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Nodal set of strongly competition systems with fractional Laplacian

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Abstract. This paper is concerned with the local structure of the nodal set of segregated configurations associated with a class of fractional singularly perturbed elliptic systems. We prove that the nodal set is a collection of smooth hyper-surfaces, up to a singular set with Hausdorff dimension not greater than $n-2$. The proof relies upon a clean-up lemma and the classical dimension reduction principle by Federer.

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1. Introduction

Let $M \geqslant 2$ be a given integer, and let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. In this paper we consider the following non-variational differential system, involving the square root of Laplacian

$$\begin{cases} (-\triangle)^{1/2} u_{i,k} = -k u_{i,k} \sum_{j \neq i} u_{j,k} & \text{in } \Omega, \\ u_{i,k} = \phi_i,\ i = 1, \ldots, M, & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where k is positive and large, the exterior boundary values ϕ_i are given smooth functions. This system can be used to describe M densities (of mass, populations, probability,..) distributed in a domain and subject to laws of fractional diffusion and Lotka–Volterra type competitive interaction [27]. The square root of Laplacian arises when the underlying Gaussian process is replaced by the Levy one, in order to allow discontinuous random walks. The action of the integro-differential operator $(-\triangle)^{1/2}$ on a smooth bounded functional $u : \mathbb{R}^n \to \mathbb{R}$ is defined by

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$$(-\triangle)^{1/2}u(x)=\gamma_n \mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-z|^{n+1}}dz \quad \text{with } \gamma_n=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}, \tag{1.2}$$

and the notation PV means that the integral is taken in the Cauchy principal value sense.

In the study of (1.1), a peculiar issue is the analysis of the behavior of the densities in the case of strong competition, i.e. when $k \to +\infty$. In such situation, one expects the formation of self-organized patterns, in which the limiting densities are spatially segregated, therefore inducing a free boundary problem. In this paper, we shall present some results concerning this topic, when the creation of the free boundary is triggered by the interplay between fractional diffusion and competitive interaction.

The study of the strong competition limits of elliptic systems is of interest in different applicative contests, from biological models for competing species to the phase-segregation phenomenon in Bose–Einstein condensation. Regarding the standard Laplace diffusion case, abroad literature is present, starting from [5,9–12,16], in a series of recent papers [2,4,21–23,28,30,32–34], also in the parabolic case [13–15,29]. Among the others, the following results are known: the uniform in k bounds for solutions of corresponding systems in Hölder spaces, the regularity of the limiting profiles and the regularity of the free boundaries in the singular limit.

Coming to the fractional diffusion case (1.1), there have been few analytical results in the literature. Verzini and Zilio [27] proved the local uniform C^α $(0<\alpha<1)$ bounds for solutions to system (1.1) (similar results concerning the systems of the relativistic version of Bose–Einstein condensation were given in [25,26], to which we refer for further details). In our recent paper [31], we gave a local uniform Lipschitz bound, using Kato inequality.

Under the results just described, we continue the study of system (1.1). We will present some results concerning the regularity of the free boundary in the singular limit. Before stating the main result, we assume throughout the paper that $\phi_i \in C^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $i=1,2,\ldots,M$, are given nonnegative functions, and assume further that $\phi_i\phi_j=0$ for all $i\neq j$. For system (1.1), we only consider nonnegative solutions, that is $u_{i,k}\geqslant 0$ for all i. Our regularity results for the nodal set associated with (1.1) can be summarized in the following theorem.

Theorem 1.1. *Let $u=(u_1,\ldots,u_M)$ be the singular limit of (1.1), whose components are all nonnegative in $\mathbb{R}^n$ and Lipschitz continuous in the interior of Ω. Assume that $u(x)\not\equiv 0$ in Ω. Let us consider the nodal set $\Gamma(u)=\{x\in\Omega : u(x)=0\}$. Then the following properties hold:*

(i) *$\mathcal{H}_{dim}(\Gamma(u))\leqslant n-1$. Moreover there exists a set $\Sigma(u)\subseteq\Gamma(u)$, relatively open in $\Gamma(u)$, such that $\mathcal{H}_{dim}(\Gamma(u)\backslash\Sigma(u))\leqslant n-2$;*
(ii) *$\Sigma(u)$ is a collection of smooth hyper-surfaces.*

For the standard Laplace diffusion case, Theorem 1.1 has been proved by Caffarelli et al. [4]. Here we extend their results to the fractional diffusion case. Our proof rests on the representation of $(-\triangle)^{1/2}$ as Dirichlet-to-Neumann operator associated to the harmonic extension to the open half-space $\mathbb{R}^{n+1}_+ :=$

$\mathbb{R}^n \times (0, +\infty)$ given by the convolution product with the Poisson kernel [7]. More precisely, given $u(x)$ defined in $\mathbb{R}^n$, extend it to $v(x, y)$ in $\mathbb{R}^{n+1}_+$ by

$$v(x, y) := \gamma_n \int_{\mathbb{R}^n} \frac{yu(z)}{(|x - z|^2 + y^2)^{(n+1)/2}} dz. \tag{1.3}$$

Then $(-\triangle)^{1/2}u(x) = -\partial_y v(x, 0)$ and $v(x, 0) = u(x)$ in $\mathbb{R}^n$. When applying the extension procedure to a solution $u_{i,k}$ of system (1.1), we end up with the following mixed boundary value problem

$$\begin{cases} \triangle v_{i,k} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu v_{i,k} := -\partial_y v_{i,k} = -k v_{i,k} \sum_{j \neq i} v_{j,k} & \text{on } \Omega \times \{0\}, \\ v_{i,k} = \phi_i, \ i = 1, 2, \ldots, M, & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.4}$$

The Proof of Theorem 1.1 follows the mainstream of [3,4,24], based upon a clean-up lemma and the classical dimension reduction principle by Federer. The original clean-up lemma is stated for the standard Laplacian diffusion case, see [4,14]. Here we generalize this lemma to the case of fractional diffusion. The strategy of the proof is the following. First using harmonic extensions defined in (1.3), the nodal set $\Gamma(u)$ is equal to $\Gamma(v(\cdot, 0))$. Here $v = (v_1, \ldots, v_M)$ is the singular limit of solutions $v_k = (v_{1,k}, \ldots, v_{M,k})$ corresponding to system (1.4). Then we establish the clean-up lemma, which implies the regularity of the regular part. Next we prove the Almgren's monotonicity formula and make a blowup to obtain the possible limits of Almgren's frequency. The main difficult in establishing this monotonicity formula is the lack of variational structure for system (1.1), and as a peculiar difference with respect to the standard Laplacian diffusion case, the singular limit does not segregate on the whole $\mathbb{R}^{n+1}_+$, but only on its boundary $\mathbb{R}^n$. However we have discovered a new vector of functions Z (see Sect. 4) adopted to our setting. Finally, using the Federer's reduction principle, we are able to prove the Hausdorff dimension estimates for the nodal sets.

2. Some preliminary results

In this section, we will discuss some preliminary facts concerning the limiting profiles of system (1.4). As mentioned in the introduction, it suffices to prove the regularity result for the nodal set $\Gamma(v(\cdot, 0)) := \{x \in \Omega : v(x, 0) = 0\}$ of the singular limit $v = (v_1, \ldots, v_M)$ associated with system (1.4). By maximum principle, we can assume that v_k are bounded in $L^\infty(\mathbb{R}^{n+1}_+)$ independently of k. Concerning the segregation property of the singular limit we have the following theorem, which has already been proved in [27].

Theorem 2.1. ([27, Proposition 5.1]) *Any sequence of $\{v_k\}_{k\in\mathbb{N}}$, $k \to +\infty$, of solutions to (1.4) admits a subsequence which converges to a limiting profile $v \in (H^1 \cap C^{0,\alpha})_{loc}(\mathbb{R}^{n+1}_+ \cup \Omega)$, for every $\alpha \in (0, 1)$. Moreover v weakly solves*

$$\begin{cases} \triangle v_i = 0 & in\ \mathbb{R}^{n+1}_+, \\ \partial_\nu v_i \leqslant 0 & in\ \Omega \times \{0\}, \\ \partial_\nu (v_i - \sum_{j\neq i} v_j) \geqslant 0 & in\ \Omega \times \{0\}, \\ v_i \partial_\nu (v_i - \sum_{j\neq i} v_j) = 0 & in\ \Omega \times \{0\}, \end{cases} \tag{2.1}$$

and $v_i(x,0)$ are locally Lipschitz continuous satisfying $v_i(x,0)v_j(x,0) = 0$ for all $i \neq j$ in Ω.

Corollary 2.1. *For every $i \neq j$ the functions $z = v_i - v_j$ are such that*

$$\begin{cases} -\triangle z^\pm \leqslant 0 & in\ \mathbb{R}^{n+1}_+, \\ \partial_\nu z^\pm \leqslant 0 & in\ \Omega \times \{0\}. \end{cases}$$

Proof. A subtraction of the equations satisfied by $v_{i,k}$ and $v_{j,k}$ yields

$$\begin{cases} -\triangle (v_{i,k} - v_{j,k}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu (v_{i,k} - v_{j,k}) = -k(v_{i,k} - v_{j,k}) \sum_{h\neq i,j} v_{h,k} & \text{in } \Omega \times \{0\}. \end{cases}$$

Now using the Kato inequality we have

$$\begin{cases} -\triangle |v_{i,k} - v_{j,k}| \leqslant 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu |v_{i,k} - v_{j,k}| = -k|v_{i,k} - v_{j,k}| \sum_{h\neq i,j} v_{h,k} & \text{in } \Omega \times \{0\}. \end{cases}$$

Taking into account of the equalities

$$z^+ = \frac{|z| + z}{2}, \quad z^- = \frac{|z| - z}{2},$$

we finally obtain that

$$\begin{cases} -\triangle (v_{i,k} - v_{j,k})^\pm \leqslant 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu (v_{i,k} - v_{j,k})^\pm = -k(v_{i,k} - v_{j,k})^\pm \sum_{h\neq i,j} v_{h,k} \leqslant 0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.2}$$

which implies the desired result after passing to the weak limit (up to a subsequence) as $k \to +\infty$. □

Remark 2.1. From the Proof of Corollary 2.1, we obtain the existence of nonnegative Radon measures $\mu^+_{i,j}$, $\mu^-_{i,j} \in \mathcal{M}(\Omega')$ such that

$$\partial_\nu (v_i - v_j)^\pm = -\mu^\pm_{i,j} \quad \text{in } \Omega' \times \{0\},$$

for every $\Omega' \Subset \Omega$. In particular,

$$\begin{cases} -\triangle (v_i - v_j) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu (v_i - v_j) = -\mu_{i,j} & \text{on } \Omega \times \{0\}, \end{cases} \tag{2.3}$$

where $\mu_{i,j} = \mu^+_{i,j} - \mu^-_{i,j}$.

Indeed, for every nonnegative $\eta \in C^\infty(\overline{\mathbb{R}^{n+1}_+})$ compactly supported in $\mathbb{R}^{n+1}_+ \cup \Omega$, we test Eq. (2.2) to find

$$\begin{aligned} 0 &\leqslant k \int_{\Omega \times \{0\}} \eta (v_{i,k} - v_{j,k})^\pm \sum_{h\neq i,j} v_{h,k} dx \\ &\leqslant \int_{\mathbb{R}^{n+1}_+} (v_{i,k} - v_{j,k})^\pm \triangle \eta dx dy + \int_{\Omega \times \{0\}} (v_{i,k} - v_{j,k})^\pm \partial_\nu \eta. \end{aligned}$$

Since v_k is uniformly bounded in $L^\infty(\mathbb{R}^{n+1}_+)$, we infer that

$$k\int_{\Omega'\times\{0\}}(v_{i,k}-v_{j,k})^{\pm}\sum_{h\neq i,j}v_{h,k}dx\leqslant C(\Omega'),$$

for any compact set $\Omega'\Subset\Omega$. Thus there exist nonnegative Radon measures $\mu^+_{i,j}$, $\mu^-_{i,j}\in\mathcal{M}(\Omega')$ such that

$$k\int_{\Omega'\times\{0\}}(v_{i,k}-v_{j,k})^{\pm}\sum_{h\neq i,j}v_{h,k}dx\rightharpoonup\mu^{\pm}_{i,j}\ \text{weak}-*\ \text{in}\ \mathcal{M}(\Omega').$$

3. A clean up result

In this section, we start to analysis the geometric properties of the nodal set. We first recall the definition of s-capacity.

Definition 3.1. ([19]) Let s be a positive number. Let E and T be Borel sets in $\mathbb{R}^n$. A measure μ is said to be admissible if the following condition holds:

$$\int_E\frac{d\mu(y)}{|x-y|^s}\leqslant 1\quad\text{for}\quad x\in T.\tag{3.1}$$

We define the relative s-capacity of the set E with respect to the set T as

$$C^T_s(E)=\sup\mu(E),$$

where the supremum is taken over all admissible measures. In particular, the relative capacity of a set E with respect to its complement will be called simply the s-capacity and denote by $C_s(E)$.

Remark 3.1. Let E_1 and E_2 be two Borel sets in $\mathbb{R}^n$. If μ is an admissible measure on E_1, then the measure μ' defined on E_2 by the equality

$$\mu'(A)=\mu(A\cap E_1)$$

is also admissible and the following equality holds:

$$\mu'(E_2)=\mu(E_1)\quad\text{for every}\quad E_2\supseteq E_1.$$

Now we state a decay property of the $\frac{1}{2}$-harmonic functions, based on a similar one for harmonic functions present in [19, Lemma 4.1, Chapter 1].

Lemma 3.1. *Let a domain $D\subset\mathbb{R}^n$ intersect the ball $B_{4R}(0)$. Denote by H the intersection of the complement of D and the ball $B_R(0)$, and by Γ the part of the boundary of D in $B_{4R}(0)$. Let $s=n-1$ and u be a continuous nonnegative solution of $(-\triangle)^{1/2}u\leqslant 0$ in D and $u=0$ on $\mathbb{R}^n\backslash D$. Then*

$$\sup_{x\in D}u(x)\geqslant\left(1+\xi\frac{C_s(H)}{R^s}\right)\sup_{x\in D\cap B_R(0)}u(x),\tag{3.2}$$

where $\xi>0$ is a constant depending on n.

Proof. Since, under the similarity transformation, the capacity of a set is multiplied by the sth power of the similarity coefficient (cf. Theorem 2.3, Chapter 1 in [19]), it suffices to consider the case $R = 1$.

In the case of $C_s(H) = 0$, the inequality holds trivially. In the case $C_s(H) > 0$, we take an arbitrary $\varepsilon > 0$, $0 < \varepsilon < C_s(H)$, and a measure μ such that

$$U(x) = \int_H \frac{d\mu(y)}{|x-y|^s} \leqslant 1 \quad \text{for } x \notin H \text{ and } \mu(H) > C_s(H) - \varepsilon.$$

Denote by $M = \sup_{x\in D} u(x)$ and set $v(x) = M(1 - U(x) + \frac{C_s(H)}{3^s})$. We also denote μ' the measure defined on $\mathbb{R}^n$ by

$$\mu'(A) = \mu(A \cap H) \text{ for every } A \subseteq \mathbb{R}^n.$$

Then by Remark 3.1,

$$U(x) = \int_H \frac{d\mu(y)}{|x-y|^s} = \int_{\mathbb{R}^n} \frac{d\mu'(y)}{|x-y|^s}.$$

In the following we let $s = n - 1$, it is not difficult to check (see [20]) that the function $U(x)$ satisfies

$$(-\triangle)^{1/2} U(x) = C(n)\mu' \quad \text{in } \mathbb{R}^n.$$

As a consequence, $(-\triangle)^{1/2}U(x) = 0$ outside H, and thus

$$(-\triangle)^{1/2} v(x) = 0 \quad \text{in } D.$$

Further, $v(x) \geqslant u(x)$ on $B_4(0)\backslash D$. We need to prove that the inequality also holds on $\mathbb{R}^n\backslash B_4(0)$. Indeed,

$$\begin{aligned} v|_{\mathbb{R}^n\backslash B_4(0)} &\geqslant M\left[1 - \frac{1}{\inf_{x\in\mathbb{R}^n\backslash B_4(0),\ y\in B_1(0)} |x-y|^s} \int d\mu + \frac{C_s(H)}{3^s}\right] \\ &\geqslant M\left[1 - \frac{1}{3^s} C_s(H) + \frac{C_s(H)}{3^s}\right] = M. \end{aligned}$$

Now we can apply the maximum principle to conclude that

$$\begin{aligned} \sup_{x\in D\cap B_1(0)} u(x) &\leqslant \sup_{x\in D\cap B_1(0)} v(x) \\ &\leqslant M\left(1 - \frac{1}{\sup_{x\in B_1(0),\ y\in B_1(0)} |x-y|^s} \int d\mu + \frac{C_s(H)}{3^s}\right) \\ &\leqslant M\left[1 - \left(\frac{1}{2^s} - \frac{1}{3^s}\right) C_s(H) + \frac{\varepsilon}{2^s}\right], \end{aligned}$$

which completes the proof of the lemma since ε was arbitrary. □

Corollary 3.1. *Under the assumptions of Lemma 3.1, we have*

$$\sup_{x\in D} u(x) \geqslant \left(1 + \eta\frac{|H|}{R^n}\right) \sup_{x\in D\cap B_R(0)} u(x), \tag{3.3}$$

where $|H|$ denotes the Lebesgue measure of the set H and $\eta > 0$ is a constant depending on n.

Proof. Once Lemma 3.1 is settled, the proof follows almost word by word the one presented in [19, Chapter 1, Lemma 4.2] for a similar situation. □

In the following, we need a technical lemma which is the fractional analogue of Lemma 12 in [4].

Lemma 3.2. *Let u be a continuous nonnegative solution of $(-\triangle)^{1/2}u \leqslant 0$ in D and $u = 0$ on $\mathbb{R}^n \backslash D$. Assume that D is "narrow" in the sense that any ball of radius h, $B_h(0)$, contained in $B_1(0)$, intersects the complement of D, $\mathcal{C}D$, satisfies*

$$\frac{|B_h \cap \mathcal{C}D|}{|B_h|} > \frac{1}{2}. \tag{3.4}$$

Then there exists $C > 0$ such that

$$\sup_{B_{1/2}(0)} u \leqslant e^{-C(1-|x|)/h} \sup_{B_1(0)} u.$$

Proof. We prove that in the ball $B_{1-kh}(0)$, $k = 1, \ldots, N$, where $N \sim h^{-1}$, we have

$$u(x) \leqslant \frac{1}{C_1} \sup_{B_{1-(k-1)h}(0)} u, \tag{3.5}$$

for some $C_1 > 1$. Indeed, combining Corollary 3.1 with (3.4) we have

$$\sup_{B_{1-kh}(0)} u(x) \leqslant \frac{1}{1 + \eta \frac{|\mathcal{C}D \cap B_{1-(k-1)h}(0)|}{|B_{1-(k-1)h}(0)|}} \sup_{B_{1-(k-1)h}(0)} u \leqslant \frac{1}{C_1} \sup_{B_{1-(k-1)h}(0)} u,$$

for some universal constant $C_1 > 1$. We can now iterate (3.5) $\mathrm{int}(\frac{1-|x|}{h})$ times, and hence

$$u(x) \leqslant \left(\frac{1}{C_1}\right)^{\frac{1-|x|}{h}} \sup_{B_1(0)} u \leqslant e^{-(\frac{1-|x|}{h}) \log C_1} \sup_{B_1(0)} u.$$

Thus the conclusion holds for $C := \log C_1 > 0$. □

In the following, we shall use the notations: for any dimension $n \geqslant 1$, we consider $(n+1)$-dimensional half ball $B_r^+(x_0, 0) := B_r(x_0, 0) \cap \{y > 0\}$, which boundary contains the spherical part $\partial^+ B_r^+ := \partial B_r \cap \{y > 0\}$ and the flat one $\partial^0 B_r := B_r \cap \{y = 0\}$. Denote by $X_0 = (x_0, 0)$ and $X = (x, y)$, we recall the following monotonicity formula by Alt et al. stated in [1].

Lemma 3.3. ([1]) *Let $u, v \in H^1_{loc}(\mathbb{R}^{n+1}) \cap C(\mathbb{R}^{n+1})$ be continuous nonegative functions such that $uv \equiv 0$. Assume moreover that $-\triangle u \leqslant 0$, $-\triangle v \leqslant 0$ in $\mathbb{R}^{n+1}$ and $u(X_0) = v(X_0) = 0$. Then the function*

$$\Phi(r) = \Phi(X_0, u, v, r) = \frac{1}{r^4} \int_{B_r^+(X_0)} \frac{|\nabla u|^2}{|X - X_0|^{n-1}} dX \int_{B_r^+(X_0)} \frac{|\nabla v|^2}{|X - X_0|^{n-1}} dX \tag{3.6}$$

is nondecreasing for $r \in (0, +\infty)$.

Lemma 3.4. *Let $v = (v_1, \ldots, v_M)$ be the singular limit in Theorem 2.1. Denote by $z := v_1 - v_2$ and assume that at $x_0 \in \Gamma(v(\cdot, 0))$*

$$\Phi(0^+) = \lim_{r \to 0^+} \Phi((x_0, 0), z^+, z^-, r) = \lambda > 0. \tag{3.7}$$

Then

(a) *the sequence $v_k(x, y) = \frac{1}{\lambda_k} v(x_0 + \lambda_k x, \lambda_k y)$, $\lambda_k \to 0$, converges to*

$$\overline{v}_1(x, 0) = \alpha x_1^+, \quad \overline{v}_2(x, 0) = \beta x_1^-, \quad \sum_{j \neq 1,2} \overline{v}_j(x, 0) = 0;$$

(b) *$\partial_\nu(\overline{v}_1 - \overline{v}_2)(x, 0) = 0$, so $\alpha = \beta$.*

Proof. Without loss of generality, we may assume $x_0 = 0$. First note that v in harmonic in $\mathbb{R}^{n+1}_+$ and $v_i(x, 0)$ is locally Lipschitz continuous in Ω, then v is locally Lipschitz continuous in $\mathbb{R}^{n+1}_+ \cup \Omega$, see for instance [8]. We now consider a blowup sequence

$$v_k(x, y) = \frac{1}{\lambda_k} v(x_0 + \lambda_k x, \lambda_k y) \quad \text{with } \lambda_k \to 0.$$

Let $z_k = v_{1,k} - v_{2,k}$ and

$$\Phi^\pm = \Phi^\pm(0, z^\pm, \lambda_k) = \frac{1}{\lambda_k^2} \int_{B^+_{\lambda_k}(0)} \frac{|\nabla z^\pm|^2}{|X|^{n-1}} dx dy.$$

In this setting, z_k is locally uniform Lipschitz bounded. Using (3.7) there exists a positive constant c_0 such that for k sufficient large

$$\Phi^\pm \geqslant c_0 > 0. \tag{3.8}$$

Define now $\widetilde{z}^\pm_k = z^\pm_k / \sqrt{\Phi^\pm}$, then $\Phi^\pm(0, \widetilde{z}^\pm_k, 1) = 1$. We claim that, up to a subsequence,

$$\widetilde{z}^\pm_k \to \widetilde{z}^\pm_0 \text{ strongly in } H^1_{\mathrm{loc}}(B^+_1(0)). \tag{3.9}$$

Indeed, by Remark 2.1 the functions $\widetilde{z}^\pm_k$ satisfy

$$\begin{cases} -\triangle \widetilde{z}^\pm_k \leqslant 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu z^\pm_k = -\widetilde{\mu}^\pm_{1,2,k} & \text{on } \Omega_k \times \{0\}, \end{cases} \tag{3.10}$$

where $\Omega_k := \Omega/\lambda_k$ and $\widetilde{\mu}^\pm_{1,2,k}$ are measures defined by

$$\widetilde{\mu}^\pm_{1,2,k}(A) = \frac{1}{\lambda_k^{n-1} \sqrt{\Phi^\pm}} \mu^\pm_{1,2}(\lambda_k A).$$

We divide the proof of (3.9) into several steps.

Step 1. There exists $C > 0$, independent of k, such that for every $0 < R < 1$,

$$\|\widetilde{z}^\pm_k\|_{H^1(B^+_R(0))} \leqslant C.$$

We choose k sufficient large that (3.8) holds and $\overline{B}^+_{\lambda_k}(0) \Subset \mathbb{R}^{n+1}_+ \cup \Omega$. Then

$$\int_{B^+_1(0)} |\nabla \widetilde{z}^\pm_k|^2 dx dy \leqslant \frac{1}{c_0} \int_{B^+_{\lambda_k}(0)} |\nabla \widetilde{z}^\pm_k|^2 dx dy \leqslant C.$$

Since $\partial_\nu \widetilde{z}_k^{\pm} \leqslant 0$ on $\Omega_k \times \{0\}$, then a standard Brezis–Kato type argument together with the H^1-boundedness yield that $\|\widetilde{z}_k^{\pm}\|_{L^\infty(B_R^+(0))} \leqslant C(R)$ for every k.

Step 2. For every given $0 < R < 1$, there exists $C > 0$, independent of k, such that $\|\widetilde{\mu}_{1,2,k}^{\pm}\|_{\mathcal{M}(\partial^0 B_R(0))} = \widetilde{\mu}_{1,2,k}^{\pm}(\partial^0 B_R(0)) \leqslant C$.

We multiply (3.10) by $\varphi(x, y)$, a cutoff function such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B_R(0)$ and $\varphi = 0$ outside $B_1(0)$, it holds after an integration by parts that

$$\int_{B_1^+(0)} \nabla \widetilde{z}_k^{\pm} \cdot \nabla \varphi dxdy \leqslant -\int_{\partial^0 B_1(0)} \varphi(x, 0) d\widetilde{\mu}_{1,2,k}^{\pm}.$$

That is

$$\widetilde{\mu}_{1,2,k}^{\pm}(\partial^0 B_R(0)) \leqslant C(R)\|\nabla \widetilde{z}_k^{\pm}\|_{L^2(B_1^+(0))} \leqslant C(R).$$

Step 3. Strong H^1_{loc} convergence.

By Steps 1. and 2. we have proved the existence of nonnegative functions $\widetilde{z}_0^{\pm}$ and nonnegative measures $\widetilde{\mu}_{1,2}^{\pm}$ such that

$$\begin{aligned}
&\widetilde{z}_k^{\pm} \to \widetilde{z}_0^{\pm} \text{ weakly in} H^1_{\text{loc}}(B_1^+(0)) \text{ and strongly in } L^2_{\text{loc}}(B_1^+(0)),\\
&\widetilde{\mu}_{1,2,k}^{\pm} \rightharpoonup \widetilde{\mu}_{1,2}^{\pm} \text{ weak} - * \text{ in } \mathcal{M}_{\text{loc}}(\partial^0 B_1(0)).
\end{aligned}$$

To obtain the strong H^1-convergence, we observe that $\widetilde{z}_k^{\pm}(x, 0)$ are locally uniform Lipschitz bounded due to the fact that $\widetilde{z}^{\pm}$ are locally Lipschitz continuous in x. Thus, from the uniqueness of the limit, we have $\widetilde{z}_k^{\pm}(x, 0) \to \widetilde{z}_0^{\pm}(x, 0)$ locally uniformly in $\partial^0 B_1(0)$. Now multiplying (3.10) by $(\widetilde{z}_k^{\pm} - \widetilde{z}_0^{\pm})\eta$, with $\eta \in C_0^\infty(B_1(0))$ yields

$$\begin{aligned}
\int_{B_1^+(0)} \nabla \widetilde{z}_k^{\pm} \cdot \nabla(\widetilde{z}_k^{\pm} - \nabla \widetilde{z}_0^{\pm})\eta dxdy \leqslant &-\int_{B_1^+(0)} \nabla \widetilde{z}_k^{\pm} \cdot \nabla \eta(\widetilde{z}_k^{\pm} - \nabla \widetilde{z}_0^{\pm})dxdy\\
&-\int_{\partial^0 B_1(0)} \eta(\widetilde{z}_k^{\pm} - \widetilde{z}_0^{\pm})d\widetilde{\mu}_{i,2,k}^{\pm} \to 0,
\end{aligned}$$

and the strong convergence follows immediately.

By (3.9) we have

$$\Phi(0, \widetilde{z}_0^{+}, \widetilde{z}_0^{-}, s) = 1 \quad \text{for every } s \leqslant 1. \tag{3.11}$$

Indeed,

$$\frac{\Phi(0, \widetilde{z}_k^{+}, \widetilde{z}_k^{-}, s)}{\Phi(0, \widetilde{z}_k^{+}, \widetilde{z}_k^{-}, t)} = \frac{\Phi(0, \widetilde{z}^{+}, \widetilde{z}^{-}, s\lambda_k)}{\Phi(0, \widetilde{z}^{+}, \widetilde{z}^{-}, t\lambda_k)} \to 1,$$

as $\lambda_k \to 0$. Using the strong H^1-convergence we have $\Phi(0, \widetilde{z}_0^{+}, \widetilde{z}_0^{-}, s) = \Phi(0, \widetilde{z}_0^{+}, \widetilde{z}_0^{-}, s) = 1$.

We now reflect evenly $\widetilde{z}_0^{\pm}$ across the hyperplane $y = 0$, defining

$$\widetilde{z}_0^{\pm}(x, y) = \begin{cases} \widetilde{z}_0^{\pm}(x, y), & y \geqslant 0,\\ \widetilde{z}_0^{\pm}(x, -y), & y < 0. \end{cases}$$

From (3.11) and Corollary 12.4 in [6], we deduce that $\widehat{z}_0^{\pm}$ must be a two-planes solution with respect to a direction transversal to the plane $y = 0$, say e_1, due to the even symmetry of $\widehat{z}_0^{\pm}$. Therefore

$$\widetilde{z}_0^{+} = \alpha x_1^{+},\ \widetilde{z}_0^{-} = \beta x_1^{-} \quad \text{for some } \alpha, \beta > 0.$$

Using the boundary segregation condition $v_1(x, 0)v_2(x, 0) = 0$ we conclude that

$$v_{1,k}(x, 0) \to \widetilde{\alpha} x_1^{+}, \quad v_{2,k}(x, 0) \to \widetilde{\beta} x_1^{-},$$

for some $\widetilde{\alpha}, \widetilde{\beta} > 0$ depending on λ. Note that $(1/\lambda_k)v_j(\lambda_k x, 0)$ is Lipschitz and supported in narrower and narrower domains, so $\overline{v}_j(x, 0) \equiv 0$.

(b) follows from the fact that $\partial_\nu(v_1 - v_2 - \sum_{j\neq 1,2})(x, 0) \geqslant 0$ and $\partial_\nu(v_2 - v_1 - \sum_{j\neq 1,2})(x, 0) \geqslant 0$, so we must have $\widetilde{\alpha} = \widetilde{\beta}$. □

Proposition 3.1. *Assume the hypotheses of Lemma 3.4 hold. Then* $\sum_{j>2} v_j(x, 0) \equiv 0$ *in a neighborhood of* x_0.

Proof. The proof follows the iteration scheme used in [4]. By the assumption of Lemma 3.4, we can assume in $B_1(0)$

$$|(v_1 - v_2) - x_1| \leqslant h_0 \quad \text{and} \quad \left|\sum_{j\neq 1,2} v_j\right| \leqslant h_0,$$

with $h_0 > 0$ small. We decompose $v_1 - v_2 = w_0 + z_0$ such that

$$\begin{cases} -\triangle w_0 = 0 & \text{in } B_1^{+}(0),\\ w_0 = v_1 - v_2 & \text{on } \partial^{+}B_1(0) := \partial B_1(0) \cap \{y \geqslant 0\},\\ \partial_\nu w_0 = 0 & \text{on } \partial^{0}B_1(0). \end{cases}$$

z_0 is the part comes from the presence of v_j, $j > 2$. Since $|w_0 - x_1|_{\partial^{+}B_1(0)} \leqslant h_0$ we have $|w_0 - (v_1 - v_2)| \leqslant 2h_0$. Reflect evenly w_0 across the hyperplane $y = 0$ and use the standard interior estimate give that

$$|\nabla_x(w_0 - x_1)| = |(\nabla_x w_0) - e_1| \leqslant Ch_0 \text{ in } \partial^{0}B_{1/2}(0),$$

where we have used the fact that $\partial_y w_0(x, 0) = 0$. We want to show that z_0 can be controlled by $\sum_{j>2} v_j$. This can be seen by the equations they satisfy,

$$\begin{cases} -\triangle z_0 = \triangle w_0 - \triangle(v_1 - v_2) = 0 & \text{in } B_1^{+}(0),\\ z_0 = (v_1 - v_2) - w_0 = 0 & \text{on } \partial^{+}B_1(0),\\ -\partial_\nu \sum_{j>2} v_j \geqslant \partial_\nu z_0 \geqslant \partial_\nu \sum_{j>2} v_j & \text{on } \partial^{0}B_1(0). \end{cases} \tag{3.12}$$

Let $\beta_0 := \sum_{j>2}(\sup_{\overline{B_1^{+}}} v_j - v_j)$ then

$$\begin{cases} -\triangle \beta_0 = 0 & \text{in } B_1^{+}(0),\\ \beta_0 \geqslant 0 & \text{on } \partial^{+}B_1(0),\\ \partial_\nu \beta_0 = -\partial_\nu \sum_{j>2} v_j & \text{on } \partial^{0}B_1(0). \end{cases} \tag{3.13}$$

By maximum principle we have

$$-Ch_0 \leqslant -\beta_0 \leqslant z_0 \leqslant \beta_0 \leqslant Ch_0 \quad \text{in } \overline{B_1^{+}}(0).$$

Therefore, for a small number h_0 to be chosen, we have the starting hypothesis: in $B_1^+(0)$, we have

(a) $|w_0 - (v_1 - v_2)| \leqslant h_0$,
(b) $|\nabla_x w(x,0) - e_1| \leqslant h_0$,
(c) $\mathrm{supp} v_j(x,0)$ for $j > 2$ is contained in the h-neighborhood of the Lipschitz level surface $w_0(x,0) = 0$.

Note that from hypothesis (c), $\sum_{j>2} v_j(x,0)$ fulfills the assumptions of Lemma 3.2. Thus in the ball $\partial^0 B_{1-s}(0)$ we have

$$\sum_{j>2} v_j(x,0) \leqslant h_0 e^{-cs/h_0} \leqslant h_0^2, \tag{3.14}$$

provided h_0 is small enough and $s = h_0^{1/2}/2$. In particular, if we decompose $v_1 - v_2 = w_1 + z_1$ such that w_1 satisfies

$$\begin{cases} -\triangle w_1 = 0 & \text{in } B_{1-s}^+(0), \\ w_1 = v_1 - v_2 & \text{on } \partial^+ B_{1-s}(0), \\ \partial_\nu w_1 = 0 & \text{on } \partial^0 B_{1-s}(0), \end{cases}$$

then $z_1 \leqslant h_0^2$. Therefore

$$|w_1 - (v_1 - v_2)| = |z_1| \leqslant h_0^2 \quad \text{in } \overline{B}_{1-s}^+(0). \tag{3.15}$$

In the following, we need a decay estimate concerning the comparison of the gradients of w_1 and w_0. Reflect evenly w_1 and w_0 across the hyperplane $y = 0$ and take a cut-off function $\eta = 1$ in $B_{1-2s}(0)$, $\eta \equiv 0$ outside $B_{1-s}(0)$, we obtain that

$$\int_{B_{1-s}(0)} |\nabla(w_1 - w_0)|^2 \eta^2 dxdy \leqslant C \int_{B_{1-s}(0)} \eta^2 (v_1 - v_0)^2 |\nabla \eta|^2 dxdy \leqslant C h_0, \tag{3.16}$$

where we have used the fact that $|w_1 - w_0| = |(v_1 - v_2) - w_0| \leqslant h_0$ on $\partial B_{1-s}(0)$ and both are harmonic in $B_{1-s}(0)$. Note also that $\nabla(w_1 - w_0)$ is harmonic in $B_{1-s}(0)$, combining (3.16) with standard Moser iteration gives that

$$\sup_{B_{1-2s}(0)} |\nabla(w_1 - w_0)| \leqslant C h_0^{1/2}.$$

In particular, by the even symmetry we have

$$\sup_{\partial^0 B_{1-2s}(0)} |\nabla_x (w_1 - w_0)(x,0)| \leqslant C h_0^{1/2}.$$

Finally, we define the iteration (starting with $R_0 = 1$)

$$h_k = h_{k-1}^2, \quad R_k = R_{k-1} - h_{k-1}^{1/2}. \tag{3.17}$$

We know that if h_0 is small enough, then h_k will converge to 0 very fast, and $\lim_{k\to+\infty} R_k \geqslant 1/2$. In each ball $B_{R_k}^+(0)$ we decompose $v_1 - v_2 = w_k + z_k$, where

$$\begin{cases} -\triangle w_k = 0 & \text{in } B_{R_k}^+(0), \\ w_k = v_1 - v_2 & \text{on } \partial^+ B_{R_k}(0), \\ \partial_\nu w_k = 0 & \text{on } \partial^0 B_{R_k}(0). \end{cases}$$

Then in $B^+_{R_k}(0)$, we have

(a) $|w_k - (v_1 - v_2)| \leqslant h_k$,
(b) $|\nabla_x (w_k - w_{k-1})(x,0)| \leqslant h_{k-1}^{1/2}$,
(c) $|\nabla_x w_k(x,0) - e_1| \leqslant \sum_{j=1}^k h_{j-1}^{1/2} \leqslant 1/4$,
(d) the level surface $w_k(x,0)$ is Lipschitz with Lipschitz constant less than 1.

This completes the proof Proposition 3.1. □

4. Almgren monotonicity formula

In this section we prove Almgren's monotonicity formula that plays a crucial role in establishing the Hausdorff dimension of the singular set. First, we know that

Lemma 4.1. *Let $\Omega' \Subset \Omega$ be fixed and define $z = v_i - v_j$ for every $i \neq j$. Then for every $x_0 \in \Gamma(v(\cdot,0)) \cap \Omega'$, the following properties hold:*

(i) *if $\Phi((x_0,0), z^+, z^-, 0^+) = 0$, then $|\nabla z(x_0,0)| = 0$;*
(ii) *if $\Phi((x_0,0), z^+, z^-, 0^+) \neq 0$, then $|\nabla z(x_0,0)| \neq 0$.*

Proof. To show the first part, we assume by contradiction that there exist $x_k \to x_0$ and $r_k \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{r_k^{n+1}} \int_{B^+_{r_k}(x_k,0)} |\nabla z|^2 dxdy = a \quad \text{with } a \in (0,+\infty] \tag{4.1}$$

We set $L_k = \frac{1}{r_k^{n+1}} \int_{B^+_{r_k}(x_k,0)} |\nabla z|^2 dxdy$ and consider the sequence of functions

$$z_k(x,y) = \frac{1}{L_k r_k} z(x_k + r_k x, r_k y) \quad \text{for } x \in \frac{\Omega' - x_k}{r_k}, \quad y \geqslant 0.$$

By (4.1) we have $\|\nabla z_k\|_{L^2(B_1^+)(0)} = 1$ for every n, and

$$\frac{1}{r^{n+1}} \int_{B_r^+(0)} |\nabla z_k|^2 dxdy = \frac{\frac{r_k^{n+1}}{r^{n+1}} \int_{B^+_{rr_k}(0)} |\nabla z|^2 dxdy}{\left(\frac{1}{r_k^{n+1}} \int_{B^+_{r_k}(x_k,0)} |\nabla z|^2 dxdy\right)^2}.$$

Since $\lim_{k\to\infty} \frac{1}{r_k^{n+1}} \int_{B^+_{r_k}(x_k,0)} |\nabla z|^2 dxdy = a > 0$, then for r_k small enough we have

$$\frac{1}{r_k^{n+1}} \int_{B^+_{r_k}(x_k,0)} |\nabla z|^2 dxdy > \frac{a}{2}.$$

Note also that $z \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Hence for every $1 < r < r_k^{-1} \mathrm{dist}(\Omega', \partial\Omega)$, we have

$$\frac{1}{r^{n+1}} \int_{B_r^+(0)} |\nabla z_k|^2 dxdy \leqslant C. \tag{4.2}$$

This implies that the sequence $\{z_k\}$ admits a nontrivial weak limit $\overline{z} \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$.

On the other hand, from the semi-continuity of $\Phi((x_0,0),z^+,z^-,r)$, given $\varepsilon>0$, there exist δ and τ such that

$$\Phi((x_n,0),z^+,z^-,\delta)\leqslant\varepsilon \quad \text{for } x_n\in\partial^0B_\tau(x_0),$$

that is

$$\frac{1}{\delta^{n+1}}\int_{B_\delta^+(x_n,0)}|\nabla z^+|^2dxdy\cdot\frac{1}{\delta^{n+1}}\int_{B_\delta^+(x_n,0)}|\nabla z^-|^2dxdy\leqslant\varepsilon.$$

Scaling to (z_k^+,z_k^-) we can distinguish among three different cases.
Case 1. Both $\|\nabla z^+\|_{L^2(B_R^+(0))}$ and $\|\nabla z^-\|_{L^2(B_R^+(0))}$ are infinitesimal for every $R>0$. In this situation, we have that there exists $c\geqslant 0$ such that $z_k\to c$. Since by even extension z_k is subharmonic and $z_k(0,0)=0$, we infer that $c\geqslant 0$.
Case 2. There exists $c>0$ such that $\|\nabla z^+\|_{L^2(B_R^+(0))}\geqslant c>0$ while $\|\nabla z^-\|_{L^2(B_R^+(0))}\to 0$. Testing the equation

$$\begin{cases}-\triangle z_k^+\leqslant 0 & \text{in } B_R^+(0),\\ \partial_\nu z_k^+\leqslant 0 & \text{on } \partial^0B_R(0),\end{cases}$$

with z_k^+ we obtain that in the limit $z_k^+\rightharpoonup\overline{z}\neq 0$, and thus $z_k^-\to 0$ strongly in $H^1(B_R^+(0))$. Reflecting evenly across $y=0$ and using again the subharmonicity of z as before, we conclude that $z_k\to c\geqslant 0$.
Case 3. There exists $c>0$ such that $\|\nabla z^-\|_{L^2(B_R^+(0))}\geqslant c>0$ while $\|\nabla z^+\|_{L^2(B_R^+(0))}\to 0$. Reasoning as in the previous case, we obtain that $z_k^+\to 0$ strongly in $H^1(B_R^+(0))$, thus $z_k\to\overline{z}=0$ by reflecting evenly across $y=0$ and using the subharmonicity of z.

In any case, $z_k^-\to 0$ and $z_k\rightharpoonup\overline{z}\geqslant 0$ in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ for all $i\neq j$. This implies in particular that $z(x,0)=(v_i-v_j)(x,0)=0$ in $H^{1/2}_{\mathrm{loc}}(\mathbb{R}^n)$.

So we conclude that if we extend $\overline{z}$ oddly across $\{y=0\}$, we obtain a harmonic function defined on $\mathbb{R}^{n+1}$ for which $\Phi(0,\overline{z}^+,\overline{z}^-,r)\leqslant C$ for all $r\geqslant 1$. From the Morrey inequality, we have that for any $X\in\mathbb{R}^{n+1}$, $|X|\geqslant 1$,

$$|\overline{z}(X)-\overline{z}(0)|\leqslant C|X|^{1-\frac{n+1}{2}}\|\nabla\overline{z}\|_{L^2(2|X|)},$$

in contradiction with the fact that $\overline{z}$ is harmonic in $\mathbb{R}^{n+1}$ and nontrivial, thanks to the classical Liouville theorem.

On the other hand, if $\Phi((x_0,0),z^+,z^-,0^+)\neq 0$ for some (v_i-v_j) and $x_0\in\Gamma(v(\cdot,0))$, then according to the clean up lemma $\partial_\nu z(x,0)=0$ in a neighborhood of x_0. In particular, $|\nabla_x z(x,0)|\neq 0$. $\square$

We can now prove Almgren's monotonicity formula adapted to our setting. As mentioned in the introduction, due to the lack of variational structure, Almgren's frequency formula does not valid for the singular limit v in current situation. Nevertheless, thanks to Lemma 4.1, we can prove the Almgren's monotonicity formula for a vector function Z, whose nodal set coincide with that of v. Precisely, we denote by Z the vector function with $\frac{M(M-1)}{2}$ components

$$Z:=(z_{1,2},z_{1,3},\cdots,z_{M-1,M}),\quad \text{where } z_{i,j}=v_i-v_j,\ i<j. \tag{4.3}$$

In this setting, the nodal set $\Gamma(v(.,0))$ coincides with the set

$$\Gamma(Z(\cdot,0)) := \{x \in \Omega : z_{i,j}(x,0) = 0, \forall i < j\}.$$

Define for every $x_0 \in \Omega' \Subset \Omega$ and $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ the quantity

$$E(r) = E(x_0, Z, r) = \frac{1}{r^{n-1}} \int_{B_r^+(x_0,0)} \sum_{i<j} |\nabla z_{i,j}|^2 dxdy$$

$$H(r) = H(x_0, Z, r) := \frac{1}{r^n} \int_{S_r^+(x_0,0)} \sum_{i<j} z_{i,j}^2 d\sigma,$$

then both E and H are absolutely continuous functions on r. We define the Almgren's frequency as

$$N(x_0, Z, r) = \frac{E(x_0, Z, r)}{H(x_0, Z, r)}.$$

Lemma 4.2. (Pohozaev identity) *Let Z be given in (4.3). Then for every $x_0 \in \Omega'$ and $r \in (0, dist(x_0, \partial\Omega))$, the following identity holds*

$$(1-n) \int_{B_r^+(x_0,0)} \sum_{i<j} |\nabla z_{i,j}|^2 dxdy = r \int_{\partial^+ B_r^+(x_0,0)} \sum_{i<j} \Big[2(\partial_\nu z_{i,j})^2 - |\nabla z_{i,j}|^2\Big] d\sigma. \tag{4.4}$$

Proof. Without loss of generality, we may assume $(x_0, 0) = (0, 0)$. By using the following Pohozaev type identity

$$div \left\{(X, \nabla z)\nabla z - X\frac{|\nabla z|^2}{2}\right\} + \left(\frac{n-1}{2}\right) |\nabla z|^2 = (X, \nabla z)\triangle z.$$

for $z = z_{i,j}$ and integrating it over $B_r^+ := B_r^+(0)$, we have

$$\begin{aligned}(1-n) \int_{B_r^+} |\nabla z_{i,j}|^2 dxdy - r \int_{\partial^+ B_r^+} (2(\partial_\nu z_{i,j})^2 - |\nabla z_{i,j}|^2) d\sigma \\ = \int_{\partial^0 B_r} 2(x, \nabla_x z_{i,j})\partial_\nu z_{i,j} dx.\end{aligned}$$

We claim that the last term in above equality equals to 0, which finishes the proof of the lemma by summing the identities for $i < j$. Indeed, Fix $\varepsilon > 0$ and define the set $S_\varepsilon = \{x \in \Omega' \backslash \Gamma(v(x,0)) : \sum_{i<j} |\nabla z_{i,j}(x,0)| \leqslant \varepsilon\}$. Hence

$$\lim_{\varepsilon\to 0^+} \sum_{i<j} \left| \int_{\partial^0 B_r \cap S_\varepsilon} (x, \nabla_x z_{i,j})\partial_\nu z_{i,j} dx \right| \leqslant \lim_{\varepsilon\to 0^+} \sum_{i<j} \varepsilon r \int_{\partial^0 B_r \cap S_\varepsilon} |\partial_\nu z_{i,j}| dx = 0. \tag{4.5}$$

Fix $x \in \Gamma(v(\cdot,0)) \cap (\Omega' \backslash (\Gamma(v(\cdot,0)) \cup S_\varepsilon))$, then we have $|\nabla z_{i,j}(x,0)| \neq 0$ for some $i < j$. By Lemma 4.1, $\Phi((x,0), z_{i,j}^+, z_{i,j}^-, 0^+) \neq 0$. Then Proposition 3.1 implies that in a small neighborhood of x there exist exactly two components $v_i(x,0)$, $v_j(x,0)$. Hence $\partial_\nu z_{i,j}(x,0) = 0$ in $\partial^0 B_\gamma(x,0)$ and

$$\sum_{i<j} \int_{\partial^0 B_\gamma(x,0) \cap (\Omega' \backslash S_\varepsilon)} (x, \nabla_x z_{i,j})\partial_\nu z_{i,j} dx = 0. \tag{4.6}$$

Therefore

$$\sum_{i<j}\int_{\partial^0 B_r\cap(\Omega'\setminus S_\varepsilon)}(x,\nabla_x z_{i,j})\partial_\nu z_{i,j}dx=0,$$

and by combining this with (4.5) we complete the proof of the lemma. $\square$

Theorem 4.1. *Let $\widetilde{\Omega}\Subset\Omega$. For every $x_0\in\widetilde{\Omega}$, the Almgren frequency function $N(x_0,Z,r)$ is nondecreasing for $r\in(0,dist(x_0,\partial\Omega))$, and the limit $N(x_0,Z,0^+)$ $:=\lim_{r\to 0^+}N(x_0,Z,r)$ exists and is finite. Moreover,*

$$\frac{d}{dr}\log H(r)=\frac{2N(r)}{r}. \tag{4.7}$$

Proof. We follow the same proof as for the harmonic functions case, see [18]. We compute the derivative of $E(r)$ and use (4.4), to obtain

$$\begin{aligned}E'(r)&=\frac{1-n}{r^n}\int_{B_r^+(x_0,0)}\sum_{i<j}|\nabla z_{i,j}|^2dxdy+\frac{1}{r^{n-1}}\int_{\partial^+B_r^+(x_0,0)}\sum_{i<j}|\nabla z_{i,j}|^2d\sigma\\&=\frac{2}{r^{n-1}}\int_{\partial^+B_r^+(x_0,0)}\sum_{i<j}(\partial_\nu z_{i,j})^2d\sigma.\end{aligned} \tag{4.8}$$

Now, if we multiply system (2.3) by $z_{i,j}$ and integrate by parts in $B_r^+(x_0,0)$ we can rewrite E as

$$E(r)=\frac{1}{r^{n-1}}\int_{\partial^+B_r^+(x_0,0)}\sum_{i<j}z_{i,j}\partial_\nu z_{i,j}d\sigma.$$

Concerning the derivative of $H(r)$, we find

$$H'(r)=\frac{2}{r^n}\int_{\partial^+B_r^+(x_0,0)}\sum_{i<j}z_{i,j}\partial_\nu z_{i,j}d\sigma.$$

Thus, by performing a direct computation, (4.7) holds whenever $H(x_0,Z,r)>0$ for $r\in(0,\mathrm{dist}(x_0,\partial\Omega))$, as well as

$$\begin{aligned}\frac{d}{dr}N(r)&=\frac{E'H-H'E}{H^2}\\&=\frac{2}{r^{2n-1}H^2}\left[\int_{\partial^+B_r(x_0,0)}\sum_{i<j}\partial_\nu z_{i,j}^2d\sigma\int_{\partial^+B_r(x_0,0)}\sum_{i<j}z_{i,j}^2d\sigma\right.\\&\qquad\left.-\left(\int_{\partial^+B_r(x_0,0)}\sum_{i<j}z_{i,j}\partial_\nu z_{i,j}\right)^2d\sigma\right]\geqslant 0,\end{aligned} \tag{4.9}$$

by the Cauchy–Schwarz inequality. The only thing left to prove is that $H(x_0,Z,r)>0$ for every $x\in\widetilde{\Omega}$ and $0<r<\mathrm{dist}(x_0,\partial\Omega)$. Since $v(x,0)\not\equiv 0$ in Ω, we can suppose without loss of generality that $v(x,0)\not\equiv 0$ in $\widetilde{\Omega}$, and thus $Z(x,0)\not\equiv 0$ in $\widetilde{\Omega}$. Now we have the following claim.

Claim. $\Gamma(v(\cdot,0))$ *has empty interior.*

Assume not, and let $x_1 \in \Gamma(v(\cdot,0))$ be such that $d_1 := \mathrm{dist}(x_1, \partial\Gamma(v(\cdot,0))) < \mathrm{dist}(x_1, \partial\Omega)$ (recall that we are assuming that $Z(x,0)$ is not identically zero in $\widetilde{\Omega}$). We have

$$H(x_1, Z, r) > 0 \quad \text{for } r \in (d_1, d_1+\delta),$$

for some small $\delta > 0$. In fact, if there exist $\varepsilon > 0$ such that $H(d_1+\varepsilon) = 0$, we would have $Z = 0$ on $\partial^+ B_{d_1+\varepsilon}(x_1, 0)$ and

$$\begin{cases} -\triangle z_{i,j}^{\pm} \leqslant 0 & \text{in } B_{d_1+\varepsilon}^{+}(x_1,0), \\ \partial_\nu z_{i,j}^{\pm} \leqslant 0 & \text{on } \partial^0 B_{d_1+\varepsilon}(x_1,0). \end{cases} \tag{4.10}$$

Then by multiplying above equation by $z_{i,j}^{\pm}$ and integrating by parts in $B_{d_1+\varepsilon}^{+}$ $(x_1, 0)$ we obtain $z_{i,j} \equiv 0$ in $\overline{B_{d_1+\varepsilon}^{+}(x_1,0)}$ for every $i<j$, in contradiction with the definition of d_1. By what we have done so far $H(r) = H(x_1, Z, r)$ verifies , in $(d_1, d_1+\delta)$, the initial value problem

$$\begin{cases} H'(r) = a(r)H(r) & \text{for } r \in (d_1, d_1+\delta), \\ H(d_1) = 0, \end{cases}$$

with $a(r) = 2N(r)/r$, which is continuous also at d_1 by the monotonicity of $N(r)$. Then by uniqueness $H(r) \equiv 0$ for $r > d_1$, a contradiction with the definition of d_1.

If there were $x_0 \in \widetilde{\Omega}$ and $r_0 \in (0, dist(x_0, \partial\Omega))$ such that $H(x_0, Z, r_0) = 0$, then similar as in the proof of above Claim., we would have $Z \equiv 0$ in $\overline{B_{r_0}^{+}(x_0,0)}$, which contradicts the fact that $\Gamma(v(\cdot,0))$ has empty interior. $\square$

Remark 4.1. As observed in the above proof, $\Gamma(v(\cdot,0))$ has empty interior unless $v(x,0) \equiv 0$ (and thus $Z(x,0) \equiv 0$).

Corollary 4.1. *Given $\widetilde{\Omega} \Subset \Omega$ there exists $C > 0$ such that*

$$H(x_0, Z, r_2) \leqslant H(x_0, Z, r_1) \left(\frac{r_2}{r_1}\right)^{2C},$$

for every $x_0 \in \widetilde{\Omega}$ and $0 < r_1 < r_2 < dist(\widetilde{\Omega}, \partial\Omega)$.

Proof. For $\widetilde{\Omega}$ fixed, we let $\widetilde{r}$ be positive constant such that $r_2 \leqslant \widetilde{r} < \mathrm{dist}(\widetilde{\Omega}, \partial\Omega)$. Let also $C := \sup_{x_0 \in \widetilde{\Omega}} |N(x_0, Z, \widetilde{r})| < +\infty$. Then

$$\begin{aligned} \frac{d}{dr} \log H(x_0, Z, r) &= \frac{2}{r} N(x_0, Z, r) \\ &\leqslant \frac{2}{r} N(x_0, Z, \widetilde{r}) \leqslant \frac{2C}{r}, \end{aligned}$$

for every $0 < r < \widetilde{r}$. Now we integrate between r_1 and r_2, $0 < r_1 < r_2 < \widetilde{r}$, obtaining

$$\frac{H(x_0, Z, r_2)}{H(x_0, Z, r_1)} \leqslant \left(\frac{r_2}{r_1}\right)^{2C},$$

as desired. $\square$

Corollary 4.2. *For each $x_0 \in \Gamma(v(\cdot,0))$ we have $N(x_0, Z, 0^+) \geqslant 1$.*

Proof. Suppose not. Since the limit $N(x_0, Z, 0^+)$ exists, we obtain the existence of $\overline{r}$ and $\varepsilon > 0$ such that $N(x_0, Z, r) \leqslant 1 - \varepsilon$ for all $0 \leqslant r \leqslant \overline{r}$. By Theorem 4.1 we have that

$$\frac{d}{dr} \log H(x_0, Z, r) \leqslant \frac{2}{r}(1 - \varepsilon).$$

Integrating this inequality between r and $\overline{r}$ ($r < \overline{r}$) yields

$$H(x_0, Z, r) \geqslant C(\overline{r}) r^{2(1-\varepsilon)}, \tag{4.11}$$

But we know, $\forall \alpha \in (0, 1)$, Z is $C^{0,\alpha}$ continuous function at $(x_0, 0)$. This means

$$H(x_0, Z, r) \leqslant C r^{2\alpha},$$

by the fact that $Z(x_0, 0) = 0$. We can choose $\alpha > 1 - \frac{\varepsilon}{2}$, then above inequality contradicts (4.11) for r small. □

Corollary 4.3. *The map $x_0 \mapsto N(x_0, Z, 0^+)$ is upper semi-continuous.*

Proof. Take a sequence $x_n \to x$ in Ω. By Theorem 4.1, for some small $r > 0$,

$$N(x_n, Z, r) \geqslant N(x_n, Z, 0^+).$$

By taking the limit superior in n and afterwards the limit as $r \to 0^+$ we obtain $N(x, Z, 0^+) \geqslant \limsup_n N(x_n, Z, 0^+)$. □

5. Compactness of the blowup sequences

The frequency formula in Theorem 4.1 above allows to study the blowups. Let $\widetilde{\Omega} \Subset \Omega$ and take some sequence $x_k \in \widetilde{\Omega}$, $t_k \downarrow 0$. We define a blowup sequence by

$$Z_k(x, y) = \frac{Z(x_k + t_k x, t_k y)}{\rho_k} \quad \text{for } x \in \Omega_k := \frac{\Omega - x_k}{t_k} \quad \text{and} \quad y > 0.$$

with

$$\rho_k^2 = \|Z(x_k + t_k \cdot, t_k \cdot)\|^2_{L^2(\partial^+ B_1^+(0))} = \frac{1}{t_k^n} \int_{\partial^+ B^+_{t_k}(x_k, 0)} Z^2 d\sigma = H(x_k, Z, t_k).$$

We observe that $\|Z_k\|_{L^2(\partial^+ B_1^+)} = 1$ and Z_k solves the system

$$\begin{cases} -\triangle z^k_{i,j} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu z^k_{i,j} = -\mu^k_{i,j} & \text{in } \Omega_k \times \{0\}, \end{cases} \tag{5.1}$$

with $\mu^k_{i,j}(E) = \frac{1}{\rho_k t_k^{n-1}} \mu_{i,j}(x_k + t_k E)$.

In this setting, for any $z_0 \in \Omega_k$ and $r \in (0, \operatorname{dist}(z_0, \partial\Omega_k))$,

$$E(z_0, Z_k, r) = \frac{1}{r^{n-1}} \int_{B_r^+(z_0, 0)} \sum_{i<j} |\nabla z_{i,j}|^2 dx dy,$$

and the following identities hold:

$$E(z_0, Z_k, r) = \frac{1}{\rho_k^2} E(x_k + t_k z_0, Z, t_k r), \;\; H(z_0, Z_k, r) = \frac{1}{\rho_k^2} H(x_k + t_k z_0, Z, t_k r), \tag{5.2}$$

and hence

$$N(z_0, Z_k, r) = N(x_k + t_k z_0, Z, t_k r). \tag{5.3}$$

We observe that $(\Omega_k - x_k)/t_k \to \mathbb{R}^n$ as $k \to +\infty$ because $\operatorname{dist}(x_k, \partial\Omega) \geqslant \operatorname{dist}(\widetilde{\Omega}, \partial\Omega) > 0$ for every k. In the remaining part of this section we will prove the following convergence result and present some of its main consequence.

Theorem 5.1. *Under previous notations, there exists $\overline{Z} \in C^{0,1}_{loc}(\overline{\mathbb{R}^{n+1}_+}) \cap H^1_{loc}(\mathbb{R}^{n+1}_+)$ such that, up to a subsequence, $Z_k \to \overline{Z}$ in $C^{0,\alpha}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ for every $0 < \alpha < 1$ and strongly in $H^1_{loc}(\mathbb{R}^{n+1}_+)$. More precisely, there exists $\overline{\mu}_{i,j} \in \mathcal{M}_{loc}(\mathbb{R}^n)$ (the set of measures μ on $\mathbb{R}^n$ which are Radon measures when restricted to any compact set), concentrated on on $\Gamma(\overline{Z}(\cdot,0))$, such that $\mu^k_{i,j} \to \overline{\mu}_{i,j}$ weak $-\star$ in $\mathcal{M}_{loc}(\mathbb{R}^n)$, $\overline{Z}$ solves*

$$\begin{cases} -\triangle \overline{z}_{i,j} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu \overline{z}_{i,j} = -\overline{\mu}_{i,j} & \text{in } \mathbb{R}^n \times \{0\}, \end{cases} \tag{5.4}$$

and it holds

$$(1-n)\int_{B^+_r(x_0,0)} \sum_{i<j} |\nabla \overline{z}_{i,j}|^2 dxdy = r\int_{\partial^+ B^+_r(x_0,0)} \sum_{i<j} \left[2(\partial_\nu \overline{z}_{i,j})^2 - |\nabla \overline{z}_{i,j}|^2\right] d\sigma \tag{5.5}$$

for every $(x,0) \in \mathbb{R}^n \times \{0\}$ and $r > 0$.

In the proof of Theorem 5.1, we need the following lemma.

Lemma 5.1. *For any given $R > 0$ we have $\|Z_k\|_{H^1(B^+_R(0))} \leqslant C$, independently of k.*

Proof. Let C be a positive constant such that Corollary 4.1 holds for the previously fixed domain $\widetilde{\Omega}$. We can assume, after taking k so large that t_k, $t_k R \leqslant \widetilde{r} < \operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$,

$$\begin{aligned} \int_{\partial^+ B^+_R(0)} \sum_{i<j} (z^k_{i,j})^2 d\sigma &= \frac{1}{\rho_k^2}\int_{\partial^+ B^+_R(0)} \sum_{i<j} z^2_{i,j}(x_k + t_k x, t_k y) d\sigma \\ &= \frac{1}{\rho_k^2 t_k^n}\int_{\partial^+ B^+_{t_k R}(x_k,0)} \sum_{i<j} z^2_{i,j} d\sigma \\ &= R^n \frac{H(x_k, Z, t_k R)}{H(x_k, Z, t_k)} \leqslant R^n \left(\frac{t_k R}{t_k}\right)^{2C} := C(R) R^n, \end{aligned}$$

where the last inequality follows by Corollary 4.1. Moreover,

$$\begin{aligned} &\frac{1}{R^{n-1}}\int_{B^+_R(0)} \sum_{i<j} |\nabla z^k_{i,j}|^2 dxdy \\ &= \frac{H(0, w_k, R)}{H(0, Z_k, R)} \frac{1}{R^{n-1}}\int_{B^+_R(0)} \sum_{i<j} |\nabla z^k_{i,j}|^2 dxdy \\ &\leqslant \frac{C(R)}{H(0, w_k, R)} \frac{1}{R^{n-1}}\int_{B^+_R(0)} \sum_{i<j} |\nabla z^k_{i,j}|^2 dxdy \end{aligned}$$

$$= \frac{C(R)}{H(x_k,Z,t_kR)}\frac{1}{(t_kR)^{n-1}}\int_{B^+_{t_kR}(x_k,0)}\sum_{i<j}|\nabla z_{i,j}|^2dxdy$$
$$= C(R)N(x_k,Z,t_kR)\leqslant C(R)N(x_k,Z,\widetilde{r})\leqslant C'(R),$$

where we have used identity (5.2), the continuity of the function $x\mapsto N(x,Z,\widetilde{r})$, as well as Theorem 4.1. □

Remark 5.1. Since $\partial_v(z^k_{i,j})^\pm\leqslant 0$ in $\Omega\times\{0\}$, then the even extension of $(z^k_{i,j})^\pm$ through $\{y=0\}$ is subharmonic in $\Omega_k\times\mathbb{R}$. This together with the H^1_{loc}-boundedness provided by the previous lemma yield that $\|Z_k\|_{L^\infty(B^+_R(0))}\leqslant C$, independently of k.

Lemma 5.2. *For any given $R>0$ there exists $C>0$, independent of k, such that $\|\mu_{i,k}\|_{\mathcal{M}(\partial^0B_R(0))}=\mu_{i,k}(\partial^0B_R(0))\leqslant C$ for every $i=1,\ldots,M$.*

Proof. Fixed $R>0$ and let k sufficiently large such that $\partial^0B_{2R}\subset\Omega_k\times\{0\}$. We multiply (5.1) by φ, a cutoff function such that $0\leqslant\varphi\leqslant 1$, $\varphi=1$ in $B_R(0)$ and $\varphi=0$ in $\mathbb{R}^{n+1}\backslash B_{2R}(0)$. It holds after an integrating by parts that

$$\int_{B^+_{2R}}\nabla(z^k_{i,j})^\pm\cdot\nabla\varphi dxdy\leqslant-\int_{\partial^0B_{2R}}\varphi(x,0)d(\mu^k_{i,j})^\pm.$$

That is

$$(\mu^k_{i,j})^\pm(\partial^0B_R)\leqslant C(R)\|\nabla z^k_{i,j}\|_{L^2(B^+_R)}+C(R)\|z^k_{i,j}\|_{L^\infty(\overline{B^+_R})}\leqslant\widetilde{C}(R),$$

by Lemma 5.1 and Remark 5.1. □

Up to now, we have proved the existence of a non trivial function $\overline{Z}\in H^1_{\text{loc}}(R^{n+1}_+)\cap L^\infty_{\text{loc}}(\overline{R^{n+1}_+})$ and $\overline{\mu}_{i,j}\in\mathcal{M}_{\text{loc}}(\mathbb{R}^n)$ such that, up to a subsequence,

$$Z_k\rightharpoonup\overline{Z}\text{ weakly in }H^l_{\text{loc}}(\mathbb{R}^{n+1}_+),\quad \mu^k_{i,j}\rightharpoonup\overline{\mu}_{i,j}\text{ weak-}\star\text{ in }\mathcal{M}_{\text{loc}}(\mathbb{R}^n).$$

Moreover we have in distributional sense that

$$\begin{cases}-\triangle\overline{z}_{i,j}=0 & \text{in }\mathbb{R}^{n+1}_+,\\ \partial_\nu\overline{z}_{i,j}=-\overline{\mu}_{i,j} & \text{in }\mathbb{R}^n\times\{0\}.\end{cases}$$

The next step is to prove that the convergence $V_k\to\overline{V}$ is indeed strongly in H^1_{loc} and in $C^{0,\alpha}_{\text{loc}}$. We begin with the following estimate for $H(r)$

Lemma 5.3. *Fix $R>0$. Then there exist constants $C,\overline{r},\overline{k}>0$ such that for $k\geqslant\overline{k}$ we have*

$$H(x,Z_k,r)\leqslant Cr^2,$$

for $0<r<\overline{r}$ and $(x,0)\in\partial^0B_{2R}(0)\cap\Gamma(v_k(\cdot,0))$. In particular, Z_k is uniform Lipschitz bounded in $\overline{B^+_R}(0)$.

Proof. Take k so large that $x_k+t_kx\in\widetilde{\Omega}$. Suppose moreover that $x\in\Gamma(v_k(\cdot,0))$, then Theorem 4.1 and Corollary 4.2 yield that for $x\in\partial^0B_{2R}(0)$ and $0<r<\overline{r}$,

$$\frac{d}{dr}\log\left(\frac{H(x,Z_k,r)}{r^2}\right)=\frac{2}{r}\left(N(x,Z_k,r)-1\right)\geqslant 0,$$

which implies (after integration)

$$\frac{H(x,Z_k,r)}{r^2} \leqslant \frac{H(x,Z_k,\overline{r})}{\overline{r}^2} \leqslant C'\|Z_k\|^2_{L^\infty(\overline{B^+_{2R+\overline{r}}(0)})} \leqslant C(R).$$

By the method of [3] (see p. 853 therein), this bound implies the uniform Lipschitz bound for Z_k. □

By the compact embeddings $C^{0,1}(\overline{B^+_R(0)}) \hookrightarrow C^{0,\alpha}(\overline{B^+_R(0)})$ for $\alpha \in (0,1)$ we deduce the existence of a convergence subsequence $Z_k \to \overline{Z}$ in $C^{0,\alpha}_{loc}$. Now we turn to the proof of the H^1-strong convergence, then we shall finish the proof of Theorem 5.1.

Lemma 5.4. *For every $R>0$ we have (up to a subsequence) $Z_k \to \overline{Z}$ strongly in $H^1(B^+_R(0))$.*

Proof. For every $i=1,\dots,M$, we have in distributional sense that

$$\begin{cases} -\triangle z^k_{i,j} = 0 & \text{in } B^+_{2R}, \\ \partial_\nu z^k_{i,j} = -\mu^k_{i,j} & \text{in } \partial^0 B_{2R}, \end{cases} \qquad \begin{cases} -\triangle \overline{z}_{i,j} = 0 & \text{in } B^+_{2R}, \\ \partial_\nu \overline{z}_{i,j} = -\overline{\mu}_{i,j} & \text{in } \partial^0 B_{2R}. \end{cases}$$

Now, subtracting the second equation from the first one and testing the result by $(z^k_{i,j}-\overline{z}_{i,j})\varphi$ on $B^+_{2R}(0)$, where φ is a cutoff function such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $B_R(0)$ and $\varphi = 0$ in $\mathbb{R}^{n+1}\backslash B_{2R}(0)$, we obtain

$$\begin{aligned}\int_{B^+_{2R}(0)} \varphi|\nabla(z^k_{i,j}-\overline{z}_{i,j})|^2 dxdy = &-\int_{B^+_{2R}(0)} (z^k_{i,j}-\overline{z}_{i,j})\nabla\varphi\cdot\nabla(z^k_{i,j}-\overline{z}_{i,j})dxdy \\ &-\int_{\partial^0 B_{2R}(0)} \varphi(z^k_{i,j}-\overline{z}_{i,j})d\mu^k_{i,j} \\ &+\int_{\partial^0 B_{2R}(0)} \varphi(z^k_{i,j}-\overline{z}_{i,j})d\overline{\mu}_{i,j}.\end{aligned}$$

By using the uniform convergence $v_{i,k} \to \overline{v}_i$ in $\overline{B^+_{2R}(0)}$, we have

$$\begin{aligned}&\left|\int_{B^+_{2R}(0)} (z^k_{i,j}-\overline{z}_{i,j})\nabla\varphi\cdot\nabla(z^k_{i,j}-\overline{z}_{i,j})dxdy\right| \\ &\quad\leqslant C\|z^k_{i,j}-\overline{z}_{i,j}\|_{L^\infty(\overline{B^+_{2R}(0)})}\int_{B^+_{2R}(0)}|\nabla z^k_{i,j}|^2 dxdy \\ &\quad\to 0,\end{aligned}$$

and

$$\begin{aligned}&\left|-\int_{B^0_{2R}(0)} \varphi(z^k_{i,j}-\overline{z}_{i,j})d\mu^k_{i,j} + \int_{B^0_{2R}(0)} \varphi(z^k_{i,j}-\overline{z}_{i,j})d\overline{\mu}_{i,j}\right| \\ &\quad\leqslant \|z^k_{i,j}-\overline{z}_{i,j}\|_{L^\infty(\overline{B^+_{2R}(0)})}(\mu^k_{i,j}(B^0_{2R}(0))+\overline{\mu}_{i,j}(B^0_{2R}(0))) \to 0.\end{aligned}$$

Hence,

$$\int_{B^+_{2R}(0)} \varphi|\nabla(z^k_{i,j}-\overline{z}_{i,j})|^2 dxdy \to 0,$$

which, together with the weak H^1_{loc} convergence yields the desired result. □

End the Proof of Theorem 5.1. After Lemma 5.4 we are left to prove that the identity (5.5) holds and that the measures $\overline{\mu}_{i,j}$ are concentrated on $\Gamma(\overline{Z}(\cdot,0))$ (for $i<j$).

As for the proof of (5.5), we note that Z_k satisfies the following Pohozaev identity

$$(1-n)\int_{B_r^+(0)}\sum_{i<j}|\nabla z_{i,j}^k|^2 dxdy = r\int_{\partial^+ B_r^+(0)}\sum_{i<j}[2(\partial_\nu z_{i,j}^k)^2-|\nabla z_{i,j}^k|^2]d\sigma.$$

By strong H^1_{loc} convergence,

$$(1-n)\int_{B_r^+(0)}\sum_{i<j}|\nabla z_{i,j}^k|^2 dxdy \to (1-n) \times\int_{B_r^+(0)}\sum_{i<j}|\nabla \overline{z}_{i,j}|^2 dx\ dy \quad \text{as } k\to+\infty.$$

We need to prove that

$$r\int_{\partial^+ B_r^+(0)}\sum_{i<j}(\partial_\nu z_{i,j}^k)^2 d\sigma \to r\int_{\partial^+ B_r^+(0)}\sum_{i<j}(\partial_\nu \overline{z}_{i,j})^2 d\sigma$$

and

$$r\int_{\partial^+ B_r^+(0)}\sum_{i<j}|\nabla z_{i,j}^k|^2 d\sigma \to r\int_{\partial^+ B_r^+(0)}\sum_{i<j}|\nabla \overline{z}_{i,j}|^2 d\sigma,$$

as $k\to+\infty$. We only prove the second convergence, which implies also the first one. Indeed, the strong H^1_{loc} convergence implies that

$$\int_0^R\int_{\partial^+ B_r^+(0)}\sum_{i<j}|\nabla z_{i,j}^k-\nabla\overline{z}_{i,j}|^2 d\sigma dr\to 0.$$

So that $\int_{\partial^+ B_r^+(0)}|\nabla z_{i,j}^k|^2 d\sigma \to \int_{\partial^+ B_r^+(0)}|\nabla \overline{z}_{i,j}|^2 d\sigma$ for *a.e.* r and there exists an integrable function $f\in L^1(0,R)$ such that, up to a subsequence

$$\int_{\partial^+ B_r^+(0)}|\nabla z_{i,j}^k|^2 d\sigma \leqslant f(r) \quad \text{for } a.e.\ r\in(0,R),$$

for every $i<j$. We can use the Dominated Convergence Theorem. Since every subsequence of $\{Z_k\}$ admits a convergent sub-subsequence, and the limit is the same, we conclude the convergence for the entire approximating sequence.

As for the proof of the second claim, we start by fixing an $R>0$ and by considering a cutoff function φ equal to one in $\partial^0 B_R(0)$, zero outside $\partial^0 B_{2R}$. Since

$$\int_{\partial^0 B_{2R}(0)}(z_{i,j}^k)^{\pm}\varphi d(\mu_{i,j}^k)^{\pm} = \int_{\partial^0 B_{2R}(0)\cap\Gamma(Z_k(\cdot,0))}(z_{i,j}^k)^{\pm}\varphi d(\mu_{i,j}^k)^{\pm},$$

then

$$\begin{aligned} 0 &= \lim_k \int_{\partial^0 B_{2R}(0)} (z_{i,j}^k)^\pm \varphi d(\mu_{i,j}^k)^\pm \\ &= \lim_k \int_{\partial^0 B_{2R}(0)} ((z_{i,j}^k)^\pm - \overline{z}_{i,j}^\pm)\varphi d(\mu_{i,j}^k)^\pm + \lim_k \int_{\partial^0 B_{2R}(0)} \overline{z}_{i,j}^\pm \varphi d(\mu_{i,j}^k)^\pm \\ &= \int_{\partial^0 B_{2R}(0)} \overline{z}_{i,j}^\pm \varphi d\overline{\mu}_{i,j}^\pm. \end{aligned}$$

Thus $\int_{\partial^0 B_R(0)} \overline{z}_{i,j}^\pm d\overline{\mu}_{i,j}^\pm = 0$ for every $R > 0$ and in particular $\overline{\mu}_{i,j}^\pm(K) = 0$ for every compact set $K \in ((\mathbb{R}^n \times \{0\}) \cap \{\overline{z}_{i,j}^\pm > 0\})$, which proves the second claim. □

Corollary 5.1. *Under the previous notations, suppose that one of these situations occurs:*

1. $x_k = x_0$ *for every* k,
2. $x_k \in \Gamma(Z(\cdot, 0))$ *and* $x_k \to x_0 \in \Gamma(Z(\cdot, 0))$ *with* $N(x_0, Z, 0^+) = 1$.
   Then $N(0, \overline{Z}, r) = N(x_0, Z, 0^+) =: \alpha$ *for every* $r > 0$, *and* $\overline{Z} = r^\alpha G(\theta)$, *where* (r, θ) *are the generalized polar coordinates centered at the origin.*

Proof. We divide the proof into two steps.
Step 1. $N(0, \overline{Z}, r)$ is constant. First observe that $N(0, Z_k, r) = N(x_k, Z, t_k r)$ and that Theorem 5.1 yields $\lim_k N(0, Z_k, r) = N(0, \overline{Z}, r)$. As for the right hand side, if $x_k = x_0$ for some x_0, then by Theorem 4.1, $\lim_k N(x_0, Z, t_k r) = N(x_0, Z, 0^+)$ for every $r > 0$. In the second situation, we claim that $\lim_k N(x_k, Z, t_k r) = 1$. let $\widetilde{r}$ be a positive constant such that $\widetilde{r} \leqslant \mathrm{dist}(\widetilde{\Omega}, \partial\Omega)$. For any given $\varepsilon > 0$ take $0 < \overline{r} = \overline{r}(\varepsilon) \leqslant \widetilde{r}$ such that

$$N(x_0, Z, r) \leqslant 1 + \varepsilon/2 \quad \text{for every} \quad 0 < r \leqslant \overline{r}.$$

Moreover there exists $\delta_0 > 0$ such that

$$N(x, Z, \overline{r}) \leqslant 1 + \varepsilon \text{ for } (x, 0) \in \partial^0 B_{\delta_0}(x_0, 0) \subset \widetilde{\Omega} \times \{0\}.$$

Thus, again by Theorem 4.1, we obtain

$$N(x, Z, r) \leqslant 1 + \varepsilon,$$

for every $(x, 0) \in \partial^0 B_{\delta_0}(x_0, 0)$ and $0 < r \leqslant \overline{r}$. Now the claim follows by also taking into account Corollary 4.2.
Step 2. $\overline{Z}$ is homogeneous. We first note that (4.9) holds with $z_{i,j}$ replaced by $\overline{z}_{i,j}$. Moreover, by Step 1., $\frac{d}{dr} N(0, \overline{Z}, r) = 0$. This together with the equality condition in the Cauchy–Schwarz inequality, yields that there exists $C(r) > 0$ such that $\partial_\nu \overline{Z} = C(r)\overline{Z}$. Thus

$$2C(r) = \frac{2\int_{\partial^+ B_r^+} \sum_{i<j} \overline{z}_{i,j} \partial_\nu \overline{z}_{i,j} d\sigma}{\int_{\partial^+ B_r^+} \sum_{i<j} \overline{z}_{i,j}^2 d\sigma} = \frac{d}{dr} \log H(0, \overline{Z}, r) = \frac{2}{r} N(0, \overline{Z}, r) = \frac{2}{r}\alpha,$$

and hence $C(r) = \frac{\alpha}{r}$ and $\overline{Z}(X) = r^\alpha G(\theta)$ in $\mathbb{R}_+^{n+1}$. □

Proposition 5.1. *Let Z be defined in (4.3). Then there exists $\delta_n > 0$ (depending only on the dimension) such that, for every $x_0 \in \Gamma(Z(\cdot, 0))$, either*

$$N(x_0, Z, 0^+) = 1 \text{ or } N(x_0, Z, 0^+) \geqslant 1 + \delta_n.$$

Proof. Consider a blowup sequence at a fixed center $(x_0, 0) \in \Gamma(Z(\cdot, 0))$

$$Z_k(x, y) = \frac{Z(x_0 + t_k x, t_k y)}{\rho_k} \text{ with } \rho_k^2 = H(x_0, Z, t_k) \quad \text{and} \quad t_k \downarrow 0.$$

By Theorem 5.1 and Corollary 5.1 (case 1.) we have that, up to a subsequence, $Z_k \to \overline{Z}$ uniformly, where

$$\overline{Z} = r^\alpha G(\theta), \quad \text{with } \alpha = N(x_0, Z, 0^+).$$

Moreover, after an even extension across $\{y = 0\}$ (still denote by $\overline{Z}$), $\overline{Z}$ is harmonic in $\mathbb{R}^{n+1} \backslash \Gamma(\overline{Z}(\cdot, 0))$, and

$$- \triangle_{\mathbb{S}^n} g_{i,j} = \lambda g_{i,j} \quad \text{in } \partial B_1(0) \backslash \Gamma(\overline{Z}(\cdot, 0)), \tag{5.6}$$

with $\lambda = \alpha(\alpha + n - 1)$. Moreover, each component $\overline{z}_{i,j}$ is harmonic in the open set $\{\overline{z}_{i,j} > 0\}$, which implies that on every given connected component $A \subseteq \{g_{i,j} > 0\} \subseteq \partial B_1(0)$, $\lambda = \lambda_1(A)$ (the first eigenvalue). We divide the rest of the proof into several steps.

Step 1. $\forall (x_0, 0) \in \Gamma(\overline{Z}(\cdot, 0)) \backslash \{(0, 0)\}$ we have

$$N(0, \overline{Z}, 0^+) \geqslant N(x_0, \overline{Z}, 0^+).$$

Moreover, if $N(x_0, \overline{Z}, 0^+) = N(0, \overline{Z}, 0^+)$, then $\overline{Z}$ is constant along the direction parallel to $(x_0, 0)$, that is $\overline{Z}(x + tx_0, y) = \overline{Z}(x, y)$, for every $t > 0$ and $(x, y) \in \mathbb{R}^{n+1}_+$.

Take, for every $\lambda > 0$, the rescaled function $\overline{Z}_{0,\lambda} = \overline{Z}(\lambda x, \lambda y) = \lambda^\alpha \overline{Z}(x, y)$ $= \lambda^\alpha \overline{Z}(x, y)$. By the definition of N we obtain that for every $r > 0$

$$N(x_0, \overline{Z}, r) = N(x_0, \lambda^\alpha \overline{Z}, r) = N(x_0, \overline{Z}_{0,\lambda}, r) = N(\lambda x_0, \overline{Z}, \lambda r).$$

Therefore, $N(\lambda x_0, \overline{Z}, 0^+) = N(x_0, \overline{Z}, 0^+)$ and the first result follows from the upper semi-continuity of the function $x \mapsto N(x, \overline{Z}, 0^+)$.

Moreover, if $N(x_0, \overline{Z}, 0^+) = N(0, \overline{Z}, 0^+)$, then the above discussion in fact shows, $\forall r > 0$, $N(x_0, \overline{Z}, r) \equiv \alpha$. So, $\overline{Z}$ is homogeneous with respect to $(x_0, 0)$. Let $X = (x_0, 0)$, $Y = (x, y))$, then

$$\overline{Z}(X + \lambda Y) = \lambda^\alpha \overline{Z}(X + Y), \quad \forall Y.$$

Recalling that $\overline{Z}$ is also homogeneous with respect to $(0, 0)$, we get, $\forall t > 0$,

$$\overline{Z}(Y + tX) = \lambda^{-\alpha} \overline{Z}(\lambda Y + tX) = \overline{Z}(Y + \lambda^{-1} tX), \quad \forall Y.$$

Here $\lambda > 0$ can be choose arbitrarily, too. By taking $\lambda \to +\infty$ and using the continuity of $\overline{Z}$, we get $\overline{Z}(Y + tX) = \overline{Z}(Y)$, $\forall Y$, that is, $\overline{Z}(x + tx_0, y) = \overline{Z}(x, y)$, $\forall (x, y)$, $t > 0$.

Step 2. If there are at least 3 connected components of $\{\overline{Z}(\cdot, 0) > 0\}$, then there exists a universal constant $\delta(n) > 0$ such that $N(0, \overline{Z}, 0^+) \geqslant 1 + \delta(n)$.

Since there at least 3 components of $\{\overline{Z}(\cdot, 0) > 0\}$, then one of them, denote by $\mathfrak{C}$, must satisfy $\mathcal{H}^{n-1}(\mathfrak{C}) \leqslant \mathcal{H}^{n-1}(\partial B_1^0(0))/3$. By the isoperimetric inequality (see for instance [17]), if we choose a half space $H_0 = \{x_n \geqslant 0\} \cap \{y =$

$0\}$, then $\lambda_1(\mathfrak{C}) > \lambda_1(H_0)$. Because the first eigenvalue of the half space H_0 is n, there exists $\gamma > 0$ such that $\lambda_1(\mathfrak{C}) = n + \gamma$, and thus $\alpha = \sqrt{(\frac{n-1}{2})^2 + \lambda_1(\mathfrak{C})} - \frac{n-1}{2} \geqslant 1 + \delta(n)$, for some $\delta(n) > 0$.

Step 3. Let $n = 1$. Given $x_0 \in \Gamma(v(\cdot, 0))$ and let $\overline{Z}$ be the blowup limit at $(x_0, 0)$. Suppose that there are at most two connected components of $\{\overline{Z}(\cdot, 0) > 0\}$. Then $\exists \delta > 0$ such that, either $N(x_0, Z, 0^+) = 1$ or $N(x_0, Z, 0^+) \geqslant 1 + \delta$.

Suppose first that there is only one nontrivial component of $\overline{Z}$, say $\overline{z}_{1,2}$, such that $z_{1,2}(\cdot, 0) \not\equiv 0$. Since $z_{1,2}^+$ is subharmonic, then the strong maximum principle together with the fact that $\overline{z}_{1,2}(0, 0) = 0$ implies that $\overline{z}_{1,2} \equiv 0$, a contradiction. So there are exactly two components of $\{\overline{Z}(\cdot, 0) > 0\}$.

If we blow up $\overline{Z}$ at $(z_0, 0) \in \Gamma(Z(\cdot, 0))$ (here $z_0 \neq 0$), we obtain a limit a limit $\widehat{Z}$, which satisfies

$$\begin{cases} -\triangle \widehat{z}_{i,j} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu \widehat{z}_{i,j} = -\widehat{\mu}_{i,j} & \text{in } \mathbb{R}^n \times \{0\}. \end{cases}$$

Suppose that there at least three connected components of $\{\widehat{Z}(\cdot, 0) > 0\}$, then we can apply *Step 2.* to obtain a universal constant $\delta > 0$ such that $N(0, \widehat{Z}, 0^+) \geqslant 1 + \delta$. Then by *Step 1.*,

$$N(0, \overline{Z}, 0^+) \geqslant N(z_0, \overline{Z}, 0^+) = N(0, \widehat{Z}, 0^+) \geqslant 1 + \delta.$$

So in the following we assume $\forall (z, 0) \in \Gamma(v(\cdot, 0))$ and $z \neq 0$, and any blowup limit $\widehat{Z}$ at $(z, 0)$, $\{\widehat{Z}(\cdot, 0) > 0\}$ has at most two connected components. Because $(0, 0) \in \Gamma(\widehat{Z}(\cdot, 0))$, there are exactly two connected components, $\{x > 0\}$ and $\{x < 0\}$. By the uniqueness of the first eigenfunction,

$$\widehat{Z} = (x_1^+ - x_1^-, 0, \ldots) \quad \text{or} \quad \widehat{Z} = (x_1^+, x_1^-, 0, \ldots).$$

So after an even extension we have $-\triangle \overline{z}_{1,2} = 0$ or $-\triangle(\overline{z}_{1,2} - \overline{z}_{1,3}) = 0$ in $\mathbb{R}^2$ and $\alpha \in \mathbb{N}$.

Step 4. We assume moreover that there are at most two connected components of $\{\overline{Z}(\cdot, 0) > 0\}$ and $\forall (z_0, 0) \in \Gamma(v(\cdot, 0)) \backslash \{(0, 0)\}$,

$$N(z_0, \overline{Z}, 0^+) = 1.$$

Then either $\alpha = 1$ or $\alpha \geqslant 2$.

Indeed, $\forall (z, 0) \in \Gamma(Z(\cdot, 0)) \backslash \{(0, 0)$, if we blow up $\overline{Z}$ at $(z_0, 0)$ to obtain a $\widehat{Z}$, then after a normalization,

$$\widehat{Z} = (x_1^+ - x_1^-, 0, \ldots) \quad \text{or} \quad \widehat{Z} = (x_1^+, x_1^-, 0, \ldots).$$

Thus similar to the previous step, we obtain that $\alpha \in \mathbb{N}$, and we can finish the proof.

Conclusion. Assume this Proposition is valid for dimension $n - 1$. (*Step 3.* says that this theorem holds for dimension 1). We only need to consider the case when the condition of *Step 4.* is not satisfied, that is, $\exists (x, 0) \in \Gamma(Z(\cdot, 0))$, $x \neq 0$, such that $N(x, \overline{Z}, 0^+) > 1$. We can blow up $\overline{Z}$ at $(x, 0)$ to obtain a limit $\widehat{Z}$. We claim that $\forall (z_0, 0) \in \Gamma_{\widehat{V}(\cdot, 0)}$, either $N(z_0, (\widehat{Z}, 0^+) \geqslant 1 + \delta(n-1)$ or $N(z, (\widehat{Z}, 0^+) = 1$. This is because if we blow up $\widehat{Z}$ at $(z_0, 0)$, the blowup limit can be reduced to $\mathbb{R}^n$, and we can apply the induction assumption on $n-1$. So we can apply *Step 4.* (if there is no $(z_0,0) \in \Gamma(\widehat{Z}(\cdot,0))$, $N(z_0,\widehat{Z},0^+) > 1$) or *Step 1.* (if $\exists (z_0,0) \in \Gamma(\widehat{Z}(\cdot,0))$, $N(z_0,\widehat{Z},0^+) \geqslant 1+\delta(n-1)$) to obtain that $N(0,\widehat{Z},0^+) \geqslant \min\{1+\delta(n-1),2\}$ (since $N(0,\widehat{Z},0^+) = N(z_0,\overline{Z},0^+) > 1$). Then $N(z_0,\overline{Z},0^+) \geqslant \min\{1+\delta(n-1),2\}$. Now using *Step 1.* again, we get $N(0,\overline{Z},0^+) \geqslant \min\{1+\delta(n-1),2\}$. □

6. Proof of Theorem 1.1

This section is devoted the proof of Theorem 1.1. To begin with, we decompose $\Gamma(v(\cdot,0))$ in two parts.

Definition 6.1. We define the regular part of the $\Gamma_{v(x,0)}$ by

$$\Sigma(v(\cdot,0)) := \{x \in \Gamma(v(\cdot,0)) : \text{there exists a pair } (i,j) \text{ such that } \Phi((x_0,0),(v_i-v_j)^+,(v_i-v_j)^-,0^+) \neq 0\},$$

and the singular part by

$$S(v(\cdot,0)) = \Gamma(v(\cdot,0)) \backslash \Sigma(v(\cdot,0)).$$

Lemma 6.1. *Let Z be defined in (4.3). Then*

$$N(x_0,Z,0^+)=1 \textit{ for } x_0 \in \Sigma(v(\cdot,0)), \quad \textit{and} \quad N(x_0,Z,0^+)>1 \textit{ for } x_0 \in S(v(\cdot,0)).$$

Proof. Suppose that $x_0 \in \Sigma(v(\cdot,0))$. Without loss of generality, we take the pair (v_1,v_2) such that

$$\Phi((x_0,0),(v_1-v_2)^+,(v_1-v_2)^-,0^+) = \lambda \neq 0.$$

Now define

$$v_k = (v_1^k,\ldots,v_M^k) \text{ with } v_i^k(x,y) = v_i(x_0+\lambda_k x,\lambda_k y)/\lambda_k,$$

and let $Z_k = Z(x_0+\lambda_k x,\lambda_k y)/\lambda_k$. Then Theorem 5.1 and Corollary 5.1 (case 1.) yield that, up to a subsequence, $Z_k \to \overline{Z}$ uniformly, where

$$\overline{Z} = r^\alpha G(\theta), \quad \text{with } \alpha = N(x_0,Z,0^+).$$

From Lemma 3.4 we have $\overline{z}_{1,2} = (\overline{v}_1-\overline{v}_2) = c_0(\lambda)x_1$ and $\sum_{j>2}\overline{Z}_j(x,0) \equiv 0$. This together with Corollary 5.1 implies that $\alpha = N(x_0,Z,0^+) = 1$.

Next, suppose that $x_0 \in S(v(\cdot,0))$. Then according to Lemma 4.1 $\lim_{x\to x_0}|\nabla Z(x,0)| = 0$. By Proposition 5.1, we assume by contradiction that

$$N(x_0,Z,0^+) = 1.$$

Consider a blowup sequence at $(x_0,0) \in \Gamma(Z(\cdot,0))$:

$$Z_k(x,y) = \frac{Z(x_0+t_k x,t_k y)}{\rho_k} \text{ with } \rho_k^2 = H(x_0,Z,t_k) \text{ and } t_k \downarrow 0.$$

Then

$$|\nabla Z_k(0)| = t_k|\nabla Z(x_0,0)|/\rho_k \equiv 0. \tag{6.1}$$

By Theorem 5.1 and Corollary 5.1 (case 1.) again, the blowup limit has the form $\overline{z}_{i,j} = \alpha(x_1^+-x_1^-)$ for some $i<j$ and $\sum_{h\neq i,j}\overline{v}_h \equiv 0$. In particular, $|\nabla\overline{z}_{i,j}(0,0)| \neq 0$ Applying the clean-up lemma we have $\partial_\nu z_{i,j} = 0$ in a small

neighborhood of x_0. As a consequence, the convergence of Z_k to $\overline{Z}$ is of class C^m for every $m \geqslant 1$. Moreover, (6.1) can be passed to the limit, obtaining $|\nabla \overline{z}_{i,j}(0,0)| = 0$, a contradiction. $\square$

Proposition 6.1. *$S(v(\cdot,0))$ is closed in Ω and $\Sigma(v(\cdot,0))$ is a smooth hypersurface.*

Proof. Since $N(x, Z, 0^+)$ is upper semi-continuous in x and $N(x, Z, 0^+) = 1$ or $N(x, Z, 0^+) \geqslant 1 + \delta_n$ for some $\delta_n > 0$, it is then easy to see that $S(v(\cdot, 0))$ is closed in Ω. The proof of the last statement immediately follows from the clean-up lemma established in Sect. 3. $\square$

To complete the proof of Theorem 1.1, we need to prove the Hausdorff dimension estimates for the nodal and singular sets. Since $\Gamma(v(\cdot, 0))$ coincide with $\Gamma(Z(\cdot, 0))$ and by appropriate translation and scaling, it suffices to show that:

Proposition 6.2. *Assume that $\partial^0 B_2(0) \Subset \Omega \times \{0\}$. Then*

$$\mathcal{H}_{dim}(\Gamma(Z(\cdot,0)) \cap \partial^0 B_1(0)) \leqslant n-1, \tag{6.2}$$

$$\mathcal{H}_{dim}(S(Z(\cdot,0)) \cap \partial^0 B_1(0)) \leqslant n-2. \tag{6.3}$$

The remainder of this section is devoted to the proof of Proposition 6.2. As in [3,4,15,24], the idea is to apply a version of the so called Federer's Reduction Principle.

Theorem 6.1. (Federer's Reduction Principle) *Let $\mathcal{F} \subseteq (L^\infty_{loc}(\mathbb{R}^n))^M$, and define, for any given $V \in \mathcal{F}$, $x_0 \in \mathbb{R}^n$ and $t > 0$, the rescaled and the translated function*

$$V_{x_0,t} := V(x_0 + t\cdot).$$

We say that $V_n \to V$ in $\mathcal{F}$ if and only if $V_n \to V$ uniformly on every compact set of $\mathbb{R}^n$. Assume that $\mathcal{F}$ satisfies the following conditions:

(A1) (*Closure under rescaling, translation and normalization*) *Given any $|x_0| \leqslant 1-t$, $0 < t < 1$, $\rho > 0$ and $V \in \mathcal{F}$, we have that also $\rho \cdot V_{x_0,t} \in \mathcal{F}$.*

(A2) (*Existence of a homogeneous "blowup"*) *Given any $|x_0| < 1$, $t_k \downarrow 0$ and $V \in \mathcal{F}$, there exists a sequence $\rho_k \in (0, +\infty)$, a real number $\alpha \geqslant 0$ and a function $\overline{Z} \in \mathcal{F}$ homogeneous of degree α such that if we define $V_k(x) = V(x_0 + t_k x)/\rho_k$, then*

$$V_k \to \overline{Z} \text{ in } \mathcal{F}, \text{ up to a subsequence.}$$

(A3) (*Singular set hypotheses*) *There exists a map $\mathfrak{S} : \mathcal{F} \to \mathcal{D}$ (where $D := \{A \subset \mathbb{R}^n : A \cap B_1(0)$ is relatively closed in $B_1(0)\}$) such that*

(i) *Given $|x_0| < 1 - t$, $0 < t < 1$ and $\rho > 0$, it holds*

$$\mathfrak{S}(\rho \cdot V_{x_0,t}) = (\mathfrak{S}(V))_{x_0,t} := \frac{\mathfrak{S}(V) - x_0}{t}.$$

(ii) *Given* $|x_0| < 1$, $t_k \downarrow 0$ *and* V, $\overline{Z} \in \mathcal{F}$ *such that there exists* $\rho_k > 0$ *satisfying* $V_k = \rho_k V_{x_0,t} \to \overline{Z}$ *in* $\mathcal{F}$, *the following "continuity" holds:*

$$\forall \varepsilon > 0\ \exists k(\varepsilon) > 0 : k \geqslant k(\varepsilon) \Rightarrow \mathfrak{S}(V_k) \cap B_1(0) \subseteq \{x \in \mathbb{R}^n : dist(x, \mathfrak{S}(\overline{Z})) < \varepsilon\}.$$

Then, if we define

$$\begin{aligned} d = \max\{&dimL : L \textit{ is a vector subspace of } \mathbb{R}^n \textit{ and there exist } V \in \mathcal{F} \textit{ and} \\ &\alpha > 0 \textit{ such that } \mathfrak{S}(V) \neq 0 \textit{ and } V_{y,t} = t^\alpha V,\ \forall y \in L, t > 0\}, \end{aligned} \tag{6.4}$$

either $\mathfrak{S}(V) \cap B_1(0) = \emptyset$ *for every* $V \in \mathcal{F}$, *or else* $\mathcal{H}_{dim}(\mathfrak{S}(V) \cap B_1(0)) \leqslant d$ *for every* V *in* $\mathcal{F}$. *Moreover in the latter case there exist a function* $V \in \mathcal{F}$, *a dimension subspace* $L \leqslant \mathbb{R}^n$ *and a real number* $\alpha \geqslant 0$ *such that*

$$V_{y,t} = t^\alpha V,\ \forall y \in L,\ t > 0, \quad \textit{and} \quad \mathfrak{S}(V) \cap B_1(0) = L \cap B_1(0).$$

If $d = 0$ *then* $\mathfrak{S}(V) \cap B_\rho(0)$ *is a finite set for each* $V \in \mathcal{F}$ *and* $0 < \rho < 1$.

Proof of Proposition 6.2. We apply the Federer's Reduction Principle to the following class of functions:

$$\begin{aligned} \mathcal{F} = \{&Z \in (L^\infty_{loc}(\overline{\mathbb{R}^{n+1}_+}))^{\frac{M(M-1)}{2}} : \text{there exists some domain } \Omega \text{ such that} \\ &\partial^0 B_2(0) \subset \Omega \times \{0\} \text{ and } Z \text{ fulfills the assumption of Theorem 5.1}\}. \end{aligned}$$

Let us start by checking (A1) and (A2). Hypothesis (A1) is immediately satisfied by (5.2). Moreover, let $|x_0| < 1$, $t_k \downarrow 0$ and $Z \in \mathcal{F}$, and choose $\rho_k = \|Z(x_0 + t_k x, t_k y)\|_{L^2(\partial^+ B_1(0))}$. Theorem 5.1 and Corollary 5.1 (Case 1) yields the existence of $\overline{Z} \in \mathcal{F}$ such that (up to a subsequence) $w_k \to \overline{Z}$ in $\mathcal{F}$ and $\overline{Z}$ is a homogeneous function of degree $\alpha = N(x_0, Z, 0^+) \geqslant 1$. Hence also (A2) holds. Next we choose the map $\mathfrak{S}$ according to different situations.
Proof of (6.2). Define $\mathfrak{S} : \mathcal{F} \to \mathcal{D}$ by $\mathfrak{S}(Z) = \Gamma(Z(\cdot, 0))$. Then $\Gamma(Z(\cdot, 0)) \cap \partial^0 B_1(0)$ is obviously closed in $\partial^0 B_1(0)$ by the continuity of Z. It is quite straightforward to check hypothesis (A3)-(i), and the local uniform convergence consider in $\mathcal{F}$ clearly yields (A3)-(ii). Therefore in order to end the proof of (6.2) the only thing left to prove is that the integer associated to $\mathfrak{S}$ (defined in (6.4)) is less than or equal to $n - 1$. Suppose by contradiction that $d = n$; then this would imply the existence of $Z \in \mathcal{F}$ with $\mathfrak{S}(Z) = \mathbb{R}^n$, i.e., $Z(x, 0) \equiv 0$, which contradicts our assumption on Z. Thus $d \leqslant n - 1$.
Proof of (6.3). Define $\mathfrak{S} : \mathcal{F} \to \mathcal{D}$ by $\mathfrak{S}(Z) = S(v(\cdot, 0)) = S(Z(\cdot, 0)) := \{x \in \Omega : N(x, Z, 0^+) > 1\}$ (which belongs to $\mathcal{D}$ since $S(Z(\cdot, 0))$ is closed in Ω). The map satisfies (A3)-(i) thanks to identity (5.3), more precisely,

$$\begin{aligned} (x, 0) \in \mathfrak{S}(Z_{x_0,t}/\rho) &\Leftrightarrow N(x, Z_{x_0,t}/\rho, 0^+) > 1 \\ &\Leftrightarrow N(x_0 + tx, Z, 0^+) > 1 \\ &\Leftrightarrow (x_0 + tx, 0) \in \mathfrak{S}(Z). \end{aligned}$$

As for (A3)-(ii), take $Z_k, \overline{Z} \in \mathcal{F}$ as stated. Then in particular $Z_k \to \overline{Z}$ uniformly in $\overline{B_2^+(0)}$ and by arguing as in the proof of Lemma 5.4 it is easy to obtain the strongly convergence in $H^1(B^+_{3/2}(0))$. Suppose now that (A3)-(ii) does not hold; then there exists a sequence $(x_k, 0) \in B^0_1(0)$ ($x_k \to x$, up to a subsequence, for some x) and $\overline{\varepsilon} > 0$ such that $N(x_k, Z_k, 0^+) \geqslant 1 + \delta_n$ and $\mathrm{dist}(x_k, \mathfrak{S}(\overline{Z})) \geqslant \overline{\varepsilon}$. But then for small r we obtain

$$N(x_k, Z_k, r) \geqslant N(x_k, Z_k, 0^+) \geqslant 1 + \delta,$$

and hence $N(x, \overline{Z}, 0^+) \geqslant 1 + \delta$, a contradiction.

Finally, let us prove that $d \leqslant n-2$. If $d = n-1$ then we would have the existence of a function Z, homogeneous with respect to every point in $\mathbb{R}^{n-1} \times \{0\}$ such that $S(Z(\cdot, 0)) = \mathbb{R}^{n-1} \times \{0\}$. Now if we take a blowup sequence centred at $(0, 0)$, namely, $Z(t_k x, t_k y)/\rho_k$, we obtain at the limit a nonzero function $\overline{Z} = r^\alpha G(\theta)$ with $\alpha = N(0, Z, 0^+) > 1$. Then $\overline{Z}(z_0 + \lambda x, \lambda y) = \lambda^\alpha \overline{Z}(x, y)$ whenever $z_0 \in \mathbb{R}^{n-1} \times \{0\}$, $(x, y) \in \mathbb{R}^{n+1}_+$. We prove that $\Gamma(\overline{Z}(\cdot, 0)) = \mathbb{R}^{n-1} \times \{0\}$, which leads to a contradiction since Hopf's Lemma implies $\alpha = 1$ (by possibly using the equation for $\overline{Z}^+$). Since $\overline{Z}(x, y) = \lim_k Z(t_k x, t_k y)/\rho_k$ and $S(Z(\cdot, 0)) = \mathbb{R}^{n-1} \times \{0\}$, it is obvious that $\mathbb{R}^{n-1} \times \{0\} \subseteq \Gamma(\overline{Z}(\cdot, 0))$. If there were $z \in \Gamma(\overline{Z}(\cdot, 0)) \backslash (\mathbb{R}^{n-1} \times \{0\})$, then since $\overline{Z}$ is homogeneous with respect to every point on $\mathbb{R}^{n-1} \times \{0\}$, we would have that either $\mathbb{R}^{n-1} \times [0, +\infty)$ or $\mathbb{R}^{n-1} \times (-\infty, 0]$ would be contained in $\Gamma(\overline{Z}(\cdot, 0))$, in contradiction with Remark 4.1. □

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