Multiple normalized solutions of Chern–Simons–Schrödinger system

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Abstract. In this paper, we consider the following equation

$$-\Delta u + \omega u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2} u = 0 \quad \text{in } \mathbb{R}^2,$$

for $p > 2$ and $\lambda > 0$, which appeared in Byeon et al. (J Funct Anal 263(6):1575–1608, 2012) to find the standing wave solutions of the Chern–Simons–Schrödinger system. By using the minimax theorem, we get the multiplicity results for the L^2 -normalized solutions to the equation, and thus there are multiple L^2 -normalized solutions of the Chern–Simons–Schrödinger system.

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1. Introduction

We investigate the following Chern–Simons gauged nonlinear Schrödinger system

$$iD_0\phi + (D_1D_1 + D_2D_2)\phi = -\lambda|\phi|^{p-2}\phi, \quad (1.1)$$

$$\partial_0 A_1 - \partial_1 A_0 = -Im(\bar{\phi}D_2\phi), \quad (1.2)$$

$$\partial_0 A_2 - \partial_2 A_0 = Im(\bar{\phi}D_1\phi), \quad (1.3)$$

$$\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \quad (1.4)$$

where ϕ is a complex scalar field and $A_\mu : \mathbb{R}^{2+1} \rightarrow \mathbb{R}$ is the gauge field, $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative for $\mu = 0, 1, 2$, $\lambda > 0$ and $p > 2$ is a constant.

The system (1.1)–(1.4) conserves the charge and the energy

$$M(t) := \|\phi(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 = M(0), \quad (1.5)$$

$$E(t) := \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^2} |D_j \phi(t, \cdot)|^2 dx - \lambda \frac{1}{p} \int_{\mathbb{R}^2} |\phi(t, \cdot)|^p dx = E(0). \quad (1.6)$$

The system (1.1)–(1.4) is invariant under the following gauge transformation

$$\phi \rightarrow \phi e^{i\chi}, A_\mu \rightarrow A_\mu - \partial_\mu \chi, \quad (1.7)$$

where $\chi : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ is an arbitrary C^∞ function.

This model was proposed to study vortex solutions, which carry both electric and magnetic charges. It has applications in high-temperature superconductivity, we refer the reader to [9, 15, 16] for the physical background. For the mathematical study of this model, see e.g. [4, 5, 10–12, 21].

In [5, 11, 21], the authors consider the standing wave solution to (1.1)–(1.4) for $p > 2$ of the form

$$\begin{aligned} \phi(t, x) &= u(|x|)e^{i\omega t}, A_0(t, x) = k(|x|) \\ A_1(t, x) &= \frac{x_2}{|x|^2} h(|x|), A_2(t, x) = -\frac{x_1}{|x|^2} h(|x|), \end{aligned} \quad (1.8)$$

where $\omega > 0$ is a given frequency and u, k, h are real valued functions on $[0, \infty)$ with $h(0) = 0$. Note the form of A_1 and A_2 satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Substituting the ansatz (1.8) into the system (1.1)–(1.4), we get the following elliptic equation

$$-\Delta u + \omega u + \left(\xi + \int_{|x|}^\infty \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2} u = 0 \quad \text{in } \mathbb{R}^2, \quad (1.9)$$

where $h(s) = \int_0^s \frac{1}{2} u^2(l) dl$ and $\xi = \lim_{|x| \rightarrow \infty} k(|x|)$.

Equation (1.9) is the the Euler–Lagrange equation of the functional

$$\begin{aligned} J(u) &:= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + (\omega + \xi) u^2 + \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx \\ &\quad - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx, u \in H_r^1(\mathbb{R}^2), \end{aligned} \quad (1.10)$$

Here $H_r^1(\mathbb{R}^2)$ denotes the set of radially symmetric functions in $H^1(\mathbb{R}^2)$.

In view of the transformation (1.7), by choosing $\chi = ct + n\pi$ for c real and n integer, we see, when

$$(\phi, A_0, A_1, A_2) = \left(u(|x|)e^{i\omega t}, k(|x|), \frac{x_2}{|x|^2} h(|x|), -\frac{x_1}{|x|^2} h(|x|) \right)$$

is a solution of (1.1)–(1.4), then so is

$$(\tilde{\phi}, \tilde{A}_0, A_1, A_2) = \left((-1)^{|n|} u(|x|)e^{i(\omega+c)t}, k(|x|) - c, \frac{x_2}{|x|^2} h(|x|), -\frac{x_1}{|x|^2} h(|x|) \right).$$

We see that if we obtain a standing wave with a certain frequency, then we can get a standing wave with any frequency, and we say that (ϕ, A_0, A_1, A_2)

and $(\tilde{\phi}, \tilde{A}_0, A_1, A_2)$ are gauge equivalent as the standing wave of the form (1.8). Correspondingly, if the two standing wave $(ue^{i\omega t}, A_0, A_1, A_2)$ and $(\tilde{u}e^{i\tilde{\omega}t}, \tilde{A}_0, \tilde{A}_1, \tilde{A}_2)$ are gauge equivalent, and

$$\begin{aligned} -\Delta u + \omega u + \left(\xi + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2} u &= 0, \\ -\Delta \tilde{u} + \tilde{\omega} \tilde{u} + \left(\tilde{\xi} + \int_{|x|}^{\infty} \frac{\tilde{h}(s)}{s} \tilde{u}^2(s) ds \right) \tilde{u} + \frac{\tilde{h}^2(|x|)}{|x|^2} \tilde{u} - \lambda |\tilde{u}|^{p-2} \tilde{u} &= 0, \\ h(s) = \int_0^s \frac{t}{2} u^2(t) dt, \tilde{h}(s) = \int_0^s \frac{t}{2} \tilde{u}^2(t) dt, \end{aligned}$$

for certain ξ and $\tilde{\xi}$, then we have $\tilde{\xi} + \tilde{\omega} = \xi + \omega$. So to seek the non gauge equivalent standing wave, we will choose the boundary values ξ and $\tilde{\xi}$ with $\omega + \xi \neq \tilde{\omega} + \tilde{\xi}$. In this paper, we just choose $\xi = \tilde{\xi} = 0$, and correspondingly $\omega \neq \tilde{\omega}$.

We call a solution u of the system (1.1)–(1.4) is normalized if for a fixed constant $c > 0$, $\|u\|_{L^2(\mathbb{R}^2)} = c$. In view of the mass conservation law (1.5), it is natural to seek solutions of these type.

Recently, there are many papers on the Schrödinger–Poisson–Slater system

$$-\Delta u + \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^3, \quad (1.11)$$

when $2 < p \leq 6$, see e.g. [1, 2, 6–8, 13, 17, 20, 22, 26]. In [7, 8, 13], for p in some ranges and $c > 0$, by using the concentration-compactness method of Lions [18, 19], the authors obtained the minimizers on

$$\tilde{S}(c) := \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)} = c\}$$

for the functional

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \iint_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx,$$

and thus obtained normalized solutions of (1.11). When $\frac{10}{3} < p < 6$, in [20], by using the methods developed for nonlinear Schrödinger equation in [3, 14], the author obtained there are infinitely many normalized solutions to the Schrödinger–Poisson–Slater system. Since our equation is similar to the Schrödinger–Poisson–Slater system in many parts, so we can use some ideas developed for it.

Now we focus on to seek the normalized solutions to (1.1)–(1.4).

The Eq. (1.9) now becomes

$$-\Delta u + \omega u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2} u = 0 \quad \text{in } \mathbb{R}^2, \quad (1.12)$$

and the solution u is obtained as a critical point of the functional

$$J(u) := \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx, \quad (1.13)$$

on the constraint

$$\text{for } c > 0, S(c) := \{u \in H_r^1(\mathbb{R}^2) : \|u\|_{L^2(\mathbb{R}^2)} = c\}. \quad (1.14)$$

In [5], by considering the minimization of $J(u)$ on $S(c)$, the authors obtained the following results: when $2 < p \leq 3$, there exists a positive minimizer of $J(u)$ on $S(c)$; when $3 < p < 4$ and c is small, there also exists a positive minimizer of $J(u)$ on $S(c)$. Here, by using the minimax theorems, we obtain the following multiplicity results when $p > 2$ and $p \neq 4$.

Theorem 1.1. (1) *If $2 < p \leq 3$, then the Eq. (1.11) has infinitely many distinct pairs of radial solutions $(\omega_n, \pm u_n)$, satisfying $\|u_n\|_{L^2} = c$, $\omega_n > 0$, for each $n \in \mathbb{N}$ and $c > 0$.*

(2) *If $3 < p < 4$, then there exists $c_0 > 0$, such that for any $c \in (0, c_0)$, the Eq. (1.11) has infinitely many distinct pairs of radial solutions $(\omega_n, \pm u_n)$, $n \in \mathbb{N}$, satisfying $\|u_n\|_{L^2} = c$, $\omega_n > 0$.*

(3) *If $p > 4$, then there exists $c_1 > 0$, such that for any $c \in (0, c_1)$, the Eq. (1.11) has infinitely many distinct pairs of radial solutions $(\omega_n, \pm u_n)$, $n \in \mathbb{N}$, satisfying $\|u_n\|_{L^2} = c$, $\omega_n > 0$, and $\|u_n\|_{H^1} \rightarrow \infty$, as $n \rightarrow \infty$.*

In the Sect. 2, we will make some preparations which will be used in Sect. 3. In Sect. 3, in the $2 < p < 4$ case, we will use the Krasnoselski genus to construct critical points; In the $p > 4$ case, as for the nonlinear Schrödinger equation in [3] and Schrödinger–Poisson–Slater system in [20], we will use linking type theorem in [3] and the auxiliary function approach in [14] to construct bounded Palais–Smale sequences of (1.12). The linking type theorem in [3], is a variant of the fountain theorem, for more on the linking type theorem, we refer the reader to [3, 25]. As in [3, 14, 20], we define the auxiliary $J(H(u, m)) : S(c) \times \mathbb{R} \rightarrow \mathbb{R}$, where $H(u, m)(x) = e^m u(e^m x)$ (see Sect. 3.1.2). The virtue of $J(H(u, m))$ compared to $J(u) : S(c) \rightarrow \mathbb{R}$, is that we have an extra dimension $\mathbb{R}$, and because of this, for the corresponding P-S sequences, they have an extra restriction $\partial_m J(H(u, m)) \rightarrow 0$, which is related to Pohožaev identity. By combining this extra information, we can show that the constructed P–S sequences are bounded. Then we consider the convergence of the Palais–Smale sequences, and prove Theorem 1.1.

Notation: Throughout the article we will use the letter C to denote various constants whose exact value may change from line to line. Recall that a sequence (u_m) in a Banach space V is a Palais–Smale sequence (abbreviated as P–S sequence) for E if $|E(u_m)| \leq c$, uniformly in m , while $\|DE(u_m)\| \rightarrow 0$ as $m \rightarrow \infty$, where $DE(u_m)$ is the Fréchet derivative of E at u_m .

2. Preliminaries

Let us recall the Pohožaev identity Proposition 2.3 and compactness lemma Lemma 3.2 in [5].

Lemma 2.1. (Pohožaev identity) *Let b, c and d be real constants and $u \in H_r^1$ be a weak solution of the equation:*

$$\Delta u + bu + c \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + c \frac{h^2(|x|)}{|x|^2} u + d|u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^2, \quad (2.1)$$

where $h(s) = \int_0^s \frac{t}{2} u^2(t) dt$, then we have

$$b \int_{\mathbb{R}^2} u^2 dx + 2c \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx + \frac{2d}{p} \int_{\mathbb{R}^2} |u|^p dx = 0. \quad (2.2)$$

Let $c(u) = \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx$, with $h(s) = \int_0^s \frac{t}{2} u^2(t) dt$, then we have

Lemma 2.2. Suppose that a sequence $\{u_n\}$ converge weakly to a function u in H_r^1 as $n \rightarrow \infty$. Then for each $\varphi \in H_r^1$, $c(u_n)$, $c'(u_n)\varphi$ and $c'(u_n)u_n$ converges up to a subsequence to $c(u)$, $c'(u)\varphi$, and $c'(u)u$, respectively, as $n \rightarrow \infty$.

Lemma 2.3.

$$\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx \leq C \|u\|_{L^2}^4 \|\nabla u\|_{L^2}^2.$$

Proof. By using Hölder's inequality and Gagliardo–Nirenberg inequality, we have

$$\left(\frac{1}{|x|} \int_{B_{|x|}} u^2 dy \right)^2 \leq \int_{B_{|x|}} u^4 dy \leq \int_{\mathbb{R}^2} u^4 dx \leq C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2,$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx &\leq C \int_{\mathbb{R}^2} u^2 \left(\frac{1}{|x|} \int_{B_{|x|}} u^2 dy \right)^2 dx \\ &\leq C \|u\|_{L^2}^4 \|\nabla u\|_{L^2}^2. \end{aligned}$$

□

Lemma 2.4. If $u \in H_r^1(\mathbb{R}^2)$ is a weak solution of

$$-\Delta u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2}u = \mu u \quad \text{in } \mathbb{R}^2, \quad (2.3)$$

with $\mu \geq 0$.

- (1) If $2 < p \leq 3$, then $u \equiv 0$.
- (2) If $3 < p < 4$, then there exists a constant $c_0 > 0$, such that, when $\|u\|_2^2 \leq c_0$ and $J(u) < 0$, $u \equiv 0$.
- (3) If $p > 4$, then there exists a constant $c_1 > 0$, such that, when $\|u\|_2^2 \leq c_1$, $u \equiv 0$.

Proof. Multiply (2.3) by u and integrate by parts, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^2} |u|^2 dx + 3 \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx - \lambda \int_{\mathbb{R}^2} |u|^p dx = 0. \quad (2.4)$$

By Lemma 2.1, we have the Pohozaev identity for (2.3),

$$\mu \int_{\mathbb{R}^2} |u|^2 dx - 2 \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx + \frac{2\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx = 0. \quad (2.5)$$

By combining (2.4) and (2.5), we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx - \lambda \frac{p-2}{p} \int_{\mathbb{R}^2} |u|^p dx = 0, \quad (2.6)$$

and

$$\frac{2}{p} \|\nabla u\|_{L^2}^2 + \left(1 - \frac{2}{p}\right) \mu \int_{\mathbb{R}^2} |u|^2 dx + \left(\frac{6}{p} - 2\right) \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx = 0. \quad (2.7)$$

When $2 < p \leq 3$, by (2.7), we have

$$0 \leq \left(1 - \frac{2}{p}\right) \mu \int_{\mathbb{R}^2} |u|^2 dx = -\frac{2}{p} \|\nabla u\|_{L^2}^2 + \left(2 - \frac{6}{p}\right) \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx < 0,$$

when $u \neq 0$, contradiction.

By (2.6), we have $0 \geq \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{p-2}{p} \lambda \|\nabla u\|_{L^2}^{p-2} \|u\|_{L^2}^{p(1-\gamma)}$, where $\gamma = 2(\frac{1}{2} - \frac{1}{p})$. Correspondingly,

$$\|\nabla u\|_{L^2}^{(4-p)} \leq C(p) \|u\|_{L^2}^{p(1-\gamma)}. \quad (2.8)$$

When $3 < p < 4$, and $\|u\|_{L^2} \leq c$, with c sufficiently small.

If $\mu = 0$, then we can use Proposition 4.1 in [5] to get a contradiction if $u \neq 0$ with $J(u) < 0$.

If $\mu > 0$, then

$$\begin{aligned} 0 &< \left(1 - \frac{2}{p}\right) \mu \int_{\mathbb{R}^2} |u|^2 dx \\ &= -\frac{2}{p} \|\nabla u\|_{L^2}^2 + \left(2 - \frac{6}{p}\right) \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx \\ &\leq \left(2 - \frac{6}{p}\right) \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx \leq C \|u\|_{L^2}^4. \end{aligned} \quad (2.9)$$

To derive the last inequality above, we used the boundedness of $\|\nabla u\|_{L^2}$, which can be seen from (2.8). We see that when $\|u\|_{L^2}$ is sufficiently small, the above inequality (2.9) will lead to a contradiction.

When $p > 4$, and $\|u\|_{L^2} \leq c$, with c sufficiently small. By Lemma 2.3, we have

$$\begin{aligned} &\left(1 - \frac{2}{p}\right) \mu \int_{\mathbb{R}^2} |u|^2 dx \\ &= -\frac{2}{p} \|\nabla u\|_{L^2}^2 + \left(2 - \frac{6}{p}\right) \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^2 dx \\ &\leq -\frac{2}{p} \|\nabla u\|_{L^2}^2 + C \left(2 - \frac{6}{p}\right) \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^4. \end{aligned} \quad (2.10)$$

When $\|u\|_{L^2}$ is sufficiently small, the right hand side of (2.10) is negative, while the left hand side of (2.10) is nonnegative, a contradiction. So we complete the proof of the Lemma. $\square$

3. Solutions with fixed L^2 norm

3.1. The minimax structure

3.1.1. $2 < p < 4$. In this case, by Proposition 2.4, (4.6) and (4.7) in [5], we have

$$J(u) \geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - C \int_{\mathbb{R}^2} |u|^2 dx,$$

so $J(u)$ is bounded from below on $S(c)$. Also we have the transformation $v \rightarrow v_t(x) \equiv tv(tx)$, under which the L^2 norm is invariant. Moreover

$$J(v_t) = t^2 \left(\frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \frac{v^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v^2(s) ds \right)^2 dx - t^{p-4} \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx \right). \quad (3.1)$$

We have, $\forall z \in S(c)$, there exists a unique $T = T(z) > 0$ such that

$$J(z_{T(z)}) = 0, \quad \forall 0 < t < T(z), J(z_t) < 0, \quad \text{and} \quad \forall t > T(z), J(z_t) > 0.$$

For any $k \in \mathbb{N}$, we choose E_k a k -dimensional subspace of H_r^1 , such that $E_k \subset E_{k'}$ for $k < k'$ and set $S_k = E_k \cap S(c)$. In particular, we have $\forall z \in S_k$, $\exists T'_z \equiv \frac{1}{2}T(z) > 0$, such that $J(z_{T'_z}) < 0$ and the map $z \rightarrow T'_z$ is continuous. We see the map $S_k \ni z \rightarrow \tau(z) := z_{T'_z}$ is odd and continuous. For any closed and symmetric set $A \subset H_r^1 \setminus \{0\}$, let $\gamma(A)$ denotes the Krasnoselski genus of A . By the discussion above, we have $\gamma(\tau(S_k)) \geq k$, and $\tau(S_k) \subset \{u \in H_r^1 : J(u) < 0\}$. Therefore, setting

$$d_k = \inf_{\gamma(A) \geq k, A \subset S(c)} \sup\{J(u) : u \in A\}, \quad k \in \mathbb{N}.$$

One has that $d_k < 0$, $k \in \mathbb{N}$. Then the Lusternik–Schnirelmann theory yields P–S sequences $\pm v_k^n$ such that

$$J(\pm v_k^n) \rightarrow d_k < 0, \quad J'|_{S(c)}(\pm v_k^n) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty, \quad k \in \mathbb{N}.$$

In Sect. 3.2, we will show that these P–S sequences converge. It is well-known that $-\infty < d_1 \leq d_2 \leq \dots \leq d_k \leq \dots < 0$, and if two of these numbers coincide, then the corresponding critical set is infinite, see e.g. [23]. So there will be infinitely many solutions in this case.

3.1.2. $p > 4$. Let $\{V_n\} \subset H_r^1(\mathbb{R}^2)$ be a strictly increasing sequence of finite-dimensional linear subspace in $H_r^1(\mathbb{R}^2)$, such that $\cup_n V_n$ is dense in $H_r^1(\mathbb{R}^2)$. We denote by $V_n^\perp$ the orthogonal space of V_n in $H_r^1(\mathbb{R}^2)$. Then

Lemma 3.1. ([3], Lemma 2.1). *Assume that $p \in (4, +\infty)$, then we have*

$$\mu_n := \inf_{u \in V_{n-1}^\perp} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) dx}{\left(\int_{\mathbb{R}^2} |u|^p dx \right)^{2/p}} = \inf_{u \in V_{n-1}^\perp} \frac{\|u\|_{H^1}^2}{\|u\|_{L^p(\mathbb{R}^2)}^2} \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty. \quad (3.2)$$

Now fix $c > 0$ and for each $n \in N^+$, we define

$$\rho_n := L^{-\frac{1}{p-2}} \mu_n^{\frac{1}{p-2}}, \quad \text{with} \quad L = \max_{x>0} \frac{(x^2 + c^2)^{p/2}}{x^p + c^p},$$

and $B_n := \{u \in V_{n-1}^\perp \cap S(c) : \|\nabla u\|_{L^2} = \rho_n\}$. Define also

$$b_n := \inf_{u \in B_n} J(u).$$

Then we have

Lemma 3.2. *Let $p \in (4, \infty)$, then $b_n \rightarrow +\infty$ as $n \rightarrow \infty$.*

Proof. For any $u \in B_n$

$$\begin{aligned} J(u) &= \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s) ds \right)^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx \\ &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{p\mu_n} (\|\nabla u\|_{L^2}^2 + c^2)^{\frac{p}{2}} \\ &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{L}{p\mu_n} (\|\nabla u\|_{L^2}^p + c^p) \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \rho_n^2 - \frac{L}{p\mu_n} c^p. \end{aligned} \quad (3.3)$$

Since $p > 2$, so by Lemma 3.1, we have $b_n \rightarrow +\infty$ as $n \rightarrow +\infty$. $\square$

In the following, we assume that $H \equiv L^2(\mathbb{R}^2)$ and $E \equiv H^1(\mathbb{R}^2)$ with their scalar products denoted by $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_E$, the corresponding norms are $\|\cdot\|_H$ and $\|\cdot\|_E$. Also we let $E \times \mathbb{R}$ be equipped with the scalar product $\langle \cdot, \cdot \rangle_{E \times \mathbb{R}} = \langle \cdot, \cdot \rangle_E + \langle \cdot, \cdot \rangle_{\mathbb{R}}$, and corresponding norm $\|\cdot\|_{E \times \mathbb{R}} = \{\|\cdot\|_E^2 + |\cdot|_{\mathbb{R}}^2\}^{\frac{1}{2}}$, and $\mathbf{E} = E \times \mathbb{R}$, $\|\cdot\|_{\mathbf{E}} = \|\cdot\|_{E \times \mathbb{R}}$ for short. We let $H : E \times \mathbb{R} \rightarrow E$, $H(u, m)(x) = e^m u(e^m x)$.

Lemma 3.3. *Let $u \in S(c)$ be arbitrary but fixed, then we have*

- (1) $\|\nabla H(u, m)\|_{L^2} \rightarrow 0$ and $J(H(u, m)) \rightarrow 0$ as $m \rightarrow -\infty$.
- (2) $\|\nabla H(u, m)\|_{L^2} \rightarrow +\infty$ and $J(H(u, m)) \rightarrow -\infty$ as $m \rightarrow +\infty$.

Proof. We have

$$\|H(u, m)\|_{L^2} = c \quad \text{and} \quad \|\nabla H(u, m)\|_{L^2} = e^m \|\nabla u\|_{L^2}.$$

Now we have

$$\begin{aligned} &J(H(u, m)) \\ &= \frac{1}{2} \|\nabla H(u, m)\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{H(u, m)^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} H(u, m)^2 ds \right)^2 dx \\ &\quad - \frac{\lambda}{p} \int_{\mathbb{R}^2} |H(u, m)|^p dx \\ &= \frac{1}{2} e^{2m} \|\nabla u\|_{L^2}^2 + \frac{1}{2} e^{2m} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2 ds \right)^2 dx - \frac{\lambda}{p} e^{(p-2)m} \int_{\mathbb{R}^2} |u|^p dx. \end{aligned}$$

We see, as $m \rightarrow -\infty$, $\|\nabla H(u, m)\|_{L^2} \rightarrow 0$ and $J(H(u, m)) \rightarrow 0$. Since $p-2 > 2$, so as $m \rightarrow +\infty$, $\|\nabla H(u, m)\|_{L^2} \rightarrow +\infty$ and $J(H(u, m)) \rightarrow -\infty$. $\square$

Lemma 3.4. *There exists ρ_0, ρ'_0 , such that when $0 < \rho_0 < \rho'_0 < \rho_n$,*

$$0 < \sup_{u \in A} J(u) < \inf_{u \in B} J(u) =: b_0,$$

with $A = \{u \in S(c), \|\nabla u\|_{L^2}^2 \leq \rho_0\}$, $B = \{u \in S(c), \|\nabla u\|_{L^2}^2 = \rho'_0\}$.

Proof. By Lemma 2.3, we have

$$J(u) \leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^4. \quad (3.4)$$

We see when $\|\nabla u\|_{L^2} \rightarrow 0$, $J(u) \rightarrow 0$. On the other hand, by the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - C \|\nabla u\|_{L^2}^{p\gamma} \|u\|_{L^2}^{p(1-\gamma)} \\ &\geq C \|\nabla u\|_{L^2}^2 (1 - \|\nabla u\|_{L^2}^{p\gamma-2} \|u\|_{L^2}^{p(1-\gamma)}), \end{aligned} \quad (3.5)$$

with $\gamma = 2(\frac{1}{2} - \frac{1}{p})$. We see that when $\|\nabla u\|_{L^2}$ is sufficiently small, $J(u) \geq C \|\nabla u\|_{L^2}^2$. By combining these two facts, we prove the Lemma. $\square$

By Lemma 3.3, there exists $m_n > 0$ such that

$$\bar{\gamma}_n : [0, 1] \times (S \cap V_n) \rightarrow S, \bar{\gamma}_n(t, u) = H(u, 2m_n t - m_n),$$

satisfies:

$$\begin{aligned} |\nabla(\bar{\gamma}_n(0, u))|_2 &< \rho_0 < \rho_n, |\nabla(\bar{\gamma}_n(1, u))|_2 > \rho_n, \quad \text{and} \\ 0 &< J(\bar{\gamma}_n(0, u)) < \max\{b_0, b_n\}, J(\bar{\gamma}_n(1, u)) < b_n, \end{aligned}$$

uniformly for $u \in S \cap V_n$. Now we define

$$\begin{aligned} \Gamma_n &= \{\gamma : [0, 1] \times (S \cap V_n) \rightarrow S \mid \gamma \text{ is continuous, odd in } u \text{ and such that} \\ &\quad \forall u : \gamma(0, u) = \bar{\gamma}_n(0, u), \gamma(1, u) = \bar{\gamma}_n(1, u)\} \end{aligned}$$

Clearly, we have $\tilde{\gamma}_n \in \Gamma_n$. And in [3], the authors prove the following Lemma:

Lemma 3.5. *For every $\gamma \in \Gamma_n$, there exists $(t, u) \in [0, 1] \times (S \cap V_n)$ such that $\gamma(t, u) \in B_n$.*

By Lemma 3.5, we have

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{t \in [0, 1], u \in S \cap V_n} J(\gamma(t, u)) \geq b_n = \inf_{u \in B_n} J(u) \rightarrow \infty. \quad (3.6)$$

Clearly, by Lemmas 3.3 and 3.4, we have $c_n \geq b_0 > 0$. Now we shall prove that $\{c_n\}$ is a set of critical values of J restricted to $S(c)$.

We fix n from now on.

Proposition 3.6. *Given $c > 0$, there exists a Palais–Smale sequence $\{v_k\} \subset S(c)$ satisfying $J(v_k) \rightarrow c_n$, $J'|_{S(c)}(v_k) \rightarrow 0$, as $k \rightarrow \infty$, and*

$$\|\nabla v_k\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{|v_k|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v_k^2 ds \right)^2 dx - \frac{\lambda(p-2)}{p} \int_{\mathbb{R}^2} |v_k|^p dx \rightarrow 0. \quad (3.7)$$

To prove Proposition 3.6, following [14], we introduce the auxiliary functional

$$\tilde{J} : S(c) \times \mathbb{R} \rightarrow \mathbb{R}, (u, m) \rightarrow J(H(u, m)),$$

and define

$\tilde{\Gamma}_n := \{\tilde{\gamma} : [0, 1] \times (S \cap V_n) \rightarrow \mathbb{R} \times S \mid \tilde{\gamma} \text{ is continuous, odd in } u, \text{ and such that } H \circ \tilde{\gamma} \in \Gamma_n\}$.

Clearly, for any $\gamma \in \Gamma_n$, $\tilde{\gamma} : H(\gamma, 0) \in \tilde{\Gamma}_n$. Define

$$\tilde{c}_n := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{t \in [0, 1], u \in S(c) \cap V_n} \tilde{J}(\tilde{\gamma}(t, u)). \quad (3.8)$$

Then we have $\tilde{c}_n = c_n$. Indeed, this can from the fact that there are maps

$$\Phi : \Gamma_n \rightarrow \tilde{\Gamma}_n, \gamma \rightarrow [(0, \gamma) : (t, u) \rightarrow (0, \gamma(t, u))],$$

and

$$\Psi : \tilde{\Gamma}_n \rightarrow \Gamma_n, \tilde{\gamma} \rightarrow [H \circ \tilde{\gamma} : (t, u) \rightarrow H(\tilde{\gamma}(t, u))],$$

satisfy

$$\tilde{J}(\Phi(\gamma)(t, u)) = J(\gamma(t, u)) \quad \text{and} \quad J(\Psi(\tilde{\gamma})(t, u)) = \tilde{J}(\tilde{\gamma}(t, u)).$$

In [14], by using the Ekeland's variational principle, the author establish the following Lemma.

Lemma 3.7. *Let $\epsilon > 0$, and $\tilde{\gamma}_0 \in \tilde{\Gamma}_n$ be such that*

$$\max_{t \in [0, 1], u \in S(c) \cap V_n} \tilde{J}(\tilde{\gamma}_0(t, u)) \leq \tilde{c}_n + \epsilon,$$

then there exists a pair of $(u_0, m_0) \in S(c) \times \mathbb{R}$ such that:

- (1) $\tilde{J}(u_0, m_0) \in [\tilde{c}_n - \epsilon, \tilde{c}_n + \epsilon]$.
- (2) $\min_{t \in [0, 1], u \in S(c) \cap V_n} \|(u_0, m_0) - \tilde{\gamma}_0(t, u)\|_E \leq \sqrt{\epsilon}$.
- (3) $\|\tilde{J}'|_{S(c) \times \mathbb{R}}(u_0, m_0)\|_{\mathbf{E}^* \times \mathbf{E}} \leq 2\sqrt{\epsilon}$, i.e.

$$|\langle \tilde{J}'(u_0, m_0), z \rangle_{\mathbf{E}^* \times \mathbf{E}}| \leq 2\sqrt{\epsilon} \|z\|_E,$$

holds for all $z \in \tilde{T}_{(u_0, m_0)} := \{(z_1, z_2) \in \mathbf{E}, \langle u_0, z_1 \rangle_{L^2} = 0\}$.

Proof of the Proposition 3.6. From the definition of c_n , we have, for each $k \in \mathbb{N}$, there exists a $\gamma_k \in \Gamma_n$ such that

$$\max_{t \in [0, 1], u \in S(c) \cap V_n} J(\gamma_k(t, u)) \leq c_n + \frac{1}{k}.$$

Since $\tilde{c}_n = c_n$, $\tilde{\gamma}_k = (\gamma_k, 0) \in \tilde{\Gamma}_n$ satisfies that

$$\max_{t \in [0, 1], u \in S(c) \cap V_n} \tilde{J}(\tilde{\gamma}_k(t, u)) \leq \tilde{c}_n + \frac{1}{k}.$$

By Lemma 3.7, we know there exists a sequence $\{u_k, m_k\} \subset S(c) \times \mathbb{R}$ such that

$$(1) \quad \tilde{J}(u_k, m_k) \in \left[c_n - \frac{1}{k}, c_n + \frac{1}{k} \right]. \quad (3.9)$$

$$(2) \quad \min_{t \in [0, 1], u \in S(c) \cap V_n} \|(u_k, m_k) - (\gamma_k(t, u), 0)\|_{\mathbf{E}} \leq \frac{1}{\sqrt{k}}. \quad (3.10)$$

$$(3) \quad \|\tilde{J}'|_{S(c) \times \mathbb{R}}(u_k, m_k)\|_{\mathbf{E}^*} \leq \frac{2}{\sqrt{k}}, \text{ i.e.}$$

$$|\langle \tilde{J}'(u_k, m_k), z \rangle_{\mathbf{E}^* \times \mathbf{E}}| \leq \frac{2}{\sqrt{k}} \|z\|_E, \quad (3.11)$$

holds for all $z \in \tilde{T}_{(u_k, m_k)} := \{(z_1, z_2) \in \mathbf{E}, \langle u_k, z_1 \rangle_{L^2} = 0\}$.

Now we let $v_k = H(u_k, m_k)$. Since $J(v_k) = J(H(u_k, m_k)) = \tilde{J}(u_k, m_k)$, so we have $J(v_k) \rightarrow c_n$, as $k \rightarrow \infty$, and we prove (1).

Let $\partial_m \tilde{J}(u_k, m_k) \equiv \langle \tilde{J}'(u_k, m_k), (0, 1) \rangle_{\mathbf{E}^* \times \mathbf{E}}$, then we have $\partial_m \tilde{J}(u_k, m_k) \rightarrow 0$ as $k \rightarrow \infty$.

By calculating, we have

$$\begin{aligned} & \partial_m \tilde{J}(u_k, m_k) \\ &= e^{2m_k} \int_{\mathbb{R}^2} |\nabla u_k|^2 dx + e^{2m_k} \int_{\mathbb{R}^2} \frac{u_k^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u_k^2 ds \right)^2 dx \\ & \quad - \frac{\lambda}{p} e^{(p-2)m_k} (p-2) \int_{\mathbb{R}^2} |u_k|^p dx \\ &= \int_{\mathbb{R}^2} |\nabla v_k|^2 dx + \int_{\mathbb{R}^2} \frac{v_k^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v_k^2 ds \right)^2 dx - \lambda \frac{p-2}{p} \int_{\mathbb{R}^2} |v_k|^p dx, \end{aligned}$$

and we get (3.7).

Now let $w \in T_{v_k}$, we have

$$\begin{aligned} & \langle J'(v_k), w \rangle_{E^* \times E} \\ &= \int_{\mathbb{R}^2} \nabla v_k \nabla w(x) dx + \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h_1(s)}{s} v_k^2 ds v_k + \frac{h_1^2(|x|)}{|x|^2} v_k \right) w dx \\ & \quad - \lambda \int_{\mathbb{R}^2} |v_k|^{p-2} v_k w dx \\ &= e^{2m_k} \int_{\mathbb{R}^2} \nabla u_k(e^{m_k} x) \nabla w(x) dx + e^{5m_k} \\ & \quad \times \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h_2(s)}{s} u_k^2(e^{m_k} x) ds u_k(e^{m_k} x) \right. \\ & \quad \left. + \frac{h_2^2(|x|)}{|x|^2} u_k(e^{m_k} x) \right) w dx - \lambda e^{(p-1)m_k} \int_{\mathbb{R}^2} |u_k(e^{m_k} x)|^{p-2} u_k(e^{m_k} x) w dx \\ &= e^{m_k} \int_{\mathbb{R}^2} \nabla u_k(x) \nabla w(e^{-m_k} x) dx \end{aligned}$$

$$\begin{aligned}
& + e^{m_k} \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h_3(s)}{s} u_k^2 ds u_k + \frac{h_3^2(|x|)}{|x|^2} u_k \right) w(e^{-m_k} x) dx \\
& - \lambda e^{(p-3)m_k} \int_{\mathbb{R}^2} |u_k|^{p-2} u_k w(e^{-m_k} x) dx,
\end{aligned}$$

where in the above, $h_1(s) = \int_0^s \frac{t}{2} v_k^2(t) dt$, $h_2(s) = \int_0^s \frac{t}{2} u_k^2(e^{m_k} t) dt$, $h_3(s) = \int_0^s \frac{t}{2} u_k^2(t) dt$. Thus setting $\tilde{w}(x) \equiv e^{-m_k} w(e^{-m_k} x)$, we have

$$\langle J'(v_k), w \rangle_{E^* \times E} = \langle \tilde{J}'(u_k, m_k), (\tilde{w}, 0) \rangle_{\mathbf{E}^* \times \mathbf{E}}. \quad (3.12)$$

Since

$$\begin{aligned}
& (\tilde{w}, 0) \in \tilde{T}(u_k, m_k) \\
& \Leftrightarrow \langle \tilde{w}, u_k \rangle_{L^2} = 0 \Leftrightarrow \int_{\mathbb{R}^2} u_k(x) e^{-m_k} w(e^{-m_k} x) dx = 0 \\
& \Leftrightarrow \int_{\mathbb{R}^2} e^{m_k} u_k(e^{m_k} x) w(x) dx = 0 \\
& \Leftrightarrow \langle v_k, w \rangle_{L^2} = 0 \Leftrightarrow w \in T_{v_k},
\end{aligned}$$

so we have $(\tilde{w}, 0) \in \tilde{T}(u_k, m_k)$. By (3.11) and (3.12), we have $|\langle J'(v_k), w \rangle_{E^* \times E}| \leq \frac{2}{\sqrt{k}} \|(\tilde{w}, 0)\|_{\mathbf{E}}$. In the following we will show that for k sufficiently large, $\|(\tilde{w}, 0)\|_{\mathbf{E}} \leq 2\|w\|_E$, then we conclude Proposition 3.6. Indeed we have

$$|m_k| = |m_k - 0| \leq \min_{t \in [0, 1], u \in S(c) \cap V_n} \|(v_k, m_k) - (\gamma_k(t, u), 0)\|_{\mathbf{E}} \leq \frac{1}{\sqrt{k}}.$$

So for k sufficiently large,

$$\begin{aligned}
& \|(\tilde{w}, 0)\|_{\mathbf{E}}^2 \\
& = \|\tilde{w}\|_{H^1}^2 = \int_{\mathbb{R}^2} |\tilde{w}(x)|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{w}(x)|^2 dx \\
& = \int_{\mathbb{R}^2} |\tilde{w}(x)|^2 dx + e^{-2m_k} \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx \\
& \leq 2\|w\|_E^2.
\end{aligned}$$

□

3.2. Compactness

Since the boundedness of the P-S sequences is easy for $2 < p < 4$, and to get the conclusions in the following Lemma 3.8 for the case $2 < p < 4$ is similar to the $p > 4$ case, we will mainly discuss in the case $p > 4$.

By (3.7), we have

$$\|\nabla v_k\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{|v_k|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v_k^2 ds \right)^2 dx - \frac{\lambda(p-2)}{p} \int_{\mathbb{R}^2} |v_k|^p dx \rightarrow 0. \quad (3.13)$$

Also

$$\tilde{J}(v_k) = \frac{1}{2} \|\nabla v_k\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|v_k|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v_k^2 ds \right)^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |v_k|^p dx \rightarrow c_n. \quad (3.14)$$

Since

$$\int_{\mathbb{R}^2} |v_k|^p dx = \frac{2p}{\lambda(p-4)} \left((3.14) - \frac{1}{2} \times (3.13) \right),$$

so there exists constants c_1 and c_2 , such that

$$0 < c_1 \leq \int_{\mathbb{R}^2} |v_k|^p dx \leq c_2. \quad (3.15)$$

By using (3.13) again, $\frac{1}{2} \|\nabla v_k\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{|v_k|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} v_k^2 ds \right)^2 dx$ is also bounded consequently. So up to a subsequence, there exists $v_n \in E$, such that $v_k \rightharpoonup v_n$ weakly in E as $k \rightarrow \infty$. By the compact embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $2 < p < \infty$ ([24]) and (3.15), we know that v_n is nontrivial. Now note that by Proposition 3.6, there exists $C > 0$ independent of $k \in N$ such that

$$\left| \left\langle J'(v_k) - \frac{1}{\|v_k\|_{L^2}^2} \langle J'(v_k), v_k \rangle v_k, z \right\rangle_{E^* \times E} \right| \leq \frac{C}{\sqrt{k}} \|z\|_E \quad \text{for all } z \in E,$$

we choose a subsequence of $\{v_k\}$ in Proposition 3.6 such that $\|J'|_{S(c)}(v_k)\|_{E^*} \leq \frac{C}{\sqrt{k}}$. Indeed for all $z \in E$,

$$\left\langle J'(v_k) - \frac{1}{\|v_k\|_{L^2}^2} \langle J'(v_k), v_k \rangle v_k, z \right\rangle_{E^* \times E} = \langle J'(v_k), P_{v_k} z \rangle_{E^* \times E},$$

where P_{v_k} denotes the orthogonal projector defined in E on the tangent space T_{v_k} . Since

$$\begin{aligned} & \|P_{v_k} z\|_E \\ &= \left\| z - \frac{1}{\|v_k\|_{L^2}^2} \langle z, v_k \rangle v_k \right\|_E \\ &\leq \|z\|_E + \|z\|_H \|v_k\|_H \|v_k\|_E / \|v_k\|_{L^2}^2 \\ &\leq \|z\|_E (1 + \|v_k\|_E / \|v_k\|_{L^2}). \end{aligned}$$

So, $\langle J'(v_k), P_{v_k} z \rangle_{E^* \times E} \leq \frac{C}{\sqrt{k}} \|P_{v_k} z\|_E \leq \frac{C}{\sqrt{k}} \|z\|_E$.

Now

$$\begin{aligned} & \left\langle J'(v_k) - \frac{1}{\|v_k\|_{L^2}^2} \langle J'(v_k), v_k \rangle v_k, z \right\rangle_{E^* \times E} \\ &= \int_{\mathbb{R}^2} \nabla v_k(x) \nabla z(x) dx + \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_k^2 ds v_k + \frac{h^2(|x|)}{|x|^2} v_k \right) z dx \\ &\quad - \lambda \int_{\mathbb{R}^2} |v_k|^{p-2} v_k z dx - \lambda_k \int_{\mathbb{R}^2} v_k(x) z(x) dx, \end{aligned}$$

with

$$\begin{aligned}\lambda_k &= \frac{1}{\|v_k\|_{L^2}^2} \left\{ \|\nabla v_k\|_{L^2}^2 + \int_{\mathbb{R}^2} \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_k^2(s) ds \right) v_k^2 \right. \\ &\quad \left. + \frac{h^2(|x|)}{|x|^2} v_k^2 dx - \lambda \int_{\mathbb{R}^2} |v_k|^p dx \right\} \\ &= \frac{1}{\|v_k\|_{L^2}^2} \left\{ \|\nabla v_k\|_{L^2}^2 + \int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} v_k^2 dx - \lambda \int_{\mathbb{R}^2} |v_k|^p dx \right\},\end{aligned}$$

which is bounded. So up to a subsequence $\lambda_k \rightarrow \omega_n$. By the weak convergence of $\{v_k\}$ and Lemma 2.2, we get the following Lemma:

Lemma 3.8. *Assume that $p > 2$ and $p \neq 4$, let $\{v_k\} \subset S(c)$ be obtained in Proposition 3.6 and in Sect. 3.1.1. Then there exists $\omega_n \in \mathbb{R}$ and $v_n \in H_r^1(\mathbb{R}^2)$, such that, up to a subsequence*

- (1) $v_k \rightharpoonup v_n \neq 0$ in $H_r^1(\mathbb{R}^2)$,
- (2) $-\Delta v_k + \omega_n v_k + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_k^2(s) ds \right) v_k + \frac{h^2(|x|)}{|x|^2} v_k - \lambda |v_k|^{p-2} v_k \rightarrow 0$ in $H_r^{-1}(\mathbb{R}^2)$,
- (3) $-\Delta v_n + \omega_n v_n + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_n^2(s) ds \right) v_n + \frac{h^2(|x|)}{|x|^2} v_n - \lambda |v_n|^{p-2} v_n = 0$ in $H_r^{-1}(\mathbb{R}^2)$.

Now, by the principle of symmetric criticality (see e.g. section 1.6 in [25]) and (3) of Lemma 3.8, v_n is a weak solution of

$$-\Delta u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u + \frac{h^2(|x|)}{|x|^2} u - \lambda |u|^{p-2} u = -\omega_n u \quad \text{in } \mathbb{R}^2. \quad (3.16)$$

If $-\omega_n \geq 0$, i.e. $\omega_n \leq 0$, by our assumptions on c as in Theorem 1.1, we can use Lemma 2 and deduce $v_n \equiv 0$, which contradicts to (1) of Lemma 3.8, so $\omega_n > 0$.

Now, since $\omega_n > 0$, by (2) of Lemma 3.8,

$$-\Delta v_k + \omega_n v_k \rightarrow - \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_k^2 ds \right) v_k - \frac{h^2(|x|)}{|x|^2} v_k + \lambda |v_k|^{p-2} v_k, \quad \text{in } H_r^{-1}(\mathbb{R}^2),$$

$$v_k \rightarrow (-\Delta + \omega_n)^{-1} \left(- \left(\int_{|x|}^{\infty} \frac{h(s)}{s} v_k^2 ds \right) v_k - \frac{h^2(|x|)}{|x|^2} v_k + \lambda |v_k|^{p-2} v_k \right) \quad \text{in } E.$$

By the uniqueness of the limit, we have $v_k \rightarrow v_n$ in $H_r^1(\mathbb{R}^2)$, in particular $\|v_n\|_{L^2(\mathbb{R}^2)} = c$, thus $v_n \in S(c)$, and by combining with (3.16), we complete the proof of Theorem 1.1.

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