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Systems of integro-PDEs with interconnected obstacles and multi-modes switching problem driven by Lévy process

Said Hamadène and Xuxhe Zhao

Abstract. In this paper we show existence and uniqueness of the solution in viscosity sense for a system of nonlinear m variational integral-partial differential equations with interconnected obstacles whose coefficients $(f_i)_{i=1,\ldots,m}$ depend on $(u_j)_{j=1,\ldots,m}$. From the probabilistic point of view, this system is related to optimal stochastic switching problem when the noise is driven by a Lévy process. The switching costs depend on (t, x). As a by-product of the main result we obtain that the value function of the switching problem is continuous and unique solution of its associated Hamilton–Jacobi–Bellman system of equations. The main tool we used is the notion of systems of reflected BSDEs with oblique reflection driven by a Lévy process.

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1. Introduction

In this paper, we study the existence and uniqueness of a solution to the system of integro-partial differential equations (IPDEs in short) of the following form: $\forall i = 1, \ldots, m,$

$$\begin{cases} min\{u_i(t,x) - \max_{j\neq i}(u_j(t,x) - g_{ij}(t,x)); \\ -\partial_t u_i(t,x) - \mathcal{L}u_i(t,x) - f_i(t,x,(u_1,u_2,\ldots,u_m)(t,x))\} = 0, \\ \quad (t,x) \in [0,T) \times \mathbb{R}, \\ u_i(T,x) = h_i(x) \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a generator associated with a stochastic differential equation whose noise is driven by a Lévy process $L := (L_t)_{t\leq T}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\leq T}, \mathbf{P})$ and then $\mathcal{L}$ is a non local operator [see (3.22) for its definition].

This system is related to a stochastic optimal switching problem since a particular case is actually its associated Hamilton–Jacobi–Bellman system.

Let us describe briefly the stochastic optimal switching problem. Let $(t,x) \in [0,T] \times \mathbb{R}$ and $(X_s^{t,x})_{s\leq T}$ be the solution of the following standard stochastic differential equation:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_{s-}^{t,x})dL_s, \quad \forall s \in [t,T] \quad \text{and} \quad X_s^{t,x} = x \quad \text{for } s \leq t.$$

Next let $(a_s)_{s\in[0,T]}$ be the following pure jump process:

$$a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1},\theta_j]}(s),\ \forall s \leq T,$$

where $\{\theta_j\}_{j\geq 0}$ is an increasing sequence of stopping times with values in $[0,T]$ and $(\alpha_j)_{j\geq 0}$ are random variables with values in $A := \{1,\ldots,m\}$ (the set of modes to which the controller can switch) such that for any $j \geq 0$, α_j is $\mathcal{F}_{\theta_j}-$measurable. The pair $\Upsilon = ((\theta_j)_{j\geq 0}, (\alpha_j)_{j\geq 0})$ is called a strategy of switching and when it satisfies $\mathbf{P}[\theta_n < T, \forall n \geq 0] = 0$ it is moreover said admissible. Finally we denote by $\mathcal{A}_t^i$ the set of admissible strategies such that $\alpha_0 = i$ and $\theta_0 = t$.

Assume next that for any $i = 1,\ldots,m$, $f_i(t,x,(y_i)_{i=1,\ldots,m}) = f_i(t,x)$, i.e., f_i does not depend on $(y_i)_{i=1,m}$. Let Υ be an admissible strategy of $\mathcal{A}_t^i$ with which one associates a payoff given by:

$$J^a(t,x) = J(\Upsilon)(t,x) := \mathbf{E}\Bigg[\int_t^T f_{a(s)}(s, X_s^{t,x})ds - \sum_{j\geq 1} g_{\alpha_{j-1},\alpha_j}(\theta_j, X_{\theta_j}^{t,x})\mathbb{1}_{\{\theta_j<T\}} + h_{a_T}(X_T^{t,x})\Bigg] \tag{1.2}$$

where $f_{a(s)}(s, X_s^{t,x}) = \sum_{i\in A} f_i(s, X_s^{t,x}) 1_{[a(s)=i]}$, $s \in [t,T]$, (resp. $h_{a_T}(X_T^{t,x}) = \sum_{i\in A} h_i(X_T^{t,x}) 1_{[a_T=i]}$) is the instantaneous (resp. terminal) payoff when the strategy a (or Υ) is implemented while $g_{i\ell}$ is the switching cost function when moving from mode i to mode ℓ ($i,\ell \in A$, $i \neq \ell$). Next let us define the optimal payoff when starting from mode $i \in A$ at time t by

$$u_i(t,x) := \inf_{\Upsilon\in\mathcal{A}_t^i} J(\Upsilon)(t,x) \tag{1.3}$$

As a by-product of our general result we obtain that the value functions $(u_i(t,x))_{i\in A}$ (or optimal payoffs) of this switching problem is continuous and of polynomial growth and is the unique solution in viscosity sense of system (1.1). A similar problem has been already considered by Biswas et al. [6], however one should emphazise that in that work, the switching costs are constant and do not depend on (t,x). This latter feature makes the problem easier to handle since one can directly work with the functions u_i defined in (1.2)–(1.3).

Optimal switching problems are well documented in the literature (see e.g. [3,6–8,11,12,14,17,18,20,24,27] etc. and the references therein), especially in connection with mathematical finance, energy market, etc.

The main objective and novelty of this paper is to study system (1.1) in the general case, i.e., to allow for f_i to depend on $(u_i)_{i=1,m}$ and the switching costs g_{ij} to depend on (t, x) and to show that (1.1) has a unique solution. Our method is based on the link of (1.1) with systems of reflected BSDEs with inter-connected obstacles driven by a Lévy process, i.e., systems of the following form: $\forall j = 1, \ldots, m$, $\forall s \le T$,

$$\begin{cases} Y_s^{j,t,x} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in A}, (U_r^{j,t,x,i})_{i\ge 1})dr \\ \qquad - \sum_{i=1}^{\infty} \int_s^T U_r^{j,t,x,i} dH_r^{(i)} + K_T^{j,t,x} - K_s^{j,t,x} \\ Y_s^{j,t,x} \ge \max_{k\neq j}\{Y_s^{k,t,x} - g_{jk}(s, X_s^{t,x})\} \\ \qquad \text{and } \left[Y_s^{j,t,x} - \max_{k\neq j}\{Y_s^{k,t,x} - g_{jk}(s, X_s^{t,x})\}\right] dK_s^{j,t,x} = 0 \end{cases} \tag{1.4}$$

where $((H_s^{(i)})_{s\le T})_{i\ge 1}$ are the Teugels martingales associated with the Lévy process L. Under appropriate assumptions on the data $(f_i)_{i=1,\ldots,m}$, $(h_i)_{i=1,\ldots,m}$ and $(g_{ij})_{i,j=1,\ldots,m}$ we show existence and uniqueness of $\mathcal{F}_s$-adapted processes $((Y_s^{j,t,x}, (U_s^{j,t,x,i})_{i\ge 1}, K_s^{j,t,x})_{s\le T})_{j\in A}$ which satisfy (1.4). Additionally there exist deterministic continuous functions $(u_j(t,x))_{j\in A}$ such that:

$$\forall s \in [t, T], Y_s^{j,t,x} = u_j(s, X_s^{t,x}), \tag{1.5}$$

and we show that $(u_j(t,x))_{j\in A}$ is the unique solution of (1.1).

In the Brownian framework of noise, the link between systems of PDEs with interconnected obstacles and systems of reflected BSDEs with oblique reflection has been already stated in several papers (see e.g. [15,18], etc.). Therefore in this work we extend this link to the setting where the noise is driven by a Lévy process.

This article is organized as follows. In Sect. 2 we collect the main results on Teugels martingales. Section 3 is devoted to reflected BSDEs driven by a Lévy process (existence and uniqueness of a solution and comparison) and their connection with IPDEs with obstacle. We finally consider the system of reflected BSDEs with inter-connected obstacles (1.4) and we show existence and uniqueness of a solution of this system when, mainly, the functions $(f_i)_{i\in A}$ are Lipschitz in $((y_i)_{i\in A}, \zeta)$ and the switching costs verify the so-called non free loop property. We construct a mapping which is a contraction in an appropriate Banach space and which has a unique fixed point which provides the solution of system (1.4). Section 4 is devoted to the study of system of IPDEs (1.1). Contrarily to system of reflected BSDEs (1.4), we only consider the case when the functions f_i, $i \in A$, do not depend on ζ. We first show that this system has a solution in viscosity sense when for any $i \in A$, the function f_i is non-decreasing w.r.t. to y_k ($k \neq i$) when the other components are fixed. We then give a comparison result of subsolutions and supersolutions of system (1.1) based on Jensen–Ishii's Lemma on PDEs with non-local term [5,6]. As usual this comparison result insures continuity and uniqueness of the solution of system (1.1). Finally we provide another existence and uniqueness result of a solution for system (1.1) in the case when for any $i \in A$, f_i is decreasing w.r.t. y_k for any $k \neq i$ when the other components are fixed. This result is deeply based on the first existence and uniqueness of the solution of system (1.1) and, on the other hand, the existence and uniqueness result of a solution of system of reflected BSDEs (1.4). According to our knowledge it cannot be obtained by using PDE techniques only. At the end of this paper we give an Appendix where two complementary results are collected. The first one is related to the representation of the Y_s^j of the solution of system (1.4) as a value function of a switching problem. As for the second one, it provides an equivalent definition of the viscosity solution of system (1.1) which is somehow of local type. □

2. Preliminaries

A Lévy process is an $\mathbb{R}$-valued RCLL (for right continuous with left limits) stochastic process $L = \{L_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with stationary and independent increments ($L_0 = 0$) and stochastically continuous.

For $t \leq T$ let us set $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$ where $\mathcal{G}_t := \sigma\{L_s, 0 \leq s \leq t\}$ and $\mathcal{N}$ is the $\mathbf{P}$-null sets of $\mathcal{F}$, therefore $\{\mathcal{F}_t\}_{t\leq T}$ is complete and right continuous. Next by $\mathcal{P}$ we denote the σ-algebra of predictable processes on $[0,T] \times \Omega$ and finally for any RCLL process $(\Gamma_t)_{t\leq}$ we denote by $\Gamma_{t-} := \lim_{s\nearrow t} \Gamma_s$ and $\Delta\Gamma_t := \Gamma_t - \Gamma_{t-}$ its jump at t, $t \in (0,T]$.

We now introduce the following spaces:

(a) $\mathcal{S}^2 := \{\varphi := \{\varphi_t, 0 \leq t \leq T\}$ is an $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted RCLL process s.t. $\mathbf{E}(\sup_{0\leq t\leq T} |\varphi_t|^2) < \infty\}$; $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of non-decreasing continuous processes null at $t = 0$;

(b) $H^2 := \{\varphi := (\varphi_t)_{t\leq T}$ is an $\mathbb{R}$-valued, $\mathcal{F}_t$-progressively measurable process such that $\mathbf{E}(\int_0^T |\varphi_t|^2 dt) < \infty\}$;

(c) $\ell^2 := \{x = (x_n)_{n\geq 1}$ is an $\mathbb{R}$-valued sequence s.t. $\|x\|^2 := \sum_{i=1}^{\infty} x_i^2 < \infty\}$;

(d) $\mathcal{H}^2(\ell^2) := \{\varphi = (\varphi_t)_{t\leq T} = ((\varphi_t^n)_{n\geq 1})_{t\leq T}$ such that $\forall n \geq 1$, φ^n is a $\mathcal{P}$-measurable process and
$$\mathbf{E}\left(\int_0^T \|\varphi_t\|^2 dt\right) = \sum_{i=1}^{\infty} \mathbf{E}\left(\int_0^T |\varphi_t^i|^2 dt\right) < \infty\};$$

(e) $\mathcal{L}^2 := \{\xi$, an $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variable such that $\mathbf{E}[|\xi|^2] < \infty\}$;

(f) Π_g is the space of deterministic functions $v(t,x)$ from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$ of polynomial growth, i.e., such that for some positive constants p and C one has,
$$|v(t,x)| \leq C(1 + |x|^p), \quad \forall (t,x) \in [0,T] \times \mathbb{R};$$

(g) $\mathcal{C}_p^{1,2} := \mathcal{C}^{1,2}([0,T] \times \mathbb{R}) \cap \Pi_g$.

□

Let us now recall the Lévy–Khintchine formula of a Lévy process $(L_t)_{t\leq T}$ whose characteristic exponent is Ψ, *i.e.*,

$$\forall t \leq T \text{ and } \theta \in \mathbb{R}, \quad \mathbf{E}(e^{i\theta L_t}) = e^{t\Psi(\theta)}$$

with

$$\begin{aligned}\Psi(\theta) &= ia\theta - \frac{1}{2}\varpi^2\theta^2 + \int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{(|x|<1)})\Pi(dx)\\ &= ia\theta - \frac{1}{2}\varpi^2\theta^2 + \int_{|x|\geq 1}(e^{i\theta x} - 1)\Pi(dx) + \int_{0<|x|<1}(e^{i\theta x} - 1 - i\theta x)\Pi(dx)\end{aligned}$$

where $a \in \mathbb{R}$, $\varpi \geq 0$ and Π is a σ-finite measure on $\mathbb{R}^* := \mathbb{R} - \{0\}$ (we set $\Pi(\{0\}) = 0$ and then the domain of integration is the whole space), called the Lévy measure of L, verifying

$$\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty. \tag{2.1}$$

Moreover we assume that Π satisfies the following assumption:

$$\exists \epsilon > 0 \text{ and } \lambda > 0 \text{ such that } \int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\Pi(dx) < +\infty. \tag{2.2}$$

Conditions (2.1)–(2.2) imply that for any $i \geq 2$,

$$\int_{\mathbb{R}} |x|^i \Pi(dx) < \infty \tag{2.3}$$

and then the process $(L_t)_{t\leq T}$ have moments of any order.

Next following Nualart-Schoutens [23] we define, for every $i \geq 1$, the so-called power-jump processes $L^{(i)}$ and their compensated version $Y^{(i)}$, also called Teugels martingales, as follows: $\forall t \leq T$,

$$L_t^{(1)} = L_t \text{ and for } i \geq 2, \quad L_t^{(i)} = \sum_{s\leq t}(\Delta L_s)^i, Y_t^{(i)} = L_t^{(i)} - t\mathbf{E}(L_1^{(i)}).$$

Note that for any $i \geq 2$ and $t \leq T$, $\mathbf{E}(L_t^{(i)}) = t\int_{\mathbb{R}} x^i \Pi(dx)$ exists, i.e., is defined and belongs to $\mathbb{R}$ ([21], pp. 29).

An orthonormalization procedure can be applied to the martingales $Y^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $(H^{(i)})_{i\geq 1}$ such that each $H^{(i)}$ is a linear combination of $(Y^{(j)})_{j=1,i}$, i.e.,

$$H^{(i)} = c_{i,i}Y^{(i)} + \cdots + c_{i,1}Y^{(1)}.$$

It has been shown in [23] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\nu(dx) = x^2\Pi(dx) + \varpi^2\delta_0(dx)$ (δ_0 is the Dirac measure at 0). Specifically the polynomials $(q_i)_{i\geq 0}$ defined by, for any $i \geq 1$,

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}$$

and satisfying

$$\int_{\mathbb{R}} q_n(x)q_m(x)\nu(dx) = \delta_{nm}, \quad \forall n, m \geq 0.$$

Next let us set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x \text{ and}$$
$$\tilde{p}_i(x) = x(q_{i-1}(x) - q_{i-1}(0)) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,2}x^2.$$

Then for any $i \geq 1$ and $t \leq T$ we have:

$$\begin{aligned} H_t^{(i)} &= \sum_{0<s\leq t} \{c_{i,i}(\Delta L_s)^i + \cdots + c_{i,2}(\Delta L_s)^2\} + c_{i,1}L_t - t\mathbf{E}[c_{i,i}(L_1)^{(i)} \\ &\quad + \cdots + c_{i,2}(L_1)^{(2)}] - tc_{i,1}\mathbf{E}(L_1) \\ &= q_{i-1}(0)L_t + \sum_{0<s\leq t} \tilde{p}_i(\Delta L_s) - t\mathbf{E}\left[\sum_{0<s\leq 1} \tilde{p}_i(\Delta L_s)\right] - tq_{i-1}(0)\mathbf{E}(L_1). \end{aligned}$$

As a consequence, for any $t \leq T$ and $i \geq 1$, $\Delta H_t^{(i)} = p_i(\Delta L_t)$ for each $i \geq 1$. In the particular case of $i = 1$, we obtain

$$\begin{aligned} H_t^{(1)} &= c_{1,1}(L_t - t\mathbf{E}(L_1)) \text{ with } c_{1,1} \\ &= \left[\int_{\mathbb{R}} x^2\Pi(dx) + \varpi^2\right]^{-\frac{1}{2}} \quad \text{and} \quad \mathbf{E}[L_1] = a + \int_{|x|\geq 1} x\Pi(dx). \end{aligned} \tag{2.4}$$

Finally note that for any $i, j \geq 1$ the predictable quadratic variation process of $H^{(i)}$ and $H^{(j)}$ is $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t, \forall t \leq T$. □

Remark 2.1. If $\Pi = 0$, we are in the classical Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H^{(i)} = 0$, $i \geq 2$. On the other hand, if Π only has mass at 1, we are in the Poisson case and once more $H^{(i)} = 0$, $i \geq 2$. Both cases are degenerate ones in this Lévy process framework. □

The main result in the paper by Nualart-Schoutens [22] is the following representation property which allows for developing the BSDE theory in this Lévy framework.

Theorem 2.1. [22, pp. 118] *Let ξ be a random variable of $\mathcal{L}^2$, then there exists a process $Z = (Z^i)_{i\geq 1}$ that belongs to $\mathcal{H}^2(\ell^2)$ such that:*

$$\xi = \mathbf{E}(\xi) + \sum_{i\geq 1} \int_0^T Z_s^i dH_s^{(i)}.$$

□

3. Systems of reflected BSDEs with oblique reflection driven by a Lévy process

3.1. Reflected BSDE driven by a Lévy process and their relationship with IPDEs

As a consequence of Theorem 2.1, and as in the framework of Brownian noise only, one can study standard BSDEs or reflected ones. The result below related to existence and uniqueness of a solution for a reflected BSDE driven by a Lévy process, is proved in [26]. Indeed let us introduce a triple (f, ξ, S) that satisfies:

Assumptions (A1):

(i) ξ a random variable of $\mathcal{L}^2$ which stands for the terminal value;

(ii) f: $[0,T] \times \Omega \times \mathbb{R} \times \ell^2 \longrightarrow \mathbb{R}$ is a function such that the process $(f(t,0,0))_{t\leq T}$ belongs to H^2 and there exists a constant $\kappa > 0$ verifying

$$|f(t,y,\zeta) - f(t,y',\zeta')| \leq \kappa(|y-y'| + \|\zeta - \zeta'\|_{\ell^2}), \text{ for every } t, y, y', \zeta \text{ and } \zeta'.$$

(iii) $S := (S_t)_{0\leq t\leq T}$ is a process of $\mathcal{S}^2$ such that $S_T \leq \xi$, $\mathbf{P} - a.s.$, and whose jump times are inaccessible stopping times. This in particular implies that for any $t \leq T$, $S^p_t = S_{t-}$, where S^p is the predictable projection of S (see e.g. [9, pp. 58]) for more details on those notions.

In [26], the authors have proved the following result related to existence and uniqueness of the solution of one barrier reflected BSDEs whose noise is driven by a Lévy process.

Theorem 3.1. *Assume that the triple (f,ξ,S) satisfies Assumptions (A1). Then there exists a unique triple of processes $(Y,U,K) := ((Y_t,U_t,K_t))_{t\leq T}$ with values in $\mathbb{R}\times\ell^2\times\mathbb{R}^+$ such that:*

$$\begin{cases} (Y,U,K) \in \mathcal{S}^2 \times \mathcal{H}(\ell^2) \times \mathcal{A}^2; \\ Y_t = \xi + \int_t^T f(s,Y_s,U_s)ds + K_T - K_t - \sum_{i=1}^{\infty} \int_t^T U^i_s dH^{(i)}_s, \ \forall t \leq T; \\ Y_t \geq S_t, \ \forall\ 0 \leq t \leq T \ \text{and} \ \int_0^T (Y_t - S_t)dK_t = 0, \ \mathbf{P} - a.s. \end{cases} \tag{3.1}$$

The triple (Y,U,K) is called the solution of the reflected BSDE associated with (f,ξ,S). □

To proceed we need to compare solutions of reflected BSDEs of types (3.1). So let us consider a stochastic process $V = (V_t)_{t\leq T} = (V^i)_{i\geq 1} = ((V^i_t)_{t\leq T})_{i\geq 1}$ which belongs to $\mathcal{H}^2(\ell^2)$ and let $M := (M_t)_{t\leq T}$ be the stochastic integral defined by:

$$\forall t \leq T, \quad M_t := \sum_{i=1}^{\infty} \int_0^t V^i_s dH^{(i)}_s.$$

We next denote by $\varepsilon(M) := (\varepsilon(M)_t)_{t\leq T}$ the process that satisfies:

$$\forall t \leq T, \quad \varepsilon(M)_t = 1 + \int_0^t \varepsilon(M)_{s-} dM_s.$$

By Doléans-Dade's formula we have (see e.g. [25]):

$$\forall t \leq T, \quad \varepsilon(M)_t = \exp\left\{ M_t - \frac{1}{2}[M,M]^c_t - \sum_{0\leq s\leq t} \Delta M_s \right\} \prod_{0\leq s\leq t} \{1 + \Delta M_s\}.$$

Let us now introduce the following assumption on the process V.

Assumptions (A2): The process $V = (V^i)_{i\geq 1} = ((V^i_t)_{t\leq T})_{i\geq 1}$ verifies

$$\mathbf{P} - a.s., \quad \forall t \leq T, \quad \sum_{i=1}^{\infty} V^i_t p_i(\Delta L_t) > -1. \tag{3.2}$$

and there exists a constant C such that:

$$\sum_{i=1}^{\infty}|V_t^i|^2 \leq C, \quad d\mathbf{P}\otimes dt - a.e. \tag{3.3}$$

We then have:

Proposition 3.1. *Assume that Assumption (A2) is fulfilled. Then,* $\mathbf{P}$*-a.s., for any* $t\in[0,T]$, $\varepsilon(M)_t>0$ *and* $\varepsilon(M)$ *is a martingale of* $\mathcal{S}^2$.

Proof. First note that for any $t\leq T$,

$$\Delta M_t = \sum_{i=1}^{\infty} V_t^i \Delta H_t^{(i)} = \sum_{i=1}^{\infty} V_t^i p_i(\Delta L_t) > -1,$$

therefore for any $t\leq T$, $\varepsilon(M_t)>0$. Next by using Doléans-Dade's formula and since $d\langle H^{(i)},H^{(j)}\rangle_s=\delta_{ij}ds$, we have: $\forall t\leq T$,

$$\begin{aligned}
\varepsilon(M)_t^2 &= \varepsilon(2M+[M,M])_t\\
&= \varepsilon\left(2\sum_{i=1}^{\infty}\int_0^{\cdot} V_s^i dH_s^{(i)} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\cdot} V_s^i V_s^j d[H^{(i)},H^{(j)}]_s\right)_t\\
&= \varepsilon\left(2\sum_{i=1}^{\infty}\int_0^{\cdot} V_s^i dH_s^{(i)} + \sum_{i=1}^{\infty}\int_0^{\cdot}|V_s^i|^2 ds\right.\\
&\quad \left. + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\cdot} V_s^i V_s^j d([H^{(i)},H^{(j)}]_s - \langle H^{(i)},H^{(j)}\rangle_s)\right)_t\\
&= \varepsilon(N)_t \exp\left\{\sum_{i=1}^{\infty}\int_0^t |V_s^i|^2 ds\right\}
\end{aligned}$$

where

$$\begin{aligned}
N_t = 2\sum_{i=1}^{\infty}\int_0^t V_s^i dH_s^{(i)} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t V_s^i V_s^j d([H^{(i)},H^{(j)}]_s\\
-\langle H^{(i)},H^{(j)}\rangle_s),\ t\leq T,
\end{aligned}$$

is a local martingale. On the other hand, the quantity $\sum_{i=1}^{\infty}\int_0^T |V_s^i|^2 ds$ is bounded and $\varepsilon(N)\geq 0$, then

$$\mathbf{E}[(\varepsilon(M)_t)^2] \leq C\mathbf{E}[\varepsilon(N)_0] \leq C, \quad \forall t\leq T,$$

since $\varepsilon(N)$ is a supermartingale. It follows that $\varepsilon(M)$ is not only a local martingale but also a martingale and then by Doob's maximal inequality it belongs to $\mathcal{S}^2$. □

Remark 3.1. The result of Proposition 3.1 still holds true if instead of (3.3) we only have

$$\sum_{i=1}^{\infty}\int_0^T |V_s^i|^2 ds \le C, \quad \mathbf{P}-a.s. \tag{3.4}$$

□

Next for two processes $U^i = (U^i_k)_{k\ge 1}$, $i = 1, 2$, of $\mathcal{H}^2(\ell^2)$ we define their scalar product in $\mathcal{H}^2(\ell^2)$ which we denote by $\langle U^1, U^2\rangle^p := (\langle U^1, U^2\rangle^p_t)_{t\le T}$ as:

$$\forall t \le T, \quad \langle U^1, U^2\rangle^p_t = \sum_{k\ge 1} U^1_k(t)U^2_k(t).$$

Proposition 3.2. *Let $\xi \in \mathcal{L}^2$, $\varphi := (\varphi_s)_{s\le T} \in H^2$, $\delta := (\delta_s)_{s\le T}$ a uniformly bounded process, and finally let $V = (V^i)_{i\ge 1} \in \mathcal{H}^2(\ell^2)$ satisfying (A2). Let $(Y, U) := (Y_t, U_t)_{t\le T} \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ be the solution of the following BSDE:*

$$\forall t \le T, \quad Y_t = \xi + \int_t^T (\varphi_s + \delta_s Y_s + \langle V, U\rangle^p_s)ds - \sum_{i=1}^{\infty}\int_t^T U^i_s dH^{(i)}_s. \tag{3.5}$$

For $t \le T$, let $(X^t_s)_{s\in[t,T]}$ be the process defined as follows:

$$\forall s \in [t, T], \quad X^t_s = e^{\int_t^s \delta_r dr}\frac{\varepsilon(M)_s}{\varepsilon(M)_t}. \tag{3.6}$$

Then for any $t \le T$, Y_t satisfies:

$$Y_t = \mathbf{E}\left[X^t_T\xi + \int_t^T X^t_s\varphi_s ds|\mathcal{F}_t\right], \quad \mathbf{P}-a.s..$$

On the other hand, if $(Y', U') \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ is the solution of the BSDE:

$$Y'_t = \xi + \int_t^T f(s, Y'_s, U'_s)ds - \sum_{i=1}^{\infty}\int_t^T U'^i_s dH^{(i)}_s, \quad \forall t \le T \tag{3.7}$$

where

$$f(t, Y'_t, U'_t) \ge \varphi_t + \delta_t Y'_t + \langle V, U'\rangle^p_t, \quad d\mathbf{P}\otimes dt - a.s.$$

then for any $t \le T$,

$$Y'_t \ge \mathbf{E}\left[X^t_T\xi + \int_t^T X^t_s\varphi_s ds|\mathcal{F}_t\right], \quad \mathbf{P}-a.s..$$

Proof. First note that the processes (Y, U) and (Y', U') exist thanks to Theorem 3.1. Let us now fix $t \in [0, T]$. Since V satisfies (A2) then $\varepsilon(M) > 0$ which implies that $(X^t_s)_{s\in[t,T]}$ is defined ω by ω. On the other hand it satisfies

$$\forall s \in [t, T], \quad dX^t_s = X^t_{s-}(\delta_s ds + dM_s)$$

and since δ is uniformly bounded then as in Proposition 3.1, one can show that $\mathbf{E}[\sup_{s\in[t,T]}|X^t_s|^2] < \infty$. Now by Itô's formula, for any $s \in [t, T]$, we have

$$\begin{aligned}-d(Y_s X^t_s) &= -Y_{s-}dX^t_s - X^t_{s-}dY_s - d[Y, X^t]_s\\ &= -X^t_{s-}Y_{s-}\delta_s ds - Y_{s-}X^t_{s-}dM_s + X^t_{s-}\varphi_s ds + X^t_{s-}\delta_s Y_s ds\end{aligned}$$

$$
\begin{aligned}
&-X^t_{s-}\left\{\sum_{i\geq 1} U^i_s dH^{(i)}\right\}\\
&-X^t_{s-}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V^i_s U^j_s d([H^{(i)},H^{(j)}]_s-\langle H^{(i)},H^{(j)}\rangle_s)\right\}\\
&= X^t_s\varphi_s ds - dN_s
\end{aligned}
$$

where for any $s\in[t,T]$

$$
\begin{aligned}
dN_s &= Y_{s-}X^t_{s-}\left\{\sum_{i=1}^{\infty} V^i_s dH^{(i)}_s\right\} + X^t_{s-}\left\{\sum_{i\geq 1} U^i_s dH^{(i)}_s\right\}\\
&\quad +X^t_{s-}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V^i_s U^j_s d([H^{(i)},H^{(j)}]_s-\langle H^{(i)},H^{(j)}\rangle_s)\right\}.
\end{aligned}
$$

Note that since X^t is uniformly square integrable, $Y\in\mathcal{S}^2$, $U\in\mathcal{H}^2(\ell^2)$ and finally taking into account Assumption (A2) on V, we get that N is a uniformly integrable martingale on $[t,T]$. Therefore taking conditional expectation to obtain:

$$
Y_t = \mathbf{E}\left[X^t_T\xi + \int_t^T X^t_s\varphi_s ds|\mathcal{F}_t\right],\quad \mathbf{P}-a.s.
$$

which is the desired result.

We now focus on the second part of the claim. By Itô's formula we have: $\forall s\in[t,T]$,

$$
\begin{aligned}
-d(Y'_sX^t_s) &= -Y'_{s-}dX^t_s - X^t_{s-}dY'_s - d[Y',X^t]_s\\
&= -X^t_{s-}Y'_{s-}\delta_s ds - Y'_{s-}X^t_{s-}\left\{\sum_{i=1}^{\infty} V^i_s dH^{(i)}_s\right\} + X^t_{s-}f(s,Y'_s,U'_s)ds\\
&\quad -X^t_{s-}\left\{\sum_{i=1}^{\infty} U'^i_s dH^{(i)}_s\right\}\\
&\quad -X^t_{s-}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V^i_s U'^j_s d[H^{(i)},H^{(j)}]_s\right\}.
\end{aligned}
$$

Next since $X^t\geq 0$ and taking into account the inequality which f verifies to obtain

$$
-d(Y'_sX^t_s)\geq X^t_s\varphi_s ds - dN'_s\ \mathbf{P}-a.s.,
$$

where for any $s\in[t,T]$,

$$\begin{aligned} dN'_s &= Y'_{s-}X^t_{s-}\left\{\sum_{i=1}^{\infty} V^i_s dH^{(i)}_s\right\} - X^t_{s-}\left\{\sum_{i=1}^{\infty} U'^i_s dH^{(i)}_s\right\} \\ &\quad - X^t_{s-}\left\{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} V^i_s U'^j_s d([H^{(i)},H^{(j)}]_s - \langle H^{(i)},H^{(j)}\rangle_s)\right\}. \end{aligned}$$

But once more N' is a uniformly integrable martingale then by taking the conditional expectation we obtain:

$$Y'_t \geq \mathbf{E}\left[X^t_T\xi + \int_t^T X^t_s\varphi_s ds|\mathcal{F}_t\right], \quad \mathbf{P}-a.s.$$

which completes the proof. □

We are now ready to give a comparison result of solutions of two BSDEs of type (3.1).

Proposition 3.3. *For $i = 1, 2$, let (f_i, ξ_i) be a pair that satisfies Assumption (A1)-(i),(ii) and let $(Y^i, U^i) \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$ be the solution of the following BSDE: $\forall t \leq T$,*

$$Y^i_t = \xi_i + \int_t^T f_i(s, Y^i_s, U^i_s)ds - \sum_{j=1}^{\infty}\int_t^T U^{i,j}_s dH^{(j)}_s.$$

Assume that:

(i) *For any $U^1, U^2 \in \mathcal{H}^2(l^2)$, there exists a process $V^{U^1,U^2} = (V^{U^1,U^2}_j)_{j\geq 1}$ (which may depend on U^1 and U^2) satisfying (A2) such that f_1 verifies:*

$$f_1(t, Y^2_t, U^1_t) - f_1(t, Y^2_t, U^2_t) \geq \langle V^{U^1,U^2}, (U^1 - U^2)\rangle^p_t, \quad d\mathbf{P}\otimes dt - a.e.; \tag{3.8}$$

(ii) *$\mathbf{P} - a.s.$, $\xi_1 \geq \xi_2$ and*

$$f_1(t, Y^2_t, U^2_t) \geq f_2(t, Y^2_t, U^2_t), \quad d\mathbf{P}\otimes dt - a.e.. \tag{3.9}$$

Then $\mathbf{P}$-a.s., $Y^1_t \geq Y^2_t$, $\forall t \in [0, T]$.

Proof. Let us set $\bar{Y} = Y^1 - Y^2$, $\bar{U} = U^1 - U^2$ and $\bar{\xi} = \xi^1 - \xi^2$, then $\forall t \in [0, T]$,

$$\bar{Y}_t = \bar{\xi} + \int_t^T \{f_1(s, Y^1_s, U^1_s) - f_2(s, Y^2_s, U^2_s)\}ds - \sum_{j=1}^{\infty}\int_t^T \bar{U}^j_s dH^{(j)}_s.$$

Next let us set

$$\begin{aligned} \forall s \leq T, \quad \delta_s &= (f_1(s, Y^1_s, U^1_s) - f_1(s, Y^2_s, U^1_s)) \times (\bar{Y}_s)^{-1}\mathbb{1}_{\{\bar{Y}_s\neq 0\}} \text{ and} \\ \varphi_s &= f_1(s, Y^2_s, U^2_s) - f_2(s, Y^2_s, U^2_s). \end{aligned} \tag{3.10}$$

Then by (3.9) we have, $\varphi_s \geq 0$, $d\mathbf{P}\otimes dt - a.e.$. On the other hand $(\delta_s)_{s\in[0,T]}$ is bounded since f_1 is uniformly Lipschitz. Finally we have

$$f_1(s, Y^1_s, U^1_s) - f_2(s, Y^2_s, U^2_s) \geq \varphi_s + \delta_s\bar{Y}_s + \langle V^{U^1,U^2}, \bar{U}\rangle^p_s, \quad d\mathbf{P}\otimes ds - a.e..$$

Therefore thanks to Proposition 3.2 we get,

$$\forall t \leq T, \quad \bar{Y}_t \geq \mathbf{E}\left[X_T^t \bar{\xi} + \int_t^T X_s^t \varphi_s ds | F_s\right] \geq 0, \quad \mathbf{P} - a.s.$$

where $(X_s^t)_{s\in[t,T]}$ is defined in the same way as in (3.6) with the new processes δ and φ defined in (3.10). As X^t, $\bar{\xi}$ and φ are non-negative then for any $t \leq T$, $\bar{Y}_t \geq 0$ which implies that $\mathbf{P} - a.s., \forall t \leq T, Y_t^1 \geq Y_t^2$ since Y^1 and Y^2 are RCLL. The proof of the claim is now complete. □

Remark 3.2. Conditions (3.8) and (3.9) can be replaced respectively with

$$f_2(t, Y_t^2, U_t^1) - f_2(t, Y_t^2, U_t^2) \geq \langle V^{U^1,U^2}, (U^1 - U^2)\rangle_t^p, \quad d\mathbf{P} \otimes dt - a.e. \tag{3.11}$$

and

$$f_1(t, Y_t^1, U_t^1) \geq f_2(t, Y_t^1, U_t^1), \quad d\mathbf{P} \otimes dt - a.e.. \tag{3.12}$$

In this case, with the other properties, one can show that we have **P**-a.s., $Y^1 \geq Y^2$. □

Remark 3.3. Point (i) of Proposition 3.3 is satisfied in the following cases:

(i) f does not depend on the component ζ;

(ii) If L reduces to a Poisson process, we have $H^{(i)} \equiv 0$ for all $i \geq 2$, then Assumption (A2) reads: (a) $V = (V_t)_{t\in[0,T]}$ is bounded; (b) for any stopping time τ, such that $\triangle L_\tau \neq 0$, $V_\tau > -1$, $\mathbf{P} - a.s.$.

(iii) The generator f satisfies

$$f(t, y, \zeta) = h_1(t, y, \sum_{i\geq 1} \theta_t^i \zeta^i), \quad \forall (t, y, \zeta) \in [0, T] \times \mathbb{R} \times \ell^2$$

where the mapping $\eta \in \mathbb{R} \mapsto h_1(t, y, \eta)$ is non decreasing and uniformly Lipschitz and $((\theta_t^i)_{i\geq 1})_{t\leq T}$ satisfies

$$\sum_{i\geq 1} |\theta_t^i|^2 \leq C, \quad dt \otimes d\mathbf{P} - a.e. \quad \text{and} \quad \mathbf{P} - a.s., \forall t \leq T, \quad \sum_{i\geq 1} \theta_t^i p_i(\Delta L_t) \geq 0.$$

□

We finally provide a comparison result of solutions of reflected BSDEs of type (3.1) which will be useful in the sequel.

Proposition 3.4. *For $i = 1, 2$, let (f_i, ξ_i, S^i) be a triple which satisfies Assumption (A1) and let $(Y_t^i, K_t^i, U_t^i)_{t\leq T}$ be the solution of the RBSDE associated with (f_i, ξ_i, S^i). Assume that:*

(i) *$\mathbf{P} - a.s$, $\xi_1 \geq \xi_2$ and $\forall t \in [0, T]$, $f_1(t, y, \zeta) \geq f_2(t, y, \zeta)$ and $S_t^1 \geq S_t^2$;*

(ii) *f_1 verifies condition (3.8).*

Then **P***-a.s. for any $t \leq T$, $Y_t^1 \geq Y_t^2$.*

Proof. For $i = 1, 2$, let us consider the following sequence of processes $(Y^{i,n}, U^{i,n}) \in \mathcal{S}^2 \times \mathcal{H}^2(\ell^2)$, $n \geq 0$, that satisfy:

$$\begin{aligned} Y_t^{i,n} &= \xi_i + \int_t^T f_i(s, Y_s^{i,n}, U_s^{i,n})ds + n \int_t^T (Y_s^{i,n} - S_s^i)^- ds \\ &\quad - \sum_{j=1}^{\infty} \int_t^T U_s^{i,n,j} dH_s^{(j)}, \quad \forall t \leq T \end{aligned}$$

and let us denote by

$$f_i^n(s, y, \zeta) := f_i(s, y, \zeta) + n(y - S_s^i)^-.$$

For any $n \geq 0$, f_1^n satisfies (3.8) and $f_1^n \geq f_2^n$. Therefore using the comparison result of Proposition 3.3, we deduce that: $\forall n \geq 0$,

$$\mathbf{P} - a.s., \quad \forall t \leq T, Y_t^{1,n} \geq Y_t^{2,n}. \tag{3.13}$$

But since f_1 verifies (3.8) then we can show that for $i = 1, 2$, $Y^{i,n} \nearrow Y^i$ in $\mathcal{S}^2$ since the processes S^i do not have predictable jumps (see e.g. [16], Theorem 1.2.a, pp. 5). Thus, inequality (3.13) implies that $\mathbf{P}$-a.s., $Y^1 \geq Y^2$. □

We are now going to make a connection between reflected BSDEs and their associated IPDEs with obstacle. So let $(t, x) \in [0, T] \times \mathbb{R}$ and let $(X_s^{t,x})_{s \leq T}$ be the solution of the following standard SDE driven by the Lévy process L, i.e.,

$$X_s^{t,x} = x + \int_t^{t \vee s} b(r, X_r^{t,x})dr + \int_t^{t \vee s} \sigma(r, X_{r-}^{t,x})dL_r, \quad \forall s \leq T, \tag{3.14}$$

where we assume that the functions b and σ are jointly continuous, Lipschitz continuous w.r.t. x uniformly in t, i.e., there exists a constant $C \geq 0$ such that for any $t \in [0, T]$, x,$x' \in \mathbb{R}$,

$$|\sigma(t, x) - \sigma(t, x')| + |b(t, x) - b(t, x')| \leq C|x - x'|. \tag{3.15}$$

As a consequence, the functions $b(t, x)$ and $\sigma(t, x)$ are of linear growth. We additionally assume that σ is bounded, i.e., there exists a constant C_σ such that

$$\forall (t, x) \in [0, T] \times \mathbb{R}, |\sigma(t, x)| \leq C_\sigma. \tag{3.16}$$

Under the above conditions on b and σ, the process $X^{t,x}$ exists and is unique (see e.g. [25], pp. 249), and satisfies:

$$\forall p \geq 1, \quad \mathbf{E}\left[\sup_{s \leq T} |X_s^{t,x}|^p\right] \leq C(1 + |x|^p). \tag{3.17}$$

Next let us consider the following functions:

$$\begin{aligned} &h : x \in \mathbb{R} \mapsto h(x) \in \mathbb{R}; \\ &f : (t, x, y, \zeta) \in [0, T] \times \mathbb{R}^{1+1} \times \ell^2 \mapsto f(t, x, y, \zeta) \in \mathbb{R}; \\ &\Psi : (t, x) \in [0, T] \times \mathbb{R} \mapsto \Psi(t, x) \in \mathbb{R}, \end{aligned}$$

which we assume satisfying:

Assumptions (A3):

(i) h, Ψ and $f(t,x,0,0)$ are jointly continuous and belong to Π_g;
(ii) the mapping $(y,\zeta) \mapsto f(t,x,y,\zeta)$ is Lipschitz continuous uniformly in (t,x);
(iii) For any $x \in \mathbb{R}$, $h(x) \geq \Psi(T,x)$;
(iv) The generator f has the following form:

$$f(t,x,y,\zeta) = \underline{\mathrm{h}}\left(t,x,y,\sum_{i\geq 1}\theta_t^i\zeta^i\right),\ \forall (t,x,y,\zeta) \in [0,T]\times\mathbb{R}^{1+1}\times\ell^2$$

where the mapping $\eta \in \mathbb{R} \longmapsto \underline{\mathrm{h}}(t,x,y,\eta)$ is non decreasing, and there exists a constant $C>0$, such that $\forall t \in [0,T]$, $x,y,\eta,\eta' \in \mathbb{R}$,

$$|\underline{\mathrm{h}}(t,x,y,\eta) - \underline{\mathrm{h}}(t,x,y,\eta') \leq C|\eta-\eta'|.$$

Moreover $(\theta_t^i)_{i\geq 1}$ satisfies

$$\sum_{i\geq 1}|\theta_t^i|^2 \leq C, \quad dt\otimes d\mathbf{P}-a.e. \text{ and } \mathbf{P}-a.s., \forall t\leq T, \quad \sum_{i\geq 1}\theta_t^i p_i(\Delta L_t) \geq 0.$$

□

Next let $(t,x) \in [0,T]\times\mathbb{R}$ be fixed and let us consider the following reflected BSDE:

$$\begin{cases} (Y^{t,x},U^{t,x},K^{t,x}) \in \mathcal{S}^2\times\mathcal{H}(\ell^2)\times\mathcal{A}^2; \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x},U_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} \\ \qquad - \sum\limits_{i=1}^{\infty}\int_s^T U_r^{t,x,i}dH_r^{(i)} \\ \forall s\leq T,\ Y_s^{t,x} \geq \Psi(s,X_s^{t,x}) \text{ and } \int_0^T (Y_s^{t,x} - \Psi(s,X_s^{t,x}))dK_s^{t,x} = 0,\ \mathbf{P}-a.s. \end{cases} \tag{3.18}$$

Under assumptions (A3)-(i),(ii),(iii), the reflected BSDE (3.18) is well-posed and, thanks to Theorem 3.1, has a unique solution $(Y^{t,x},U^{t,x},K^{t,x})$. Moreover the following estimate holds true:

$$\begin{aligned} &\mathbf{E}\left[\sup_{0\leq s\leq T}|Y_s^{t,x}|^2 + \int_0^T\left\{\sum_{i\geq 1}|U_s^{t,x,i}|^2\right\}ds\right] \\ &\quad \leq C\mathbf{E}\left[|h(X_T^{t,x})|^2 + \int_0^T |f(s,X_s^{t,x},0,0)|^2 ds + \sup_{0\leq s\leq T}|\Psi(s,X_s^{t,x})|^2\right]. \end{aligned} \tag{3.19}$$

On the other hand, the quantity

$$u(t,x) = Y_t^{t,x}, \tag{3.20}$$

is deterministic, continuous and satisfies

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \forall s\in[t,T],\ Y_s^{t,x} := u(s,X_s^{t,x}).$$

Fore more details, one can see e.g. [26, pp. 1265]. Finally note that under Assumptions (A3) and by (3.19) the function u belongs also to Π_g.

We now introduce the following IPDE with obstacle:

$$\begin{cases} \min\Big\{u(t,x) - \Psi(t,x); -\partial_t u(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),\Phi(u)(t,x))\Big\} \\ \quad = 0,\ (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) = h(x), \end{cases} \tag{3.21}$$

where $\mathcal{L}$ is the generator associated with the process $X^{t,x}$ of (3.14) which has the following expression:

$$\begin{aligned} \mathcal{L}u(t,x) &= (\mathbf{E}[L_1]\sigma(t,x) + b(t,x))\partial_x u(t,x) + \frac{1}{2}\sigma(t,x)^2\varpi^2\partial^2_{xx}u(t,x) \\ &\quad + \int_{\mathbb{R}} [u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)\sigma(t,x)y]\Pi(dy) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &\Phi(u)(t,x) \\ &\quad = \Bigg(\frac{1}{c_{1,1}}\partial_x u(t,x)\sigma(t,x)\mathbb{1}_{k=1} \\ &\qquad + \int_{\mathbb{R}} (u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)y)p_k(y)\Pi(dy)\Bigg)_{k\geq 1} \end{aligned}$$

where $c_{1,1}$ is defined in (2.4).

We are going to consider solutions of (3.21) in viscosity sense whose definition is as follows:

Definition 3.1. *A continuous function $u : [0,T]\times\mathbb{R} \to \mathbb{R}$ is said to be a viscosity subsolution (resp. supersolution) of (3.21) if:*

(i) *$u(T,x) \leq h(x)$ (resp. $u(T,x) \geq h(x)$);*
(ii) *for any $(t,x) \in (0,T)\times\mathbb{R}$ and for any $\varphi \in \mathcal{C}^{1,2}_p$ such that $\varphi(t,x) = u(t,x)$ and $\varphi - u$ attains its global minimum (resp. maximum) at (t,x),*

$$\begin{aligned} \min\Big\{&u(t,x) - \Psi(t,x); -\partial_t\varphi(t,x) - \mathcal{L}\varphi(t,x) \\ &-f(t,x,\varphi(t,x),\Phi(\varphi)(t,x))\Big\} \leq 0 \ (resp.\ \geq 0). \end{aligned}$$

The function u is said to be a viscosity solution of (3.21) if it is both its viscosity subsolution and supersolution.

□

In [26], Ren-El Otmani (Theorem 5.8, pp. 1265) have shown that under Assumption (A3), the function u defined in (3.20) is a viscosity solution for (3.21).

3.2. Systems of reflected BSDEs with inter-connected obstacles driven by a Lévy process and multi-modes switching problem

We now introduce the following functions f_i, h_i and g_{ij}, $i, j \in A$:

$$\begin{aligned} f_i &: (t, x, (y^i)_{i=1,m}, \zeta) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \ell^2 \longmapsto f_i(t, x, (y^i)_{i=1,m}, \zeta) \in \mathbb{R}, \\ g_{ij} &: (t, x) \in [0, T] \times \mathbb{R} \longmapsto g_{ij}(t, x) \in \mathbb{R}, \\ h_i &: x \in \mathbb{R} \longmapsto h_i(x) \in \mathbb{R} \end{aligned} \tag{3.23}$$

which we assume satisfying:
Assumptions (A4):

(I) For any $i \in A$:
  - (i) The mapping $(t, x) \to f_i(t, x, \overrightarrow{y}, \zeta)$ is continuous uniformly with respect to $(\overrightarrow{y}, \zeta)$ where $\overrightarrow{y} = (y^i)_{i=1,m}$;
  - (ii) The mapping $(\overrightarrow{y}, \zeta) \mapsto f_i(t, x, \overrightarrow{y}, \zeta)$ is Lipschitz continuous uniformly w.r.t. (t, x);
  - (iii) $f_i(t, x, 0, 0)$ is measurable and of polynomial growth;
  - (iv) For any $U^1, U^2 \in \mathcal{H}^2(l^2)$, $X, Y \in \mathcal{S}^2$, $i \in A$, there exist $V^{U^1,U^2,i} = (V_k^{U^1,U^2,i})_{k\geq 1}$ (which may depend on U^1 and U^2) that satisfy Assumption (A2) and such that:

$$\begin{aligned} &f_i(t, X_t, Y_t, U_t^1) - f_i(t, X_t, Y_t, U_t^2) \\ &\quad \geq \langle V^{U^1,U^2,i}, (U^1 - U^2)\rangle_t^p, \quad d\mathbf{P} \otimes dt - a.e. \end{aligned} \tag{3.24}$$

  - (v) For any $i \in A$ and $k \in A_i := A - \{i\}$, the mapping $y_k \to f_i(t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m, \zeta)$ is nondecreasing whenever the other components $(t, x, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m, \zeta)$ are fixed.

(II) $\forall i, j \in A$, $g_{ii} \equiv 0$ and for $k \neq j$, $g_{jk}(t, x)$ is non-negative, continuous with polynomial growth and satisfy the following non free loop property:
For any $(t, x) \in [0, T] \times \mathbb{R}$ and for any sequence of indices $i_1, \ldots, i_k$ such that $i_1 = i_k$ and $card\{i_1, \ldots, i_k\} = k - 1$ we have

$$g_{i_1 i_2}(t, x) + g_{i_2 i_3}(t, x) + \cdots + g_{i_k i_1}(t, x) > 0.$$

(III) $\forall i \in A$, h_i is continuous with polynomial growth and satisfies the following consistency condition:

$$h_i(x) \geq \max_{j\in A_i}(h_j(x) - g_{ij}(T, x)), \quad \forall x \in \mathbb{R}.$$

We now describe precisely the switching problem. Let $\Upsilon = ((\theta_j)_{j\geq 0}, (\alpha_j)_{j\geq 0})$ be an admissible strategy and let $a = (a_s)_{s\in[0,T]}$ be the process defined by

$$\forall s \leq T, \quad a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s),$$

where $\{\theta_j\}_{j\geq 0}$ is an increasing sequence of $\mathcal{F}_t$-stopping times with values in [0,T] and for $j \geq 0$, α_j is a random variable F_{θ_j}-measurable with values in $A = \{1, \ldots, m\}$. If $\mathbf{P}[\lim_n \theta_n < T] = 0$, then the pair $\{\theta_j, \alpha_j\}_{j\geq 0}$ (or the process a) is called an admissible strategy of switching. Next we denote by

$(A_s^a)_{s\leq T}$ the switching cost process associated with an admissible strategy a, which is defined as following:

$$\forall s < T, \quad A_s^a = \sum_{j\geq 1} g_{\alpha_{j-1},\alpha_j}(\theta_j, X_{\theta_j}^{t,x})\mathbb{1}_{[\theta_j\leq s]} \quad \text{and} \quad A_T^a = \lim_{s\to T} A_s^a \tag{3.25}$$

where $X^{t,x}$ is the process given in (3.14). For $\eta \leq T$ and $i \in A$, we denote by

$\mathcal{A}_\eta^i := \{a$ admissible strategy such that $\alpha_0 = i$, $\theta_0 = \eta$ and $E[(A_T^a)^2] < \infty\}$.

Assume momentarily that for $i \in A$, the function f_i of (3.23) does not depend on $\overrightarrow{y}$ and ζ. For $t \leq T$ and a given admissible strategy $a \in \mathcal{A}_t^i$, we define the payoff $J_i^a(t,x)$ by:

$$J_i^a(t,x) := \mathbf{E}\left[\int_t^T f_{a(s)}(s, X_s^{t,x})ds + h_{a(T)}(X_T^{t,x}) - A_T^a\right]$$

where $f_{a(s)}(\dots) = f_k(\dots)$ (resp. $h_{a(T)}(.) = h_k(.)$) if at time s (resp. T) $a(s) = k$ (resp. $a(T) = k$) ($k \in A$). Finally let us define

$$J^i(t,x) := \sup_{a\in\mathcal{A}_t^i} J_i^a(t,x), \quad i = 1,\dots,m. \tag{3.26}$$

As a by-product of our main result which is given in Theorem 4.1 below, we get that the functions $(J^i(t,x))_{i=1,\dots,m}$ is the unique continuous viscosity solution of the Hamilton–Jacobi–Bellman system associated with this switching problem. □

Let $(t,x) \in [0,T] \times \mathbb{R}$ and let us consider the following system of reflected BSDEs with oblique reflection: $\forall j = 1,\dots,m$

$$\begin{cases} Y^j \in \mathcal{S}^2, \quad U^j \in \mathcal{H}^2(\ell^2), \quad K^j \in \mathcal{A}^2; \\ Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, Y_r^2, \dots, Y_r^m, U_r^j)dr \\ \qquad - \sum\limits_{i=1}^{\infty} \int_s^T U_r^{j,i} dH_r^{(i)} + K_T^j - K_s^j, \quad \forall s \leq T; \\ \forall s \leq T, \quad Y_s^j \geq \max\limits_{k\in Aj}\{Y_s^k - g_{jk}(s, X_s^{t,x})\} \quad \text{and} \\ \qquad \int_0^T \{Y_s^j - \max\limits_{k\in A_j}\{Y_s^k - g_{jk}(s, X_s^{t,x})\}\}dK_s^j = 0. \end{cases} \tag{3.27}$$

Note that the solution of this BSDE depends actually on (t,x) which we will omit for sake of simplicity, as far as there is no confusion. We then have the following result related to existence and uniqueness of the solution of (3.27).

Theorem 3.2. *Assume that Assumption* (A4) (I) (ii)–(iv), (A4) (II) *and* (A4) (III) *are fulfilled. Then system of reflected BSDE with oblique reflection* (3.27) *has a unique solution.*

Proof. The proof follows the same lines as in [7] and [15]. It will be given in two steps.
Step 1 We will first assume that the functions f_i, $i \in A$, verify (A4) (I) (ii)–(v). The other assumptions remain fixed.

Let us introduce the following standard BSDEs:

$$\begin{cases} \bar{Y} \in \mathcal{S}^2, \quad \bar{U} \in \mathcal{H}^2(\ell^2); \\ \bar{Y}_s = \max\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max\limits_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \dots, \bar{Y}_r, \bar{U}_r)dr \\ \qquad - \sum\limits_{i=1}^{\infty} \int_s^T \bar{U}_r^i dH_r^{(i)}, \quad \forall s \le T, \end{cases} \tag{3.28}$$

and

$$\begin{cases} \underline{Y} \in \mathcal{S}^2, \quad \underline{U} \in \mathcal{H}^2(\ell^2); \\ \underline{Y}_s = \min\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \min\limits_{j=1,m} f_j(r, X_r^{t,x}, \underline{Y}_r, \dots, \underline{Y}_r, \underline{U}_r)dr \\ \qquad - \sum\limits_{i=1}^{\infty} \int_s^T \underline{U}_r^i dH_r^{(i)}, \quad \forall s \le T. \end{cases} \tag{3.29}$$

Note that thanks to Theorem 1 in [23], each one of the above BSDEs has a unique solution. Next for $j \in A$ and $n \ge 1$, let us define $(Y^{j,n}, U^{j,n}, K^{j,n})$ by:

$$\begin{cases} Y^{j,n} \in \mathcal{S}^2, \quad U^{j,n} \in \mathcal{H}^2(\ell^2), \ K^{j,n} \in \mathcal{A}^2; \\ Y^{j,0} = \underline{Y} \\ Y_s^{j,n} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^{1,n-1}, \dots, \\ \qquad Y_r^{j-1,n-1}, Y_r^{j,n}, Y_r^{j+1,n-1}, \dots, Y^{m,n-1}, U_r^{j,n})dr \\ \qquad - \sum\limits_{i=1}^{\infty} \int_s^T U_r^{j,n,i} dH_r^{(i)} + K_T^{j,n} - K_s^{j,n}, \quad \forall s \le T; \\ Y_s^{j,n} \ge \max\limits_{k \in A_j} (Y_r^{k,n-1} - g_{jk}(r, X_r^{t,x})), \quad \forall s \le T; \\ \qquad \displaystyle\int_0^T \left[Y_r^{j,n} - \max_{k \in A_j} (Y_r^{k,n-1} - g_{jk}(r, X_r^{t,x})) \right] dK_r^{j,n} = 0. \end{cases} \tag{3.30}$$

By induction we can show that system (3.30) has a unique solution for any fixed $n \ge 1$ since when n is fixed, (3.30) reduces to m decoupled reflected BSDEs of the form (3.1). On the other hand it is easily seen that $(\bar{Y}, \bar{U}, 0)$ is also a solution of:

$$\begin{cases} \bar{Y}_s = \max\limits_{j=1,m} h_j(X_T^{t,x}) + \int_s^T \max\limits_{j=1,m} f_j(r, X_r^{t,x}, \bar{Y}_r, \dots, \bar{Y}_r, \bar{U}_r)dr \\ \qquad - \sum\limits_{i=1}^{\infty} \int_s^T \bar{U}_r^i dH_r^{(i)} + \bar{K}_T - \bar{K}_s, \forall s \le T; \\ \bar{Y}_s \ge \max\limits_{k \in A_j} (\bar{Y}_s - g_{jk}(s, X_s^{t,x})), \quad \forall s \le T; \\ \qquad \int_0^T \left[\bar{Y}_r - \max\limits_{k \in A_j} (\bar{Y}_s - g_{jk}(s, X_s^{t,x})) \right] d\bar{K}_r = 0. \end{cases}$$

Next since for any $i \in A$, f_i verifies Assumption A4(I)(ii)-(v), by Proposition 3.4 and an induction argument, we get that $\mathbf{P}$-a.s. for any j, n and $s \le T$, $Y_s^{j,n-1} \le Y_s^{j,n} \le \bar{Y}_s$. Then the sequence $(Y^{j,n})_{n \ge 0}$, has a limit which we denote by Y^j, for any $j \in A$. By the monotonic limit theorem in [13], $Y^j \in \mathcal{S}^2$ and there exist $U^j \in \mathcal{H}^2(\ell^2)$ and K^j a non-decreasing process of $\mathcal{S}^2$ such that: $\forall s \le T$,

$$\begin{cases} Y_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{Y_r}, U_r^j)dr - \sum_{i=1}^{\infty} \int_s^T U_r^{j,i} dH_r^{(i)} + K_T^j - K_s^j, \\ Y_s^j \geq \max_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \end{cases} \tag{3.31}$$

where for any $j \in A$, U^j is the weak limit of $(U^{j,n})_{n\geq 1}$ in $\mathcal{H}^2(\ell^2)$ and for any stopping time τ, K_τ^j is the weak limit of $K_\tau^{j,n}$ in $L^2(\Omega, \mathcal{F}_\tau, \mathbf{P})$. Finally note that K^j is predictable since the processes $K^{n,j}$ are so, for any $n \geq 1$.

Let us now consider the following RBSDE:

$$\begin{cases} \hat{Y}^j \in \mathcal{S}^2, \quad \hat{U}^j \in \mathcal{H}^2(\ell^2), \quad \hat{K}^j \in \mathcal{S}^2, \text{ non-decreasing and } \hat{K}_0^j = 0; \\ \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, Y_r^1, \ldots, Y_r^{j-1}, \hat{Y}_r^j, Y_r^{j+1}, \ldots, Y_r^m, \hat{U}_r^j)dr \\ \qquad - \sum_{i=1}^{\infty} \int_s^T \hat{U}_r^{j,i} dH_r^{(i)} + \hat{K}_T^j - \hat{K}_s^j, \forall s \leq T; \\ \hat{Y}_s^j \geq \max_{k \in A_j}(Y_s^k - g_{jk}(s, X_s^{t,x})), \quad \forall s \leq T; \\ \qquad \int_0^T \left[\hat{Y}_{r-}^j - \max_{k \in A_j}(Y_{r-}^k - g_{jk}(r, X_{r-}^{t,x})) \right] d\hat{K}_r^j = 0. \end{cases} \tag{3.32}$$

According to Theorem 3.3 in [1], this equation has a unique solution. By Tanaka-Meyer's formula (see e.g.[25], Theorem 68, pp. 216), for all $j \in A$:

$$\begin{aligned} (\hat{Y}_T^j - Y_T^j)^+ = (\hat{Y}_s^j - Y_s^j)^+ &+ \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{Y}_r^j - Y_r^j) \\ &+ \sum_{s<r\leq T} \left[\mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{Y}_r^j - Y_r^j)^- + \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j \leq 0\}} (\hat{Y}_r^j - Y_r^j)^+ \right] \\ &+ \frac{1}{2} L_t^0(\hat{Y}^j - Y^j) \end{aligned}$$

where the process $(L_t^0(\hat{Y}^j - Y^j))_{t\leq T}$ is the local time of the semi martingale $(\hat{Y}_s^j - Y_s^j)_{0\leq s\leq T}$ at 0 which is a nonnegative process. Then we have

$$\begin{aligned} (\hat{Y}_T^j - Y_T^j)^+ &\geq (\hat{Y}_s^j - Y_s^j)^+ + \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{Y}_r^j - Y_r^j) \\ &= (\hat{Y}_s^j - Y_s^j)^+ - \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} \\ &\quad \times \left[f_j(r, X_r^{t,x}, Y_r^1, \ldots, Y_r^{j-1}, \hat{Y}_r^j, Y_r^{j+1} \ldots, Y_r^m, \hat{U}_r^j) \right. \\ &\quad \left. - f_j(r, X_r^{t,x}, Y_r^1, \ldots, Y_r^j, \ldots, Y_r^m, U_r^j) \right] dr \\ &\quad - \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} d(\hat{K}_r^j - K_r^j) \\ &\quad + \sum_{i=1}^{\infty} \int_s^T \mathbb{1}_{\{\hat{Y}_{r-}^j - Y_{r-}^j > 0\}} (\hat{U}_r^{j,i} - U_r^{j,i}) dH_r^{(i)}. \end{aligned}$$

First note that by (3.32), $\int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}} d(\hat{K}^j_r - K^j_r) \le 0$. Now by Assumption (A4)(I)(iv), we obtain:

$$\begin{aligned}(\hat{Y}^j_s - Y^j_s)^+ \le \int_s^T & \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}}[f_j(r, X^{t,x}_r, Y^1_r, \dots, \hat{Y}^j_r, \dots, Y^m_r, \hat{U}^j_r) \\ & -f_j(r, X^{t,x}_r, Y^1_r, \dots, Y^j_r, \dots, Y^m_r, \hat{U}^j_r) \\ & +f_j(r, X^{t,x}_r, Y^1_r, \dots, Y^j_r, \dots, Y^m_r, \hat{U}^j_r) \\ & -f_j(r, X^{t,x}_r, Y^1_r, \dots, Y^j_r, \dots, Y^m_r, U^j_r)]dr \\ & -\sum_{i=1}^\infty \int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}}(\hat{U}^{j,i}_r - U^{j,i}_r) dH^{(i)}_r \\ \le \int_s^T & \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}} C(\hat{Y}^j_{r-} - Y^j_{r-})^+ dr \\ & +\sum_{i=1}^\infty \int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}} V_i^{U^j,\hat{U}^j,j}(r)(\hat{U}^{j,i}_r - U^{j,i}_r) dr \\ & -\sum_{i=1}^\infty \int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}}(\hat{U}^{j,i}_r - U^{j,i}_r) dH^{(i)}_r.\end{aligned}$$

Next for $t \le T$, let us set $M_t = \sum_{i=1}^\infty \int_0^t V_i^{U^j,\hat{U}^j,j}(r) dH^{(i)}_r$ and $Z_t = \sum_{i=1}^\infty \int_0^t \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}}(\hat{U}^{j,i}_r - U^{j,i}_r) dH^{(i)}_r$ (M and Z depend on j but this is irrelevant). By Proposition 3.1, $\varepsilon(M) \in \mathcal{S}^2$, $\varepsilon(M) > 0$ and $\mathbf{E}[\varepsilon(M)_T] = 1$. Then using Girsanov's Theorem [25, pp. 136], under the probability measure $d\tilde{\mathbf{P}} := \varepsilon(M)_T d\mathbf{P}$, we obtain that the process $(\tilde{Z}_t = Z_t - < M, Z >_t)_{t \le T}$ is a martingale and then

$$\begin{aligned}\mathbf{E}_{\tilde{\mathbf{P}}} \Bigg[& \sum_{i=1}^\infty \int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}} V_i^{U^j,\hat{U}^j,j}(r)(\hat{U}^{j,i}_r - U^{j,i}_r) dr \\ & -\sum_{i=1}^\infty \int_s^T \mathbb{1}_{\{\hat{Y}^j_{r-}-Y^j_{r-}>0\}}(\hat{U}^{j,i}_r - U^{j,i}_r) dH^{(i)}_r \Bigg] \\ & = -\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{Z}_T - \tilde{Z}_s) = 0.\end{aligned}$$

Thus for any $s \le T$,

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\hat{Y}^j_s - Y^j_s)^+ \le \mathbf{E}_{\tilde{\mathbf{P}}}\left[\int_s^T C(\hat{Y}^j_r - Y^j_r)^+ dr\right]$$

and finally by Gronwall's Lemma, $\forall j \in A$, $\forall s \le T$, $(\hat{Y}^j_s - Y^j_s)^+ = 0$ $\tilde{\mathbf{P}} - a.s.$ and then also $\mathbf{P} - a.s.$ since those probabilities are equivalent. It implies that $\mathbf{P}$-a.s., $\hat{Y}^j \le Y^j$ for any $j \in A$. On the other hand, since $\forall n \ge 1$, $\forall j \in A$, $Y^{j,n-1} \le Y^j$, then we have

$$\forall s \le T, \quad \max_{k \in A_j}(Y^{k,n-1}_s - g_{jk}(s, X^{t,x}_s)) \le \max_{k \in A_j}(Y^k_s - g_{jk}(s, X^{t,x}_s)).$$

Therefore by comparison, we obtain $Y^{j,n} \leq \hat{Y}^j$, and then $Y^j \leq \hat{Y}^j$ which implies $Y^j = \hat{Y}^j$, $\forall j \in A$.

Next by Itô's formula applied to $(Y^j - \hat{Y}^j)^2$ we obtain: $\forall s \in [0,T]$,

$$(Y^j_s - \hat{Y}^j_s)^2 = (Y^j_0 - \hat{Y}^j_0)^2 + 2\int_0^s (Y^j_{r-} - \hat{Y}^j_{r-}) d(Y^j_r - \hat{Y}^j_r)$$
$$+ \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \int_0^s (U^{j,i}_r - \hat{U}^{j,i}_r)(U^{j,k}_r - \hat{U}^{j,k}_r) d[H^{(i)}, H^{(k)}]_r.$$

As $Y^j = \hat{Y}^j$ and taking expectation in both-hand sides of the previous equality to obtain

$$\mathbf{E}\left[\int_0^T \sum_{i\geq 1} (U^{j,i}_r - \hat{U}^{j,i}_r)^2 dr\right] = 0.$$

It implies that $U^j = \hat{U}^j$, $dt \otimes d\mathbf{P}$ and finally $K^j = \hat{K}^j$ for any $j \in A$, i.e. $(Y^j, U^j, K^j)_{j\in A}$ verify (3.32).

Next we will show that the predictable process K^j does not have jumps. First note that since K^j is predictable then its jumping times are also predictable. So assume there exist $j_1 \in A$ and a predictable stopping time τ such that $\Delta K^{j_1}_\tau = \Delta \hat{K}^{j_1}_\tau > 0$. As Y^j verifies (3.32) and since the martingale part in this latter equation has only inaccessible jump times then $\Delta Y^{j_1}_\tau = -\Delta K^{j_1}_\tau = -\Delta \hat{K}^{j_1}_\tau < 0$. By the second equality in (3.32) we have

$$Y^{j_1}_{\tau-} = \max_{k\in A_{j_1}} (Y^k_{\tau-} - g_{j_1 k}(\tau, X^{t,x}_{\tau-})). \tag{3.33}$$

Now let $j_2 \in A_{j_1}$ be the optimal index in (3.33), i.e.,

$$Y^{j_2}_{\tau-} - g_{j_1,j_2}(\tau, X^{t,x}_\tau) = Y^{j_1}_{\tau-} > Y^{j_1}_\tau \geq Y^{j_2}_\tau - g_{j_1,j_2}(\tau, X^{t,x}_\tau).$$

Note that $g_{j_1,j_2}(\tau, X^{t,x}_{\tau-}) = g_{j_1,j_2}(\tau, X^{t,x}_\tau)$ since the stopping time τ is predictable, and the process $(X^{t,x}_s)_{t\leq s\leq T}$ does not have predictable jump times. Thus $\Delta Y^{j_2}_\tau < 0$ and once more we have,

$$Y^{j_2}_{\tau-} = \max_{k\in A_{j_2}} (Y^k_{\tau-} - g_{j_2 k}(\tau, X^{t,x}_{\tau-})). \tag{3.34}$$

We can now repeat the same argument as many times as necessary, to deduce the existence of a loop $\ell_1, \ldots, \ell_{p-1}, \ell_p = \ell_1$ $(p \geq 2)$ and $\ell_2 \neq \ell_1$ such that

$$Y^{\ell_1}_{\tau-} = Y^{\ell_2}_{\tau-} - g_{\ell_1\ell_2}(\tau, X^{t,x}_{\tau-}), \ldots, Y^{\ell_{p-1}}_{\tau-} = Y^{\ell_p}_{\tau-} - g_{\ell_{p-1}\ell_p}(\tau, X^{t,x}_{\tau-})$$

which implies that

$$g_{\ell_1\ell_2}(\tau, X^{t,x}_{\tau-}) + \cdots + g_{\ell_{p-1}\ell_p}(\tau, X^{t,x}_{\tau-}) = 0$$

which is contradictory with Assumption (A4) (II). It implies that $\Delta K^{j_1}_\tau = 0$ and then K^{j_1} is continuous since it is predictable. As j is arbitrary in A, then the processes K^j are continuous and taking into account (3.32), we deduce that the triples $(Y^j, U^j, K^j)_{j\in A}$, is a solution for system (3.27). □

Step 2 We now deal with the general case i.e. we assume that f_i, $i \in A$, do no longer satisfy the monotonicity assumption (A4) (I) (v) but (A4) (I) (ii)–(iv) solely.

Let $j \in A$ and $t_0 \in [0,T]$ be fixed. We should stress here that we do not need to take $t_0 = t$ since the result is valid for general stochastic process and not only of Markovian type as $X^{t,x}$. For $a \in \mathcal{A}^j_{t_0}$ and $\Gamma := ((\Gamma^l_s)_{s\in[0,T]})_{l\in A} \in [H^2]^m := H^2 \times \cdots \times H^2$ (m times), we introduce the unique solution of the switched BSDE which is defined by: $\forall s \in [t_0, T]$,

$$V^a_s = h_{a(T)}(X^{t,x}_T) + \int_s^T f_{a(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, N^a_r)dr - \sum_{i=1}^{\infty}\int_s^T N^{a,i}_r dH^{(i)}_r - A^a_T + A^a_s \tag{3.35}$$

where $V^a \in \mathcal{S}^2$ and $N^a \in \mathcal{H}^2(\ell^2)$ ($\overrightarrow{\Gamma}_r = (\Gamma^i_r)_{i\in A}$). First note that the solution of this equation exists and is unique since in setting, for $s \in [t_0, T]$, $\tilde{V}^a_s = V^a_s - A^a_s$ and $\tilde{h}^a_T = h_{a(T)}(X^{t,x}_T) - A^a_T$ this equation becomes standard and has a unique solution by Nualart et al.'s result (see [23], Theorem 1, pp. 765). Moreover (see Appendix, Proposition 5.1) we have the following link between the BSDEs (3.27) and (3.35),

$$Y^j_{t_0} = \text{esssup}_{a\in\mathcal{A}^j_{t_0}}(V^a_{t_0} - A^a_{t_0}) = V^{a^*}_{t_0} - A^{a^*}_{t_0} \tag{3.36}$$

for some $a^* \in \mathcal{A}^j_{t_0}$. Next let us introduce the following mapping Θ defined on $[H^2]^m$ by

$$\begin{aligned} \Theta \ &: \ [H^2]^m \to [H^2]^m \\ \Gamma = (\Gamma^j)_{j\in A} &\mapsto (Y^j)_{j\in A} \end{aligned} \tag{3.37}$$

where $(Y^j, U^j, K^j)_{j\in A}$ is the unique solution of the following system of RBSDEs:

$$\begin{cases} Y^j_s = h_j(X^{t,x}_T) + \int_s^T f_j(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^j_r)dr; \\ \qquad - \sum\limits_{i=1}^{\infty}\int_s^T U^{j,i}_r dH^{(i)}_r + K^j_T - K^j_s, \quad \forall s \le T. \\ Y^j_s \ge \max\limits_{k\in A_j}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\}, \quad \forall s \le T; \\ \qquad \int_0^T [Y^j_s - \max\limits_{k\in A_j}\{Y^k_s - g_{jk}(s, X^{t,x}_s)\}]dK^j_s = 0. \end{cases} \tag{3.38}$$

By the result proved in Step 1, Θ is well-defined. Next for $\eta \in H^2$ let us define $\|\cdot\|_{2,\beta}$ by

$$\|\eta\|_{2,\beta} := \left(\mathbf{E}\left[\int_0^T e^{\beta s}|\eta_s|^2 ds\right]\right)^{\frac{1}{2}},$$

which is a norm of H^2, equivalent to $\|.\|$ and $(H^2, \|\cdot\|_{2,\beta})$ is a Banach space. Let now Γ^1 and Γ^2 be two processes of $[H^2]^m$ and for $k = 1, 2$, let $(Y^{k,j}, U^{k,j}, K^{k,j})_{j\in A} = \Theta(\Gamma^k)$, i.e., that satisfy: $\forall s \le T$,

$$\begin{cases} Y_s^{k,j} = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{\Gamma^k_r}, U_r^{k,j})dr; \\ \qquad\qquad - \sum_{i=1}^{\infty} \int_s^T U_r^{k,j,i} dH_r^{(i)} + K_T^{k,j} - K_s^{k,j} \\ Y_s^{k,j} \geq \max_{q \in A_j} \{Y_s^{k,q} - g_{jq}(s, X_s^{t,x})\}; \\ \qquad\qquad \int_0^T \left[Y_s^{k,j} - \max_{q \in A_j} \{Y_s^{k,q} - g_{jq}(s, X_s^{t,x})\} \right] dK_s^{k,j} = 0. \end{cases}$$

Next let us define $(\hat{Y}^j)_{j \in A}$ through the following system of reflected BSDEs with oblique reflection: $\forall s \leq T$,

$$\begin{cases} \hat{Y}_s^j = h_j(X_T^{t,x}) + \int_s^T f_j(r, X_r^{t,x}, \overrightarrow{\Gamma^1_r}, \hat{U}_r^j) \vee f_j(r, X_r^{t,x}, \overrightarrow{\Gamma^2_r}, \hat{U}_r^j)dr; \\ \qquad - \sum_{i=1}^{\infty} \int_s^T \hat{U}_r^{j,i} dH_r^{(i)} + \hat{K}_T^j - \hat{K}_s^j \\ \hat{Y}_s^j \geq \max_{q \in A_j} \{\hat{Y}_s^q - g_{jq}(s, X_s^{t,x})\};\ \int_0^T \left[\hat{Y}_s^j - \max_{q \in A_j} \{\hat{Y}_s^q - g_{jq}(s, X_s^{t,x})\} \right] d\hat{K}_s^j = 0. \end{cases}$$

Recall once more that $a \in \mathcal{A}_{t_0}^j$ and let us define $V^{k,a}$, $k = 1, 2$, and $\hat{V}^a$, via BSDEs, by

$$\hat{V}_s^a = h_{a(T)}(X_T^{t,x}) + \int_s^T f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma^1_r}, \hat{N}_r^a) \vee f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma^2_r}, \hat{N}_r^a)dr$$
$$- \sum_{i=1}^{\infty} \int_s^T \hat{N}_r^{a,i} dH_r^{(i)} - A_T^a + A_s^a, \quad s \leq T,$$

and for $k = 1, 2$,

$$V_s^{k,a} = h_{a(T)}(X_T^{t,x}) + \int_s^T f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma^k_r}, N_r^{k,a})dr - A_T^a + A_s^a$$
$$- \sum_{i=1}^{\infty} \int_s^T N_r^{k,a,i} dH_r^{(i)}, \quad s \leq T.$$

By Proposition 5.1 in Appendix, we have:

$$\begin{aligned} Y_{t_0}^{k,j} &= \mathrm{esssup}_{a \in \mathcal{A}_{t_0}^j} (V_{t_0}^{k,a} - A_{t_0}^a), \quad k = 1, 2 \text{ and} \\ \hat{Y}_{t_0}^j &= \mathrm{esssup}_{a \in \mathcal{A}_{t_0}^j} (\hat{V}_{t_0}^a - A_{t_0}^a) := \hat{V}_{t_0}^{a^*} - A_{t_0}^{a^*}. \end{aligned} \tag{3.39}$$

In addition for $s \in [t_0, T]$, $f_{a(s)}$ verifies the inequality (3.24) of Assumption (A4) (I) (iv). Actually let us set $a_s = \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s)$, $s \in [t_0, T]$, and let $U^1, U^2 \in \mathcal{H}^2(l^2)$, $X, Y \in \mathcal{S}^2$. For any $s \in [t_0, T]$ we have:

$$\begin{aligned} & f_{a(s)}(s, X_s, Y_s, U_s^1) - f_{a(s)}(s, X_s, Y_s, U_s^2) \\ &\quad = [f_{\alpha_0}(s, X_s, Y_s, U_s^1) - f_{\alpha_0}(s, X_s, Y_s, U_s^2)] \mathbb{1}_{\{\theta_0 \leq s \leq \theta_1\}} \\ &\qquad + \sum_{j \geq 2} [f_{\alpha_{j-1}}(s, X_s, Y_s, U_s^1) - f_{\alpha_{j-1}}(s, X_s, Y_s, U_s^2)] \mathbb{1}_{]\theta_{j-1}, \theta_j]}(s) \\ &\quad \geq \langle V^{U^1, U^2, \alpha_0}, (U^1 - U^2) \rangle_s^p \mathbb{1}_{\{\theta_0 \leq s \leq \theta_1\}} \end{aligned}$$

$$+\sum_{j\geq 2}\langle V^{U^1,U^2,\alpha_{j-1}},(U^1-U^2)\rangle_s^p \mathbb{1}_{]\theta_{j-1},\theta_j]}(s)$$
$$=: \langle V^{U^1,U^2,a},(U^1-U^2)\rangle_s^p.$$

where for any $s\in[t_0,T]$,

$$\begin{aligned} V_s^{U^1,U^2,a} &:= (V_s^{U^1,U^2,a,i})_{i\geq 1}\\ &= V_s^{U^1,U^2,\alpha_0}\mathbb{1}_{\{\theta_0\leq s\leq\theta_1\}}\\ &\quad +\sum_{j\geq 2}V_s^{U^1,U^2,\alpha_{j-1}}\mathbb{1}_{]\theta_{j-1},\theta_j]}(s). \end{aligned}$$

But on $[t_0,T]\times\Omega$,

$$\begin{aligned} &\mathbf{P}\left\{\omega, \exists s\leq T,\text{ such that }\sum_{i=1}^{\infty}V_s^{U^1,U^2,a,i}(\omega)p_i(\Delta L_s(\omega))\leq -1\right\}\\ &\leq\sum_{j\in A}\mathbf{P}\{\omega,\exists s\leq T,\text{ such that }\sum_{i=1}^{\infty}V_s^{U^1,U^2,j,i}(\omega)p_i(\Delta L_s(\omega))\leq -1\}=0 \end{aligned}$$

which implies that

$$\mathbf{P}-a.s., \forall s\in[t_0,T],\quad \sum_{i=1}^{\infty}V_s^{U^1,U^2,a,i}(\omega)p_i(\Delta L_s(\omega))>-1.$$

On the other hand, on $[t_0,T]\times\Omega$,

$$\sum_{i=1}^{\infty}|V_s^{U^1,U^2,a,i}|^2\leq\sum_{\ell\in A}\sum_{i=1}^{\infty}|V_s^{U^1,U^2,\ell,i}|^2\leq C,\ ds\otimes d\mathbf{P}-a.e.$$

Thus the process $V^{U^1,U^2,a}$ verifies Assumption (A2) and $f_{a(s)}$ satisfies Assumption (A4) (I) (iv) on $[t_0,T]$.

Consequently, by the comparison result of Proposition 3.3, for any strategy $a\in\mathcal{A}_{t_0}^j$, $\mathbf{P}$-a.s. for any $s\in[t_0,T]$, $\hat{V}_s^a\geq V_s^{1,a}\vee V_s^{2,a}$. This combined with (3.39) leads to $Y_{t_0}^{1,j}\vee Y_{t_0}^{2,j}\leq\hat{Y}_{t_0}^j=\hat{V}_{t_0}^{a^*}-A_{t_0}^{a^*}$. We then deduce

$$V_{t_0}^{1,a^*}-A_{t_0}^{a^*}\leq Y_{t_0}^{1,j}\leq\hat{V}_{t_0}^{a^*}-A_{t_0}^{a^*}\quad\text{and}\quad V_{t_0}^{2,a^*}-A_{t_0}^{a^*}\leq Y_{t_0}^{2,j}\leq\hat{V}_{t_0}^{a^*}-A_{t_0}^{a^*}$$

which implies

$$|Y_{t_0}^{1,j}-Y_{t_0}^{2,j}|\leq|\hat{V}_{t_0}^{a^*}-V_{t_0}^{1,a^*}|+|\hat{V}_{t_0}^{a^*}-V_{t_0}^{2,a^*}|. \tag{3.40}$$

Next we first estimate the quantity $|\hat{V}_{t_0}^{a^*}-V_{t_0}^{1,a^*}|$. For $s\in[t_0,T]$ let us set $\Delta V_s^{a^*}:=\hat{V}_s^{a^*}-V_s^{1,a^*}$ and $\Delta N_s^{a^*}:=\hat{N}_s^{a^*}-N_s^{1,a^*}$. Applying Itô's Formula to the process $e^{\beta s}|\Delta V_s^{a^*}|^2$ we obtain: $\forall s\in[t_0,T]$,

$$\begin{aligned} &e^{\beta s}|\Delta V_s^{a^*}|^2+\int_s^T e^{\beta r}\|\Delta N_r^{a^*}\|^2dr\\ &\quad=-\int_s^T\beta e^{\beta r}|\Delta V_{r-}^{a^*}|^2dr-2\sum_{i=1}^{\infty}\int_s^T e^{\beta r}\Delta V_{r-}^{a^*}\Delta N_r^{a^*,i}dH_r^{(i)} \end{aligned}$$

$$+2\int_s^T e^{\beta r}\Delta V_{r-}^{a^*}[f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^2_r},\hat{N}_r^{a^*})$$
$$-f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{1,a^*})]dr$$
$$-\sum_{i=1}^\infty\sum_{l=1}^\infty\int_s^T e^{\beta r}\Delta N_r^{a^*,i}\Delta N_r^{a^*,l}d([H^{(i)},H^{(l)}]_r-\langle H^{(i)},H^{(l)}\rangle_r).$$

By the Lipschitz property of f_j, $j\in A$, and then of f_{a*} and the fact that for any $x,y\in\mathbb{R}$, $|x\vee y-y|\le|x-y|$ we have: $\forall s\in[t_0,T]$,

$$\begin{aligned}
&|f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^2_r},\hat{N}_r^{a^*})-f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{1,a^*})|\\
&\quad\le|f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{a^*})\vee f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^2_r},\hat{N}_r^{a^*})-f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{a^*})|\\
&\qquad+|f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{a^*})-f_{a^*(r)}(r,X_r^{t,x},\overrightarrow{\Gamma^1_r},\hat{N}_r^{1,a^*})|\\
&\quad\le L(|\overrightarrow{\Gamma^1_r}-\overrightarrow{\Gamma^2_r}|+\|\hat{N}_r^{a^*}-N_r^{1,a^*}\|).
\end{aligned}\tag{3.41}$$

The inequality $2xy\le\frac{1}{\beta}x^2+\beta y^2$ (for any $\beta>0$ and $x,y\in\mathbb{R}$) and (3.41) yield: $\forall s\in[t_0,T]$,

$$\begin{aligned}
e^{\beta s}|\Delta V_s^{a^*}|^2\le&-\int_s^T e^{\beta r}\|\Delta N_r^{a^*}\|^2dr-\int_s^T\beta e^{\beta r}|\Delta V_{r-}^{a^*}|^2ds\\
&-2\sum_{i=1}^\infty\int_s^T e^{\beta r}\Delta V_{r-}^{a^*}\Delta N_r^{a^*,i}dH_r^{(i)}\\
&+2L\int_s^T e^{\beta r}|\Delta V_{r-}^{a^*}|(|\overrightarrow{\Gamma^1_r}-\overrightarrow{\Gamma^2_r}|+\|\Delta N_r^{a^*}\|)dr\\
&-\sum_{i=1}^\infty\sum_{l=1}^\infty\int_s^T e^{\beta r}\Delta N_r^{a^*,i}\Delta N_r^{a^*,l}d([H^{(i)},H^{(l)}]_r-\langle H^{(i)},H^{(l)}\rangle_r)\\
\le&-\int_s^T e^{\beta r}\|\Delta N_r^{a^*}\|^2dr-\int_s^T\beta e^{\beta r}|\Delta V_{r-}^{a^*}|^2ds\\
&-2\sum_{i=1}^\infty\int_s^T e^{\beta r}\Delta V_{r-}^{a^*}\Delta N_r^{a^*,i}dH_r^{(i)}\\
&+\int_s^T\beta e^{\beta r}|\Delta V_{r-}^{a^*}|^2ds+\frac{L^2}{\beta}\int_s^T e^{\beta r}(|\overrightarrow{\Gamma^1_r}-\overrightarrow{\Gamma^2_r}|+\|\Delta N_r^{a^*}\|)^2dr\\
&-\sum_{i=1}^\infty\sum_{l=1}^\infty\int_s^T e^{\beta r}\Delta N_r^{a^*,i}\Delta N_r^{a^*,l}d([H^{(i)},H^{(l)}]_r-\langle H^{(i)},H^{(l)}\rangle_r)\\
\le&\frac{2L^2}{\beta}\int_s^T e^{\beta r}|\overrightarrow{\Gamma^1_r}-\overrightarrow{\Gamma^2_r}|^2dr-2\sum_{i=1}^\infty\int_s^T e^{\beta r}\Delta V_{r-}^{a^*}N_r^{a^*,i}dH_r^{(i)}\\
&-\sum_{i=1}^\infty\sum_{l=1}^\infty\int_s^T e^{\beta r}\Delta N_r^{a^*,i}\Delta N_r^{a^*,l}d([H^{(i)},H^{(l)}]_r-\langle H^{(i)},H^{(l)}\rangle_r),
\end{aligned}$$

for $\beta \geq 2L^2$. We deduce, in taking expectation,

$$\forall s \in [t_0, T], \quad \mathbf{E}[e^{\beta s}|\Delta \hat{V}_s^{a^*}|^2] \leq \frac{2L^2}{\beta}\mathbf{E}\left[\int_s^T e^{\beta r}|\overrightarrow{\Gamma_r^1} - \overrightarrow{\Gamma_r^2}|^2 dr\right].$$

Similarly, we get also $\forall s \in [t_0, T]$,

$$\mathbf{E}[e^{\beta s}|\hat{V}_s^{a^*} - V_s^{2,a^*}|^2] \leq \frac{2L^2}{\beta}\mathbf{E}\left[\int_s^T e^{\beta r}|\overrightarrow{\Gamma_r^1} - \overrightarrow{\Gamma_r^2}|^2 dr\right].$$

Therefore by (3.40) we obtain:

$$\mathbf{E}[e^{\beta t_0}|Y_{t_0}^{1,j} - Y_{t_0}^{2,j}|^2] \leq \frac{8L^2}{\beta}\|\Gamma^1 - \Gamma^2\|_{2,\beta}^2. \tag{3.42}$$

As t_0 is arbitrary in $[0, T]$ then by integration w.r.t. t_0 we get

$$\|\Theta(\Gamma^1) - \Theta(\Gamma^2)\|_{2,\beta} \leq \sqrt{\frac{8L^2 Tm}{\beta}}\|\Gamma^1 - \Gamma^2\|_{2,\beta}. \tag{3.43}$$

Henceforth for β large enough, Θ is contraction on the Banach space $([H^2]^m, \|.\|_{2,\beta})$, then it has a fixed point $(Y^j)_{j\in A}$ which has a version which is the unique solution of system of RBSDE (3.27). □

Remark 3.4. As a consequence of (3.42), there exists a constant $C > 0$, such that $\forall j \in A$, $s \leq T$,

$$\mathbf{E}[|Y_s^{1,j} - Y_s^{2,j}|^2] \leq C\|(Y^{1,j})_{j\in A} - (Y^{2,j})_{j\in A}\|_{2,\beta}^2. \tag{3.44}$$

This estimate will be useful later. □

Corollary 3.1. *Under Assumptions (A4), there exist deterministic lower semi-continuous functions $(u^j(t,x))_{j\in A}$ of polynomial growth such that*

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \forall s \in [t,T],\ Y_s^j = u^j(s, X_s^{t,x}), \quad \forall j \in A.$$

Proof. This is a direct consequence of the construction by induction of the solution $(Y^j, U^j, K^j)_{j\in A}$ given in Step 1. Actually by Ren et al.'s result [26], there exist deterministic continuous functions of polynomial growth $\bar{u}(t,x)$, $\underline{u}(t,x)$ and $u^{j,n}(t,x)$, $n \geq 0$ and $j \in A$, such that $\forall (t,x) \in [0,T]\times\mathbb{R}$, $\forall s \in [t,T]$

(a)

$$\bar{Y}_s = \bar{u}(s, X_s^{t,x}) \quad \text{and} \quad \underline{Y}_s = \underline{u}(s, X_s^{t,x}).$$

(b)

$$Y_s^{j,n} = u^{j,n}(s, X_s^{t,x}), \quad \forall j \in A,$$

and

$$\underline{Y} \leq Y^{j,n} \leq Y^{j,n+1} \leq \bar{Y}.$$

This yields, for any $n \geq 0$ and $(t,x) \in [0,T]\times\mathbb{R}$,

$$\underline{u}(t,x) \leq u^n(t,x) \leq u^{n+1}(t,x) \leq \bar{u}(t,x).$$

Thus $u_j(t,x) := \lim_{n\to\infty} u^{j,n}(t,x)$, $j \in A$, verify the required properties since $(Y^{j,n})_n$ converges to Y^j, $j \in A$, in $\mathcal{S}^2$. □

We now give a comparison result for solutions of systems (3.27). The induction argument allows to compare the solution of the approximating schemes, by Proposition 3.3, and then to deduce the same property for the limiting processes.

Remark 3.5. Let $(\bar{Y}^j, \bar{U}^j, \bar{K}^j)_{j\in A}$ be a solution of the system of RBSDEs (3.27) associated with $((\bar{f}_j)_{j\in A}, (\bar{g}_{jk})_{j,k\in A}, (\bar{h}_j)_{j\in A})$ which satisfy (**A4**). If for any $j, k \in A$,

$$f_j \le \bar{f}_j, \quad h_j \le \bar{h}_j, \quad g_{jk} \ge \bar{g}_{jk}$$

then for any $j \in A$, $Y^j \le \bar{Y}^j$. □

4. Existence and uniqueness of the solution for the system of IPDEs with inter-connected obstacles

This section focuses on the main result of this paper which is the proof of existence and uniqueness of a solution for the system of IPDEs introduced in the beginning of this paper (1.1). For this objective we use its link with the system of RBSDEs (3.27). However we are led to make, hereafter, the following additional assumption because, basically, the hypothesis (A4)-(iv) is either artificial in this deterministic setting or not easy to verify.
Assumption (A5): For any $i \in A$, f_i does not depend on the variable $\zeta \in \ell^2$. □

So we are going to consider the following system of IPDEs: $\forall i \in A$,

$$\begin{cases} min\{u_i(t,x) - \max_{j\in A_i}(u_j(t,x) - g_{ij}(t,x)); \\ \quad -\partial_t u_i(t,x) - \mathcal{L}u_i(t,x) - f_i(t,x,u_1(t,x),\dots,u_m(t,x))\} \\ \qquad = 0, \ (t,x) \in [0,T]\times\mathbb{R}; \\ u_i(T,x) = h_i(x) \end{cases} \tag{4.1}$$

where

$$\mathcal{L}u(t,x) = \mathcal{L}^1 u(t,x) + \mathcal{I}(t,x,u)$$

with

$$\mathcal{L}^1 u(t,x) := (\mathbf{E}[L_1]\sigma(t,x) + b(t,x))\partial_x u(t,x) + \frac{1}{2}\sigma(t,x)^2\varpi^2 D^2_{xx}u(t,x) \quad \text{and}$$

$$\mathcal{I}(t,x,u) := \int_{\mathbb{R}} [u(t,x+\sigma(t,x)y) - u(t,x) - \partial_x u(t,x)\sigma(t,x)y]\Pi(dy). \tag{4.2}$$

Note that for any $\phi \in \mathcal{C}^{1,2}_p$ and $(t,x) \in [0,T]\times\mathbb{R}$, the non-local term

$$\mathcal{I}(t,x,\phi) := \int_{\mathbb{R}} [\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y]\Pi(dy) \tag{4.3}$$

is well-defined. Actually let $\delta > 0$ and let us define, for any $q \in \mathbb{R}$,

$$\mathcal{I}^{1,\delta}(t,x,\phi) := \int_{|y|\le\delta} [\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y]\Pi(dy), \tag{4.4}$$

$$\mathcal{I}^{2,\delta}(t,x,q,u) := \int_{|y|>\delta} [u(t,x+\sigma(t,x)y) - u(t,x) - q\sigma(t,x)y]\Pi(dy). \quad (4.5)$$

By Taylor's expansion we have

$$\begin{aligned}&\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y\\ &\quad = \int_0^y \sigma(t,x)^2 D^2_{xx}\phi(t,x+\sigma(t,x)r)(y-r)dr.\end{aligned}$$

But there exists a constant C_{tx} such that for any $|r| \le \delta$, $|D^2_{xx}\phi(t,x+\sigma(t,x)r)| \le C_{tx}$ since ϕ belongs to $\mathcal{C}^{1,2}$ and σ is bounded. Therefore for $|y| \le \delta$,

$$|\phi(t,x+\sigma(t,x)y) - \phi(t,x) - \partial_x\phi(t,x)\sigma(t,x)y| \le C_{tx}|y|^2$$

which implies that $\mathcal{I}^{1,\delta}(t,x,\phi) \in \mathbb{R}$. Next for any (t,x), $\mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),\phi) \in \mathbb{R}$ since Π integrates any power function outside $[-\epsilon,\epsilon]$. Henceforth $\mathcal{I}(t,x,\phi)$ is well-defined. □

We are now going to give the definition of a viscosity solution of (4.1). First for a locally bounded function u: $(t,x) \in [0,T] \times \mathbb{R} \to u(t,x) \in \mathbb{R}$, we define its lower semi-continuous (lsc for short) envelope u_* and upper semi-continuous (usc for short) envelope u^* as following:

$$u_*(t,x) = \underline{\lim}_{(t',x')\to(t,x),\ t'<T} u(t',x'), \quad u^*(t,x) = \overline{\lim}_{(t',x')\to(t,x),\ \ t'<T} u(t',x')$$

Definition 4.1. *A function $(u_1,\ldots,u_m) : [0,T]\times\mathbb{R} \to \mathbb{R}^m$ which belongs to Π_g such that for any $i \in A$, u_i is usc (resp. lsc), is said to be a viscosity subsolution (resp. supersolution) of (4.1) if for any $i \in A$, $\varphi \in \mathcal{C}^{1,2}_p$, $u_i(T,x) \le h_i(x)$ (resp. $u_i(T,x) \ge h_i(x)$) and if $(t_0,x_0) \in (0,T)\times\mathbb{R}$ is a global maximum (resp. minimum) point of $u_i - \varphi$,*

$$\begin{aligned}min\Big\{&u_i(t_0,x_0) - \max_{j\in A_i}\{u_j(t_0,x_0) - g_{ij}(t_0,x_0)\};\ -\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0)\\ &-f_i(t_0,x_0,u_1(t_0,x_0),\ldots,u_{i-1}(t_0,x_0),u_i(t_0,x_0),\ldots,u_m(t_0,x_0))\Big\}\\ &\le 0 (resp. \ge 0).\end{aligned}$$

The function $(u_i)_{i=1}^m$ is called a viscosity solution of (4.1) if $(u_{i*})_{i=1}^m$ and $(u_i^*)_{i=1}^m$ are respectively viscosity supersolution and subsolution of (4.1).

The following result is needed later.

Lemma 4.1. *Let $(u_i)_{i=1}^m$ be a supersolution of (4.1) which belongs to Π_g, i.e. for some $\gamma > 0$ and $C > 0$,*

$$|u_i(t,x)| \le C(1+|x|^\gamma),\ \forall (t,x) \in [0,T]\times\mathbb{R} \text{ and } i \in A.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$ and $\theta > 0$, $\overrightarrow{v}(t,x) = (u_i(t,x) + \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))_{i=1}^m$ is supersolution of (4.1).

Proof. As usual wlog we assume that the functions $(u_i)_{i=1,m}$ are *lsc* and we use Definition 4.1. Let $i \in A$ be fixed and $\varphi^i \in \mathcal{C}^{1,2}_p$ such that $\varphi^i(s,y) -$

$(u_i(s,y)+\theta e^{-\lambda s}(1+|y|^{2\gamma+2}))$ has a global maximum in $(t,x)\in(0,T)\times\mathbb{R}$ and $\varphi^i(t,x)=u_i(t,x)+\theta e^{-\lambda t}(1+|x|^{2\gamma+2})$. By Definition 4.1 we have:

$$
\begin{aligned}
min\Big\{ & u_i(t,x)+\theta e^{-\lambda t}(1+|x|^{2\gamma+2})-\max_{j\in A_i}(-g_{ij}(t,x)+(u_j(t,x)\\
& +\theta e^{-\lambda t}(1+|x|^{2\gamma+2})));\\
& -\partial_t(\varphi^i(t,x)-\theta e^{-\lambda t}(1+|x|^{2\gamma+2}))-\frac{1}{2}\sigma(t,x)^2\\
\varpi^2 & D^2_{xx}(\varphi^i(t,x)-\theta e^{-\lambda t}(1+|x|^{2\gamma+2}))-(\sigma(t,x)E(L_1)+b(t,x))D_x(\varphi^i(t,x)\\
& -\theta e^{-\lambda t}(1+|x|^{2\gamma+2}))-\int_{\mathbb{R}}[\varphi^i(t,x+\sigma(t,x)y)\\
& -\theta e^{-\lambda t}|x+\sigma(t,x)y|^{2\gamma+2}-(\varphi^i(t,x)-\theta e^{-\lambda t}|x|^{2\gamma+2})\\
& -D_x(\varphi^i(t,x)-\theta e^{-\lambda t}|x|^{2\gamma+2})\sigma(t,x)y]\Pi(dy)-f_i(t,x,\overrightarrow{u})\Big\}\geq 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
-\partial_t & \varphi^i(t,x)-\mathcal{L}\varphi^i(t,x)-f_i(t,x,\overrightarrow{v}(t,x))\\
& \geq \theta\lambda e^{-\lambda t}(1+|x|^{2\gamma+2})-\frac{1}{2}\theta e^{-\lambda t}\sigma(t,x)^2\varpi^2D^2_{xx}|x|^{2\gamma+2}\\
& \quad -(\sigma(t,x)\mathbf{E}(L_1)+b(t,x))D_x(\theta e^{-\lambda t}|x|^{2\gamma+2})\\
& \quad -\int_{\mathbb{R}}(\theta e^{-\lambda t}|x+\sigma(t,x)y|^{2\gamma+2}-\theta e^{-\lambda t}|x|^{2\gamma+2}-\theta e^{-\lambda t}D_x|x|^{2\gamma+2}\sigma(t,x)y)\Pi(dy)\\
& \quad + f_i(t,x,\overrightarrow{u}(t,x))-f_i(t,x,\overrightarrow{v}(t,x))\\
& \geq \theta e^{-\lambda t}\Big\{\lambda(1+|x|^{2\gamma+2})-\frac{1}{2}\sigma(t,x)^2\varpi^2D^2_{xx}|x|^{2\gamma+2}\\
& \quad -(\sigma(t,x)E(L_1)+b(t,x))D_x|x|^{2\gamma+2}\\
& \quad -\int_{\mathbb{R}}(|x+\sigma(t,x)y|^{2\gamma+2}-|x|^{2\gamma+2}-D_x|x|^{2\gamma+2}\sigma(t,x)y)\Pi(dy)\\
& \quad +\sum_{k=1}^{m}C^{k,i}_{t,x,\theta,\lambda}(1+|x|^{2\gamma+2})\Big\}
\end{aligned}
\tag{4.6}
$$

where $C^{k,i}_{t,x,\theta,\lambda}$ is bounded by the Lipschiz constant of f_i with respect to $(y^i)_{i=1,\dots,m}$ which is independent of θ. But, since $\phi(y)=|y|^{2\gamma+2}\in\mathcal{C}^{1,2}_p$, then the non-local term is well-defined. Now let us set $\psi(\rho):=\phi(x+\rho\sigma(t,x)y)$, for $\rho,x,y\in\mathbb{R}$. First note that for any t,x,y we have

$$
\begin{aligned}
|x+\sigma(t,x)y|^{2\gamma+2}-|x|^{2\gamma+2}-D_x|x|^{2\gamma+2}\sigma(t,x)y| & =|\psi(1)-\psi(0)-D_\rho\psi(0)|\\
=\left|\int_0^1(1-\rho)\psi^{(2)}(\rho)d\rho\right| &\\
\leq C|y|^2(|x|^{2\gamma}+|y|^{2\gamma}). &
\end{aligned}
$$

Therefore by (2.3) we have

$$
\int_{\mathbb{R}}||x+\sigma(t,x)y|^{2\gamma+2}-|x|^{2\gamma+2}-D_x|x|^{2\gamma+2}\sigma(t,x)y|\Pi(dy)
$$

$$\leq C\int_{\mathbb{R}}|y|^2(|x|^{2\gamma}+|y|^{2\gamma})\Pi(dy)$$
$$\leq C(1+|x|^{2\gamma}).$$

It follows that there exists a constant $\lambda_0 \in \mathbb{R}^+$ which does not depend on θ such that if $\lambda \geq \lambda_0$ then the right-hand side of (4.6) is non-negative for any $i \in A$. Thus $\vec{v}$ is a viscosity supersolution of (4.1), which is the desired result. □

Remark 4.1. In the same way one can show that if $(u_i)_{i=1}^m$ is a viscosity subsolution of (4.1) which belongs to Π_g, i.e. for some $\gamma > 0$ and $C > 0$,

$$|u_i(t,x)| \leq C(1+|x|^\gamma), \quad \forall (t,x) \in [0,T]\times\mathbb{R} \text{ and } i \in A.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $\theta > 0$, $\overrightarrow{v}(t,x) = (u_i(t,x) - \theta e^{-\lambda t}(1+|x|^{2\gamma+2}))_{i=1}^m$ is subsolution of (4.1). □

4.1. Existence of the viscosity solution of system (4.1)

In this section we deal with the issue of existence of the viscosity solution of (4.1). Recall that $(Y^j, U^j, K^j)_{j\in A}$ is the unique solution of (3.27) and let $(u_j(t,x))_{j\in A}$ be the functions defined in Corollary 3.1.

Theorem 4.1. *Assume Assumptions* (**A4**) *and* (**A5**) *and* (3.15), (3.16) *as well, then* $(u_j(t,x))_{j\in A}$ *is a viscosity solution of* (4.1).

Proof. The proof will be divided into two steps.
Step 1 We first show that $(u_j)_{j=1}^m$ is a supersolution of (4.1). We will use Definition 4.1. Note that for all $j \in A$, as u_j is lsc, we then have $u_{j_*} = u_j$. Next let us set $u_j^n(t,x) = Y_t^{j,n,t,x}$, where $(Y^{j,n,t,x}; U^{j,n,t,x}, K^{j,n,t,x})_{j\in A}$ is the unique solution of (3.30). As pointed out in Corollary 3.1, for any $n \geq 0$, $(t,x) \in [0,T]\times\mathbb{R}$ and $s \in [t,T]$,

$$Y_s^{j,n,t,x} = u_j^n(s, X_s^{t,x}) \quad \text{and} \quad u_j^n(t,x) \nearrow u_j(t,x).$$

Additionally by induction for any $n \geq 0$, $(u_j^n)_{j\in A}$, are continuous, belong to Π_g and by Ren et al.'s result ([26], Theorem 5.8) verify in viscosity sense the following system $(n \geq 1)$: $\forall j \in A$,

$$\begin{cases} min\Big\{u_j^n(t,x) - \max_{k\in A_j}(u_j^{n-1}(t,x) - g_{jk}(t,x)); \\ -\partial_t u_j^n(t,x) - \mathcal{L}u_j^n(t,x) - f_j(t,x,(u_1^{n-1},\ldots, \\ \qquad u_{j-1}^{n-1}, u_j^n, u_{j+1}^{n-1},\ldots,u_m^{n-1})(t,x))\Big\} = 0; \\ u_j^n(T,x) = h_j(x). \end{cases} \tag{4.7}$$

First note that for any $j \in A$, u_j verifies

$$u_j(T,x) = h_j(x) \quad \text{and} \quad u_j(t,x) \geq \max_{k\in A_j}\{u_k(t,x) - g_{jk}(t,x)\},$$
$$\forall (t,x) \in [0,T]\times\mathbb{R}.$$

Now let $j \in A$, $(t,x) \in (0,T)\times\mathbb{R}$ and ϕ a function which belongs to $\mathcal{C}_p^{1,2}$ such that $u_j - \phi$ has a global minimum in (t,x) on $[0,T]\times\mathbb{R}$ (wlog we assume it strict and that $u_j(t,x) = \phi(t,x)$). Next let $\delta > 0$ and for $n \geq 0$ let (t_n, x_n)

be the global minimum of $u_j^n - \phi$ on $[0,T] \times B'(x, 2\delta C_\sigma)$ (C_σ is the constant of boundedness of the diffusion coefficient σ which appears in (3.16) and B' stands for the closure of the ball B). Therefore

$$(t_n, x_n) \to_n (t,x) \quad \text{and} \quad u_j^n(t_n, x_n) \to_n u(t,x).$$

Actually let us consider a convergent subsequence of (t_n, x_n), which we still denote by (t_n, x_n), and let set (t^*, x^*) its limit. Then

$$u_j^n(t_n, x_n) - \phi(t_n, x_n) \le u_j^n(t,x) - \phi(t,x). \tag{4.8}$$

Taking the limit wrt n and since $u_{j*} = u_j$ is lsc to obtain

$$u_j(t^*, x^*) - \phi(t^*, x^*) \le u_j(t,x) - \phi(t,x).$$

As the minimum (t,x) of $u_j - \phi$ on $[0,T] \times \mathbb{R}$ is strict then $(t^*, x^*) = (t,x)$. It follows that the sequence $((t_n, x_n))_n$ converges to (t,x). Going back now to (4.8) and sending n to infinity to obtain

$$u_{j*}(t,x) = u_j(t,x) \le \liminf_n u_j^n(t_n, x_n) \le \limsup_n u_j^n(t_n, x_n) \le u_j(t,x)$$

which implies that $u_j^n(t_n, x_n) \to_n u_j(t,x)$.

Now for n large enough $(t_n, x_n) \in (0,T) \times B(x, 2C_\sigma\delta)$ and it is the global minimum of $u_j^n - \phi$ in $[0,T] \times B(x_n, C_\sigma\delta)$. As u_j^n is a supersolution of (4.7), then by Definition 5.1 in Appendix we have

$$\begin{aligned} &-\partial_t\phi(t_n, x_n) - \mathcal{L}^1\phi(t_n, x_n) - \mathcal{I}^{1,\delta}(t_n, x_n, \phi) \ge \mathcal{I}^{2,\delta}(t_n, x_n, D_x\phi(t_n, x_n), u_j^n) \\ &+f_j(t_n, x_n, u_1^{n-1}(t_n, x_n), \ldots, u_{j-1}^{n-1}(t_n, x_n), u_j^n(t_n, x_n), \\ &\quad u_{j+1}^{n-1}(t_n, x_n), \ldots, u_m^{n-1}(t_n, x_n)). \end{aligned} \tag{4.9}$$

But there exists a subsequence of $\{n\}$ such that:

(i) for any $k \in A_j$, $(u_k^{n-1}(t_n, x_n))_n$ is convergent and then $\lim_n u_k^{n-1}(t_n, x_n) \ge u_{k*}(t,x) = u_k(t,x)$;

(ii) $(\mathcal{I}^{1,\delta}(t_n, x_n, \phi))_n \to_n \mathcal{I}^{1,\delta}(t,x,\phi)$.

Next by Fatou's Lemma and since $u_{j*} = u_j$ and $u_j \ge \phi$ we have

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{I}^{2,\delta}(t_n, x_n, D_x\phi(t_n, x_n), u_j^n) &\ge \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x), u_j) \\ &\ge \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x), \phi). \end{aligned} \tag{4.10}$$

Taking the lim inf wrt to n (through the previous subsequence) in each hand-side of (4.9), using the fact that f_j is continuous and verifies (A4) (I) (v) and finally by (4.10) to obtain:

$$\begin{aligned} &-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) - \mathcal{I}^{1,\delta}(t,x,\phi) \\ &\quad \ge \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u_j) + f_j(t,x,u_1(t,x),\ldots, \\ &\qquad u_{j-1}(t,x), u_j(t,x), u_{j+1}(t,x), \ldots, u_m(t,x)). \end{aligned}$$

As $u_j \ge \phi$ and since $\mathcal{I}(\ldots) = \mathcal{I}^{1,\delta}(\ldots) + \mathcal{I}^{2,\delta}(\ldots)$ we then obtain from the previous inequality,

$$\begin{aligned} &-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) \\ &\quad \ge \mathcal{I}(t,x,\phi) + f_j(t,x,u_1(t,x),\ldots,u_{j-1}(t,x),u_j(t,x),u_{j+1}(t,x),\ldots,u_m(t,x)) \end{aligned}$$

which means that u_j is a viscosity supersolution of

$$\begin{cases} min\{u_j(t,x) - \max_{k\in A_j}(u_k(t,x) - g_{jk}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - f_j(t,x,u_1(t,x),\ldots,u_m(t,x))\} = 0; \\ u_j(T,x) = h_j(x). \end{cases}$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity supersolution of (4.1). $\square$

Step 2 We will now show that $(u_j^*)_{j\in A}$ is a subsolution of (4.1). As a first step we are going to show that

$$\forall j \in A, \quad min\{u_j^*(T,x) - h_j(x); \quad u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} = 0.$$

By definition of u_j^* and since $u_j^n \nearrow u_j$, we have

$$min\{u_j^*(T,x) - h_j(x); \quad u_j^*(T,x) - \max_{k\in A_j}(u_k^*(T,x) - g_{jk}(T,x))\} \geq 0$$

Next suppose that for some $x_0 \in \mathbb{R}$, $\exists j > 0$, s.t.

$$min\{u_j^*(T,x_0) - h_j(x_0); \quad u_j^*(T,x_0) - \max_{k\in A_j}(u_k^*(T,x_0) - g_{jk}(T,x_0))\} = 2\epsilon.$$

We will show that leads to a contradiction. Let $(t_k,x_k)_{k\geq 1} \to (T,x_0)$ and $u_j(t_k,x_k) \to u_j^*(T,x_0)$. We can find a sequence of functions $(v^n)_{n\geq 0} \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ of compact support such that $v^n \to u_j^*$, since u_j^* is usc. On some neighborhood B_n of (T,x_0) we have,

$$\begin{aligned} \forall (t,x) \in B_n, \quad & min\{v^n(t,x) - h_j(x); \\ & v^n(t,x) - \max_{k\in A_j}(u_k^*(t,x) - g_{jk}(t,x))\} \geq \epsilon. \end{aligned} \tag{4.11}$$

Let us denote by $B_k^n := [t_k,T]\times B(x_k,\delta_n^k)$, for some $\delta_n^k \in]0,1]$ small enough such that $B_k^n \subset B_n$. Since u_j^* is of polynomial growth, there exists $c>0$, such that $|u_j^*| \leq c$ on B_n. We can then assume $v^n \geq -2c$ on B_n. Define

$$V_k^n(t,x) := v^n(t,x) + \frac{4c|x-x_k|^2}{{\delta_k^n}^2} + \sqrt{T-t}$$

Note that $V_k^n(t,x) \geq v^n(t,x)$ and

$$(u_j^* - V_k^n)(t,x) \leq -c \; \forall (t,x) \in [t_k,T]\times\partial B(x_k,\delta_k^n). \tag{4.12}$$

On the other hand, an easy calculation yields

$$\begin{aligned} & -\{\partial_t V_k^n(t,x) + \mathcal{L}V_k^n(t,x)\} \\ & \quad = -\Big\{\partial_t v^n(t,x) + \partial_t((T-t)^{\frac{1}{2}}) + \{\mathbf{E}(L_1)\sigma(t,x) + b(t,x)\} \\ & \qquad \times \left\{\partial_x v^n(t,x) + \frac{8c(x-x_k)}{(\delta_k^n)^2}\right\} \\ & \qquad + \frac{1}{2}\sigma(t,x)^2\varpi^2\left(D_{xx}^2 v^n(t,x) + \frac{8c}{(\delta_k^n)^2}\right) \\ & \qquad + \int_{\mathbb{R}}[v^n(t,x+\sigma(t,x)y) - v^n(t,x) - \partial_x v^n(t,x)\sigma(t,x)y]\Pi(dy) \end{aligned}$$

$$+\int_{\mathbb{R}}\left[\frac{4c|x-x_k+\sigma(t,x)y|^2}{(\delta_k^n)^2}-\frac{4c|x-x_k|^2}{(\delta_k^n)^2}-\frac{8c(x-x_k)}{(\delta_k^n)^2}\sigma(t,x)y\right]\Pi(dy)\Big\}.$$

Note that $\Phi(x) := \frac{4c|x-x_k|^2}{(\delta_k^n)^2} \in C^2 \cap \Pi_g$ and $v^n \in \mathcal{C}^{1,2}$ and of compact support, then the two non-local terms are bounded and $\partial_t v^n$, $\partial_x v^n$, $D^2_{xx} v^n$ are so. Since $\partial_t(\sqrt{T-t}) \to -\infty$, when $t \to T$, then we can choose t_k large enough in front of δ_k and the derivatives of v^n to ensure that

$$-\left(\partial_t V_k^n(t,x) + \mathcal{L}V_k^n(t,x)\right) \geq 0, \quad \forall (t,x) \in B_n^k. \tag{4.13}$$

Consider now the stopping time $\theta_n^k := inf\{s \geq t_k, (s, X_s^{t_k,x_k}) \in B_n^{k^c}\} \wedge T$, where $B_n^{k^c}$ is the complement of B_n^k and $\theta_k := inf\{s \geq t_k, u_j(s, X_s^{t_k,x_k}) = \max\limits_{l\in A_j}(u_l(s, X_s^{t_k,x_k}) - g_{jl}(s, X_s^{t_k,x_k}))\} \wedge T$. Applying Itô's formula with $V_k^n(t,x)$ on $[t_k, \theta_n^k \wedge \theta_k]$ yields:

$$\begin{aligned}
V_k^n(t_k,x_k) &= V_k^n(\theta_n^k \wedge \theta_k, X^{t_k,x_k}_{\theta_n^k\wedge\theta_k}) \\
&\quad - \int_{t_k}^{\theta_n^k\wedge\theta_k} [b(r, X_r^{t_k,x_k})\partial_x V_k^n(r, X_r^{t_k,x_k}) + \partial_t V_k^n(t,x)(r, X_r^{t_k,x_k})]dr \\
&\quad - \int_{t_k}^{\theta_n^k\wedge\theta_k} \sigma(r, X_{r_-}^{t_k,x_k})\partial_x V_k^n(r, X_{r_-}^{t_k,x_k})dL_r \\
&\quad -\frac{1}{2}\int_{t_k}^{\theta_n^k\wedge\theta_k} \sigma^2(r, X_r^{t_k,x_k})\varpi^2\partial^2_{xx}V_k^n(r, X_r^{t_k,x_k})dr \\
&\quad - \sum_{t_k<r\leq\theta_n^k\wedge\theta_k} \{V_k^n(r, X_r^{t_k,x_k}) - V_k^n(r, X_{r_-}^{t_k,x_k}) \\
&\quad -\sigma(r, X_{r_-}^{t_k,x_k})\partial_x V_k^n(r, X_{r_-}^{t_k,x_k})\Delta L_r\}.
\end{aligned} \tag{4.14}$$

Next let us deal with the last term of (4.14) and let us set

$$\begin{aligned}
h(s,y) = V_k^n(s, X_{s-}^{t_k,x_k} + \sigma(s, X_{s-}^{t_k.x_k})y) - V_k^n(s, X_{s-}^{t_k,x_k}) \\
-\partial_x V_k^n(s, X_{s-}^{t_k.x_k})\sigma(s, X_{s-}^{t_k.x_k})y.
\end{aligned}$$

By the mean value theorem we have

$$\begin{aligned}
h(s,y) = \frac{1}{2}\partial^2_{xx}v^n(s, X_{s-}^{t_k,x_k} + \bar{X}\sigma(s, X_{s-}^{t_k.x_k})y)(\sigma(s, X_{s-}^{t_k.x_k})y)^2 \\
+\frac{4c}{\delta_k^{n\,2}}(\sigma(s, X_{s-}^{t_k.x_k})y)^2
\end{aligned}$$

where $\bar{X}$ is a stochastic processes which is valued in $(0,1)$. As v^n is of compact support and σ is bounded then

$$\mathbf{E}\left[\int_0^T\int_{\mathbb{R}} |h(s,y)|\Pi(dy)ds\right] < \infty.$$

It follows that

$$\mathbf{E}\left[\sum_{t_k<r\leq\theta_n^k\wedge\theta_k}\{V_k^n(r,X_r^{t_k,x_k})\right.$$
$$\left.-V_k^n(r,X_{r_-}^{t_k,x_k})-\sigma(r,X_{r_-}^{t_k,x_k})\partial_x V_k^n(r,X_{r_-}^{t_k,x_k})\Delta L_r\}\right]$$
$$=\mathbf{E}\left[\int_{t_k}^{\theta_n^k\wedge\theta_k}\int_{\mathbb{R}}h(s,y)\Pi(dy)ds\right]<\infty.$$

Next going back to (4.14), taking expectation and taking into account of (4.13), (4.12) and (4.11) to obtain

$$\begin{aligned}
V_k^n(t_k,x_k) &= \mathbf{E}\left[V_k^n(\theta_n^k\wedge\theta_k,X_{\theta_n^k\wedge\theta_k}^{t_k,x_k})\right.\\
&\quad\left.-\int_{t_k}^{\theta_n^k\wedge\theta_k}(\partial_t V_k^n(r,X_r^{t_k,x_k})+\mathcal{L}V_k^n(r,X_r^{t_k,x_k}))dr\right]\\
&\geq \mathbf{E}[V_k^n(\theta_n^k,X_{\theta_n^k}^{t_k,x_k})\mathbb{1}_{\{\theta_n^k\leq\theta_k\}}+V_k^n(\theta_k,X_{\theta_k}^{t_k,x_k})\mathbb{1}_{\{\theta_n^k>\theta_k\}}]\\
&= \mathbf{E}[\{V_k^n(\theta_n^k,X_{\theta_n^k}^{t_k,x_k})\mathbb{1}_{\{\theta_n^k<T\}}+V_k^n(T,X_T^{t_k,x_k})\mathbb{1}_{\{\theta_n^k=T\}}\}\mathbb{1}_{\{\theta_n^k\leq\theta_k\}}\\
&\quad+V_k^n(\theta_k,X_{\theta_k}^{t_k,x_k})\mathbb{1}_{\{\theta_n^k>\theta_k\}}]\\
&\geq \mathbf{E}[\{(u_j^*(\theta_n^k,X_{\theta_n^k}^{t_k,x_k})+c)\mathbb{1}_{\{\theta_n^k<T\}}+(\epsilon+h_j(X_T^{t_k,x_k}))\mathbb{1}_{\{\theta_n^k=T\}}\}\mathbb{1}_{\{\theta_n^k\leq\theta_k\}}\\
&\quad+\{\epsilon+\max_{k\in A_j}(u_k^*(\theta_k,X_{\theta_k}^{t_k,x_k})-g_{jk}(\theta_k,X_{\theta_k}^{t_k,x_k}))\}\mathbb{1}_{\{\theta_n^k>\theta_k\}}]\\
&\geq \mathbf{E}[u_j(\theta_n^k\wedge\theta_k,X_{\theta_n^k\wedge\theta_k}^{t_k,x_k})]+c\wedge\epsilon\\
&= \mathbf{E}\left[u_j(t_k,x_k)-\int_{t_k}^{\theta_n^k\wedge\theta_k}f_j(s,X_s^{t_k,x_k},(u_l(s,X_s^{t_k,x_k}))_{l=1,m}ds\right]+c\wedge\epsilon
\end{aligned}$$

since the processes $(Y^j=u_j(.,X.))_{j\in A}$ stopped at time $\theta_n^k\wedge\theta_k$ solves an explicit RBSDE system with triple of data given by $((f_j)_{j\in A},(h_j)_{j\in A},(g_{i,j})_{i,j\in A})$. In addition, $dK^{j,t,x}=0$ on $[t_k,\theta_k]$. On the other hand, $(u_j)_{j\in A}\in\Pi_g$ and then taking into account (3.17) and Assumption (A4) (I) (iii), we deduce that

$$\lim_{k\to\infty}\mathbf{E}\left[\int_{t_k}^{\theta_n^k\wedge\theta_k}f_j(s,X_s^{t_k,x_k},(u_l(s,X_s^{t_k,x_k}))_{l=1,m})ds\right]=0.$$

Taking the limit in the previous inequalities yields:

$$\begin{aligned}
\lim_{k\to\infty}V_k^n(t_k,x_k) &= \lim_{k\to\infty}\{v^n(t_k,x_k)+\sqrt{T-t_k}\}=v^n(T,x_0)\\
&\geq \lim_{k\to\infty}u_j(t_k,x_k)+c\wedge\epsilon=u_j^*(T,x_0)+c\wedge\epsilon.
\end{aligned}$$

As $v^n \to u_j^*$ pointwisely, then we get a contradiction, when taking the limit in the previous inequalities, and the result follows, i.e., $\forall x \in \mathbb{R}$, $\forall j \in A$,

$$min\left\{u_j^*(T,x) - h_j(x); \quad u_j^*(T,x) - \max_{l \in A_j}(u_l^*(T,x) - g_{jl}(T,x))\right\} = 0.$$

Finally the proof of $u_j^*(T,x) = h_j(x), \forall j \in A$, is obtained in the same way as in [15, pp. 180] since the function $(g_{ij})_{i,j \in A}$ verify the non free loop property (A4) (II). □

Now let us show $(u_j^*)_{j \in A}$ is a subsolution of (4.1). First note that since $u_j^n \nearrow u_j$ and u_j^n is continuous, we have

$$u_j^*(t,x) = \limsup_{n\to\infty}{}^* u_j^n(t,x) = \overline{\lim_{n\to\infty, t'\to t, x'\to x}} u_j^n(t',x').$$

Besides $\forall j \in A$ and $n \geq 0$ we deduce from the construction of u_j^n that

$$u_j^n(t,x) \geq \max_{l \in A_j}(u_l^{n-1}(t,x) - g_{jl}(t,x))$$

and by taking the limit in n we obtain: $\forall j \in A$, $\forall x \in \mathbb{R}$,

$$u_j^*(t,x) \geq \max_{l \in A_j}(u_l^*(t,x) - g_{jl}(t,x)).$$

Next fix $j \in A$. Let $(t,x) \in (0,T) \times \mathbb{R}$ be such that

$$u_j^*(t,x) - \max_{l \in A_j}(u_l^*(t,x) - g_{jl}(t,x)) > 0. \tag{4.15}$$

We are going to use once more Definition 4.1. Let $(t,x) \in (0,T) \times \mathbb{R}^k$ and ϕ be a function of $\mathcal{C}_p^{1,2}$ such that $u_j^* - \phi$ has a global maximum at (t,x) on $[0,T] \times \mathbb{R}$ which wlog we assume strict and verifying $u_j^*(t,x) = \phi(t,x)$. Then there exist subsequences $\{n_k\}$ and $((t'_{n_k}, x'_{n_k}))_k$ such that

$$((t'_{n_k}, x'_{n_k}))_k \to_k (t,x) \quad \text{and} \quad u_j^{n_k}(t'_{n_k}, x'_{n_k}) \to_k u_j^*(t,x).$$

Let now $\delta > 0$ and (t_{n_k}, x_{n_k}) be the global maximum of $u_j^{n_k} - \phi$ on $[0,T] \times B'(x, 2\delta C_\sigma)$. Therefore

$$(t_{n_k}, x_{n_k}) \to_k (t,x) \quad \text{and} \quad u_j^{n_k}(t_{n_k}, x_{n_k}) \to_k u_j^*(t,x).$$

Actually let us consider a convergent subsequent of (t_{n_k}, x_{n_k}), which we still denote by (t_{n_k}, x_{n_k}), and let $(\bar{t}, \bar{x})$ be its limit. Then for some k_0 and for $k \geq k_0$ we have

$$u_j^{n_k}(t_{n_k}, x_{n_k}) - \phi(t_{n_k}, x_{n_k}) \geq u_j^{n_k}(t'_{n_k}, x'_{n_k}) - \phi(t'_{n_k}, x'_{n_k}). \tag{4.16}$$

Taking the limit wrt k to obtain

$$u_j^*(\bar{t}, \bar{x}) - \phi(\bar{t}, \bar{x}) \geq u_j^*(t,x) - \phi(t,x).$$

As the maximum (t,x) of $u_j^* - \phi$ on $[0,T] \times \mathbb{R}$ is strict then $(\bar{t}, \bar{x}) = (t,x)$. It follows that the sequence $((t_{n_k}, x_{n_k}))_k$ converges to (t,x). Going back now to (4.16) and taking the limit wrt k we obtain

$$u_j^*(t,x) \geq \limsup_k u_j^{n_k}(t_{n_k},x_{n_k}) \geq \liminf_k u_j^{n_k}(t_{n_k},x_{n_k}) \\ \geq \liminf_k u_j^{n_k}(t'_{n_k},x'_{n_k}) = u_j^*(t,x)$$

which implies that $u_j^{n_k}(t_{n_k},x_{n_k}) \to u_j^*(t,x)$ as $k \to \infty$.
Now for k large enough,

(i) $(t_{n_k},x_{n_k}) \in (0,T) \times B(x, 2\delta C_\sigma)$ and is the global maximum of $u_j^{n_k} - \phi$ in $[0,T] \times B(x_{n_k}, C_\sigma \delta)$;
(ii) $u_j^{n_k}(t_{n_k},x_{n_k}) > \max_{l \in A_j}(u_l^{n_k-1}(t_{n_k},x_{n_k}) - g_{jl}(t_{n_k},x_{n_k}))$.

As $u_j^{n_k}$ is a subsolution of (4.7), then by Definition 5.1 in Appendix we have

$$\begin{aligned} &-\partial_t\phi(t_{n_k},x_{n_k}) - \mathcal{L}^1\phi(t_{n_k},x_{n_k}) \\ &\quad\leq \mathcal{I}^{1,\delta}(t_{n_k},x_{n_k},\phi) \\ &\qquad + \mathcal{I}^{2,\delta}(t_{n_k},x_{n_k},D_x\phi(t_{n_k},x_{n_k}),u_j^{n_k}) \\ &\qquad + f_j(t_{n_k},x_{n_k},u_1^{n_k-1}(t_{n_k},x_{n_k}), \\ &\qquad \dots, u_{j-1}^{n_k-1}(t_{n_k},x_{n_k}), u_j^{n_k}(t_{n_k},x_{n_k}), u_{j+1}^{n_k-1}(t_{n_k},x_{n_k}), \dots, u_m^{n_k-1}(t_{n_k},x_{n_k})). \end{aligned} \tag{4.17}$$

But there exists a subsequence of $\{n_k\}$ (which we still denote by $\{n_k\}$) such that:

(i) for any $l \in A_j$, $(u_l^{n_k-1}(t_{n_k},x_{n_k}))_k$ is convergent and then $\lim_k u_l^{n_k-1}(t_{n_k},x_{n_k}) \leq u_l^*(t,x)$;
(ii) $(\mathcal{I}^{1,\delta}(t_{n_k},x_{n_k},\phi))_{n_k} \to_k \mathcal{I}^{1,\delta}(t,x,\phi)$;
(iii) $\limsup_k \mathcal{I}^{2,\delta}(t_{n_k},x_{n_k},D_x\phi(t_{n_k},x_{n_k}),u_j^{n_k}) \leq \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u_j^*)$.

Point (i) is due to the fact that u_l^n belongs uniformly to Π_g; (ii) is just the Lebesgue dominated convergence Theorem ; (iii) stems from an adaptation of Fatou's Lemma, definition of u_j^* and finally monotonicity of $\mathcal{I}^{2,\delta}$.

Going back now to (4.17) and taking the limit superior wrt k (through the previous subsequence), using the fact that f_j is continuous and verifies (A4)(I)(v) to obtain

$$\begin{aligned} &-\partial_t\phi(t,x) - \mathcal{L}^1\phi(t,x) \\ &\quad \leq \mathcal{I}^{1,\delta}(t,x,\phi) + \mathcal{I}^{2,\delta}(t,x,D_x\phi(t,x),u_j^*) + f_j(t,x,u_1^*(t,x),\dots, \\ &\qquad u_{j-1}^*(t,x), u_j^*(t,x), u_{j+1}^*(t,x), \dots, u_m^*(t,x)) \\ &\quad \leq \mathcal{I}(t,x,D_x\phi(t,x),\phi) + f_j(t,x,u_1^*(t,x),\dots, \\ &\qquad u_{j-1}^*(t,x), u_j^*(t,x), u_{j+1}^*(t,x), \dots, u_m^*(t,x)). \end{aligned}$$

This last inequality is due to that fact that $u_j^* \leq \phi$ and since $\mathcal{I}^{1,\delta} + \mathcal{I}^{2,\delta} = \mathcal{I}$. Finally combining it with (4.15) we obtain that u_j is a viscosity subsolution of

$$\begin{cases} min\{u_j(t,x) - \max_{k\in A_j}(u_k^*(t,x) - g_{jk}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - f_j(t,x,u_1^*(t,x),\dots, \\ \qquad u_{j-1}^*(t,x), u_j(t,x), u_{j+1}^*(t,x),\dots,u_m^*(t,x))\} = 0; \\ u_j(T,x) = h_j(x). \end{cases}$$

As j is arbitrary then $(u_j)_{j\in A}$ is a viscosity subsolution of (4.1). □

4.2. Uniqueness of the viscosity solution of system (4.1)

We now give a comparison result of subsolution and supersolution of system (4.1), from which we get the continuity and uniqueness of its solution.

Proposition 4.1. *Assume Assumptions* **(A4)** *fulfilled. Let* $(u_j)_{j\in A}$ *(resp.* $(w_j)_{j\in A}$*) be a subsolution (resp. supersolution) of* (4.1) *which belongs to* Π_g*. Then for any* $j \in A$*,*

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad u_j(t,x) \le w_j(t,x)$$

Proof. Let γ be a real constant such that for any $j \in A$ and $(t,x) \in [0,T]\times\mathbb{R}$,

$$|u_j(t,x)| + |w_j(t,x)| \le C(1+|x|^\gamma).$$

To begin with we additionally assume the existence of a constant λ such that $\lambda < -m.\max_{j\in A}\{C_j\}$ (C_j is the Lipschitz constant of f_j w.r.t $\overrightarrow{y}$) and for any $j \in A$ and any $t,x,y_1,\dots,y_{j-1},y_{j+1},\dots,y_m$, $y \ge y'$,

$$\begin{aligned} &f_j(t,x,y_1,\dots,y_{j-1},y,y_{j+1},\dots,y_m) \\ &\quad -f_j(t,x,y_1,\dots,y_{j-1},y',y_{j+1}\dots,y_m) \le \lambda(y-y'). \end{aligned} \tag{4.18}$$

Thanks to Lemma 4.1 and Remark 4.1, we know there exists ν large enough such that for any $\theta > 0$, $w_{j,\theta,\nu}(t,x) = w_j(t,x) + \theta e^{-\nu t}(1+|x|^{2\gamma+2})$ (resp. $u_{j,\theta,\nu}(t,x) = u_j(t,x) - \theta e^{-\nu t}(1+|x|^{2\gamma+2})$) is a supersolution (resp. subsolution). So it is enough to show that

$$\forall j \in A, \quad \forall (t,x) \in [0,T]\times\mathbb{R}, \quad u_{j,\theta,\nu}(t,x) \le w_{j,\theta,\nu}(t,x),$$

then taking limits as $\theta \to 0$, the result follows. By the growth condition there exists a constant $C > 0$ such that

$$\forall j \in A, \forall (t,x) \in [0,T]\times\mathbb{R}, \quad s.t.\ |x| \ge C, \quad u_{j,\theta,\nu}(t,x) < 0 < w_{j,\theta,\nu}(t,x). \tag{4.19}$$

Finally for the sake of simplicity we merely denote $u_{j,\theta,\nu}$ (resp. $w_{j,\theta,\nu}$) by u_j (resp. w_j).

To obtain the comparison result, we proceed by contradiction assuming that

$$\exists (t_1,x_1) \in [0,T]\times\mathbb{R}, \quad \text{such that } \max_{j\in A}(u_j(t_1,x_1) - w_j(t_1,x_1)) > 0.$$

Taking into account the values of the subsolution and the supersolution at T, there exist $(\bar{t},\bar{x}) \in [0,T[\times B(0,C)$ (wlog we assume that $\bar{t} > 0$), such that :

$$\begin{aligned} 0 &< \max_{(t,x)\in[0,T]\times\mathbb{R}} \max_{j\in A}(u_j(t,x) - w_j(t,x)) \\ &= \max_{(t,x)\in[0,T[\times B(0,C)} \max_{j\in A}(u_j(t,x) - w_j(t,x)) = \max_{j\in A}(u_j(\bar{t},\bar{x}) - w_j(\bar{t},\bar{x})). \end{aligned}$$

We now define the set $\underline{\mathrm{A}}$ as follows:

$$\underline{\mathrm{A}} := \{j \in A, u_j(\bar{t}, \bar{x}) - w_j(\bar{t}, \bar{x}) = \max_{k \in A}(u_k(\bar{t}, \bar{x}) - w_k(\bar{t}, \bar{x}))\}. \tag{4.20}$$

By the assumption **(A4)**(II), using the same argument as in [15, pp. 171], we can prove that there exists $j \in \underline{\mathrm{A}}$ such that,

$$u_j(\bar{t}, \bar{x}) > \max_{k \in A_j}(u_k(\bar{t}, \bar{x}) - g_{jk}(\bar{t}, \bar{x})). \tag{4.21}$$

Let us now take such a $j \in \underline{\mathrm{A}}$. For $\varepsilon > 0$ and $\rho > 0$, let us define

$$\Phi^j_{\varepsilon,\rho}(t, x, y) := u_j(t, x) - w_j(t, y) - \frac{|x - y|^2}{\varepsilon} - |t - \bar{t}|^2 - \rho|x - \bar{x}|^4.$$

By (4.19) and since $\lim_{|y|\to\infty} w_j(t, y) = \infty$, $\lim_{|x|\to\infty} u_j(t, x) = -\infty$, there exists a constant C' such that for any $t \in [0, T]$, $u_j(t, x) - w_j(t, y) < 0$ for any $|x| \geq C'$ or $|y| \geq C'$. It follows that for any $\varepsilon > 0$ and $\rho > 0$, there exists (t_0, x_0, y_0) such that

$$\Phi^j_{\varepsilon,\rho}(t_0, x_0, y_0) = \max_{(t,x,y)\in[0,T]\times B'(0,C')^2} \Phi^j_{\varepsilon,\rho}(t, x, y) = \max_{(t,x,y)\in[0,T]\times\mathbb{R}^2} \Phi^j_{\varepsilon,\rho}(t, x, y).$$

Note that the maximum exists since $\Phi^j_{\varepsilon,\rho}$ is *usc* and $B'(0, C')^2$ is the closure of $B(0, C')^2$. On the other hand let us point out that (t_0, x_0, y_0) depends actually on ε and ρ which we omit for sake of simplicity. We then have,

$$\begin{aligned}\Phi^j_{\varepsilon,\rho}(\bar{t}, \bar{x}, \bar{x}) &= u_j(\bar{t}, \bar{x}) - w_j(\bar{t}, \bar{x}) \\ &\leq u_j(\bar{t}, \bar{x}) - w_j(\bar{t}, \bar{x}) + \frac{|x_0 - y_0|^2}{\varepsilon} + |t_0 - \bar{t}|^2 + \rho|x_0 - \bar{x}|^4 \\ &\leq u_j(t_0, x_0) - w_j(t_0, y_0).\end{aligned} \tag{4.22}$$

The growth condition of u_j and w_j implies that
$\varepsilon^{-1}|x_0 - y_0|^2 + |t_0 - \bar{t}|^2 + \rho|x_0 - \bar{x}|^4$ is bounded and then $\lim_{\varepsilon\to 0}(x_0 - y_0) = 0$. Next by (4.22), for any subsequence $(t_{0_l}, x_{0_l}, y_{0_l})_l$ which converges to $(\tilde{t}, \tilde{x}, \tilde{x})$,

$$u_j(\bar{t}, \bar{x}) - w_j(\bar{t},\ \bar{x}) \leq u_j(\tilde{t}, \tilde{x}) - w_j(\tilde{t}, \tilde{x}),$$

since u_j is *usc* and w_j is *lsc*. By the definition of $(\bar{t}, \bar{x})$ this last inequality is an equality. Using both the definition of $\Phi^j_{\varepsilon,\rho}$ and (4.22), it implies that the sequence

$$\lim_{\varepsilon\to 0}(t_0, x_0, y_0) = (\bar{t}, \bar{x}, \bar{x}) \tag{4.23}$$

and once more from (4.22) we deduce

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}|x_0 - y_0|^2 = 0. \tag{4.24}$$

Finally classically (see e.g. [15, pp. 173]) we have also

$$\lim_{\varepsilon\to 0}(u_j(t_0, x_0), w_j(t_0, y_0)) = (u_j(\bar{t}, \bar{x}), w_j(\bar{t}, \bar{x})). \tag{4.25}$$

Next as the functions $(u_k)_{k\in A}$ are *usc* and $(g_{ij})_{i,j\in A}$ are continuous, and since the index j satisfies (4.21), there exists $r > 0$ such that for $(t, x) \in B((\bar{t}, \bar{x}), r)$

we have $u_j(t,x) > \max_{k\in A_j}(u_k(t,x) - g_{jk}(t,x))$. But by (4.25), (4.23) and once more since u_j is *usc* then there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, we have:

$$u_j(t_0,x_0) > \max_{k\in A_j}(u_k(t_0,x_0) - g_{ij}(t_0,x_0)).$$

Now for ε small enough, we are able to apply Jensen–Ishii's lemma for non local operators (see e.g. Barles et al. [5, pp. 583] or Biswas et al. [6], Lemma 4.1, pp. 64) with u_j, w_j and $\phi(t,x,y) := \frac{|x-y|^2}{\varepsilon} + |t-\bar{t}|^2 + \rho|x_0-\bar{x}|^4$ at point (t_0,x_0,y_0). For any $\delta \in (0,1)$ there are p_u^0, q_u^0, p_w^0, q_w^0, M_u^0 and M_w^0 real constants such that:

(i) $$p_u^0 - p_w^0 = \partial_t\phi(t_0,x_0,y_0), \quad q_u^0 = \partial_x\phi(t_0,x_0,y_0), \quad q_w^0 = -\partial_y\phi(t_0,x_0,y_0) \tag{4.26}$$

and

$$\begin{pmatrix} M_u^0 & 0 \\ 0 & -M_w^0 \end{pmatrix} \le \frac{4}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 12\rho|x_0-\bar{x}|^2 & 0 \\ 0 & 0 \end{pmatrix}; \tag{4.27}$$

(ii) $$\begin{aligned} &-p_u^0 - \{\sigma(t_0,x_0)\mathbf{E}(L_1) + b(t_0,x_0)\}q_u^0 - \tfrac{1}{2}\sigma(t_0,x_0)^2\varpi^2 M_u^0 \\ &\quad -f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m) - I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0)) \\ &\quad -I^{2,\delta}(t_0,x_0,q_u^0,u_j) \le 0; \end{aligned} \tag{4.28}$$

(iii) $$\begin{aligned} &-p_w^0 - \{\sigma(t_0,y_0)\mathbf{E}(L_1) + b(t_0,y_0)\}q_w^0 - \tfrac{1}{2}\sigma(t_0,y_0)^2\varpi^2 M_w^0 \\ &\quad -f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m) - I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.)) \\ &\quad -I^{2,\delta}(t_0,y_0,q_w^0,w_j) \ge 0. \end{aligned} \tag{4.29}$$

We are now going to provide estimates for the non-local terms. First let us set $\psi_\rho(t,x) := \rho|x-\bar{x}|^4 + |t-\bar{t}|^2$. By definition of (t_0,x_0,y_0), for any $d,d' \in \mathbb{R}$,

$$\begin{aligned} &u_j(t_0,x_0+d') - w_j(t_0,y_0+d) - \varepsilon^{-1}|x_0+d'-y_0-d|^2 - \psi_\rho(t_0,x_0+d') \\ &\quad \le u_j(t_0,x_0) - w_j(t_0,y_0) - \varepsilon^{-1}|x_0-y_0|^2 - \psi_\rho(t_0,x_0). \end{aligned}$$

Therefore for $z \in \mathbb{R}$, in taking $d' = \sigma(t_0,x_0)z$ and $d = \sigma(t_0,y_0)z$, we obtain

$$\begin{aligned} &u_j(t_0,x_0+\sigma(t_0,x_0)z) - u_j(t_0,x_0) - q_u^0\sigma(t_0,x_0)z \\ &\quad \le w_j(t_0,y_0+\sigma(t_0,y_0)z) - w_j(t_0,y_0) \\ &\qquad -q_w^0\sigma(t_0,y_0)z + \varepsilon^{-1}|\sigma(t_0,x_0)-\sigma(t_0,y_0)|^2 z^2 \\ &\qquad +\psi_\rho(t_0,x_0+\sigma(t_0,x_0)z) - \psi_\rho(t_0,x_0) - D_x\psi_\rho(t_0,x_0)\sigma(t_0,x_0)z. \end{aligned}$$

It implies that for any $\delta > 0$,

$$\begin{aligned} &I^{2,\delta}(t_0,x_0,q_u^0,u_j) - I^{2,\delta}(t_0,y_0,q_w^0,w_j) \\ &\quad \le C\varepsilon^{-1}|x_0-y_0|^2 + I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho) \end{aligned} \tag{4.30}$$

since $\sigma(t,x)$ is uniformly Lipschitz w.r.t. x. But it easy to check that we have

$$|I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)| \le C\rho\int_{|z|\ge\delta}\{|z|^2+|z|^4\}\Pi(dz).$$

On the other hand, since $\phi \in \mathcal{C}^2$

$$\begin{aligned}I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0)) &= \int_{|z|\leq\delta}\{\phi(t_0,x_0+\sigma(t_0,x_0)z,y_0)-\phi(t_0,x_0,y_0)\\&\quad -D_x\phi(t_0,x_0,y_0)\sigma(t_0,x_0)z\}\Pi(dz)\\&\leq \sigma(t_0,x_0)^2\int_{|z|\leq\delta}\{\varepsilon^{-1}+C\rho(1+|z|^2)\}|z|^2\Pi(dz),\end{aligned}$$

and

$$\begin{aligned}I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.)) &= \int_{|z|\leq\delta}\{-\phi(t_0,x_0,y_0+\sigma(t_0,y_0)z)+\phi(t_0,x_0,y_0)\\&\quad +D_y\phi(t_0,x_0,y_0)\sigma(t_0,y_0)z\}\Pi(dz)\\&= -\varepsilon^{-1}\sigma(t_0,y_0)^2\int_{|z|\leq\delta}|z|^2d\Pi(z).\end{aligned}$$

Therefore we have

$$\begin{aligned}&-I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0))+I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.))\\&\quad\geq -\sigma(t_0,x_0)^2\int_{|z|\leq\delta}\{\varepsilon^{-1}+C\rho(1+|z|^2)\}|z|^2\Pi(dz)\\&\qquad -\varepsilon^{-1}\sigma(t_0,y_0)^2\int_{|z|\leq\delta}|z|^2d\Pi(z). \qquad (4.31)\end{aligned}$$

Making now the difference between (4.28) and (4.29) yields

$$\begin{aligned}&-(p_u^0-p_w^0)-[(\sigma(t_0,x_0)\mathbf{E}(L_1)+b(t_0,x_0))q_u^0-(\sigma(t_0,y_0)\mathbf{E}(L_1)+b(t_0,y_0))q_w^0]\\&\quad-\frac{1}{2}\varpi^2[\sigma(t_0,x_0)^2M_u^0-\sigma(t_0,y_0)^2M_w^0]-[f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m)\\&\quad-f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m)]-I^{1,\delta}(t_0,x_0,\phi(t_0,.,y_0))\\&\quad+I^{1,\delta}(t_0,y_0,-\phi(t_0,x_0,.))-I^{2,\delta}(t_0,x_0,q_u^0,u_j)+I^{2,\delta}(t_0,y_0,q_w^0,w_j)\leq 0.\end{aligned}$$

Taking now into account (4.30) and (4.31) we get

$$\begin{aligned}&-(p_u^0-p_w^0)-[(\sigma(t_0,x_0)\mathbf{E}(L_1)+b(t_0,x_0))q_u^0-(\sigma(t_0,y_0)\mathbf{E}(L_1)+b(t_0,y_0))q_w^0]\\&\quad-\frac{1}{2}\varpi^2[\sigma(t_0,x_0)^2M_u^0-\sigma(t_0,y_0)^2M_w^0]-[f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m)\\&\quad-f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m)]-\sigma(t_0,x_0)^2\int_{|z|\leq\delta}\{\varepsilon^{-1}+C\rho(1+|z|^2)\}|z|^2\Pi(dz)\\&\quad-\varepsilon^{-1}\sigma(t_0,y_0)^2\int_{|z|\leq\delta}|z|^2d\Pi(z)\\&\quad-C\varepsilon^{-1}|x_0-y_0|^2-I^{2,\delta}(t_0,x_0,D_x\psi_\rho(t_0,x_0),\psi_\rho)\leq 0.\end{aligned}$$

Next by using the properties satisfied by p_u^0, q_u^0, p_w^0, q_w^0, M_u^0 and M_w^0 and sending δ to 0 to obtain the existence of a constant $C_{\varepsilon,\rho}$ such that for any fixed ρ we have $\limsup_{\varepsilon\to 0} C_{\varepsilon,\rho}\leq 0$ and

$$-\{f_j(t_0,x_0,(u_k(t_0,x_0))_{k=1}^m)-f_j(t_0,y_0,(w_k(t_0,y_0))_{k=1}^m)\}$$
$$\le C_{\varepsilon,\rho}+\rho C\int_{\mathbb{R}}\{|z|^2+|z|^4\}\Pi(dz). \tag{4.32}$$

Next since f_j is Lipschitz w.r.t. $(y_k)_{k=1}^m$ and by condition (4.18) we have

$$-\lambda(u_j(t_0,x_0)-w_j(t_0,y_0))-\sum_{k\in A_j}\Upsilon^{j,k}_{\varepsilon,\rho}(u_k(t_0,x_0)-w_k(t_0,y_0))$$
$$\le C_{\varepsilon,\rho}+C\rho\int_{\mathbb{R}}\{|z|^2+|z|^4\}\Pi(dz),$$

where $\Upsilon^{j,k}_{\varepsilon,\rho}$ stands for the increment rate of f_j with respect to y_k ($k\neq j$), which, by monotonicity condition (A4) (I) (v) on f_j, is non-negative and bounded by C_j. Thus

$$\begin{aligned}-\lambda(u_j(t_0,x_0)-w_j(t_0,y_0))&\le\sum_{k\in A_j}\Upsilon^{j,k}_{\varepsilon,\rho}(u_k(t_0,x_0)-w_k(t_0,y_0))^++C_{\varepsilon,\rho}\\&\quad+C\rho\int_{\mathbb{R}}\{|z|^2+|z|^4\}\Pi(dz)\\&\le C_j\sum_{k\in A_j}(u_k(t_0,x_0)-w_k(t_0,y_0))^++C_{\varepsilon,\rho}\\&\quad+C\rho\int_{\mathbb{R}}\{|z|^2+|z|^4\}\Pi(dz).\end{aligned}$$

Taking the limit superior in both hand-sides as $\varepsilon\to0$, once again u_k (resp. w_k) is *usc* (resp. *lsc*) and $j\in\underline{A}$, we get

$$-\lambda(u_j(\bar t,\bar x)-w_j(\bar t,\bar x))\le C_j\sum_{k\in A_j}(u_k(\bar t,\bar x)-w_k(\bar t,\bar x))^+$$
$$+C\rho\int_{\mathbb{R}}\{|z|^2+|z|^4\}\Pi(dz),$$

finally take $\rho\to0$ to obtain,

$$-\lambda(u_j(\bar t,\bar x)-w_j(\bar t,\bar x))\le C_j\sum_{k\in A_j}(u_k(\bar t,\bar x)-w_k(\bar t,\bar x))^+$$
$$\le(m-1)C_j(u_j(\bar t,\bar x)-w_j(\bar t,\bar x)).$$

But this is contradictory since $u_j(\bar t,\bar x)-w_j(\bar t,\bar x)>0$ and $-\lambda>(m-1)C_j$. Henceforth for any $j\in A$, $u_j\le w_j$.

We now consider the general case. Let $(u_j)_{j\in A}$ (resp. $(w_j)_{j\in A}$) be a subsolution (resp. supersolution) of (4.1). Denote $\tilde u_j(t,x)=e^{\lambda t}u_j(t,x)$ and $\tilde w_j(t,x)=e^{\lambda t}w_j(t,x)$. Then it is easy to show that $(\tilde u_j)_{j\in A}$ (resp. $(\tilde w_j)_{j\in A}$) is a subsolution (resp. supersolution) of the following system of variational inequalities which is similar to (4.1):

$$\begin{cases}min\{\tilde u_j(t,x)-\max\limits_{k\in A_j}(\tilde u_k(t,x)-e^{\lambda t}g_{jk}(t,x));\\-\partial_t\tilde u_j(t,x)-\mathcal{L}\tilde u_j(t,x)+\lambda\tilde u_j(t,x)-e^{\lambda t}f_j(t,x,(e^{-\lambda t}\tilde u_k)_{k=1}^m)\}=0;\\\tilde u_j(T,x)=e^{\lambda T}h_j(x).\end{cases} \tag{4.33}$$

Next let us set

$$F_j(t,x,\overrightarrow{y}) := -\lambda y_j + e^{\lambda t} f_j(t,x,(e^{-\lambda t}y_k)_{k=1}^m)$$

with λ is chosen such that $\lambda \geq m(1+\max_{k\in A} C_k)$ where C_k is the Lipschitz constant of f_k w.r.t. to $(y_k)_{k=1}^m$. Then we can mimic the proof of Step 1 to obtain that $\forall j \in A, \tilde{u}_j \leq \tilde{w}_j$ which yields also $u_j \leq w_j$ for any $j \in A$. The proof is now complete. □

As a by-product we have:

Theorem 4.2. *Under Assumptions* (**A4**), (**A5**), *and* (3.15), (3.16) *as well, the system of variational inequalities with inter-connected obstacles* (4.1) *has a unique continuous viscosity solution with polynomial growth.* □

In the case when the functions f_j, $j \in A$, do not depend on $\vec{y}$, by the characterization (3.35)–(3.36) (see also Remark 5.1), we deduce that the functions $(u_j(t,x))_{j\in A}$ are nothing but $(J^j(t,x))_{j\in A}$. Thus, as a by product of Theorem 4.2, we have:

Corollary 4.1. *Assume that:*

(i) *For any $i = 1,\ldots,m$, f_i is jointly continuous and of polynomial growth ;*
(ii) *For any $i,j \in A$, g_{ij} (resp. h_i) satisfy (A4) (II) (resp. (A4) (III)).*
Then the value functions $(J^j(t,x))_{j\in A}$ defined in (3.26) *are continuous, belong to Π_g and is the unique viscosity solution of the Hamilton–Jacobi–Bellman system associated with the stochastic optimal switching problem which is: $\forall j \in A$,*

$$\begin{cases} min\{u_j(t,x) - \max_{\ell\in A_j}(u_\ell(t,x) - g_{j\ell}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - f_j(t,x)\} = 0,\ (t,x) \in [0,T]\times\mathbb{R}; \\ u_j(T,x) = h_j(x). \end{cases} \tag{4.34}$$

□

4.3. Second existence and uniqueness result

In this section we consider the issue of existence and uniqueness of a solution for the systems of IPDEs (4.1) when the functions $(-f_j)_{j\in A}$ verify (A4) (I). This turns into assuming that $(f_j)_{j\in A}$ verify, instead of (A4) (I) (v), the following:

(A4)(†): For any $j \in A$, for any $k \neq j$, the mapping $y_k \to f_j(t,x,y_1,\ldots,y_{k-1}, y_k,y_{k+1},\ldots,y_m)$ is nonincreasing whenever the other components $(t,x,y_1,\ldots, y_{k-1},y_{k+1},\ldots,y_m)$ are fixed.

The other assumptions on $(-f_j)_{j\in A}$ remain the same.

Theorem 4.3. *Assume that Assumptions (A4) (II)–(III), (A5) are fulfilled and $(-f_j)_{j\in A}$ verify (A4) (I). Then the system of IPDEs* (4.1) *has a continuous and of polynomial growth solution which is moreover unique.*

Proof. We first focus on the issue of existence.

For any $j \in A$ and $\lambda \in \mathbf{R}$ let us define F_j by:

$$F_j(t,x,y^1,\ldots,y^m) := e^{\lambda t} f_j(t,x,e^{-\lambda t}y^1,\ldots,e^{-\lambda t}y^m) - \lambda y^j.$$

Since f_j is uniformly Lipschitz w.r.t. $(y_k)_{k=1,m}$ then F_j is so and for λ large enough, F_j satisfies:

For any $k = 1, m$, the mapping $y_k \to F_j(t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m)$ is **nonincreasing** whenever the other components $(t, x, y_1, \cdots, y_{k-1}, y_{k+1}, \ldots, y_m)$ are fixed.

Let us now consider the following iterative Picard sequence : $\forall j \in A$, $Y^{j,0} = 0$ and for $n \geq 1$, define:

$$(Y^{1,n}, \ldots, Y^{m,n}) = \Theta((Y^{1,n-1}, \ldots, Y^{m,n-1}))$$

where Θ is the mapping defined in (3.37)–(3.38) where f_j is replaced with F_j. By (3.43), the sequence $(Y^{j,n})_{j\in A}$ converges in $([H^2]^m, \|.\|_{2,\beta})$ to the unique solution $(Y^j)_{j\in A}$ of the system of RBSDEs associated with

$$((F_j(s, X_s^{t,x}, y^1, \ldots, y^m))_{j\in A}, \quad (e^{\lambda T} h_j(X_T^{t,x}))_{j\in A}, \quad (e^{\lambda t} g_{jk}(s, X_s^{t,x}))_{j,k\in A}).$$

So using an induction argument on n and Theorem 4.2, there exist deterministic continuous functions with polynomial growth $(u_j^n)_{j\in A}$ such that: for any $n \geq 0$ and $j \in A$,

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad \forall s \in [t,T], \quad Y_s^{j,n} = u_j^n(s, X_s^{t,x}). \tag{4.35}$$

By (3.44), take $s = t$ we obtain

$$\begin{aligned}\forall j, n, q, t \leq T, x \in \mathbb{R}, \quad &|u_j^n(t,x) - u_j^q(t,x)| = \mathbf{E}[|Y_t^{j,n} - Y_t^{j,q}|^2] \\ &\leq C\|(Y^{j,n-1})_{j\in A} - (Y^{j,q-1})_{j\in A}\|_{2,\beta}^2.\end{aligned}$$

Thus for any $j \in A$, $(u_j^n)_{n\geq 0}$ is of Cauchy type and converges pointwisely to a deterministic function u_j. But $(Y^j)_{j\in A} = \Theta((Y^j)_{j\in A})$, then once more by (3.44), we also have:

$$\forall s \in [0,T], \quad \mathbf{E}[|Y_s^j - Y_s^{j,m}|^2] \leq C\|(Y^j)_{j\in A} - (Y^{j,m-1})_{j\in A}\|_{2,\beta}^2. \tag{4.36}$$

By (4.35) we then obtain

$$\forall j \in A, \quad \forall s \in [t,T], \quad \mathbf{P}-a.s., \quad Y_s^j = u_j(s, X_s^{t,x}). \tag{4.37}$$

Next as Θ is a contraction then, by induction on n we have

$$\forall n, q \geq 0, \quad \|(Y^{j,n+q})_{j\in A} - (Y^{j,n})_{j\in A}\|_{2,\beta} \leq \frac{C_\Theta^n}{1 - C_\Theta}\|(Y^{j,1})_{j\in A}\|_{2,\beta}$$

where $C_\Theta \in]0,1[$ is the constant of contraction of Θ. Since the norms $\|.\|$ and $\|.\|_{2,\beta}$ are equivalent, then there exists a constant C_1 such that:

$$\forall n, q \geq 0, \quad \|(Y^{j,n+q})_{j\in A} - (Y^{j,n})_{j\in A}\| \leq C_1 C_\Theta^n \|(Y^{j,1})_{j\in A}\|.$$

Take now the limit as q goes to $+\infty$ and in the view of (4.36) and (4.37), if we take $s = t$ we deduce that :

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad |u_j(t,x) - u_j^n(t,x)| \leq C_2\|(Y^{j,1})_{j\in A}\|.$$

But it is easy to check that $\|(Y^{j,1})_{j\in A}\|(t,x)$ is of polynomial growth (by (3.28)–(3.29) and since $\mathbf{E}[\sup_{s\leq T} |X_s^{t,x}|^\gamma]$ is of polynomial growth for any $\gamma \geq$

0). Therefore for any $j \in A$, u_j is of polynomial growth, i.e., belongs to Π_g since u_j^n is so. We will now show the continuity of u_j. For any $j \in A$, let us set

$$\bar{Y}_s^{j,0} = C(1 + |X_s^{t,x}|^p), \quad s \le T,$$

where C and p are related to polynomial growth of $(u_j)_{j\in A}$, i.e.,

$$\forall j \in A,\ |u_j(t,x)| \le C(1 + |x|^p), \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Next for any $n \ge 1$ and $j \in A$ let us set

$$(\bar{Y}^{1,n}, \dots, \bar{Y}^{m,n}) = \Theta((\bar{Y}^{1,n-1}, \dots, \bar{Y}^{m,n-1})).$$

As Θ is a contraction then once more the sequence $((\bar{Y}^{j,n})_{j\in A})_{n\ge 0}$ converges in $([H^2]^m, \|.\|_{2,\beta})$ to $(Y^{j,t,x})_{j\in A}$ the unique solution of the system of RBSDEs associated with

$$((F_j(s, X_s^{t,x}, y^1, \dots, y^m))_{j\in A}, (e^{\lambda T} h_j(X_T^{t,x}))_{j\in A}, (e^{\lambda t} g_{jk}(s, X_s^{t,x}))_{j,k\in A}).$$

By the definition of $\bar{Y}^{j,0}$, we have

$$\mathbf{P} - a.s., \quad \forall j \in A, s \in [t,T], \quad Y_s^{j,t,x} \le \bar{Y}_s^{j,0}$$

and taking into account of **(A4)(†)** we obtain

$$\forall j \in A, \forall s \in [t,T],\ F_j(s, X_s^{t,x}, Y_s^{1,t,x}, \dots, Y_s^{m,t,x}) \ge F_j(s, X_s^{t,x}, \bar{Y}_s^{1,0}, \dots, \bar{Y}_s^{m,0}).$$

Next by the comparison result of Remark 3.5 and since $(\bar{Y}^{j,1})_{j\in A} = \Theta((\bar{Y}^{j,0})_{j\in A})$, $(Y^{j,t,x})_{j\in A} = \Theta((Y^{j,t,x})_{j\in A})$ we get

$$\forall j \in A, \quad s \in [t,T], \quad \bar{Y}_s^{j,1} \le Y_s^{j,t,x}.$$

Now by an induction argument we obtain, for any $n \ge 0$ and $j \in A$,

$$\forall s \in [t,T], \quad \bar{Y}_s^{j,2n+1} \le Y_s^{j,t,x} \le \bar{Y}_s^{j,2n}. \tag{4.38}$$

In the same way as previously there exist deterministic continuous functions $\bar{u}_j^n$ with polynomial growth such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad s \in [t,T], \quad \bar{Y}_s^{j,n} = \bar{u}_j^n(s, X_s^{t,x}).$$

Moreover for any $j \in A$, the sequence $(\bar{u}_j^n)_n$ converges pointwisely to u and by (4.38) we have

$$\forall j \in A, \quad \forall (t,x), \quad u_j(t,x) = \lim_n \nearrow \bar{u}_j^{2n+1}(t,x) = \lim_n \searrow \bar{u}_j^{2n}(t,x).$$

Therefore, u_j, $j \in A$, is both *lsc* and *usc* and then continuous. Finally as $(Y^{j,t,x})_{j\in A} = \Theta((Y^{j,t,x})_{j\in A})$ and $\forall j \in A,\ Y_s^{j,t,x} = u_j(s, X_s^{t,x})$, $s \in [t,T]$, with u_j a deterministic continuous function with polynomial growth, then $(u_j)_{j\in A}$ is a viscosity solution of the following system of IPDEs:

$$\begin{cases} min\{u_j(t,x) - \max_{\ell\in A_j}(u_\ell(t,x) - e^{\lambda t} g_{j\ell}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}u_j(t,x) - F_j(t,x,u_1(t,x), \dots, u_m(t,x))\} = 0, \\ \qquad (t,x) \in [0,T] \times \mathbb{R}; \\ u_j(T,x) = e^{\lambda T} h_j(x), \end{cases} \tag{4.39}$$

thus $(e^{-\lambda t} u_j)_{j\in A}$ is a viscosity solution of the system of IPDEs (4.1) with polynomial growth.

Let us now deal with the issue of uniqueness. Let $(\bar{u}_j)_{j\in A}$ be another solution of (4.1) which belongs to Π_g and $(\bar{Y}^j)_{j\in A} \in [H^2]^m$ such that for any $j \in A, s \in [t,T]$,

$$\bar{Y}_s^{j,t,x} = \bar{u}_j(s, X_s^{t,x}).$$

Define $(\tilde{Y}^{j,t,x})_{j\in A}$ as follow:

$$(\tilde{Y}^{j,t,x})_{j\in A} = \Theta((\bar{Y}^{j,t,x})_{j\in A}).$$

Then there exist $(\tilde{u}_j)_{j\in A}$ deterministic continuous functions with polynomial growth $(\tilde{u}_j)_{j\in A}$ such that:

$$\forall j \in A, s \in [t,T], \quad \tilde{Y}_s^{j,t,x} = \tilde{u}_j(s, X_s^{t,x}).$$

Moreover $(\tilde{u}_j)_{j\in A}$ is the unique viscosity solution of the following system of IPDEs: $\forall j \in A$

$$\begin{cases} min\{\tilde{u}_j(t,x) - \max_{k\in A_j}(\tilde{u}_k(t,x) - g_{jk}(t,x)); \\ -\partial_t u_j(t,x) - \mathcal{L}\tilde{u}_j(t,x) - f_j(t,x,(\bar{u}_k(t,x))_{k\in A})\} = 0\,; \\ \tilde{u}_j(T,x) = h_j(x). \end{cases} \tag{4.40}$$

Note that it is $(\bar{u}_k(t,x))_{k\in A}$ inside the arguments of f_j and not $(\tilde{u}_k(t,x))_{k\in A}$. As $(\bar{u}_j)_{j\in A}$ is also a solution of (4.40), then by uniqueness of Theorem 4.2 we obtain $\tilde{u}_j = \bar{u}_j$, for any $j \in A$. Therefore

$$(\bar{Y}^{j,t,x})_{j\in A} = \Theta((\bar{Y}^{j,t,x})_{j\in A}).$$

As $(Y^j)_{j\in A}$ is the unique fixed point of Θ in $[H^2]^m$, we then have

$$\forall j \in A, s \in [t,T], \quad \bar{Y}_s^{j,t,x} = Y_s^j.$$

It follows that $\forall j \in A, \bar{u}_j = u_j$. Finally $(u_j(t,x))_{j\in A}$ is the unique continuous with polynomial growth functions viscosity solution of the system of IPDEs (4.1). □

5. Appendix

5.1. Representation of the value function of the stochastic optimal switching problem

Let $\Upsilon := (\theta_n, \alpha_n)_{n\geq 0}$ be an admissible strategy of switching and let $a = (a_s)_{s\in[0,T]}$ be the process defined by

$$\forall s \leq T, \quad a_s := \alpha_0 \mathbb{1}_{\{\theta_0\}}(s) + \sum_{j=1}^{\infty} \alpha_{j-1} \mathbb{1}_{]\theta_{j-1}\theta_j]}(s). \tag{5.1}$$

Let $t_0 \in [0,T]$ and $\Gamma := ((\Gamma_s^j)_{s\in[0,T]})_{j\in A} \in [H^2]^m$. Let us define the pair of processes $(V^a, N^a) := (V_s^a, N_s^a)_{s\in[0,T]}$ as the solution of the following BSDE:

$$\begin{cases} V^a \in \mathcal{S}^2, \quad N^a \in \mathcal{H}^2(l^2) \\ V_s^a = h_{a(T)}(X_T^{t,x}) + \int_s^T \mathbb{1}_{\{r\geq t_0\}} f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, N_r^a)dr \\ \qquad - \sum_{i=1}^{\infty} \int_s^T N_r^{a,i} dH_r^{(i)} - A_T^a + A_s^a,\ s \in [0,T], \end{cases} \tag{5.2}$$

where $\overrightarrow{\Gamma_r} = (\Gamma^k_r)_{k\in A}$ and A^a is the cumulative switching cost associated with the strategy a or Υ [see (3.25) for its definition]. This BSDE is not a standard one, but in assuming that $\mathbf{E}[(A^a_T)^2] < \infty$ and by setting $\bar{V}^a = V^a - A^a$, it becomes a standard one and then it has a unique solution. Note that V^a is RCLL since A^a is so.

Proposition 5.1. *Under Assumption* **(A4)** *(I) (ii)–(iv), (II) and (III), the solution of BSDE* (5.2) *satisfies:* $\forall j \in A$,

$$Y^{\Gamma,j}_{t_0} = esssup_{a\in\mathcal{A}^j_{t_0}}(V^a_{t_0} - A^a_{t_0}), \quad \mathbf{P}-a.s. \tag{5.3}$$

where $(Y^{\Gamma,j})_{j\in A}$ *is the first component of the solution of the BSDE* (3.38). *Thus the solution of* (3.38) *is unique. Moreover there exists* $a^* \in \mathcal{A}^j_{t_0}$ *such that* $Y^{\Gamma,j}_{t_0} = V^{a^*}_{t_0} - A^{a^*}_{t_0}$.

Proof. Let $(Y^{\Gamma,j}, U^{\Gamma,j}, K^{\Gamma,j})_{j\in A}$ be the solution of the system (3.38). Let $a \in \mathcal{A}^j_{t_0}$ and let us define

$$\tilde{K}^a_T = (K^{\Gamma,j}_{\theta_1} - K^{\Gamma,j}_{t_0}) + \sum_{n\geq 1}(K^{\Gamma,\alpha_n}_{\theta_{n+1}} - K^{\Gamma,\alpha_n}_{\theta_n}) \text{ and}$$

$$\forall i \geq 1 \text{ and } r \leq T, U^{a,i}_r = \sum_{n\geq 0} U^{\Gamma,\alpha_n,i}_r \mathbb{1}_{[\theta_n \leq r < \theta_{n+1}[} \text{ and } U^a := (U^{a,i})_{i\geq 1}.$$

Therefore

$$\begin{aligned}
Y^{\Gamma,j}_{t_0} &= Y^{\Gamma,j}_{\theta_1} + \int_{t_0}^{\theta_1} f_j(r, X^{t,x}_r, \overrightarrow{\Gamma_r}, U^{\Gamma,j}_r)dr - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_1} U^{\Gamma,j,i}_r dH^{(i)}_r + (K^{\Gamma,j}_{\theta_1} - K^{\Gamma,j}_{t_0}) \\
&\geq (Y^{\Gamma,\alpha_1}_{\theta_1} - g_{j,\alpha_1}(\theta_1, X^{t,x}_{\theta_1}))1_{[\theta_1<T]} + 1_{[\theta_1=T]}h_{\alpha_0}(X^{t,x}_T) \\
&\quad + \int_{t_0}^{\theta_1} f_{a(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma_r}, U^a_r)dr \\
&\quad - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_1} U^{a,i}_r dH^{(i)}_r + (K^{\Gamma,j}_{\theta_1} - K^{\Gamma,j}_{t_0}) \\
&= Y^{\Gamma,\alpha_1}_{\theta_2}1_{[\theta_1<T]} + \int_{t_0}^{\theta_2} f_{a(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma_r}, U^a_r)dr \\
&\quad - \sum_{i=1}^{\infty}\int_{t_0}^{\theta_2} U^{a,i}_r dH^{(i)}_r + (K^{\Gamma,j}_{\theta_1} - K^{\Gamma,j}_{t_0}) + (K^{\Gamma,\alpha_1}_{\theta_2} - K^{\Gamma,\alpha_1}_{\theta_1}) \\
&\quad - g_{j,\alpha_1}(\theta_1, X^{t,x}_{\theta_1})1_{[\theta_1<T]} + 1_{[\theta_1=T]}h_{\alpha_0}(X^{t,x}_T).
\end{aligned}$$

Repeat now this procedure as many times as necessary and since a is an admissible strategy (i.e. $\mathbf{P}[\theta_n < T, \forall n \geq 0] = 0$) we obtain:

$$\begin{aligned}
Y^{\Gamma,j}_{t_0} &\geq h_{a(T)}(X^{t,x}_T) + \int_{t_0}^{T} f_{a(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma_r}, U^a_r)dr \\
&\quad - \sum_{i=1}^{\infty}\int_{t_0}^{T} U^{a,i}_r dH^{(i)}_r - A^a_T + \tilde{K}^a_T.
\end{aligned} \tag{5.4}$$

As $\tilde{K}_T^a \geq 0$ and by (5.2) we have

$$
\begin{aligned}
Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a &\geq \int_{t_0}^T (f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^a) - f_{a(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, N_r^a))dr \\
&\quad - \sum_{i=1}^{\infty} \int_{t_0}^T (U_r^{a,i} - N_r^{a,i})dH_r^{(i)} \\
&\geq \int_{t_0}^T \langle V^{U^a,N^a,a}, U^a - N^a \rangle_s^p ds - \sum_{i=1}^{\infty} \int_{t_0}^T (U_r^{a,i} - N_r^{a,i})dH_r^{(i)}
\end{aligned}
$$

Next by Girsanov's Theorem [25, pp. 136], under the probability measure $d\tilde{\mathbf{P}} := \varepsilon(\sum_{i=1}^{\infty} \int_{t_0}^{\cdot} V_r^{U^a,N^a,a,i} dH_r^{(i)})_T d\mathbf{P}$, $(M_t := \int_{t_0}^t \langle V^{U^a,N^a,a}, U^a - N^a \rangle_s^p ds - \sum_{i=1}^{\infty} \int_{t_0}^t (U_r^{a,i} - N_r^{a,i}) dH_r^{(i)})_{t \in [t_0,T]}$ is a martingale, and by taking conditional expectation of $Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a$, we obtain

$$
\begin{aligned}
\mathbf{E}_{\tilde{\mathbf{P}}}[Y_{t_0}^{\Gamma,j} - V_{t_0}^a + A_{t_0}^a | \mathcal{F}_{t_0}] \geq \mathbf{E}_{\tilde{\mathbf{P}}} \Bigg[&\int_{t_0}^T \langle V^{U^a,N^a,a,i}, U^a - N^a \rangle_s^p ds \\
&- \sum_{i=1}^{\infty} \int_{t_0}^T (U_r^{a,i} - N_r^{a,i}) dH_r^{(i)} | \mathcal{F}_{t_0} \Bigg] = 0.
\end{aligned}
$$

Thus $Y_{t_0}^{\Gamma,j} \geq V_{t0}^a - A_{t0}^a$, $\tilde{\mathbf{P}} - a.s.$ and then, since $\mathbf{P}$ and $\tilde{\mathbf{P}}$ are equivalent, for any $a \in \mathcal{A}_{t_0}^j$,

$$
Y_{t_0}^{\Gamma,j} \geq V_{t_0}^a - A_{t_0}^a, \ \mathbf{P} - a.s.. \tag{5.5}
$$

Next let us consider a^* the strategy defined by $a^*(r) = \alpha_0^* \mathbb{1}_{\{t_0\}}(r) + \sum_{k=1}^{\infty} \alpha_{k-1}^* \mathbb{1}_{]\theta_{k-1}^* \theta_k^*]}(r)$, $r \leq T$, where $\theta_0^* = t_0$, $\alpha_0^* = j$ and for $n \geq 0$,

$$
\theta_{n+1}^* = \inf \left\{ r \geq \theta_n^*, \ Y_r^{\Gamma,\alpha_n^*} = \max_{k \in A_{\alpha_n^*}} (Y_r^{\Gamma,k} - g_{\alpha_n^*,k}(r, X_r^{t,x})) \right\} \wedge T,
$$

and

$$
\alpha_{n+1}^* = arg \max_{k \in A_{\alpha_n^*}} \left\{ Y_{\theta_{n+1}^*}^{\Gamma,k} - g_{\alpha_n^*,k}(\theta_{n+1}^*, X_{\theta_{n+1}^*}^{t,x}) \right\}.
$$

Let us show that $a^* \in \mathcal{A}_{t_0}^j$. We first prove that $\mathbf{P}[\theta_n^* < T, \ \forall n \geq 0] = 0$. We proceed by contradiction assuming that $\mathbf{P}[\theta_n^* < T, \ \forall n \geq 0] > 0$. By definition of θ_n^*, we then have

$$
\mathbf{P} \left[Y_{\theta_{n+1}^*}^{\Gamma,\alpha_n^*} = Y_{\theta_{n+1}^*}^{\Gamma,\alpha_{n+1}^*} - g_{\alpha_n^*,\alpha_{n+1}^*}(\theta_{n+1}^*, X_{\theta_{n+1}^*}^{t,x}), \ \alpha_{n+1}^* \in A_{\alpha_n^*}, \ \forall n \geq 0 \right] > 0.
$$

But A is finite, then there is a loop $i_0, i_1, \ldots, i_k, i_0$ ($i_1 \neq i_0$) of elements of A and a subsequence $(n_q(\omega))_{q \geq 0}$ such that:

$$\mathbf{P}\left[Y^{\Gamma,i_l}_{\theta^*_{n_{q+l}}} = Y^{\Gamma,i_{l+1}}_{\theta^*_{n_{q+l}}} - g_{i_l,i_{l+1}}(\theta^*_{n_{q+l}}, X^{t,x}_{\theta^*_{n_{q+l}}}),\right.$$
$$\left. l = 1, \ldots, k,\ (i_{k+1} = i_0), \quad \forall q \geq 0\right] > 0. \tag{5.6}$$

Next let us consider $\theta^* = \lim_{n\to\infty} \theta^*_n$ and $\Theta = \{\theta^*_n < \theta^*, \forall n \geq 0\}$. Thanks to the non free loop property $\mathbf{P}[(\theta^* < T) \cap \Theta^c] = 0$ and then θ^* is an accessible stopping time (see e.g. [10, pp. 214], for more details). But for any $j \in A$, the process Y^j has only inaccessible jump times and θ^* is accessible, therefore for any $j \in A$, $\Delta Y^j_{\theta^*} = 0, \mathbf{P} - a.s.$. Going back to (5.6) and take the limit w.r.t. q to obtain:

$$\mathbf{P}[g_{i_0,i_1}(\theta^*, X^{t,x}_{\theta^*}) + \cdots + g_{i_k,i_0}(\theta^*, X^{t,x}_{\theta^*}) = 0] > 0,$$

which contradicts the non free loop property. We then have $\mathbf{P}[\theta^*_j < T,\ \forall j \geq 0] = 0$.

Now it remains to prove that $\mathbf{E}[(A^{a^*}_T)^2] < \infty$ and a^* is optimal in $\mathcal{A}^j_{t_0}$ for the switching problem (5.3). Since $(Y^{\Gamma,j})_{j\in A}$ solves the RBSDE (3.38) and by the definition of a^*, it yields:

$$Y^{\Gamma,j}_{t_0} = Y^{\Gamma,j}_{\theta^*_1} + \int_{t_0}^{\theta^*_1} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta^*_1} U^{a^*,k}_r dH^{(k)}_r \tag{5.7}$$

since $K^{\Gamma,j}_r - K^{\Gamma,j}_{\theta^*_0} = 0$ holds for any $r \in [t_0, \theta^*_1]$. But

$$Y^{\Gamma,j}_{\theta^*_1} = (Y^{\Gamma,\alpha^*_1}_{\theta^*_1} - g_{j\alpha^*_1}(\theta^*_1, X^{t,x}_{\theta^*_1}))1_{[\theta^*_1<T]} + h_j(X^{t,x}_T)1_{[\theta^*_1=T]}$$

then

$$\begin{aligned}
Y^{\Gamma,j}_{t_0} &= (Y^{\Gamma,\alpha^*_1}_{\theta^*_1} - g_{j\alpha^*_1}(\theta^*_1, X^{t,x}_{\theta^*_1}))1_{[\theta^*_1<T]} + h_j(X^{t,x}_T)1_{[\theta^*_1=T]}\\
&\quad + \int_{t_0}^{\theta^*_1} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta^*_1} U^{a^*,k}_r dH^{(k)}_r\\
&= Y^{\Gamma,\alpha^*_1}_{\theta^*_1}1_{[\theta^*_1<T]} + h_j(X^{t,x}_T)1_{[\theta^*_1=T]}\\
&\quad + \int_{t_0}^{\theta^*_1} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{t_0}^{\theta^*_1} U^{a^*,k}_r dH^{(k)}_r - A^{a^*}_{\theta^*_1}.
\end{aligned} \tag{5.8}$$

But we can do the same for the quantity $Y^{\Gamma,\alpha^*_1}_{\theta^*_1}1_{[\theta^*_1<T]}$ to obtain

$$\begin{aligned}
Y^{\Gamma,\alpha^*_1}_{\theta^*_1}1_{[\theta^*_1<T]} &= Y^{\Gamma,\alpha^*_1}_{\theta^*_2}1_{[\theta^*_2<T]} + h_{\alpha^*_1}(X^{t,x}_T)1_{[\theta^*_2=T]}1_{[\theta^*_1<T]}\\
&\quad + \int_{\theta^*_1}^{\theta^*_2} f_{a^*(r)}(r, X^{t,x}_r, \overrightarrow{\Gamma}_r, U^{a^*}_r)dr - \sum_{k=1}^{\infty}\int_{\theta^*_1}^{\theta^*_2} U^{a^*,k}_r dH^{(k)}_r.
\end{aligned}$$

Substitute now this equality in the previous one and since α^*_2 is the optimal index at θ^*_2 to obtain:

$$
\begin{aligned}
Y_{t_0}^{\Gamma,j} &= (Y_{\theta_2^*}^{\Gamma,\alpha_2^*} - g_{\alpha_1^*\alpha_2^*}(\theta_2^*, X_{\theta_2^*}^{t,x}))1_{[\theta_2^*<T]} + h_{\alpha_1^*}(X_T^{t,x})1_{[\theta_2^*=T]}1_{[\theta_1^*<T]} \\
&\quad + h_j(X_T^{t,x})1_{[\theta_1^*=T]} + \int_{t_0}^{\theta_2^*} f_{a^*(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^{a^*})dr \\
&\quad - \sum_{k=1}^{\infty} \int_{t_0}^{\theta_2^*} U_r^{a^*,k} dH_r^{(k)} - A_{\theta_1^*}^{a^*} \\
&= Y_{\theta_2^*}^{\Gamma,\alpha_2^*} 1_{[\theta_2^*<T]} + h_{\alpha_1^*}(X_T^{t,x})1_{[\theta_2^*=T]}1_{[\theta_1^*<T]} + h_j(X_T^{t,x})1_{[\theta_1^*=T]} \\
&\quad + \int_{t_0}^{\theta_2^*} f_{a^*(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^{a^*})dr - \sum_{k=1}^{\infty} \int_{t_0}^{\theta_2^*} U_r^{a^*,k} dH_r^{(k)} - A_{\theta_2^*}^{a^*}.
\end{aligned}
\tag{5.9}
$$

Repeating this procedure as many times as necessary and since $\mathbf{P}[\theta_j^* < T,\ \forall j \geq 0] = 0$ to get

$$
Y_s^{\Gamma,j} = h_{a^*(T)}(X_T^{t,x}) + \int_{t_0}^{T} f_{a^*(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^{a^*})dr - \sum_{k=1}^{\infty} \int_{t_0}^{T} U_r^{a^*,k} dH_r^{(k)} - A_T^{a^*}.
\tag{5.10}
$$

Now since $\Gamma \in [H^2]^m$, $U^{a^*} \in \mathcal{H}^2(\ell^2)$ and $Y^{\Gamma,j} \in \mathcal{S}^2$, we deduce from (5.10) that $\mathbf{E}[(A_T^{a^*})^2] < \infty$. Next by (5.2),

$$
\begin{aligned}
V_{t_0}^{a^*} - A_{t_0}^{a^*} - Y_{t_0}^{\Gamma,j} &= \int_{t_0}^{T} f_{a^*(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, N_r^{a^*})dr \\
&\quad - \int_{t_0}^{T} f_{a^*(r)}(r, X_r^{t,x}, \overrightarrow{\Gamma}_r, U_r^{a^*})dr \\
&\quad - \sum_{k=1}^{\infty} \int_{t_0}^{T} (N_r^{a^*} - U_r^{a^*,k}) dH_r^{(k)} \\
&\geq \int_{t_0}^{T} \langle V^{N^{a^*},U^{a^*},a^*}, N^{a^*} - U^{a^*} \rangle_r^p dr \\
&\quad - \sum_{k=1}^{\infty} \int_{t_0}^{T} (N_r^{a^*,k} - U_r^{a^*,k}) dH_r^{(k)}.
\end{aligned}
$$

Once more using Girsanov's Theorem, as in the bulk of the proof of Theorem 3.2, to obtain $\mathbf{E}_{\tilde{\mathbf{P}}}[V_{t_0}^{a^*} - A_{t_0}^{a^*} - Y_{t_0}^{\Gamma,i} | \mathcal{F}_{t_0}] \geq 0$ and then $V_{t_0}^{a^*} - A_{t_0}^{a^*} - Y_{t_0}^{\Gamma,i} \geq 0, \mathbf{P} - a.s.$ Taking now into account (5.5) leads to the desired result. □

Remark 5.1. As a by product of (5.3) we have also:

$$
\forall j \in A, \quad \mathbf{E}[Y_{t_0}^{\Gamma,j}] = \sup_{a \in \mathcal{A}_{t_0}^j} \mathbf{E}[V_{t_0}^a - A_{t_0}^a].
$$

□

5.2. Other equivalent definitions of viscosity solution of IPDEs

The following definition is an equivalent one for the solution of the IPDE (3.21) in the case when f does not depend on the component ζ. Basically it is an adaptation to our framework, which is of evolution type, of Definitions 1 and 2 given in [5] in the stationary case.

Definition 5.1. *Assume that the function f of IPDE (3.21) does not depend on ζ. Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function which belongs to Π_g. It is said a viscosity subsolution (resp. supersolution) of (3.21) if:*

(i) *$u(T,x) \le h(x)$ (resp. $u(T,x) \ge h(x)$), $\forall x \in \mathbb{R}$;*
(ii) *for any $(t,x) \in (0,T) \times \mathbb{R}$, $\delta > 0$ and a function $\varphi \in \mathcal{C}_p^{1,2}$ such that $u(t,x) = \varphi(t,x)$ and $u - \varphi$ has a global maximum (resp. minimum) at (t,x) on $[0,T] \times B(x, C_\sigma \delta)$, we have:*

$$\min\Big\{u(t,x) - \Psi(t,x); -\partial_t\varphi(t,x) - \mathcal{L}^1\varphi(t,x) \\ -\mathcal{I}^{1,\delta}(t,x,\varphi) - \mathcal{I}^{2,\delta}(t,x,u,D_x\varphi(t,x)) - f(t,x,u(t,x))\Big\} \le 0 \ (resp. \ \ge 0).$$

The function u is said to be a viscosity solution of (3.21) if it is both its viscosity subsolution and supersolution. □

Proposition 5.2. *If f does not depend on ζ then Definitions (3.1) and (5.1) are equivalent.*

Proof. We prove it only for the subsolution property since the supersolution one is similar. Let u be a subsolution of equation (3.21) according to Definition 5.1. Then for any $x_0 \in \mathbb{R}$ we have $u(T,x_0) \le h(x_0)$. Next let $(t_0,x_0) \in (0,T)\times\mathbb{R}$ and $\varphi \in \mathcal{C}_p^{1,2}$ such that $u - \varphi$ has a global maximum at (t_0,x_0) in $[0,T] \times \mathbb{R}$. If we set $\bar\varphi(t,x) := \varphi(t,x) + u(t_0,x_0) - \varphi(t_0,x_0)$, then $\bar\varphi$ belongs also to $\mathcal{C}_p^{1,2}$ and $u - \bar\varphi$ has a global maximum at (t_0,x_0) in $[0,T] \times \mathbb{R}$ and finally verifies $\bar\varphi(t_0,x_0) := u(t_0,x_0)$. Applying Definition 5.1 with $\bar\varphi$ yields:

$$min\Big\{u(t_0,x_0) - \Psi(t_0,x_0); -\partial_t\varphi(t_0,x_0) - \mathcal{L}^1\varphi(t_0,x_0) - \mathcal{I}^{1,\delta}(t_0,x_0,\varphi) \\ -\mathcal{I}^{2,\delta}(t_0,x_0,u(t_0,x_0),D_x\varphi(t_0,x_0)) - f(t_0,x_0,u(t_0,x_0))\Big\} \le 0$$

for any $\delta > 0$. Next since $(t_0,x_0) \in (0,T) \times \mathbb{R}$ is a global maximum point of $u - \varphi$, we then have

$$u(t_0,x_0 + \sigma(t_0,x_0)y) - u(t_0,x_0) \le \varphi(t_0,x_0 + \sigma(t_0,x_0)y) - \varphi(t_0,x_0)$$

which implies that $I^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),u) \le I^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi)$ and then

$$min\{u(t_0,x_0) - \Psi(t_0,x_0); -\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0))\} \le 0$$

which means that u is a subsolution for (3.21) according to Definition 3.1.

We are going now to show that if u is a subsolution of (3.21) according to Definition 3.1 then it is a subsolution according to Definition 5.1. Once more let us consider a continuous function u which belong to Π_g which is a subsolution of (3.21) according to Definition 3.1. Then for all $x_0 \in \mathbb{R}$, $u(T,x_0) \le h(x_0)$. Next let us fix $\delta > 0$, $(t_0,x_0) \in (0,T) \times \mathbb{R}$ and finally let us consider $\varphi \in \mathcal{C}_p^{1,2}$

such that $u-\varphi$ has a global maximum at (t_0,x_0) on $[0,T]\times B(x_0,C_\sigma\delta)$ and $u(t_0,x_0)=\varphi(t_0,x_0)$. There exists a function $\tilde\varphi$ which belongs to $\mathcal{C}_p^{1,2}$ such that $u-\tilde\varphi$ attains a global maximum in (t_0,x_0) on $[0,T]\times\mathbb{R}$ and satisfying $\tilde\varphi(s,y)=\varphi(s,y)$, for any (s,y) such that $|(s,y)-(t_0,x_0)|<\frac{C_\sigma\delta}{2}$. Consequently we have also

$$\begin{aligned}\partial_t\tilde\varphi(t_0,x_0)&=\partial_t\varphi(t_0,x_0),\quad D_x\tilde\varphi(t_0,x_0)=D_x\varphi(t_0,x_0),\\ D^2_{xx}\tilde\varphi(t_0,x_0)&=D^2_{xx}\varphi(t_0,x_0),\quad u(t_0,x_0)=\tilde\varphi(t_0,x_0).\end{aligned}\tag{5.11}$$

Next for any $\epsilon>0$, there exists φ_ϵ element of $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ such that $u\le\varphi_\epsilon\le\tilde\varphi$ and $\varphi_\epsilon\to u$ as $\epsilon\to0$, a.e. (see e.g. Lemma 4.7 in [19] or [2]). It implies that $u-\varphi_\epsilon$ and $\varphi_\epsilon-\tilde\varphi$ have a global maximum at (t_0,x_0) on $[0,T]\times\mathbb{R}$. Therefore, on the one hand, we have

$$\begin{aligned}&\partial_t\varphi_\epsilon(t_0,x_0)=\partial_t\tilde\varphi(t_0,x_0),\ D_x\varphi_\epsilon(t_0,x_0)\\ &\quad=D_x\tilde\varphi(t_0,x_0),\ D^2_{xx}\varphi_\epsilon(t_0,x_0)\le D^2_{xx}\tilde\varphi(t_0,x_0)\end{aligned}\tag{5.12}$$

and, on the other hand, by Definition 3.1 it holds

$$\begin{aligned}&min\Big\{u(t_0,x_0)-\Psi(t_0,x_0);-\partial_t\varphi_\epsilon(t_0,x_0)-\mathcal{L}^1\varphi_\epsilon(t_0,x_0)\\ &\quad-\mathcal{I}(t_0,x_0,\varphi_\epsilon)-f(t_0,x_0,u(t_0,x_0))\Big\}\le0.\end{aligned}\tag{5.13}$$

Recall now the definition of $\mathcal{L}^1$ in (4.2) and taking into account of (5.11) and (5.12) to obtain

$$\mathcal{L}^1\varphi_\epsilon(t_0,x_0)\le\mathcal{L}^1\varphi(t_0,x_0).\tag{5.14}$$

On the other hand

$$\begin{aligned}\mathcal{I}(t_0,x_0,\varphi_\epsilon)&=\mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi_\epsilon)+\mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi_\epsilon(t_0,x_0),\varphi_\epsilon)\\ &\le\mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\tilde\varphi)+\mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon)\\ &=\mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi)+\mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon).\end{aligned}\tag{5.15}$$

Plug now (5.14) and (5.15) in (5.13) to obtain

$$\begin{aligned}&min\Big\{u(t_0,x_0)-\Psi(t_0,x_0);\\ &\quad-\partial_t\varphi(t_0,x_0)-\mathcal{L}^1\varphi(t_0,x_0)-\mathcal{I}^{1,\frac{\delta}{2}}(t_0,x_0,\varphi)-\mathcal{I}^{2,\frac{\delta}{2}}(t_0,x_0,D_x\varphi(t_0,x_0),\varphi_\epsilon)\\ &\quad-f(t_0,x_0,u(t_0,x_0))\Big\}\le0.\end{aligned}\tag{5.16}$$

Take now the limit as $\epsilon\to0$ in (5.16), using the Lebesgue dominated convergence theorem and by the following inequality (which is valid since $u\le\varphi$ in $[0,T]\times B(x_0,C_\sigma\delta)$ and $u(t_0,x_0)=\varphi(t_0,x_0)$)

$$\begin{aligned}&\int_{\frac{\delta}{2}<|z|\le\delta}(\varphi(t_0,x_0+\sigma(t_0,x_0)z)-\varphi(t_0,x_0)-D_x\varphi(t_0,x_0)\sigma(t_0,x_0)z\}d\Pi(z)\\ &\quad\ge\int_{\frac{\delta}{2}<|z|\le\delta}(u(t_0,x_0+\sigma(t_0,x_0)z)-u(t_0,x_0)-D_x\varphi(t_0,x_0)\sigma(t_0,x_0)z\}d\Pi(z)\end{aligned}$$

we obtain

$$min\Big\{u(t_0,x_0)-\Psi(t_0,x_0);-\partial_t\varphi(t_0,x_0)-\mathcal{L}^1\varphi(t_0,x_0)$$
$$-\mathcal{I}^{1,\delta}(t_0,x_0,\varphi)-\mathcal{I}^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),u)-f(t_0,x_0,u(t_0,x_0))\Big\}\leq 0$$

which is the desired result. □

Similarly, there is another equivalent definition for system of IPDEs (4.1) which is:

Definition 5.2. *A function* $(u_1,\ldots,u_m):[0,T]\times\mathbb{R}\to\mathbb{R}^m\in\Pi_g$ *such that for any* $i\in A$, u_i *is usc (resp. lsc), is said to be a viscosity subsolution (resp. supersolution) of (4.1) if for any* $i\in A$,

(i) $u_i(T,x_0)\leq h_i(x_0)$ *(resp.* $u_i(T,x)\geq h_i(x)$*),* $\forall x_0\in\mathbb{R}$*;*
(ii) *for any* $(t_0,x_0)\in(0,T)\times\mathbb{R}$, $\delta>0$ *and a function* $\varphi\in\mathcal{C}_p^{1,2}$ *such that* $u_i(t_0,x_0)=\varphi(t_0,x_0)$ *and* $u_i-\varphi$ *has a global maximum (resp. minimum) at* (t_0,x_0) *on* $[0,T]\times B(x_0,C_\sigma\delta)$*, we have*

$$min\Big\{u_i(t_0,x_0)-\max_{j\in A_i}(u_j(t_0,x_0)-g_{ij}(t_0,x_0));-\partial_t\varphi(t_0,x_0)-\mathcal{L}^1\varphi(t_0,x_0)$$
$$-\mathcal{I}^{1,\delta}(t_0,x_0,\varphi)-I^{2,\delta}(t_0,x_0,D_x\varphi(t_0,x_0),u_i)$$
$$-f_i(t_0,x_0,u_1(t_0,x_0),\ldots,u_{i-1}(t_0,x_0),u_i(t_0,x_0),\ldots,u_m(t_0,x_0))\Big\}$$
$$\leq 0\ (resp.\geq 0).$$

The functions $(u_i)_{i=1}^m$ *is called a viscosity solution of (4.1) if* $(u_{i*})_{i=1}^m$ *and* $(u_i^*)_{i=1}^m$ *are respectively viscosity supersolution and viscosity subsolution of (4.1).*

We then have the following result whose proof is just an adaptation of the previous one and then is left for the reader.

Proposition 5.3. *Definitions (5.2) and (4.1) are equivalent.* □

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Said Hamadène and Xuxhe Zhao
Université du Maine, LMM
Avenue Olivier Messiaen
72085 Le Mans Cedex 9
France
e-mail: hamadene@univ-lemans.fr

Xuxhe Zhao
School of Mathematics and Statistics
Xidian University
Xi'an 710071
People's Republic of China
e-mail: sosmall129@hotmail.com

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