Stability tests for second order linear and nonlinear delayed models

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Abstract. For the nonlinear second order Lienard-type equations with time-varying delays

$$\ddot{x}(t) + \sum_{k=1}^m f_k(t, x(t), \dot{x}(g_k(t))) + \sum_{k=1}^l s_k(t, x(h_k(t))) = 0,$$

global asymptotic stability conditions are obtained. The results are based on the new sufficient stability conditions for relevant linear equations and are applied to derive explicit stability conditions for the nonlinear Kaldor–Kalecki business cycle model. We also explore multistability of the sunflower non-autonomous equation and its modifications.

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1. Introduction

The second order delay differential equation

$$\ddot{x} + f(t, x(t), \dot{x}(t - \tau)) + g(t, x(t), x(t - \tau)) = 0 \quad (1.1)$$

has a more than 65-year history of study, and was used to examine aftereffects in mechanics, physics, biology, medicine and economics (see, for example [18]). Recently, these models have been used to mimic regenerative vibrations in a milling process, a balancing motion and chatter vibrations. For example, a one degree of freedom milling equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = -\alpha[x(t) - x(t - \tau(t))] \quad (1.2)$$

was introduced in [33]. The milling model with several delays

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \sum_{k=1}^p \alpha_k [x(t) - x(t - \tau_k)]^k = 0$$

was recently studied, mostly numerically, in [13, 19, 20]. The following milling models with variable parameters were derived and examined in [18, 28, 29, 32–34]:

$$\begin{aligned} \ddot{x}(t) + a\dot{x}(t) + b(t)x(t) &= c(t)x(t - \tau(t)), \\ \ddot{x}(t) + a\dot{x}(t) + bx(t) + \sum_{k=1}^p \alpha_k(t)[x(t) - x(t - \tau_k(t))]^k &= 0. \end{aligned} \quad (1.3)$$

In economics, the well-known Kaldor–Kalecki business cycle model expressed as the delayed system of two nonlinear equations [14], in some cases can be reduced to the second order equation (see, for example, [30])

$$\ddot{x}(t) + [\alpha - \beta p'(x(t))]\dot{x}(t) + \gamma[p(x(t)) - \eta x(t)] + \delta p(x(t - \tau)) = 0. \quad (1.4)$$

Here $p(x)$ is a frequently used in mathematical economics sigmoid function [14], e.g. $p(x) = \frac{A}{1+e^{-bx}} - \frac{A}{2}$, and all coefficients are nonnegative constants.

Different techniques were applied to study second-order delay equations in [1, 6, 7, 11, 15, 16, 21, 23, 26] and [31, 35–38]. Characteristic quasipolynomials were broadly used for local stability analysis of autonomous models, (see, for example, [18]). The fixed point technique for second order differential and functional equations was pioneered by Burton [8, 9]. In the paper [10] explicit and easily-verifiable tests were obtained for the autonomous model

$$\ddot{x}(t) = p_1\dot{x}(t) + p_2\dot{x}(t - \tau) + q_1x(t) + q_2x(t - \tau). \quad (1.5)$$

Theorem 1.1. [10] *Assume that at least one of the following conditions holds: (1) $p_1p_2 > 0, q_1 > 0, q_2 > 0$ or (2) $p_1 > 0, p_2 > 0, q_1 > 0, q_2 < 0$. Then Eq. (1.5) is unstable.*

Theorem 1.2. [10] *Assume $p_1 = p_2 = 0, q_2 > 0$ and denote $B = \tau^2 q_1, D = \tau^2 q_2$. Equation (1.5) is asymptotically stable if and only if $q_1 < 0$ and there exists $k \in \mathbb{N}$ such that*

$$2k\pi < \sqrt{-B} < (2k+1)\pi, \quad D < \min \{ -(2k)^2\pi^2 - B, (2k+1)^2\pi^2 + B \}.$$

Example 1.3. The second-order delay equation

$$\ddot{x}(t) = -49x(t) + 7x(t - 1) \quad (1.6)$$

is asymptotically stable by Theorem 1.2. Based on the algorithmic tests presented in [10], the equation

$$\ddot{x}(t) = 0.6\dot{x}(t) + 0.3\dot{x}(t - 1) - 2x(t) + x(t - 1) \quad (1.7)$$

is asymptotically stable. It is interesting to note that Eqs. (1.6) and (1.7) without delays are unstable. This illustrates a very interesting feature of second-order delay differential equations, i.e. delays may improve asymptotic properties of a given equation, whereas delays in first-order linear equations have mostly destabilizing effects or do not change stability of the model.

Several stability tests for non-autonomous linear models with variable delays

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0, \quad (1.8)$$

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + a_1(t)\dot{x}(g(t)) + b_1(t)x(h(t)) = 0, \quad (1.9)$$

were obtained in our recent paper [3], under the assumptions: a , a_1 , b and b_1 are Lebesgue measurable and essentially bounded functions on $[0, \infty)$; $a(t) \geq a_0 > 0$, $b(t) \geq b_0 > 0$, $0 \leq t - h(t) \leq \tau$, $0 \leq t - g(t) \leq \delta$, $a^2(t) \geq 4b(t)$, $\int_{g(t)}^t a(s)ds < 1/e$. Below $\|\cdot\|$ is the norm in the space $\mathbf{L}_\infty[t_0, \infty)$.

Theorem 1.4. [3, Theorem 5.1] *If for some $t_0 \geq 0$*

$$\delta \left\| \frac{a}{b} \right\| \left(\|a\| \left\| \frac{b}{a} \right\| + \|b\| \right) + \tau \left\| \frac{b}{a} \right\| < 1,$$

then Eq. (1.8) is exponentially stable.

Theorem 1.5. [3, Theorem 5.3] *Suppose for some $t_0 \geq 0$*

$$\left\| \frac{a_1}{a} \right\| < 1, \quad \left\| \frac{a_1}{b} \right\| \frac{\left\| \frac{b}{a} \right\| + \left\| \frac{b_1}{a} \right\|}{1 - \left\| \frac{a_1}{a} \right\|} + \left\| \frac{b_1}{b} \right\| < 1,$$

then Eq. (1.9) is exponentially stable.

In the present paper, a specially designed substitution transforms linear second order equations into a system, with a further application of the M-matrix method. This and the linearization techniques are used to devise new global stability tests for nonlinear non-autonomous models. These results are explicit, easily verifiable and can be applied to a general class of second order non-autonomous equations. Some of the theorems of the present paper complement our earlier results [2, 3], as well as the tests obtained in recent papers [10, 11, 15].

The paper is organized as follows. Section 2 contains stability results for linear second order non-autonomous equations with several delays. To illustrate efficiency of the results obtained each stability test is accompanied by numerical examples. In Sect. 3 the tests for linear models are applied to nonlinear Lienard-type equations of the second order. Applications incorporate a global stability test for the non-autonomous business cycle model. Section 4 includes the study of bounds and multistability properties for the sunflower model and its generalizations. In particular, sufficient conditions for convergence to one of an infinite number of equilibrium points are presented, and existence of unbounded linearly growing solutions is illustrated. Final remarks are presented in Sect. 5.

2. Stability tests for linear Lienard equations

The technique in this section involves parlaying a second order equation into two first order equations. Consider a linear equation of the second order

$$\ddot{x}(t) + \sum_{k=1}^m a_k(t) \dot{x}(h_k(t)) + \sum_{k=1}^m b_k(t) \int_{g_k(t)}^t \dot{x}(s) ds + \sum_{k=1}^m c_k(t) x(r_k(t)) = 0. \quad (2.1)$$

Together with Eq. (2.1), for any $t_0 \geq 0$ we consider the initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t \leq t_0. \quad (2.2)$$

Henceforth, we assume that the following assumptions are satisfied:

- (a1) $a_i, b_i, c_i, i = 1, \dots, m$ are Lebesgue measurable and essentially bounded on $[0, \infty)$;
- (a2) h_i, g_i, r_i are Lebesgue measurable functions, $h_i(t) \leq t, g_i(t) \leq t, r_i(t) \leq t$, $\lim_{t \rightarrow \infty} h_i(t) = \infty, \lim_{t \rightarrow \infty} g_i(t) = \infty, \lim_{t \rightarrow \infty} r_i(t) = \infty, i = 1, \dots, m$;
- (a3) φ and ψ are Borel measurable bounded functions.

We assume that conditions (a1)–(a3) hold for all equations considered in the paper.

Definition 2.1. A function $x : \mathbb{R} \rightarrow \mathbb{R}$ with locally absolutely continuous on $[t_0, \infty)$ derivative $\dot{x}$ is called a *solution* of problem (2.1), (2.2) if it satisfies Eq. (2.1) for almost every $t \in [t_0, \infty)$ and equalities (2.2) for $t \leq t_0$.

We quote a useful lemma that will play a major role in the proofs. Below, the upper k is an index, not a power.

Lemma 2.2. [5] *Consider the system*

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^m \sum_{k=1}^{l_{ij}} b_{ij}^k(t)x_j(h_{ij}^k(t)), \quad i = 1, \dots, m, \quad (2.3)$$

where $a_i(t) \geq \alpha_i > 0$, $|b_{ij}^k(t)| \leq L_{ij}^k$, $t - h_{ij}^k(t) \leq \sigma_{ij}^k$.

If the matrix $B = (b_{ij})_{i,j=1}^m$, with $b_{ii} = 1 - \left(\sum_{k=1}^{l_{ii}} L_{ii}^k \right) / \alpha_i$, $b_{ij} = - \left(\sum_{k=1}^{l_{ij}} L_{ij}^k \right) / \alpha_i$, $i \neq j$, is an M -matrix, then system (2.3) is uniformly exponentially stable.

We recall that a matrix $B = (b_{ij})_{i,j=1}^m$ is a (nonsingular) M -matrix if $b_{ij} \leq 0$, $i \neq j$ and one of the following equivalent conditions holds: either there exists a positive inverse matrix $B^{-1} > 0$ or all the principal minors of the matrix B are positive.

Further proofs will also require the following lemma.

Lemma 2.3. *Consider the system*

$$\begin{aligned} \dot{x}_i(t) = & -a_i(t)x_i(t) \\ & + \sum_{j=1}^m \sum_{k=1}^{l_{ij}} \left(c_{ij}^k(t)x_j(g_{ij}^k(t)) + d_{ij}^k(t) \int_{h_{ij}^k(t)}^t x_j(s) ds \right), \quad i = 1, \dots, m, \end{aligned} \quad (2.4)$$

where $a_i(t) \geq \alpha_i > 0$, $|d_{ij}^k(t)| \leq L_{ij}^k$, $|c_{ij}^k(t)| \leq C_{ij}^k$, $t - h_{ij}^k(t) \leq \sigma_{ij}^k$, $t - g_{ij}^k(t) \leq \tau$. If the matrix $B = (b_{ij})_{i,j=1}^m$, with $b_{ii} = 1 - \sum_{k=1}^{l_{ii}} (L_{ii}^k \sigma_{ii}^k + C_{ii}^k) / \alpha_i$, $b_{ij} = -\sum_{k=1}^{l_{ij}} (L_{ij}^k \sigma_{ij}^k + C_{ij}^k) / \alpha_i$, $i \neq j$, is an M-matrix, then system (2.4) is exponentially stable.

Proof. Let $x(t)$ be a solution of (2.4). Since $x_j(t)$ are continuous then for any i, j, k and t there exists $p_{ij}^k(t) \in (h_{ij}^k(t), t)$ such that $x_j(p_{ij}^k(t))(t - h_{ij}^k(t)) = \int_{h_{ij}^k(t)}^t x_j(s) ds$.

Thus x_j are solutions of system (2.3) with $b_{ij}^k(t)x_j(h_{ij}^k(t))$ being replaced by $c_{ij}^k x_j(g_{ij}^k(t)) + d_{ij}^k(t)(t - h_{ij}^k(t))x_j(p_{ij}^k(t))$. We have $|c_{ij}^k(t)| \leq C_{ij}^k$, $|d_{ij}^k(t)(t - h_{ij}^k(t))| \leq L_{ij}^k \sigma_{ij}^k$, $i \neq j$. The application of Lemma 2.2 validates the proof. $\square$

Note that a different proof of Lemma 2.3 involves application of the Halanay-type inequalities (see, for example [22]).

To examine the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)x(h_k(t)) = 0 \quad (2.5)$$

we assume

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau.$$

Theorem 2.4. *Suppose at least one of the following conditions holds:*

- (1) $B \leq \frac{a^2}{4}$, $\sum_{k=1}^m C_k < b - \frac{a}{2}(A - a)$,
- (2) $b \geq \frac{a}{2}(A - \frac{a}{2})$, $\sum_{k=1}^m C_k < \frac{a^2}{2} - B$.

Then Eq. (2.5) is exponentially stable.

Proof. Substituting $\dot{x} = -\frac{a}{2}x + y$, $\ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into Eq. (2.5), we arrive at

$$\begin{aligned} \dot{x} = & -\frac{a}{2}x + y \\ \dot{y} = & \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \right] x(t) - \sum_{k=1}^m c_k(t)x(h_k(t)) - \left(a(t) - \frac{a}{2} \right) y(t). \end{aligned} \quad (2.6)$$

Condition (1) yields $\frac{a}{2}(a(t) - \frac{a}{2}) - b(t) \geq \frac{a^2}{4} - B \geq 0$, $\frac{a}{2}(a(t) - \frac{a}{2}) - b(t) \leq \frac{a}{2}(A - \frac{a}{2}) - b$. Hence the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a} \left(\frac{a}{2} (A - \frac{a}{2}) - b + \sum_{k=1}^m C_k \right) & 1 - \frac{2}{a} \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 Eq. (2.5) is exponentially stable.

If condition (2) holds then $b(t) - \frac{a}{2}(a(t) - \frac{a}{2}) \geq b - \frac{a}{2}(A - \frac{a}{2}) \geq 0$, $b(t) - \frac{a}{2}(a(t) - \frac{a}{2}) \leq B - \frac{a}{2}(a - \frac{a}{2}) = B - a^2/4$, and the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a}\left(B - \frac{a^2}{4} + \sum_{k=1}^m C_k\right) & 1 \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 Eq. (2.5) is exponentially stable. $\square$

Remark 2.5. Application of the classical substitution $\dot{x} = y$ is not useful in our stability investigation, since for the system obtained after this substitution, the matrix B in Lemma 2.2 is not an M-matrix. For Eq. (2.5) with constant a and b , $|c(t)| \leq C$ and $m = 1$

$$\ddot{x}(t) + a\dot{x}(t) + b(t)x(t) + c(t)x(h(t)) = 0, \quad (2.7)$$

we compare two substitutions

$$\dot{x}(t) = -\lambda x(t) + y(t), \quad \lambda > 0, \quad (2.8)$$

and $\dot{x}(t) = -\frac{a}{2}x(t) + y(t)$. By Theorem 2.4 Eq. (2.7) is exponentially stable, if at least one of the following conditions holds:

- (1) $b \leq \frac{a^2}{4}, C < b$,
- (2) $b > \frac{a^2}{4}, C < \frac{a^2}{2} - b$.

Whereas application of (2.8) by the same token yields a slight improvement: equation (2.7) is exponentially stable, if at least one of the following conditions holds:

(1), (2) or

(3) $b \leq \frac{a^2}{4}, C < \frac{a^2}{2} - b$ and the following two intervals have a nonempty intersection

$$\left[\frac{a - \sqrt{a^2 - 4b}}{2}, \frac{a + \sqrt{a^2 - 4b}}{2} \right] \cap \left[a - \sqrt{a^2 - 2(b + C)}, a + \sqrt{a^2 - 2(b + C)} \right] \neq \emptyset.$$

Implementation of (2.8) for Eq. (2.5) with nonconstant coefficients $a(t)$ and $b(t)$ will produce a more complicated condition (3). Trading-off these options, we prefer the substitution $\dot{x}(t) = -\frac{a}{2}x(t) + y(t)$.

The following numerical examples illustrate the application of Theorem 2.4.

Example 2.6. Consider the delay equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + cx(t - h|\sin t|) = 0. \quad (2.9)$$

- (a) $a = 3, b = 1.1, c = -0.8, h = 2$. Condition (1) of Theorem 2.4 holds, condition (2) does not hold. Equation (2.9) is asymptotically stable.
- (b) $a = 2, b = 1.1, c = -0.8, h = 2$. Condition (2) of Theorem 2.4 holds, condition (1) does not hold. Equation (2.9) is asymptotically stable.
- (c) $a = 0.1, b = 1.5, c = -1.45, h = 2$. Conditions of Theorem 2.4 do not hold, and numerical simulations suggest that the Eq. (2.9) is unstable.

Let us note that in (c) the coefficient of the non-delay term exceeds the one of the delayed term: $|c(t)| < b(t)$. This is in contrast to the result for the equation with the second derivative omitted

$$a\dot{x}(t) + b(t)x(t) + c(t)x(h(t)) = 0,$$

which is exponentially stable if $a > 0$, $b(t) \geq b_0 > 0$, $|c(t)| < b(t)$, $t - \tau \leq h(t) \leq t$ for $\tau > 0$.

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)\dot{x}(h_k(t)) = 0, \quad (2.10)$$

where

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) \leq B, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau.$$

Theorem 2.7. Suppose that at least one of the following conditions holds:

- (1) $B \leq \frac{a^2}{4}$, $\sum_{k=1}^m C_k < \frac{2b-a(A-a)}{2a}$,
- (2) $b \geq \frac{a}{2} \left(A - \frac{a}{2}\right)$, $\sum_{k=1}^m C_k < \frac{a^2-2B}{2a}$.

Then Eq. (2.10) is exponentially stable.

Proof. The substitution $\dot{x} = -\frac{a}{2}x + y$, $\ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into Eq. (2.10) yields

$$\begin{aligned} \dot{x} &= -\frac{a}{2}x + y \\ \dot{y} &= \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \right] x(t) + \frac{a}{2} \sum_{k=1}^m c_k(t)x(h_k(t)) \\ &\quad - \sum_{k=1}^m c_k(t)y(h_k(t)) - \left(a(t) - \frac{a}{2} \right) y(t). \end{aligned} \quad (2.11)$$

If condition (1) holds, we have $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \geq \frac{a^2}{4} - B \geq 0$, $\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - b(t) \leq \frac{a}{2} \left(A - \frac{a}{2} \right) - b$. Hence the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a} \left[\frac{a}{2} \left(A - \frac{a}{2} \right) - b + \frac{a}{2} \sum_{k=1}^m C_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m C_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 Eq. (2.10) is exponentially stable.

If the inequalities in (2) hold then $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \geq b - \frac{a}{2} \left(A - \frac{a}{2} \right) \geq 0$, $b(t) - \frac{a}{2} \left(a(t) - \frac{a}{2} \right) \leq B - \frac{a}{2} \left(a - \frac{a}{2} \right) = B - a^2/4$. Thus the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \\ -\frac{2}{a} \left[B - \frac{a^2}{4} + \frac{a}{2} \sum_{k=1}^m C_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m C_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.2 Eq. (2.10) is exponentially stable. □

Example 2.8. Consider the equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + c\dot{x}(t - h|\sin t|) = 0. \quad (2.12)$$

To illustrate Theorem 2.7, we examined:

- (a) $a = 2.1$, $b = 1$, $c = -0.4$, $h = 2$. Condition (1) of Theorem 2.7 holds, condition (2) does not hold. Equation (2.12) is asymptotically stable.
- (b) $a = 4$, $b = 5$, $c = -0.7$, $h = 2$. Condition (2) of Theorem 2.7 holds, condition (1) does not hold. Equation (2.12) is asymptotically stable.
- (c) $a = 1$, $b = 1.5$, $c = -0.8$, $h = 2$. Conditions of Theorem 2.7 do not hold, and numerical simulations suggest that the Eq. (2.12) is unstable. Hence, in general, the conditions $a(t) \geq a_0 > 0$, $b(t) \geq b_0 > 0$, $m = 1$, $|c(t)| < a(t)$ are not sufficient for stability of Eq. (2.10).

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + \sum_{k=1}^m b_k(t)x(h_k(t)) = 0, \quad (2.13)$$

where $0 < a \leq a(t) \leq A$, $0 < b_k \leq b_k(t) \leq B_k$, $t - h_k(t) \leq \tau_k$.

Theorem 2.9. *Suppose at least one of the following conditions holds:*

1. $\sum_{k=1}^m B_k \leq \frac{a^2}{4}$, $\frac{a}{2}(A - a) < \sum_{k=1}^m b_k - a \sum_{k=1}^m B_k \tau_k$,
2. $\sum_{k=1}^m b_k \geq \frac{a}{2}(A - \frac{a}{2})$, $\sum_{k=1}^m B_k(1 + a\tau_k) < \frac{a^2}{2}$.

Then Eq. (2.13) is exponentially stable.

Proof. With the substitution $\dot{x} = -\frac{a}{2}x + y$, $\ddot{x} = -\frac{a}{2}\dot{x} + \dot{y}$ into Eq. (2.13), we arrive at

$$\begin{aligned} \dot{x} &= -\frac{a}{2}x + y \\ \dot{y} &= \left[\frac{a}{2} \left(a(t) - \frac{a}{2} \right) - \sum_{k=1}^m b_k(t) \right] x(t) \\ &\quad + \sum_{k=1}^m b_k(t) \int_{h_k(t)}^t \left[-\frac{a}{2}x(s) + y(s) \right] ds - \left(a(t) - \frac{a}{2} \right) y(t). \end{aligned} \quad (2.14)$$

If condition (1) holds, we have $\frac{a}{2}(a(t) - \frac{a}{2}) - \sum_{k=1}^m b_k(t) \geq \frac{a^2}{4} - \sum_{k=1}^m B_k \geq 0$, $\frac{a}{2}(a(t) - \frac{a}{2}) - \sum_{k=1}^m b_k(t) \leq \frac{a}{2}(A - \frac{a}{2}) - \sum_{k=1}^m b_k$. Hence the off-diagonal entries of the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m b_k \tau_k \\ -\frac{2}{a} \left[\frac{a}{2} \left(A - \frac{a}{2} \right) - \sum_{k=1}^m b_k + \frac{a}{2} \sum_{k=1}^m B_k \tau_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m B_k \tau_k \end{pmatrix}$$

are non-positive, and the inequalities in (1) yield that it is an M-matrix. By Lemma 2.3 Eq. (2.13) is exponentially stable. Assumption (2) implies $\sum_{k=1}^m b_k(t) - \frac{a}{2}(a(t) - \frac{a}{2}) \geq \sum_{k=1}^m b_k - \frac{a}{2}(A - \frac{a}{2}) \geq 0$, $\sum_{k=1}^m b_k(t) - \frac{a}{2}(a(t) - \frac{a}{2}) \leq \sum_{k=1}^m B_k - \frac{a}{2}(a - \frac{a}{2}) = \sum_{k=1}^m B_k - a^2/4$, therefore the matrix

$$\begin{pmatrix} 1 & -\frac{2}{a} \sum_{k=1}^m B_k \tau_k \\ -\frac{2}{a} \left[\sum_{k=1}^m B_k - \frac{a^2}{4} + \frac{a}{2} \sum_{k=1}^m B_k \tau_k \right] & 1 - \frac{2}{a} \sum_{k=1}^m B_k \tau_k \end{pmatrix}$$

is an M-matrix. By Lemma 2.3 Eq. (2.13) is exponentially stable. $\square$

Corollary 2.10. *Suppose $a(t) \equiv a > 0$, $b_k(t) \equiv b_k > 0$, and at least one of the following conditions holds:*

1. $\sum_{k=1}^m b_k \leq \frac{a^2}{4}$, $\sum_{k=1}^m b_k(1 - a\tau_k) > 0$,
2. $\sum_{k=1}^m b_k \geq \frac{a^2}{4}$, $\sum_{k=1}^m b_k(1 + a\tau_k) < \frac{a^2}{2}$.

Then Eq. (2.13) is exponentially stable.

Example 2.11. Consider the equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t - h|\sin t|) = 0. \quad (2.15)$$

To illustrate Theorem 2.9, we consider numerical examples:

- (a) $a = 2$, $b = 0.9$, $h = 0.4$. Condition (1) of Theorem 2.9 holds, condition (2) does not hold. Equation (2.15) is asymptotically stable.
- (b) $a = 2$, $b = 1.1$, $h = 0.4$. Condition (2) of Theorem 2.9 holds, condition (1) does not hold. Equation (2.15) is asymptotically stable.
- (c) $a = 1$, $b = 1.1$, $h = 2.5$. Conditions of Theorem 2.9 do not hold, and numerical simulations suggest that the Eq. (2.14) is unstable.

Consider the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \sum_{k=1}^m c_k(t) [x(t) - x(h_k(t))], \quad (2.16)$$

where $0 < a \leq a(t) \leq A$, $0 < b \leq b_k(t) \leq B$, $|c_k(t)| \leq C_k$, $t - h_k(t) \leq \tau_k$.

Theorem 2.12. Suppose at least one of the following conditions holds:

1. $B \leq \frac{a^2}{4}$, $\sum_{k=1}^m C_k \tau_k < \frac{2b - a(A - a)}{2a}$,
2. $b \geq \frac{a}{2} (A - \frac{a}{2})$, $\sum_{k=1}^m C_k \tau_k < \frac{a^2 - 2B}{2a}$.

Then Eq. (2.16) is exponentially stable.

Proof. After rewriting Eq. (2.16) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \sum_{k=1}^m c_k(t) \int_{h_k(t)}^t \dot{x}(s) ds,$$

we apply the same argument as in the proof of Theorem 2.7. $\square$

Theorem 2.4 gives delay-independent stability conditions for Eq. (2.5). The following statement contains delay-dependent stability conditions for this equation.

Theorem 2.13. Assume that

$$0 < a \leq a(t) \leq A, \quad 0 < b \leq b(t) + \sum_{k=1}^m c_k(t) \leq B, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau_k$$

and at least one of the conditions of Theorem 2.12 holds. Then Eq. (2.5) is exponentially stable.

Proof. Rewrite Eq. (2.5) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + \left(b(t) + \sum_{k=1}^m c_k(t) \right) x(t) = \sum_{k=1}^m c_k(t) \int_{h_k(t)}^t \dot{x}(s) ds.$$

The end of the proof is a straightforward imitation of the proof of Theorem 2.7. $\square$

3. Stability tests for nonlinear Lienard equations

In this section we examine several nonlinear delay differential equations of the second order which have the following general form

$$\ddot{x}(t) + \sum_{k=1}^m f_k(t, x(p_k(t)), \dot{x}(g_k(t))) + \sum_{k=1}^l s_k(t, x(h_k(t))) = 0, \quad (3.1)$$

with the following initial function

$$x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t \leq t_0, \quad t_0 \geq 0 \quad (3.2)$$

where $f_k(t, u_1, u_2)$, $k = 1, \dots, m$, $s_k(t, u)$, $k = 1, \dots, l$ are Caratheodory functions which are measurable in t and continuous in all the other arguments, condition (a2) holds for delay functions p_k, g_k, h_k ; φ and ψ are Borel measurable bounded functions.

The definition of the solution of initial value Problem (3.1), (3.2) is the same as for Problem (2.1), (2.2). We will assume that the initial value problem has a unique global solution on $[t_0, \infty)$ for all nonlinear equations considered in this section [1, 16].

Definition 3.1. Suppose the number K is an equilibrium of Eq. (3.1). We will say that K is a *global attractor* of this equation if for any solution x of the problem we have $\lim_{t \rightarrow \infty} x(t) = K$.

Theorem 3.2. Consider the equation

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) + \sum_{k=1}^m s_k(t, x(t), x(h_k(t))) = 0, \quad (3.3)$$

where

$$\begin{aligned} f(t, v, 0) = 0, \quad s(t, 0) = 0, \quad s_k(t, v, 0) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A, \\ 0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad \left| \frac{s_k(t, v, u)}{u} \right| \leq C_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau. \end{aligned}$$

If at least one of the following conditions holds:

1. $B \leq \frac{a_0^2}{4}$, $\sum_{k=1}^m C_k < b_0 - \frac{a_0}{2}(A - a_0)$,
2. $b_0 \geq \frac{a_0}{2}(A - \frac{a_0}{2})$, $\sum_{k=1}^m C_k < \frac{a_0^2}{2} - B$,

then zero is a global attractor for all solutions of Problem (3.3), (3.2).

Proof. Suppose x is a fixed solution of Problems (3.3), (3.2). Rewrite Eq. (3.3) in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) + \sum_{k=1}^m c_k(t)x(h_k(t)) = 0,$$

$$\text{where } a(t) = \begin{cases} \frac{f(t, x(t), \dot{x}(t))}{\dot{x}(t)}, & \dot{x}(t) \neq 0, \\ a_0, & \dot{x}(t) = 0, \end{cases} \quad b(t) = \begin{cases} \frac{s(t, x(t))}{x(t)}, & x(t) \neq 0, \\ b_0, & x(t) = 0, \end{cases} \quad c_k(t) = \begin{cases} \frac{s_k(t, x(t), x(h_k(t)))}{x(h_k(t))}, & x(h_k(t)) \neq 0, \\ 0, & x(h_k(t)) = 0. \end{cases}$$

Hence the function x is a solution of the linear equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) + \sum_{k=1}^m c_k(t)y(h_k(t)) = 0, \quad (3.4)$$

which is exponentially stable by Theorem 2.4. Thus for any solution y of Eq. (3.4) we have $\lim_{t \rightarrow \infty} y(t) = 0$. Since x is a solution of (3.4), we have $\lim_{t \rightarrow \infty} x(t) = 0$. $\square$

The previous proof is readily adapted to the proof of the following theorems.

Theorem 3.3. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) + \sum_{k=1}^m s_k(t, x(t), \dot{x}(h_k(t))) = 0, \quad (3.5)$$

where

$$\begin{aligned} f(t, v, 0) = 0, \quad s(t, 0) = 0, \quad s_k(t, v, 0) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A, \\ 0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad \left| \frac{s_k(t, v, u)}{u} \right| \leq C_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau. \end{aligned}$$

Suppose at least one of the following conditions holds:

1. $B \leq \frac{a_0^2}{4}$, $\sum_{k=1}^m C_k < \frac{2b_0 - a_0(A - a_0)}{2a_0}$,
2. $b_0 \geq \frac{a_0}{2} \left(A - \frac{a_0}{2} \right)$, $\sum_{k=1}^m C_k < \frac{a_0^2 - 2B}{2a_0}$.

Then zero is a global attractor for all solutions of Problems (3.5), (3.2).

Theorem 3.4. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + \sum_{k=1}^m s_k(t, x(h_k(t)), \dot{x}(t)) = 0, \quad (3.6)$$

where

$$\begin{aligned} f(t, v, 0) = 0, \quad s_k(t, 0, u) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A, \\ 0 < b_k \leq \frac{s_k(t, v, u)}{v} \leq B_k, \quad u \neq 0, \quad t - h_k(t) \leq \tau. \end{aligned}$$

Suppose at least one of the following conditions holds:

1. $\sum_{k=1}^m B_k \leq \frac{a_0^2}{4}$, $\frac{a_0}{2} (A - a_0) < \sum_{k=1}^m b_k - a_0 \sum_{k=1}^m B_k \tau_k$,
2. $\sum_{k=1}^m b_k \geq \frac{a}{2} \left(A - \frac{a_0}{2} \right)$, $\sum_{k=1}^m B_k (1 + a_0 \tau_k) < \frac{a_0^2}{2}$.

Then zero is a global attractor for all solutions of Problems (3.6), (3.2).

Theorem 3.5. *Consider the equation*

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) + s(t, x(t)) = \sum_{k=1}^m c_k(t)(x(t) - x(h_k(t))), \quad (3.7)$$

where

$$\begin{aligned} f(t, v, 0) &= 0, \quad s(t, 0) = 0, \quad 0 < a_0 \leq \frac{f(t, v, u)}{u} \leq A, \\ 0 < b_0 &\leq \frac{s(t, u)}{u} \leq B, \quad u \neq 0, \quad |c_k(t)| \leq C_k, \quad t - h_k(t) \leq \tau_k. \end{aligned}$$

Suppose at least one of the following conditions holds:

1. $B \leq \frac{a_0^2}{4}$, $\sum_{k=1}^m C_k \tau_k < \frac{2b_0 - a_0(A - a_0)}{2a_0}$,
2. $b_0 \geq \frac{a_0}{2} \left(A - \frac{a_0}{2}\right)$, $\sum_{k=1}^m C_k \tau_k < \frac{a_0^2 - 2B}{2a_0}$.

Then zero is a global attractor for all solutions of Problem (3.7), (3.2).

Example 3.6. To illustrate Part (2) of Theorem 3.4, consider the equation

$$\ddot{x}(t) + (1.9 + 0.1 \sin x(t))\dot{x}(t) + (1.1 + 0.1 \cos x(t))x(t - 0.19 \sin^2 t) = 0. \quad (3.8)$$

We have $m = 1$, $a_0 = 1.8$, $A = 2$, $b_0 = 1$, $B = 1.2$, $\tau = 0.19$; therefore, all conditions of the theorem hold, hence zero is a global attractor for all solutions of Eq. (3.8).

Motivated by model (1.4), consider a generalized Kaldor–Kalecki model

$$\ddot{x}(t) + [\alpha(t) - \beta(t)p'(x(t))]\dot{x}(t) + s(t, x(t)) = p(x(t)) - p(x(h(t))), \quad (3.9)$$

where α, β are locally essentially bounded functions, s is a Caratheodory function, p is a locally absolutely continuous nondecreasing function,

$$\begin{aligned} 0 < \alpha_0 &\leq \alpha(t) \leq \alpha_1, \quad 0 < \beta_0 \leq \beta(t) \leq \beta_1, \\ |p'(t)| &\leq C, \quad \alpha_0 - \beta_1 C > 0, \quad 0 < b_0 \leq \frac{s(t, u)}{u} \leq B, \quad u \neq 0 \quad t - h(t) \leq \tau. \end{aligned}$$

Denote $a_0 = \alpha_0 - \beta_1 C$.

Theorem 3.7. Suppose at least one of the following conditions holds:

1. $B \leq \frac{a_0^2}{4}$, $C\tau < \frac{2b_0 - a_0(\alpha_1 - a_0)}{2a_0}$,
2. $b \geq \frac{a_0}{2} \left(\alpha_1 - \frac{a_0}{2}\right)$, $C\tau < \frac{a_0^2 - 2B}{2a_0}$.

Then zero is a global attractor for all solutions of Problem (3.9), (3.2).

Proof. Suppose x is a fixed solution of Problem (3.9), (3.2). There exists a function $\xi(t)$ such that $p(x(t)) - p(h(x(t))) = p'(\xi(t))(x(t) - x(h(t)))$. Denote $\alpha(t) - \beta(t)p'(x(t)) = a(t)$, $p'(\xi(t)) = c(t)$. Hence x is a solution of the following equation

$$\ddot{y}(t) + a(t)\dot{y}(t) + s(t, y(t)) = c(t)(y(t) - y(h(t))). \quad (3.10)$$

Since $p'(x) \geq 0$ then $0 < \alpha_0 - \beta_1 C \leq a(t) \leq \alpha_1$. Equation (3.10) has a form (3.7) with $f(t, x(t), \dot{x}(t)) = a(t)\dot{x}(t)$, $m = 1$. All conditions of Theorem 3.5 hold, hence for any solution of (3.10) we have $\lim_{t \rightarrow \infty} y(t) = 0$. Then also $\lim_{t \rightarrow \infty} x(t) = 0$. $\square$

4. Sunflower model and its modifications

The sunflower equation was introduced in 1967 by Israelson and Johnson in [17] as a model for the geotropic circumnutations of *Helianthus annuus* and studied in [12, 24, 27]. Historically, it was derived from the following first order delay equation

$$\dot{u} + \frac{b}{\tau} e^{a(1-t/\tau)} \int_{-\infty}^{t-\tau} e^{as/\tau} \sin u(s) ds = 0. \quad (4.1)$$

Taking the derivative of (4.1) we arrive at the sunflower equation

$$\ddot{x} + \frac{a}{\tau} \dot{x} + \frac{b}{\tau} \sin x(t - \tau) = 0, \quad (4.2)$$

for which evidently the results of the previous section are not applicable.

Remark 4.1. It is interesting to note that a non-delayed version of (4.2)

$$\ddot{x} + a\dot{x} + b \sin x(t) = 0, \quad (4.3)$$

has a long history (see, for example, [25]). However, many important questions for delayed model (4.2) are still left unanswered.

Consider a generalization of model (4.1)

$$\frac{du}{dt} + b \int_{-\infty}^{h(t)} K(t, s) \sin u(s) ds = 0, \quad (4.4)$$

with the initial conditions

$$u(t) = \varphi(t), \quad t \leq 0, \quad (4.5)$$

under the following assumptions:

- (b1) $h(t) \leq t - \tau$ for some $\tau > 0$;
- (b2) $K(\cdot, \cdot)$ is Lebesgue measurable, $K(t, s) \geq 0$, there exists $a > 0$ such that $K(t, s) \leq \frac{1}{\tau} \exp\left\{-\frac{a}{\tau}(t - s - \tau)\right\}$ and $\int_0^\infty \int_{-\infty}^{h(t)} K(t, s) ds dt = \infty$;
- (b3) $\varphi : [-\infty, 0] \rightarrow \mathbb{R}$ is a continuous bounded function.

Theorem 4.2. Suppose that (b1)–(b3) hold, $b > 0$ and the characteristic equation

$$\lambda^2 \tau - a\lambda + be^{\lambda\tau} = 0 \quad (4.6)$$

has a positive root $\lambda_0 > 0$. Then any solution of (4.4), (4.5) with the initial conditions satisfying either $\varphi(t) \in (2\pi k, 2\pi k + \pi)$, $k \in \mathbb{N}$, or $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, $k \in \mathbb{N}$, together with $|\varphi(t) - 2\pi k| \leq \varphi(0)e^{-\lambda_0 t}$, $t < 0$, tends to $2\pi k$ as $t \rightarrow \infty$.

Moreover, for $\varphi(t) \in (2\pi k, 2\pi k + \pi)$ the solution is monotone decreasing, while for $\varphi(t) \in (2\pi k - \pi, 2\pi k)$ it is monotone increasing.

Proof. First assume that $\varphi(t) \in (0, \pi)$, $t \leq 0$, u is a solution of (4.4). Let us prove that

- (i) $u(t)$ is a positive and non-increasing function;

(ii) $u(t)$ satisfies the inequality

$$u(t) \geq u(0)e^{-\lambda_0 t}, \quad t \geq 0; \quad (4.7)$$

(iii) $u(t)$ tends to zero as $t \rightarrow \infty$.

Denote $u(t) = \varphi(t)$ for $t \leq 0$ as well, then by the assumptions of the theorem, $u(t) \leq u(0)e^{-\lambda_0 t}$, $t < 0$.

We start verifying (ii) by induction. First, we prove that $u(t) \geq u(0)e^{-\lambda_0 t}$ for $t \in [0, \tau]$, and then proceed to any segment $[n\tau, (n+1)\tau]$. In the inequalities below, we use the estimates of K in (b2), the fact that $\sin u \leq u$ for $u > 0$ and $u(t) \leq u(0)e^{-\lambda_0 t}$ for $t < 0$ to evaluate the derivative of u on $[0, \tau]$:

$$\begin{aligned} \frac{du}{dt} &= -b \int_{-\infty}^{h(t)} K(t, s) \sin(u(s)) ds \geq -b \int_{-\infty}^{t-\tau} K(t, s) \varphi(s) ds \\ &\geq -\varphi(0) \frac{b}{\tau} \int_{-\infty}^{t-\tau} \exp\left\{-\frac{a}{\tau}(t-s-\tau)\right\} e^{-\lambda_0 s} ds \\ &= -\varphi(0) \frac{b}{\tau} \exp\left\{-\frac{a}{\tau}(t-\tau)\right\} \int_{-\infty}^{t-\tau} \exp\left\{\left(\frac{a}{\tau} - \lambda_0\right)s\right\} ds \\ &= -\varphi(0) \frac{b}{a - \lambda_0 \tau} e^{-\lambda_0(t-\tau)} = -\varphi(0) \lambda_0 e^{-\lambda_0 t}, \end{aligned}$$

since $a - \lambda_0 \tau = \frac{b}{\lambda_0} e^{\lambda_0 \tau}$ by (4.6).

First, by integrating $u'(t) \geq -u(0)\lambda_0 \exp(-\lambda_0 t)$, we find that $u(t) \geq u(0) \exp(-\lambda_0 t)$ on $I = [0, \tau]$. Thus $u(\tau) \geq u(0) \exp(-\lambda_0 \tau)$ and therefore $u'(t) \geq -u(\tau)\lambda \exp(-\lambda_0(t-\tau))$. After integration the latter inequality between $t \in I$ and τ and using $u(0) \leq u(\tau)e^{-\lambda_0 \tau}$ for $t \leq 0$, we obtain that

$$u(t) \leq u(\tau)e^{-\lambda_0(t-\tau)}, \quad t \in (-\infty, \tau]. \quad (4.8)$$

Consider further the initial problem with a shifted initial point $t_0 = \tau$ instead of $t_0 = 0$, we get the same estimate as in (4.8) for any $t \in (-\infty, n\tau]$ by induction. Hence,

$$u(t) \geq u(n\tau)e^{-\lambda_0(t-n\tau)} \geq u(0)e^{-\lambda_0 t} > 0, \quad [n\tau, (n+1)\tau],$$

and the induction step proves (4.7) and justifies (ii).

Thus the solution is positive for any t . From non-negativity of K in (4.4), the solution is non-increasing and thus does not exceed $\varphi(0)$ for $t \geq 0$, which justifies (i). If $\varphi(t) \in (0, \pi)$, the positive non-increasing solution satisfies $u(t) \in (0, \pi)$ for any $t \geq 0$.

Since u is non-increasing for $t \geq 0$ and positive there is $\lim_{t \rightarrow \infty} u(t) = d$. Assuming $d > 0$ we obtain from $\int_0^\infty \int_{-\infty}^{h(t)} K(t, s) ds dt = \infty$ in (b2) that $\lim_{t \rightarrow \infty} u(t) = -\infty$, which is a contradiction, thus (iii) is also valid.

A similar process proves the case $\varphi(t) \in (-\pi, 0)$. If $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, we apply the same argument to $u - 2\pi k$. $\square$

Note that sharp conditions when all solutions of characteristic equation (4.6) have positive real parts can be found in [27, Lemma 3.1, p. 470].

Corollary 4.3. *Let*

$$\tau < \frac{a^2}{4b} e^{-a/2} \quad (4.9)$$

and $|\varphi(t) - 2\pi k| \leq \varphi(0)e^{-\lambda_0 t}$, $t < 0$, then any solution of (4.4), (4.5) with the initial conditions satisfying $\varphi(t) \in (2\pi k, 2\pi k + \pi)$, $k \in \mathbb{N}$, is monotone decreasing and tends to $2\pi k$ as $t \rightarrow \infty$. Any solution with $\varphi(t) \in (2\pi k - \pi, 2\pi k)$, $k \in \mathbb{N}$ tends to $2\pi k$ as $t \rightarrow \infty$.

Proof. Let $f(\lambda) = \tau\lambda^2 - a\lambda + be^{\lambda\tau}$, then $f(0) = b > 0$. Inequality (4.9) implies $f(a/(2\tau)) = -a^2/(4\tau) + be^{a/2} < 0$, so Eq. (4.6) has a positive solution. We invoke Theorem 4.2 to conclude the proof. $\square$

The following example illustrates that conditions (b1)–(b3) do not guarantee boundedness of the solutions of Eq. (4.4) with the generalized kernel.

Example 4.4. Let $a = \frac{1}{3} \ln(\frac{4}{\pi})$, $b = 2$, $\tau = \pi$,

$$K(t, s) = \begin{cases} \frac{1}{4}, & t \in [(2k-1)\pi, (2k+1)\pi], \quad s \in [(2k-3)\pi, (2k-2)\pi], \\ 0, & t \in [(2k-1)\pi, (2k+1)\pi], \quad s \notin [(2k-3)\pi, (2k-2)\pi]. \end{cases}$$

Then obviously $K(t, s) = 0$ for $s > t - \pi = t - \tau$, and also for $t - s > 4\pi$. The exponential estimate has the form

$$0 \leq K(t, s) \leq \frac{1}{\pi} e^{-\frac{1}{3\pi} \ln(4/\pi)(t-s-\pi)} = \frac{1}{\pi} \left(\frac{4}{\pi}\right)^{-(t-s-\pi)/(3\pi)},$$

but as $t - s - \pi \leq 3\pi$ whenever $K(t, s) \neq 0$, the right-hand side is not less than $\frac{1}{\pi} \left(\frac{4}{\pi}\right)^{-1} = \frac{1}{4}$, thus $K(t, s)$ has an exponential estimate as in (b2). Further, $u(t) = t$ is an unbounded solution of (4.4). In fact, let $u(t) = t$, $t \in [-\pi, \pi]$. Then for $t \in [\pi, 3\pi]$ we have $\frac{du}{dt} = -2 \int_{-\pi}^0 \frac{1}{4} \sin(t) dt = 1$, so $u(t) = t$ on $[-\pi, 3\pi]$. Due to the periodicity of the sine function and K , we have $\frac{du}{dt} \equiv 1$. Thus the solution is a linear function $u(t) = t$ and it is unbounded.

In the following theorem we will prove that for a non-autonomous case the solution of the sunflower equation is bounded by a linear function.

Consider the non-autonomous sunflower equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t) \sin x(h(t)) = 0. \quad (4.10)$$

Theorem 4.5. *Suppose $a(t) \geq a_0 > 0$, $|b(t)| \leq b_0$. For any solution $x(t)$ of Eq. (4.10) we have the estimates*

$$|x(t)| \leq |x(t_0)| + \left(|\dot{x}(0)| + \frac{b_0}{a_0} \right) t, \quad |\dot{x}(t)| \leq |\dot{x}(0)| + \frac{b_0}{a_0}.$$

Proof. Denote $\dot{x} = y$, $f(t) = b(t) \sin x(h(t))$, where $|f(t)| \leq b_0$. Then $\dot{y}(t) + a(t)y(t) + f(t) = 0$, hence $y(t) = y(0) + \int_0^t e^{-\int_s^t a(\tau) d\tau} f(s) ds$. Then

$$\begin{aligned} |\dot{x}(t)| &\leq |\dot{x}(0)| + \int_0^t e^{-a_0(t-s)} |f(s)| ds \leq |\dot{x}(0)| + \frac{b_0}{a_0}, \\ x(t) &= x(0) + \int_0^t \dot{x}(s) ds, \quad |x(t)| \leq |x(t_0)| + \left(|\dot{x}(0)| + \frac{b_0}{a_0} \right) t. \end{aligned}$$

□

Local stability conditions for Eq. (4.10) one can find in the following theorem.

Theorem 4.6. *Suppose $0 < a \leq a(t) \leq A$, $0 < b \leq b(t) \leq B$, $t - h(t) \leq \tau$ and at least one of the following conditions hold:*

1. $B \leq \frac{a^2}{4}$, $\frac{a}{2}(A - a) < b - aB\tau$,
2. $b \geq \frac{a}{2}(A - \frac{a}{2})$, $B(1 + a\tau) < \frac{a^2}{2}$.

Then any equilibrium $x(t) = 2k\pi$, $k = 0, \dots$ of Eq. (4.10) is locally asymptotically stable. For any equilibrium $x(t) = (2k + 1)\pi$, $k = 0, \dots$ the linearized equation associated with Eq. (4.10) at this equilibrium is not asymptotically stable.

Proof. For the equilibrium $x(t) = 2k\pi$, the linearization of Eq. (4.10) has the form

$$\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(h(t)) = 0,$$

which is exponentially stable by Theorem 2.9.

It is well known (see, for example [4]) that exponential stability of a linearized equation implies asymptotic stability of the nonlinear equation, in our case Eq. (4.10).

For the equilibrium $x(t) = (2k + 1)\pi$, the linearized equation for (4.10) has the form

$$\ddot{y}(t) + a(t)\dot{y}(t) - b(t)y(h(t)) = 0. \quad (4.11)$$

Denote by $y_0(t)$ the solution of Eq. (4.11) with the following initial conditions

$$y(t) = 0, \quad t < 0; \quad y(0) = 0, \quad \dot{y}(0) = 1.$$

We will prove that

$$y_0(t) > 0, \quad \dot{y}_0(t) > 0, \quad t > 0. \quad (4.12)$$

Initial conditions imply that for some interval $(0, t_0]$ conditions (4.12) hold. Suppose there exists $c > t_0$ such that $y_0(t) > 0$, $\dot{y}_0(t) > 0$, $0 < t < c$ and $y_0(c) = 0$ or $\dot{y}_0(c) = 0$. We have $\dot{y}(0) = 1$ and

$$\dot{y}_0(t) = e^{-\int_0^t a(\tau) d\tau} + \int_0^t e^{-\int_s^t a(\tau) d\tau} b(s) y_0(h(s)) ds,$$

hence

$$\dot{y}_0(c) = e^{-\int_0^c a(\tau) d\tau} + \int_0^c e^{-\int_s^c a(\tau) d\tau} b(s) y_0(h(s)) ds > 0.$$

Since $\dot{y}_0(t) > 0$ for $t \in [0, c]$, we have $y(t) > 0$ for $t \in (0, c]$ and $y_0(c) > 0$, which leads to a contradiction.

Hence $y_0(t)$ does not tend to zero, and thus linearized equation (4.11) is not asymptotically stable in the following sense: there is $\varepsilon > 0$ such that once $y(t) \in ((2k + 1)\pi - \varepsilon, (2k + 1)\pi + \varepsilon)$ for $t \leq 0$, for the solution y of the

linearized equation the value $|y(t) - (2k + 1)\pi|$ is increasing in t , as long as $y(t) \in ((2k + 1)\pi - \varepsilon, (2k + 1)\pi + \varepsilon)$. $\square$

5. Concluding remarks

The technique of reduction of a high-order linear differential equation to a system by the substitution $x^{(k)} = y_{k+1}$ is quite common. However, this substitution does not depend on the parameters of the original equation, and therefore does not offer new insight from a qualitative analysis point of view. Instead, we proposed a substitution which exploits the parameters of the original model. By using that approach, a broad class of the second order non-autonomous linear equations with delays was examined and explicit easily-verifiable sufficient stability conditions were obtained. There is a natural extension of this approach to stability analysis of higher-order models. For the nonlinear second order non-autonomous equations with delays we applied the linearization technique and the results obtained for linear models. Our stability tests are applicable to some milling models, e.g. models (1.2) and (1.3), and to a non-autonomous Kaldor–Kalecki business cycle model. Several numerical examples illustrate the application of the stability tests. We suggest that a similar technique can be developed for higher order linear delay equations, with or without non-delay terms. For a non-autonomous version of a classical sunflower model, we verified that the derivative is bounded and thus the solution has a linear bound. Example 4.4 illustrates the existence of an unbounded linearly growing solution for the generalized sunflower equation. We also obtained sufficient conditions under which a solution tends to one of the infinite number of the equilibrium points.

Solution of the following problems will complement the results of the present paper:

1. In all stability conditions obtained, we used lower and upper bounds of the coefficients and the delays. It is interesting to obtain stability conditions in an integral form, for instance, in the assumptions of Theorem 2.9 replace the term $a\tau_k$ by, generally, a smaller term $\int_{h_k(t)}^t a(s) ds$.
2. Apply the technique used in the paper to examine delay differential equations of higher order.
3. Is it possible to generalize Theorem 4.2 to the case when the initial function $\varphi(t) \in (2\pi k - \pi, 2\pi k + \pi)$ and characteristic equation (4.6) has a solution with a positive real part?
4. Establish necessary stability conditions for the equations considered in this paper.
5. For the sunflower equation and its modifications establish set of conditions to guarantee boundedness of all solutions.

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