On invariant tori of vector field under weaker non-degeneracy condition

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Abstract. In this paper we prove the persistence of invariant tori for analytic perturbation of constant vector field under weaker non-degeneracy condition. In the proof we introduce a parameter q and make the steps of KAM iteration infinitely small in the speed of function $q^n \epsilon$, $0 < q < 1$, rather than super exponential function.

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1. Introduction and main results

In this paper we consider a vector field with parameters $X = N + P$, where $N = \omega(\xi)$ denotes a constant vector on the n -torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, describing uniform rotational motions with frequencies $\omega(\xi) = (\omega_1(\xi), \dots, \omega_n(\xi))$, $P(\theta; \xi)$ is a small perturbation, the parameters ξ vary in some bounded closed connected domain Π . Suppose the frequencies $\omega(\xi)$ satisfy

$$|\langle k, \omega(\xi) \rangle| \geq \frac{\alpha}{\Delta(|k|)}, \quad 0 \neq k \in \mathbb{Z}^n, \quad (1.1)$$

with some $\alpha > 0$ and some Rüssmann approximation function Δ . These are continuous, increasing, unbounded function $\Delta : [1, +\infty) \rightarrow [1, +\infty)$ such that $\Delta(1) = 1$ and

$$\int_1^{+\infty} \frac{\ln \Delta(t)}{t^2} < \infty.$$

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When the frequencies $\omega(\xi)$ satisfy the non-degeneracy condition:

$$\det \left(\frac{\partial \omega}{\partial \xi} \right) \neq 0, \quad \forall \xi \in \Pi, \quad (1.2)$$

and non-resonant condition (1.1), Pöschel [2] proved that the majority of invariant tori of constant vector field can persist under a small perturbation. Note that the frequencies themselves are chosen as parameters in the paper [2]. Indeed, by the non-degeneracy condition (1.2), the frequency map

$$\xi \rightarrow \omega(\xi)$$

is a diffeomorphism. Hence, instead of $\xi \in \Pi$ we may introduce the frequencies ω as independent parameters. See [1, 2] for more details.

In this paper we are mainly concerned with the existence of invariant tori for analytic perturbation of constant vector field under weaker non-degeneracy condition:

$$\text{rank} \left\{ \omega(\xi), \frac{\partial^\beta \omega}{\partial \xi^\beta} \mid \forall \beta \in \mathbb{Z}_+^n, |\beta| \leq n-1 \right\} = n, \forall \xi \in \Pi. \quad (1.3)$$

In order to state our results, we now describe setting more precisely. Usually, denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of integers and positive integers.

Define

$$D(s) = \{ \theta \in \mathbb{C}^n / 2\pi \mathbb{Z}^n \mid |\text{Im} \theta| \leq s \}$$

and denote a complex neighborhood of Π by

$$\Pi_h = \{ \xi \in \mathbb{C}^n \mid \text{dist}(\xi, \Pi) \leq h \}.$$

Now the function $f(\theta; \xi)$ is real analytic on $D(s) \times \Pi_h$. We expand $f(\theta; \xi)$ as Fourier series with respect to θ ,

$$f(\theta; \xi) = \sum_{k \in \mathbb{Z}^n} f_k(\xi) e^{i \langle k, \theta \rangle},$$

then define

$$\|f\|_{D(s) \times \Pi_h} = \sum_{k \in \mathbb{Z}^n} \|f_k\|_h e^{s|k|},$$

where

$$\|f_k\|_h = \sup_{\xi \in \Pi_h} |f_k(\xi)|.$$

Finally, with any approximation function Δ we define another function Λ by setting $\Lambda(t) = t\Delta(t)$.

Theorem 1.1. *Suppose the vector field $X = N + P = \omega(\xi) + P(\theta; \xi)$ is real analytic on $D(s) \times \Pi_h$ and satisfies that*

$$\|P\|_{D(s) \times \Pi_h} = \epsilon < \frac{h}{16} \leq \frac{\alpha}{32\Lambda(\tau)},$$

where τ is so large that

$$32 \int_{\tau}^{\infty} \frac{\ln \Lambda(t)}{t^2} dt < \frac{s}{2}.$$

Then there exists a non-empty Cantor set $\Pi_* \subset \Pi$ and a family of transformations $\Phi^*(\cdot; \xi) : D(s_*) \rightarrow D(s), \forall \xi \in \Pi_*$, satisfying

$$|\Phi^* - id|_{D(s_*) \times \Pi_*} \leq 200e^2 \frac{\Lambda(\tau)\epsilon}{\alpha},$$

where $s_* \geq s/2$ is a positive constant.

Under these transformations, the vector X is transformed into $X_* = \omega_*(\xi)$, whose frequencies satisfy that

$$|\omega_*(\xi) - \omega(\xi)|_{\Pi_*} \leq 16\epsilon,$$

and

$$|\langle k, \omega_*(\xi) \rangle| \geq \frac{99}{100} \frac{\alpha}{\Delta(|k|)}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}.$$

Moreover, we have $\text{meas}(\Pi \setminus \Pi_*) \leq c(\text{diam}\Pi)^{n-1} \alpha^{\frac{1}{n-1}}$.

2. Proof of the main results

In this section we will prove our Theorem 1.1. Under weaker non-degeneracy condition (1.3), the frequencies can not be regarded as independent parameters. Moreover, the frequencies may have some drifts at each step, rather than under the non-degeneracy condition (1.2), the frequencies remain unchanged. So the previous methods in [1–3] are not valid. The key is to control the parameters in the small divisors. Combining some technique in [2, 3, 5–7], we introduce an artificial parameter q and make the steps of KAM iteration infinitely small in the speed of function $q^n \epsilon, 0 < q < 1$, rather than super exponential function. We first give one KAM step. In the following the norm $\|\cdot\|_{s;h}$ indicates $\|\cdot\|_{D(s) \times \Pi_h}$ for simplicity. Let $T = \max_{\xi \in \Pi_h} |\partial\omega/\partial\xi|$, where $|\partial\omega/\partial\xi| = \max_{1 \leq i \leq n} |\partial\omega_i/\partial\xi|_2$.

Lemma 2.1. (KAM step). *Consider the real analytic vector field $X = N + P$ on $D(s) \times \Pi_h$, where $N = \omega(\xi)$ is a constant vector field, $P = P(\theta; \xi)$ is a small perturbation. Let $0 < \sigma < s/2$ and $\tau \geq 1$. Set $a = 1 - e^{-\tau\sigma}$. Suppose $\forall \xi \in \Pi$, the following small divisors condition holds:*

$$|\langle k, \omega(\xi) \rangle| \geq \frac{\alpha}{\Delta(|k|)}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad 0 < |k| \leq \tau. \quad (2.4)$$

Let

$$\max_{\xi \in \Pi_h} \left| \frac{\partial\omega}{\partial\xi} \right| \leq T, \quad h = \frac{\alpha}{2\Lambda(\tau)T}.$$

Suppose that

$$\|P\|_{s;h} \leq \epsilon < \frac{b\alpha}{2\Lambda(\tau)}, \quad (2.5)$$

where $0 < b \leq 1/2$ is a positive constant.

Let

$$s_+ = s - \sigma, \quad q = (1 - a + a^2b)(1 + b)e^a,$$

where $0 < a < 1, 0 < b \leq 1/2$ are positive constants, $0 < q < 1$.

Then for any $\xi \in \Pi_h$, there exists a transformation $\Phi(\cdot; \xi) : D(s_+) \rightarrow D(s_-)$ such that

$$X_+ = X \circ \Phi = N_+ + P_+,$$

where $N_+ = \omega_+(\xi)$ with

$$|\omega_+(\xi) - \omega(\xi)| \leq a\epsilon, \forall \xi \in \Pi_h,$$

the new perturbation P_+ satisfies

$$\|P_+\|_{s_+, h} \leq \epsilon_+ = q\epsilon.$$

Let $\alpha_+ = \alpha - a\Lambda(\tau)\epsilon$. Denote by

$$R_k^+ = \left\{ \xi \in \Pi \mid |\langle k, \omega_+(\xi) \rangle| < \frac{\alpha_+}{\Delta(|k|)}, \forall \tau < |k| \leq \tau_+ \right\}$$

and $\Pi_+ = \Pi \setminus R_k^+$. Then $\forall \xi \in \Pi_+$, it follows that

$$|\langle k, \omega_+(\xi) \rangle| \geq \frac{\alpha_+}{\Delta(|k|)}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad 0 < |k| \leq \tau_+, \quad (2.6)$$

where

$$\tau_+ = \Lambda^{-1}(\Lambda_+), \quad \Lambda_+ = \frac{\Lambda(\tau)}{\lambda q}.$$

Let

$$T_+ = T + \frac{a\epsilon}{(1 - \lambda q)h}, \quad h_+ = \frac{\alpha_+}{2\Lambda(\tau_+)T_+},$$

where $\lambda > 1$ is a constant.

If $h_+ \leq \lambda q h$, then $\max_{\xi \in \Pi_{h_+}} |\partial \omega / \partial \xi| \leq T_+$. Moreover, we have

$$\|P_+\|_{s_+, h_+} \leq \epsilon_+ = q\epsilon.$$

Thus the above result also holds for the new vector field X_+ in place of X .

Proof of Lemma 2.1. The above lemma is actually one KAM step. We divide the proof into several parts.

A. Truncation. P is real analytic, expanding P as Fourier series $P = \sum_k P_k(\xi)e^{i\langle k, \theta \rangle}$. Let $P = \tilde{P} + \hat{P}$ with

$$\hat{P} = \sum_{|k| > \tau} P_k e^{i\langle k, \theta \rangle} + (1 - a) \sum_{|k| \leq \tau} P_k e^{|k|\sigma} e^{i\langle k, \theta \rangle}.$$

In view of $e^{-\tau\sigma} = 1 - a$, we have

$$\|\hat{P}\|_{s-\sigma; h} \leq (1 - a)\|P\|_{s; h} \leq (1 - a)\epsilon.$$

On the other hand, the polynomial rest

$$\tilde{P} = \sum_{|k| \leq \tau} \tilde{P}_k e^{i\langle k, \theta \rangle}, \quad \tilde{P}_k = (1 - (1 - a)e^{|k|\sigma})P_k,$$

is bounded. Indeed, with $\tilde{\sigma} = \frac{\sigma(1-a)}{a}$,

$$\begin{aligned} \|\tilde{P}\|_{s+\tilde{\sigma};h} &= \sum_{0 \leq |k| \leq \tau} (1 - (1-a)e^{|k|\sigma}) \|P_k\|_h e^{|k|(s+\tilde{\sigma})} \\ &\leq \sup_{0 \leq t \leq \tau} (1 - (1-a)e^{t\sigma}) e^{t\tilde{\sigma}} \sum_{0 \leq |k| \leq \tau} \|P_k\|_h e^{|k|s} = a\epsilon, \end{aligned} \quad (2.7)$$

as the function under the sup is monotonically decreasing for $0 \leq t \leq \tau$ and equals a at $t = 0$.

B. Extension of small divisors estimate. For any $\xi \in \Pi_h$, there is a parameter $\xi_0 \in \Pi$ with $|\xi - \xi_0|_2 < h = \frac{\alpha}{2\Lambda(\tau)T}$, hence, in view of $\Lambda(\tau) = \tau\Delta(\tau)$,

$$|\langle k, \omega(\xi) - \omega(\xi_0) \rangle| \leq |k| |\omega(\xi) - \omega(\xi_0)| \leq \tau Th = \frac{\alpha}{2\Delta(\tau)}.$$

As $\omega(\xi_0)$ satisfies (2.4), all relevant divisors thus admit the lower bound

$$\begin{aligned} |\langle k, \omega(\xi) \rangle| &\geq |\langle k, \omega(\xi_0) \rangle| - |\langle k, \omega(\xi) - \omega(\xi_0) \rangle| \\ &\geq \frac{\alpha}{\Delta(|k|)} - \frac{\alpha}{2\Delta(\tau)} \geq \frac{\alpha}{2\Delta(\tau)}. \end{aligned} \quad (2.8)$$

Thus the small divisors condition (2.4) also holds for all ξ in the neighborhood Π_h of Π .

C. Construction of the transformation. The coordinate transformation Φ is generated by the time-1 map of the flow F_t of a vector field F , which satisfies that

$$\begin{aligned} \Phi^*(N + P) &= F_t^*(N + \tilde{P} + \hat{P})|_{t=1} \\ &= N + [N, F] + \int_0^1 (1-t) F_t^*[[N, F], F] dt \\ &\quad + \tilde{P} + \int_0^1 F_t^*[\tilde{P}, F] dt + F_1^* \hat{P}, \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket of two vector fields. The point is to find F which solves the homological equation

$$[N, F] + \tilde{P} = P^0, \quad (2.9)$$

where P^0 denotes the mean value of $\tilde{P}$. By (2.8), for any $\xi \in \Pi_h$ the linearized equation (2.9) is solved by

$$F = \sum_{0 < |k| \leq \tau} \frac{\tilde{P}_k}{i\langle k, \omega(\xi) \rangle} e^{i\langle k, \theta \rangle}.$$

Then with $a = 1 - e^{-\tau\sigma} \leq \tau\sigma$, the estimates (2.7) and (2.8), we get

$$\|F\|_{s+\tilde{\sigma}} \leq 2\Delta(\tau)\alpha^{-1} \|\tilde{P}\|_{s+\tilde{\sigma}} \leq 2\Delta(\tau)\alpha^{-1} a\epsilon = 2\Lambda(\tau)\sigma\alpha^{-1}\epsilon. \quad (2.10)$$

By hypothesis (2.5), $\|F\|_{s+\tilde{\sigma}} \leq \sigma$, so the vector field F generates a flow F_t , which satisfies for $0 \leq t \leq 1$,

$$F_t : D(s - 2\sigma) \rightarrow D(s - \sigma), \quad \|F_t - id\|_{s-2\sigma} \leq 2\Lambda(\tau)\sigma\alpha^{-1}\epsilon.$$

Therefore, the vector field X is transformed into $X_+ = N_+ + P_+$, where $N_+ = N + P^0 = \omega(\xi) + P_0(\xi)$ is a new constant vector field,

$$\begin{aligned} P_+ &= \int_0^1 (1-t) F_t^* [[N, F], F] dt + \int_0^1 F_t^* [\tilde{P}, F] dt + F_1^* \hat{P} \\ &= \int_0^1 F_t^* [P_t, F] dt + F_1^* \hat{P} \end{aligned}$$

is the new perturbation of the constant vector field N_+ , $P_t = t\tilde{P} + (1-t)P^0$.

D. Estimates of the new frequencies. Let $\alpha_+ = \alpha - a\Lambda(\tau)\epsilon$. We have

$$\begin{aligned} |\langle k, \omega_+(\xi) \rangle| &\geq |\langle k, \omega(\xi) \rangle| - |\langle k, \omega_+(\xi) - \omega(\xi) \rangle| \\ &\geq \frac{\alpha}{\Delta(|k|)} - a\tau\epsilon \\ &\geq \frac{1}{\Delta(|k|)} (\alpha - a\Lambda(\tau)\epsilon) = \frac{\alpha_+}{\Delta(|k|)}, \end{aligned}$$

$\forall \xi \in \Pi, 0 < |k| \leq \tau$. Then by the definition of R_k^+ , it follows that (2.6) holds.

Suppose $h_+ \leq \lambda q h$, then by Cauchy's estimates we have

$$\left| \frac{\partial(\omega_+(\xi) - \omega(\xi))}{\partial \xi} \right| \leq \frac{a\epsilon}{(1-\lambda q)h}, \quad \forall \xi \in \Pi_{h_+}.$$

Let $T_+ = T + \frac{a\epsilon}{(1-\lambda q)h}$. Then $\max_{\xi \in \Pi_{h_+}} |\partial \omega_+(\xi) / \partial \xi| \leq T_+$.

E. Estimates of the new perturbation. To estimate P_+ we note that $\|P_t\|_{s+\tilde{\sigma}} \leq t\|\tilde{P}\|_{s+\tilde{\sigma}} + (1-t)\|P^0\|_{s+\tilde{\sigma}} \leq a\epsilon$ and

$$(s + \tilde{\sigma}) - (s - \sigma) = \frac{1-a}{a}\sigma + \sigma = \frac{\sigma}{a}.$$

Lemma 3.1 of Appendix and the abbreviation $b = 2\Lambda(\tau)\alpha^{-1}\epsilon$ thus yield

$$\|[P_t, F]\|_{s-\sigma} \leq \frac{a}{\sigma} \|P_t\|_{s+\tilde{\sigma}} \|F\|_{s+\tilde{\sigma}} \leq 2a^2\Lambda(\tau)\alpha^{-1}\epsilon^2 = a^2b\epsilon.$$

In view of (2.10) we can apply Lemma 3.2 of Appendix with $r = s - \sigma$ and $\eta = 1/a$ to obtain

$$\int_0^1 \|F_t^* [P_t, F]\|_{s-2\sigma} dt \leq (1+b)e^a \|[P_t, F]\|_{s-\sigma} \leq a^2b(1+b)e^a\epsilon.$$

Similarly,

$$\|F_1^* \hat{P}\|_{s-2\sigma} \leq (1+b)e^a \|\hat{P}\|_{s-\sigma} \leq (1-a)(1+b)e^a\epsilon.$$

Both estimates together yield the stated estimates of P_+ .

Setting the parameters and iteration. Now we choose some suitable parameters so that the above iteration can go on infinitely.

We can always choose $0 < a < 1$ and $0 < b \leq 1/2$ so that

$$q = (1-a+a^2b)(1+b)e^a < 1,$$

and we can even make q as close to 1 as we wish.

It then suffices to choose for ϵ and Λ geometric sequences with the same base q , namely

$$\epsilon_n = \epsilon_0 q^n, \quad \Lambda_n = \frac{\Lambda_0}{(\lambda q)^n},$$

where $\lambda > 1$ is a constant, we assume $\Lambda_0 \geq \Lambda(1) = \Delta(1)$. Next, let $\tau_n = \Lambda^{-1}(\Lambda_n)$ and define other parameters through

$$\alpha_{n+1} = \alpha_n - a\Lambda(\tau_n)\epsilon_n, \quad h_n = \frac{\alpha_n}{2\Lambda(\tau_n)T_n},$$

$$T_{n+1} = T_n + \frac{a\epsilon_n}{(1 - \lambda q)h_n},$$

$$1 - a = e^{-\tau_n \sigma_n}, \quad s_{n+1} = s_n - \sigma_n.$$

As we will see in a moment, s_n has a positive limit for Λ_0 sufficiently large.

Let

$$\Pi_n = \left\{ \xi \in \Pi_{n-1} \mid |\langle k, \omega_n(\xi) \rangle| \geq \frac{\alpha_n}{\Delta(|k|)}, \forall \tau_{n-1} < |k| \leq \tau_n \right\}. \quad (2.11)$$

Denote $\Pi_{h_n} = \{\xi \in \mathbb{C}^n \mid \text{dist}(\xi, \Pi_n) \leq h_n\}$ and $D_n = D(s_n)$ for simplicity.

For any $\xi \in \Pi_{h_n}$, there is a parameter $\xi_0 \in \Pi_n$ with $|\xi - \xi_0| < h_n = \frac{\alpha_n}{2\Lambda(\tau_n)T_n}$, hence, in view of $\Lambda(\tau_n) = \tau_n \Delta(\tau_n)$,

$$|\langle k, \omega_n(\xi) - \omega_n(\xi_0) \rangle| \leq |k| |\omega_n(\xi) - \omega_n(\xi_0)| \leq \tau_n T_n h_n = \frac{\alpha_n}{2\Delta(\tau_n)}.$$

As $\omega_n(\xi_0)$ satisfies (2.11), all relevant divisors thus admit the lower bound

$$\begin{aligned} |\langle k, \omega_n(\xi) \rangle| &\geq |\langle k, \omega_n(\xi_0) \rangle| - |\langle k, \omega_n(\xi) - \omega_n(\xi_0) \rangle| \\ &\geq \frac{\alpha_n}{\Delta(|k|)} - \frac{\alpha_n}{2\Delta(\tau_n)} \geq \frac{\alpha_n}{2\Delta(\tau_n)}. \end{aligned}$$

Thus the small divisors condition (2.11) also holds for all ξ in the neighborhood Π_{h_n} of Π_n .

By the Step Lemma 2.1, we have a sequence of transformations $\Phi_n = F_{nt}|_{t=1}$, such that for any $\xi \in \Pi_{h_n}$, $\Phi_n : D_n \times \Pi_{h_n} \rightarrow D_{n-1} \times \Pi_{h_{n-1}}$, satisfying

$$|\Phi_n - id|_{D_n \times \Pi_{h_n}} \leq 2\Lambda(\tau_n)\alpha_n^{-1}\sigma_n\epsilon_n,$$

and

$$|D\Phi_n - Id|_{D_n \times \Pi_{h_n}} \leq 2a\Lambda(\tau_n)\alpha_n^{-1}\epsilon_n.$$

Thus the transformation $\Phi^n = \Phi_0 \circ \Phi_1 \cdots \Phi_n$ is well defined in $D_n \times \Pi_{h_n}$ and is seen to take $X_0 = N_0 + P_0 = \omega_0(\xi) + P_0(\theta; \xi)$ into

$$X_n = N_n + P_n = \omega_n(\xi) + P_n(\theta; \xi),$$

satisfying

$$\|P_n\|_{D_n \times \Pi_{h_n}} \leq \epsilon_n = \epsilon_0 q^n,$$

and

$$|\omega_{n+1}(\xi) - \omega_n(\xi)|_{D_n \times \Pi_{h_n}} \leq a\epsilon_n.$$

In the following we will check the assumptions in the Step Lemma 2.1 to ensure that KAM step is valid for all $n \geq 0$.

By the definition of h_n , $h_n = \frac{\alpha_n}{2\Lambda(\tau_n)T_n}$, it follows that

$$\frac{h_{n+1}}{h_n} = \frac{\alpha_{n+1}}{\alpha_n} \frac{T_n}{T_{n+1}} \frac{\Lambda(\tau_n)}{\Lambda(\tau_{n+1})} \leq \frac{\Lambda_n}{\Lambda_{n+1}} = \lambda q.$$

So we have $h_{n+1} \leq \lambda q h_n$.

By the definition of α_n , we have

$$\begin{aligned} \alpha_{n+1} &= \alpha_n - a\Lambda(\tau_n)\epsilon_n = \alpha_n - \frac{a\Lambda_0\epsilon_0}{\lambda^n} \\ &\geq \alpha_0 - a\Lambda_0\epsilon_0 \sum_{0 \leq n \in \mathbb{Z}} \frac{1}{\lambda^n} \\ &\geq \alpha - \frac{\lambda a\Lambda_0\epsilon_0}{\lambda - 1}. \end{aligned}$$

Since $\epsilon_0 \leq \frac{\alpha(\lambda-1)^2}{\lambda^2 a\Lambda_0}$, we obtain $\alpha/\lambda \leq \alpha_n \leq \alpha, \forall n \geq 1$.

Suppose $\max_{\xi \in \Pi_{h_n}} |\partial\omega_n/\partial\xi| \leq T_n$. Let $T_{n+1} = T_n + \frac{a\epsilon_n}{(1-\lambda q)h_n}$. Then we have $\max_{\xi \in \Pi_{h_{n+1}}} |\partial\omega_{n+1}/\partial\xi| \leq T_{n+1}$.

By iteration,

$$\begin{aligned} T_{n+1} &= T_n + \frac{a\epsilon_n}{(1-\lambda q)h_n} \leq T_n \left(1 + \frac{2\lambda a\Lambda_0\epsilon_0}{(1-\lambda q)\alpha} \frac{1}{\lambda^n} \right) \\ &\leq T_0 \prod_{n \geq 0} \left(1 + \frac{2\lambda a\Lambda_0\epsilon_0}{(1-\lambda q)\alpha} \frac{1}{\lambda^n} \right). \end{aligned}$$

By $\epsilon_0 \leq \frac{(1-\lambda q)\alpha}{2\lambda a\Lambda_0}$, then $T_0 \leq T_n \leq e^{\frac{\lambda}{\lambda-1}} T_0, \forall n \geq 1$.

Lemma 2.2. (Iterative Lemma). *Consider the real analytic vector field $X_0 = N_0 + P_0 = \omega_0(\xi) + P_0(\theta; \xi)$, satisfying*

$$\|P_0\|_{D_0 \times \Pi_{h_0}} \leq \epsilon_0 \leq \min \left\{ \frac{\alpha(\lambda-1)^2}{\lambda^2 a\Lambda_0}, \frac{(1-\lambda q)\alpha}{2\lambda a\Lambda_0}, \frac{\alpha b}{2\Lambda_0} \right\},$$

with Λ_0 sufficiently large. Then for each $n \geq 1$ there exists a parameter and coordinate transformation

$$\Phi^n = \Phi_0 \circ \cdots \circ \Phi_n : D(s_n) \times \Pi_{h_n} \rightarrow D(s_0) \times \Pi_{h_0},$$

which transforms the vector field X into $X_n = N_n + P_n = \omega_n(\xi) + P_n(\theta; \xi)$ such that

$$\|P_n\|_{D_n \times \Pi_{h_n}} \leq \epsilon_n = \epsilon_0 q^n.$$

Moreover,

$$|\Phi^{n+1} - \Phi^n|_{D_{n+1} \times \Pi_{h_{n+1}}} \leq c\Lambda(\tau_{n+1})\alpha_{n+1}^{-1}\sigma_{n+1}\epsilon_{n+1}. \quad (2.12)$$

Proof. This follows by applying the Step Lemma 2.1 repeatedly. Just note that $\Lambda(\tau_n) = \Lambda_n$ and

$$2\epsilon_n\Lambda_n\alpha_n^{-1} \leq 2\epsilon_0\Lambda_0\alpha_0^{-1} \leq b, \quad \forall n \geq 1.$$

In the following the estimate of Φ^n is given. First by Step Lemma, we observe that

$$|\Phi_n - id|_{D_n \times \Pi_{h_n}} \leq 2\Lambda(\tau_n)\alpha_n^{-1}\sigma_n\epsilon_n,$$

and

$$|D\Phi_n - Id|_{D_n \times \Pi_{h_n}} \leq 2a\Lambda(\tau_n)\alpha_n^{-1}\epsilon_n,$$

where D denotes the Jacobian with respect to θ .

We then have

$$\begin{aligned} |\Phi^{n+1} - \Phi^n|_{D_{n+1} \times \Pi_{h_{n+1}}} &= |\Phi^n \circ \Phi_{n+1} - \Phi^n|_{D_{n+1} \times \Pi_{h_{n+1}}} \\ &\leq |D\Phi^n|_{D_n \times \Pi_{h_n}} |\Phi_{n+1} - id|_{D_{n+1} \times \Pi_{h_{n+1}}} \\ &\leq c\Lambda(\tau_{n+1})\alpha_{n+1}^{-1}\sigma_{n+1}\epsilon_{n+1}, \end{aligned}$$

provided that $D\Phi^n$ is uniformly bounded on $D_n \times \Pi_{h_n}$.

In fact by induction we have $D\Phi^n = D\Phi_0 \circ \dots \circ D\Phi_n$, with the Jacobians evaluated at different points, and

$$\begin{aligned} |D\Phi^n| &= |D\Phi_0 \circ \dots \circ D\Phi_n| \leq \prod_{n \geq 0} (1 + 2a\Lambda(\tau_n)\alpha_n^{-1}\epsilon_n) \\ &\leq \prod_{n \geq 0} \left(1 + \frac{2a\Lambda_0\epsilon_0}{\alpha} \frac{1}{\lambda^{n-1}} \right) \leq \exp \left\{ \frac{2a\lambda^2\Lambda_0\epsilon_0}{(\lambda-1)\alpha} \right\} \leq e^{2(\lambda-1)}, \end{aligned}$$

which is uniformly bounded.

Convergence of iteration We first verify that the s_n tend to a positive limit. Indeed, let $t = \Lambda^{-1}(\Lambda_0(\lambda q)^{-x})$, we have

$$\sum_{n \geq 0} \frac{1}{\tau_n} \leq \int_0^\infty \frac{dx}{\Lambda^{-1}(\Lambda_0(\lambda q)^{-x})} = \frac{1}{\ln(\lambda q)^{-1}} \int_{\tau_0}^\infty \frac{d\Lambda(t)}{t\Lambda(t)}.$$

Integrating by parts and requiring $\Lambda(\tau_0) \geq (\lambda q)^{-1}$ we get

$$\sum_{n \geq 0} \frac{1}{\tau_n} \leq \frac{1}{\ln(\lambda q)^{-1}} \int_{\tau_0}^\infty \frac{\ln \Lambda(t)}{t^2} dt.$$

It follows that

$$\sum_{n \geq 0} \sigma_n = \sum_{n \geq 0} \frac{\ln(1-a)^{-1}}{\tau_n} \leq \frac{\ln(1-a)}{\ln \lambda q} \int_{\tau_0}^\infty \frac{\ln \Lambda(t)}{t^2} dt.$$

Hence by choosing τ_0 sufficiently large, we can achieve that $\sum_{n \geq 0} \sigma_n \leq s/2$ and thus $s_n \rightarrow s_* \geq s/2$.

By the estimates of Iterative Lemma 2.2, Φ^n and $\omega_n(\xi)$ respectively satisfy

$$|\Phi^{n+1} - \Phi^n|_{D_{n+1} \times \Pi_{h_{n+1}}} \leq c\Lambda(\tau_{n+1})\alpha_{n+1}^{-1}\sigma_{n+1}\epsilon_{n+1},$$

and

$$|\omega_{n+1}(\xi) - \omega_n(\xi)|_{\Pi_{h_n}} \leq a\epsilon_n, \quad (2.13)$$

which ensures the weaker non-degeneracy condition (1.3) also holds at each step.

Note that $s_n \rightarrow s_* \geq s/2$, $h_n \rightarrow 0$, as $n \rightarrow \infty$. Thus the mappings Φ^n converge uniformly on

$$\bigcap_{n \geq 0} D_n \times \Pi_{h_n} = D(s_*) \times \Pi_*,$$

to a mapping Φ^* , which is real analytic on $D(s_*)$ and uniformly continuous on Π_* . Moreover, we have

$$|\Phi^* - id|_{D(s_*) \times \Pi_*} \leq \frac{\lambda^2 e^{2(\lambda-1)} \Lambda_0 \epsilon_0 \sigma}{(\lambda-1)\alpha},$$

$$|\omega_*(\xi) - \omega(\xi)|_{\Pi_*} \leq \frac{a\epsilon_0}{1-q},$$

and

$$|\langle k, \omega_*(\xi) \rangle| \geq \frac{\alpha_*}{\Delta(|k|)}, \forall k \in \mathbb{Z}^n \setminus \{0\}, \forall \xi \in \Pi_*,$$

where $\alpha_* = \lim_{n \rightarrow \infty} \alpha_n$, and $\alpha/\lambda \leq \alpha_* \leq \alpha$.

For the statement of the theorem, we choose $a = 1/4$, $b = 1/64$, and $\lambda = 100/99$, which results in

$$q \approx \frac{49}{50}, \quad \frac{\ln(1-a)}{\ln \lambda q} \leq 32.$$

Thus the proof of Theorem 1.1 is complete. $\square$

Lemma 2.3. (Estimates of measure for parameters). *Suppose the frequencies $\omega(\xi)$ satisfy the Brjuno–Rüssmann’s non-resonant condition (2.4) and weaker non-degeneracy condition (1.3) on Π . If the approximation function is chosen such that*

$$\sum_{0 \neq k \in \mathbb{Z}^n} \left(\frac{1}{|k| \Delta(|k|)} \right)^{\frac{1}{n-1}} < \infty, \quad (2.14)$$

then $\text{meas}(\Pi \setminus \Pi_*) \leq c(\text{diam} \Pi)^{n-1} \alpha^{\frac{1}{n-1}}$.

Proof. In the following we estimate the Lebesgue measure of the set Π_* , on which the small divisors condition holds in KAM iteration. By the KAM step, we have

$$\Pi \setminus \Pi_* = \bigcup_{n \geq 0} R_k^n,$$

where

$$R_k^n = \left\{ \xi \in \Pi_{n-1} \mid |\langle k, \omega_n(\xi) \rangle| < \frac{\alpha_n}{\Delta(|k|)}, \forall \tau_{n-1} < |k| \leq \tau_n \right\},$$

with $\Pi_{-1} = \Pi$, $\tau_{-1} = 0$.

By the weaker non-degeneracy condition (1.3) and the estimate (2.13), if τ_0 is sufficiently large, then for all $n \geq 0$, the frequencies $\omega_n(\xi)$ also satisfy (1.3). So according to Lemma 3.4, we have

$$\begin{aligned} \text{meas}(R_k^n) &\leq c(\text{diam}\Pi)^{n-1} \sum_{\tau_{n-1} < |k| \leq \tau_n} \left(\frac{\alpha_n}{|k|\Delta(|k|)} \right)^{\frac{1}{n-1}} \\ &\leq c(\text{diam}\Pi)^{n-1} \alpha^{\frac{1}{n-1}} \sum_{\tau_{n-1} < |k| \leq \tau_n} \left(\frac{1}{|k|\Delta(|k|)} \right)^{\frac{1}{n-1}}. \end{aligned}$$

Since the condition (2.14), we deduce

$$\begin{aligned} \text{meas}(\Pi \setminus \Pi_*) &\leq c(\text{diam}\Pi)^{n-1} \alpha^{\frac{1}{n-1}} \sum_{0 \neq k \in \mathbb{Z}^n} \left(\frac{1}{|k|\Delta(|k|)} \right)^{\frac{1}{n-1}} \\ &\leq c(\text{diam}\Pi)^{n-1} \alpha^{\frac{1}{n-1}}. \end{aligned}$$

□

Remark. From power function $t^\zeta, \zeta > n(n-1) - 1$, to exponent function $e^{t^{1-\gamma}}, \gamma > 0$, can all be chosen as a Brjuno–Rüssmann approximation function.

Appendix

In this section we formulate some lemmas which have been used in the previous section. For detailed proofs we refer to [2, 4].

Lemma 3.1. *Let U and V be analytic vector fields on the torus $\mathbb{T}^n$. Then for $0 < r < \min\{u, v\}$,*

$$\| [U, V] \|_r \leq \frac{1}{e} \left(\frac{1}{u-r} + \frac{1}{v-r} \right) \|u\|_u \|v\|_v.$$

Lemma 3.2. *Suppose the vector fields F and V are analytic on the torus $\mathbb{T}^n$. If $b = \sigma^{-1} \|F\|_{r+\eta\sigma} \leq 1/2$ with $0 < \sigma < r$ and $\eta > 0$, then*

$$\|F_t^* V\|_{r-\sigma} \leq (1+bt)e^{1/\eta} \|V\|_r, \quad 0 \leq t \leq 1.$$

Lemma 3.3. *Suppose $f(x)$ is m -th differentiable function on the closure $\bar{I}$ of I , where $I \subset \mathbb{R}$ is an interval. Let $I_h = \{x \mid |f(x)| < h, x \in I\}$, $h > 0$. If $|f^{(m)}(x)| \geq d > 0$ for all $x \in I$, where d is a constant, then*

$$\text{meas}(I_h) \leq ch^{\frac{1}{m}},$$

where $c = 2(2 + 3 + \dots + m + d^{-1})$.

Lemma 3.4. *Suppose the mapping $g : x \in \bar{\Pi} \rightarrow (g_1(x), g_2(x), \dots, g_n(x))$ is analytic on $\bar{\Pi}$, where Π is a bounded connected domain in $\mathbb{R}^n$ and $\bar{\Pi}$ is its closure. Let*

$$R_k = \left\{ x \mid k_1 g_1(x) + k_2 g_2(x) + \dots + k_n g_n(x) < \frac{\alpha}{\Delta(|k|)} \right\}, \quad \forall 0 \neq k \in \mathbb{Z}^n,$$

where $\tau > n(n-1) - 1$. If for all $x \in \bar{\Pi}$,

$$\text{rank} \left\{ g, \frac{\partial^\beta g}{\partial x^\beta} \mid \forall \beta \in \mathbb{Z}_+^n, |\beta| \leq n-1 \right\} = n, \quad (3.1)$$

then for any $\alpha > 0$ sufficiently small, $\text{meas}(R_k) \leq c(\text{diam}\Pi)^{n-1} \left(\frac{\alpha}{|k|\Delta(|k|)} \right)^{\frac{1}{n-1}}$, where c is independent of α .

Proof. For completeness, we give the proof, which follows from [4]. By the non-degeneracy condition (3.1), it follows that for $\xi \in \Pi$ there exist n integers, $1 \leq r_1, r_2, \dots, r_n \leq n-1$, and n directions, $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ such that

$$\text{rank} \{ D_{v_1}^{r_1} \omega(\xi), D_{v_2}^{r_2} \omega(\xi), \dots, D_{v_n}^{r_n} \omega(\xi) \} = n, \quad (3.2)$$

where $D_v^r \omega = \frac{d^r}{dt^r} \omega(\xi + tv)|_{t=0}$ are r -th direction derivatives of ω at ξ along v . There exists a neighborhood of ξ , $\Pi_\xi \subset \Pi$, such that (3.1) holds on its closure $\bar{\Pi}_\xi$.

Since Π is compact, we can choose such finite neighborhoods to cover Π , so without loss of generality we suppose that (3.2) holds for all $\xi \in \Pi$.

Let the matrix $A(\xi) = (D_{v_1}^{r_1} \omega(\xi), D_{v_2}^{r_2} \omega(\xi), \dots, D_{v_n}^{r_n} \omega(\xi))$. Since $\det(A(\xi)) \neq 0$ for all $\xi \in \Pi$, there exists $c_1 > 0$ such that for $\forall(\xi, \nu) \in \Pi \times S$, $|A(\xi)\nu| \geq c_1$, where $S = \{\nu \in \mathbb{R}^n, |\nu| = \sum_{j=1}^n |\nu_j| = 1\}$, the norm of the vector $A(\xi)\nu$ is the same as that of ν . Thus $\forall(\xi, \nu) \in \Pi \times S$, there exists a neighborhood of ξ in Π , Π_ξ , and a neighborhood of ν in S , S_ν such that for some i ,

$$|\langle D_{v_i}^{r_i} \omega(\xi'), \nu' \rangle| \geq \frac{c_1}{2n}, \quad \forall(\xi', \nu') \in \Pi_\xi \times S_\nu.$$

Since the sets $\{\Pi_\xi \times S_\nu | (\xi, \nu) \in \Pi \times S\}$ covers the compact set $\Pi \times S$, there exists finite covers: $\Pi_1 \times S_1, \dots, \Pi_N \times S_N$, such that $\bigcup_{i=1}^N \Pi_i \times S_i \supset \Pi \times S$ and for $(\xi, \nu) \in \Pi_i \times S_i$,

$$|\langle D_{\bar{v}_i}^{\bar{r}_i} \omega(\xi), \nu \rangle| \geq \frac{c_1}{2n},$$

where $\bar{r}_i \in \{r_1, r_2, \dots, r_n\}$ and $\bar{v}_i \in \{v_1, v_2, \dots, v_n\}$, $i = 1, 2, \dots, N$.

Now fix $0 \neq k$ and suppose $\frac{k}{|k|} \in S_i$. Then for $\xi \in \Pi_i$,

$$\left| \left\langle D_v^r \omega(\xi), \frac{k}{|k|} \right\rangle \right| \geq \frac{c_1}{2n},$$

where we write $\bar{r}_i$ and $\bar{v}_i$ as r and v to simplify notations, and $1 \leq r \leq n-1$. Let $f(\xi) = \langle k, \omega(\xi) \rangle$. Then

$$\frac{1}{|k|} |D_v^r f(\xi)| \geq \frac{c_1}{2n}.$$

Let

$$R_{kv}^i = \left\{ t \in \mathbb{R} \left| |f(\xi + tv)| < \frac{\alpha}{\Delta(|k|)}, \xi \in \Pi_i, \xi + tv \in \Pi_i \right. \right\},$$

and

$$R_k^i = \left\{ \xi \in \mathbb{R}^n \left| |f(\xi)| < \frac{\alpha}{\Delta(|k|)}, \xi \in \Pi_i \right. \right\}.$$

Since for $\xi + tv \in \Pi_i$, we have

$$\frac{1}{|k|} \left| \frac{d^r}{dt^r} f(\xi + tv) \right| = \frac{1}{|k|} |D_v^r f(\xi)| \geq \frac{c_1}{2n}.$$

By Lemma 3.3, it follows that

$$\text{meas}(R_{kv}^i) \leq c_2 \left(\frac{\alpha}{|k|\Delta(|k|)} \right)^{\frac{1}{r}}.$$

So,

$$\text{meas}(R_k^i) \leq c_2(\text{diam}\Pi)^{n-1} \left(\frac{\alpha}{|k|\Delta(|k|)} \right)^{\frac{1}{r}}.$$

Since $\frac{k}{|k|}$ belongs at most to N sets $S_1, S_2, \dots, S_N$, it follows that

$$\text{meas}(R_k) \leq c_2 N (\text{diam}\Pi)^{n-1} \left(\frac{\alpha}{|k|\Delta(|k|)} \right)^{\frac{1}{n-1}}.$$

The proof of Lemma 3.4 is complete. $\square$

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