\alpha\beta}(x_2,z)| \le h(|x_1-x_2|)$ for some modulus h for any $x,x_1,x_2 \in \Omega, z \in \mathbb{R}^n, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$, then the nonlocal operator $I[x,u]$ satisfies the following properties:

(1) $I[x,u]$ is well defined as long as $u \in C^{1,1}(x)$ and $u \in L^1\big(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}}\big)$.

(2) If $u \in C^2(\Omega) \cap L^1\big(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}}\big)$, then $I(x,u)$ is continuous in Ω as a function of x.

Thus $I[x,u]$ falls into the class of nonlocal operators considered in [13,23,25] which was a little more general. Moreover the definition of viscosity sub/supersolutions in [13,23,25] was slightly different from Definition 2.2 as they allowed viscosity sub/supersolutions to be unbounded (as long as they are in the domain of definition of the nonlocal operator I) and they did not require them to be uniformly continuous.

We say that the nonlocal operator I above is uniformly elliptic with respect to a class $\mathcal{L}$ of linear nonlocal operators if

$$M^-_{\mathcal{L}}(u-v)(x) \le I[x,v] - I[x,u] \le M^+_{\mathcal{L}}(u-v)(x).$$

The norm $\|I\|$ of a nonlocal operator I is defined in the following way.

Definition 6.1.

$$\|I\| := \sup\left\{ \frac{|I[x,u]|}{1+M} : x \in \Omega, u \in C^{1,1}(x), \|u\|_{L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}})} \le M, \right.$$
$$\left. |u(x+z) - u(x) - Du(x)\cdot z| \le M|z|^2 \quad \text{for any } z \in B_1(0) \right\}.$$

The following classes of linear nonlocal operators $\mathcal{L}_0(\sigma)$ and $\mathcal{L}_\kappa(\sigma), 0 < \kappa < 1$ were introduced in [13,25]. Let $0 < \lambda \le \Lambda$ be fixed constants. A linear nonlocal operator $L \in \mathcal{L}_0(\sigma)$ if

$$Lu = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]K(z)dz, \tag{6.3}$$

where the kernel K is symmetric and satisfies for all $z \in \mathbb{R}^n\backslash\{0\}$

$$(2-\sigma)\frac{\lambda}{|z|^{n+\sigma}} \le K(z) \le (2-\sigma)\frac{\Lambda}{|z|^{n+\sigma}}. \tag{6.4}$$

Since K is symmetric, we have

$$Lu = \int_{\mathbb{R}^n} [u(x+z) + u(x-z) - 2u(x)]K(z)dz.$$

Lemma 6.1. *The class $\mathcal{L}_0(\sigma)$ satisfies (H3) for any $0 < \sigma < 2$.*

Proof. We will be using the form of L in (6.3). Let R be such that $R^{\frac{3}{2}} > \max\{3R, 1+R\}$ and $\Omega \subset B_R(0)$. We define $\varphi(x) = \min(R^3, |x|^2)$ (see Assumption 5.1 in [12]). By the definition of R, the fact that K is symmetric, we now have for every $x \in \Omega(\subset B_R(0))$

$$\begin{aligned} L\varphi(x) \geq & \int_{B_1(0)} |z|^2 K(z)dz + \int_{1\leq|z|<R^{\frac{3}{2}}-R} (|z|^2 + 2x\cdot z)K(z)dz \\ & + \int_{\{\varphi(x+z)<R^3\}\cap\{|z|\geq R^{\frac{3}{2}}-R\}} (|z|^2 - 2|x||z|)K(z)dz \\ & + \int_{\{\varphi(x+z)=R^3\}\cap\{|z|\geq R^{\frac{3}{2}}-R\}} (R^3 - R^2)K(z)dz \\ \geq & (2-\sigma)\lambda \int_{B_1(0)} |z|^{-n-\sigma+2}dz. \end{aligned}$$

□

The class $\mathcal{L}_\kappa(\sigma)$ is a subclass with $\mathcal{L}_0(\sigma)$ with kernels K such that

$$[K]_{C^\kappa(B_\rho)} \leq \Lambda(2-\sigma)\rho^{-n-\sigma-\kappa} \quad \text{if } B_{2\rho} \subset \mathbb{R}^n\backslash\{0\}$$

for any concentric balls $B_\rho, B_{2\rho}$ of radii $\rho, 2\rho > 0$. We will sometimes write $K \in \mathcal{L}_k(\sigma)$. We notice that the classes $\mathcal{L}_0(\sigma)$ and $\mathcal{L}_\kappa(\sigma)$ have scale σ. A class $\mathcal{L} \subset \mathcal{L}_0(\sigma)$ has scale σ if whenever a nonlocal operator with kernel $K(z)$ is in $\mathcal{L}$, then the one with kernel $\nu^{n+\sigma}K(\nu z)$ is also in $\mathcal{L}$ for any $\nu < 1$. The following definition of a distance between two nonlocal operators takes scaling of order σ into account.

Definition 6.2. For any $0 < \sigma < 2$ and any nonlocal operator I, we define the rescaled operator

$$I_{\mu,\nu}[x, u] = \nu^\sigma \mu I[\nu x, \mu^{-1}u(\nu^{-1}\cdot)].$$

The norm of scale σ is defined as

$$\|I^{(1)} - I^{(2)}\|_\sigma = \sup_{\nu<1} \|I^{(1)}_{1,\nu} - I^{(2)}_{1,\nu}\|.$$

The following regularity theorems for nonlocal equations were proved in [13, 23, 25]. We only state their simplified versions which are suitable for our purposes.

Theorem 6.3. (Theorem 2.6 of [13]) *Assume that $0 < \sigma_0 < \sigma < 2$. Let u solve*

$$M^+_{\mathcal{L}_0} u \geq -C_0 \quad \text{in } B_1(0),$$
$$M^-_{\mathcal{L}_0} u \leq C_0 \quad \text{in } B_1(0)$$

in the viscosity sense for some $C_0 \geq 0$. Then there exists a constant $0 < r < 1$, depending only on λ, Λ, n and σ_0, such that $u \in C^r(B_{\frac{1}{2}}(0))$ and

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \leq C\left(\|u\|_{L^\infty(B_1(0))} + \|u\|_{L^1\left(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma_0}}\right)} + C_0\right)$$

for some constant $C > 0$ which depends on σ_0, λ, Λ and n.

Theorem 6.4. (Theorem 4.1 of [23]) *Assume* $1 < \sigma_0 < \sigma < 2$*. Let* $I = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} I_{\alpha\beta}$ *be a nonlocal operator such that* $\{I_{\alpha\beta,x_0} : \alpha \in \mathcal{A}, \beta \in \mathcal{B}, x_0 \in B_1(0)\} \subset \mathcal{L}_0(\sigma)$*. Denote* $I_{x_0} = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} I_{\alpha\beta,x_0}$*. There exist constants* $r > 1, \eta > 0$ *such that if for every* $x_0 \in B_{\frac{1}{2}}(0)$*,*

$$\|I - I_{x_0}\|_\sigma < \eta,$$

and u *is a viscosity solution of*

$$-I[x, u] = f(x) \quad in\ B_1(0)$$

for a bounded continuous function f*, then* $u \in C^r(B_{\frac{1}{2}}(0))$ *and*

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \le C\left(\|u\|_{L^\infty(B_1(0))} + \|u\|_{L^1\left(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma_0}}\right)} + \|f\|_{L^\infty(B_1(0))}\right)$$

for some absolute constant $C > 0$*.*

Theorem 6.5. (Theorem 1.2 and Remark 1.3 of [25]) *Let* $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ *be a class of linear nonlocal operators*

$$I_\alpha[x, u] = \int_{\mathbb{R}^n} [u(x + z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x) \cdot z]K_\alpha(x, z)dz$$

such that $\{I_{\alpha,x_0} : \alpha \in \mathcal{A}, x_0 \in B_1(0)\} \subset \mathcal{L}_\kappa(\sigma)$ *for some* $\kappa > 0$ *and* $0 < \sigma < 2$*. Suppose that for all* $x_1, x_2 \in B_1(0), z \in \mathbb{R}^n\setminus\{0\}, \alpha \in \mathcal{A}$*,*

$$|K_\alpha(x_1, z) - K_\alpha(x_2, z)| \le |x_1 - x_2|^\theta \frac{\Lambda(2 - \sigma)}{|z|^{n+\sigma}}.$$

Then there exists $\bar{r} > 0$ *such that if* $\kappa \in (0, \bar{r}], \theta \in (0, \kappa)$ *and* u *is a viscosity solution of*

$$-I[x, u] = -\inf_{\alpha\in\mathcal{A}} I_\alpha[x, u] = 0 \quad in\ B_1(0),$$

then $u \in C^{\sigma+\theta}(B_{\frac{1}{2}}(0))$ *and*

$$\|u\|_{C^{\sigma+\theta}(B_{\frac{1}{2}}(0))} \le C\|u\|_{L^\infty(\mathbb{R}^n)}$$

for some absolute constant $C > 0$*.*

Theorem 6.6. (Theorem 5.2 of [13]) *Assume* $1 < \sigma_0 < \sigma < 2$*. Let* $I^0 = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} I^0_{\alpha\beta}$ *be a nonlocal operator such that* $\{I^0_{\alpha\beta}\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} \subset \mathcal{L}$*, where* $\mathcal{L} \subset \mathcal{L}_0(\sigma)$ *has scale* σ *and interior* $C^{\bar{r}}$ *estimates for some* $\bar{r} > 1$*. Let* $I = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} I_{\alpha\beta}$ *be a nonlocal operator uniformly elliptic with respect to* $\mathcal{L}_0(\sigma)$*. Then for every* $r < \min\{\bar{r}, \sigma_0\}$ *there is* $\eta > 0$ *such that if*

$$\|I^0 - I\|_\sigma < \eta$$

and u *is a viscosity solution of*

$$-I[x, u] = f(x) \quad in\ B_1(0)$$

for a bounded and continuous function f*, then* $u \in C^r(B_{\frac{1}{2}}(0))$ *and*

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \le C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1(0))})$$

for some absolute constant $C > 0$*.*

Corollary 6.1. *Let $0 < \sigma < 2$ and let u be a viscosity solution of (6.2) in $B_1(0)$, where $\gamma \geq 0$, $f \in C(\overline{B_1(0)})$ and $I[x,u] = \inf_{\alpha\in\mathcal{A}}(2-\sigma)\int_{\mathbb{R}^n}[u(x+z)-u(x)-\mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]\frac{a_\alpha(x,z)}{|z|^{n+\sigma}}dz$. Assume that $a_\alpha(x,\cdot)$ is symmetric, $\lambda \leq a_\alpha(x,z) \leq \Lambda$, $\frac{a_\alpha(x,\cdot)}{|\cdot|^{n+\sigma}} \in \mathcal{L}_\kappa(\sigma)$ and $|a_\alpha(x_1,z)-a_\alpha(x_2,z)| \leq C|x_1-x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $x, x_1, x_2 \in B_1(0)$, $z \in \mathbb{R}^n\backslash\{0\}$, and some constants $\kappa > 0, \theta > \max\{0, 1-\sigma\}$. Then, for any $r < \sigma$, $u \in C^r(B_{\frac{1}{2}}(0))$.*

Proof. For $0 < \sigma \leq 1$, since $\lambda \leq a_\alpha(x,z) \leq \Lambda$ for any $x \in B_1(0)$ and $z \in \mathbb{R}^n$, it follows that the family of Lévy measures $\{\frac{a_\alpha(x,z)}{|z|^{n+\sigma}}dz\}_{x,\alpha}$ satisfies (6.1) (see Example 1 in [6]). Thus, by Theorem 6.2, the proof is complete for the case $0 < \sigma \leq 1$.

For $\sigma > 1$, if we fix $x_0 \in B_{\frac{1}{2}}(0)$, then the operator $I_{\alpha,x_0}u = (2-\sigma)\int_{\mathbb{R}^n}[u(x+z)-u(x)-\mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]\frac{a_\alpha(x_0,z)}{|z|^{n+\sigma}}dz$ is in $\mathcal{L}_\kappa(\sigma)$. Thus, by Theorem 6.5, it has interior $C^{\overline{r}}$ estimates for some $\overline{r} > \sigma$. By the Hölder continuity of $a_\alpha(\cdot,z)$ for fixed $z \in \mathbb{R}^n\backslash\{0\}$, we can find a small ball $B_{r_0}(x_0)$ such that $|a_\alpha(x,z)-a_\alpha(x_0,z)| < \eta$. Thus, by a simple calculation (see the proof of Theorem 6.1 in [13]), we can derive that $\|I - I_{x_0}\|_\sigma < C\eta$ in $B_{r_0}(x_0)$ where C is a positive constant and $I_{x_0} = \inf_{\alpha\in\mathcal{A}} I_{\alpha,x_o}$. Finally, we apply Theorem 6.6 with $I^0 = I_{x_0}$ and $f := f - \gamma u$, scaled in $B_{r_0}(x_0)$. ☐

Corollary 6.2. *Let $0 < \sigma < 2$. Let u be a viscosity solution of*

$$\gamma u - \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\{I_{\alpha\beta}[x,u]\} = f(x) \quad \text{in } B_1(0),$$

where $\gamma \geq 0$, $f \in C(\overline{B_1(0)})$ and $I_{\alpha\beta}[x,u] = (2-\sigma)\int_{\mathbb{R}^n}[u(x+z)-u(x)-\mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]\frac{a_{\alpha\beta}(x,z)}{|z|^{n+\sigma}}dz$. Assume that $a_{\alpha\beta}(x,\cdot)$ is symmetric, $\lambda \leq a_{\alpha\beta}(x,z) \leq \Lambda$, and $|a_{\alpha\beta}(x_1,z)-a_{\alpha\beta}(x_2,z)| \leq |x_1-x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $x, x_1, x_2 \in B_1(0)$, $z \in \mathbb{R}^n\backslash\{0\}$ and some constant $\theta > \max\{0, 1-\sigma\}$. Then, if $\sigma > 1$, $u \in C^r(B_{\frac{1}{2}}(0))$, where r is from Theorem 6.4, and if $\sigma \leq 1$, $u \in C^r(B_{\frac{1}{2}}(0))$ for every $r < \sigma$.

Proof. For $0 < \sigma \leq 1$, the proof is the same as for Corollary 6.1. For $\sigma > 1$, by the Hölder continuity of $a_{\alpha\beta}(\cdot,z)$ for fixed $z \in \mathbb{R}^n\backslash\{0\}$, we can find a small ball $B_{r_0}(x_0)$ such that $|a_{\alpha\beta}(x,z)-a_{\alpha\beta}(x_0,z)| < \eta$. Thus, like in the proof of Corollary 6.1, we can obtain $\|I - I_{x_0}\|_\sigma < C\eta$ in $B_{r_0}(x_0)$ for some constant $C > 0$. We then apply Theorem 6.4 with $f := f - \gamma u$, scaled in $B_{r_0}(x_0)$. ☐

7. Applications

In this section, we provide several concrete applications when we have uniqueness of viscosity solutions.

7.1. Nonlinear convex equations with variable coefficients

Theorem 7.1. *Let Ω be a bounded domain. Consider the following nonlinear nonlocal equations*

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \{-I_\alpha[x,u]\} = f(x) \quad in\ \Omega, \tag{7.1}$$

where $\gamma \geq 0$, $0 < \sigma < 2$, $f \in C(\overline{\Omega})$ and $\mathcal{I}_\alpha[x,u] = (2-\sigma)\int_{\mathbb{R}^n}[u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]\frac{a_\alpha(x,z)}{|z|^{n+\sigma}}dz$. Assume that $a_\alpha(x,\cdot)$ is symmetric, $\lambda \leq a_\alpha(x,z) \leq \Lambda$, $\frac{a_\alpha(x,\cdot)}{|\cdot|^{n+\sigma}} \in \mathcal{L}_\kappa(\sigma)$ and $|a_\alpha(x_1,z) - a_\alpha(x_2,z)| \leq C|x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $x, x_1, x_2 \in \Omega$, $z \in \mathbb{R}^n \backslash \{0\}$ and some $\kappa > 0, \theta > 0$. Suppose that $\theta > \max\{0, 1-\sigma\}$. Then, if u is a viscosity solution of (7.1), v is a viscosity supersolution (respectively, subsolution) of (7.1) and $u \leq v$ (respectively, $u \geq v$) in Ω^c, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.

Proof. The theorem follows from Theorem 4.2, Corollary 6.1, and Lemma 6.1 since we can take r arbitrarily close to σ. □

7.2. Nonlinear non-convex equations with variable coefficients

Theorem 7.2. *Let Ω be a bounded domain. Consider the following nonlinear nonlocal equations*

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-\mathcal{I}_{\alpha\beta}[x,u]\} = f(x) \quad in\ \Omega, \tag{7.2}$$

where $\gamma \geq 0$, $0 < \sigma < 2$, $f \in C(\overline{\Omega})$ and $I_{\alpha\beta}[x,u] = (2-\sigma)\int_{\mathbb{R}^n}[u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x)\cdot z]\frac{a_{\alpha\beta}(x,z)}{|z|^{n+\sigma}}dz$. Assume that $a_{\alpha\beta}(x,\cdot)$ is symmetric, $\lambda \leq a_{\alpha\beta}(x,z) \leq \Lambda$ and $|a_{\alpha\beta}(x_1,z) - a_{\alpha\beta}(x_2,z)| \leq C|x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $x, x_1, x_2 \in \Omega$ and $z \in \mathbb{R}^n \backslash \{0\}$. Then, if u is a viscosity solution of (7.2), v is a viscosity supersolution (respectively, subsolution) of (7.2) and $u \leq v$ (respectively, $u \geq v$) in Ω^c, we have:

(i) *For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*
(ii) *For $1 < \sigma < 2$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, where $r < 2$ is given by Corollary 6.2, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*

Proof. The theorem follows from Theorem 4.2, Corollory 4.2, Lemma 6.1, and Corollary 6.2. □

7.3. General nonlocal uniformly elliptic equations with respect to $\mathcal{L}_0$

Theorem 7.3. *Let Ω be a bounded domain and $1 \geq \sigma > 0$. Suppose that the nonlinearity G in (1.1) is continuous and uniformly elliptic with respect to $\mathcal{L}_0$, and satisfies (1.3) with $\gamma \geq 0$ and (H1). Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Suppose that u is a viscosity solution of (1.1), v is a viscosity supersolution (respectively, subsolution) of (1.1) and $u \leq v$ (respectively, $u \geq v$) in Ω^c. Then, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$ and $\theta > 1 - r$, where $r < 1$ is given by Theorem 6.3, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*

Proof. The theorem follows from Corollary 4.1(i), Theorem 6.3, and Lemma 6.1. □

7.4. General nonlocal equations with a family of Lévy measures satisfying (6.1)

Theorem 7.4. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (1.1) is continuous and satisfies (1.3) with $\gamma > 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2), and there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathcal{S}^{n-1}$, $\eta, \delta \in (0,1)$, we have (6.1). If u is a viscosity solution of (1.1), v is a viscosity supersolution (respectively, subsolution) of (1.1) and $u \leq v$ (respectively, $u \geq v$) in Ω^c, then:*

- (i) *For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*
- (ii) *For $1 < \sigma < 2$, if $0 < \theta \leq 1$ and $\sigma < 2 - \frac{1}{1+\theta}$, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*

Proof. The theorem follows from Theorems 3.1, 6.1 and Corollary 3.1. □

Theorem 7.5. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (1.1) is continuous and uniformly elliptic with respect to $\mathcal{L}_0$, and satisfies (1.3) with $\gamma = 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2) and, there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathcal{S}^{n-1}$, $\eta, \delta \in (0,1)$, we have (6.1). If u is a viscosity solution of (1.1), v is a viscosity supersolution (respectively, subsolution) of (1.1) and $u \leq v$ (respectively, $u \geq v$) in Ω^c, then:*

- (i) *For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*
- (ii) *For $1 < \sigma < 2$, if $0 < \theta \leq 1$ and $\sigma < 2 - \frac{1}{1+\theta}$, we have $u \leq v$ (respectively, $u \geq v$) in $\mathbb{R}^n$.*

Proof. This theorem follows from Theorems 4.1, 6.1, Corollary 4.1, and Lemma 6.1. □

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