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Homogenization of diffusion problems with a nonlinear interfacial resistance

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Abstract. In this paper, we consider a stationary heat problem on a two-component domain with an ε-periodic imperfect interface, on which the heat flux is proportional via a nonlinear function to the jump of the solution, and depends on a real parameter γ. Homogenization and corrector results for the corresponding linear case have been proved in Donato et al. (J Math Sci 176(6):891–927, 2011), by adapting the periodic unfolding method [see (Cioranescu et al. SIAM J Math Anal 40(4):1585–1620, 2008), (Cioranescu et al. SIAM J Math Anal 44(2):718–760, 2012), (Cioranescu et al. Asymptot Anal 53(4):209–235, 2007)] to the case of a two-component domain. Here, we first prove, under natural growth assumptions on the nonlinearities, the existence and the uniqueness of a solution of the problem. Then, we study, using the periodic unfolding method, its asymptotic behavior when $\varepsilon \to 0$. In order to describe the homogenized problem, we complete some convergence results of Donato et al. (J Math Sci 176(6):891–927, 2011) concerning the unfolding operators and we investigate the limit behaviour of the unfolded Nemytskii operators associated to the nonlinear terms. According to the values of the parameter γ we have different limit problems, for the cases $\gamma < -1$, $\gamma = -1$ and $\gamma \in \,]-1, 1]$. The most relevant case is $\gamma = -1$, where the homogenized matrix differs from that of the linear case, and is described in a more complicated way, via a nonlinear function involving the correctors.

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1. Introduction

In the present paper we investigate the thermal conduction in a composite material of two components in the presence of a nonlinear interfacial thermal resistance, which is described by nonlinear transmission conditions on the interface.

More precisely, we study the periodic homogenization of the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_1^\varepsilon) + h_1^\varepsilon(x, u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon \nabla u_2^\varepsilon) + h_2^\varepsilon(x, u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = -A^\varepsilon \nabla u_2^\varepsilon . n_2^\varepsilon & \text{on } \Gamma^\varepsilon, \\ -A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = \varepsilon^{\gamma+1} h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0 & \text{on } \partial\Omega \cap \partial\Omega_i^\varepsilon, \ i = 1, 2, \end{cases} \tag{1.1}$$

where $\Omega = \Omega_1^\varepsilon \cup \overline{\Omega}_2^\varepsilon$ is a two-component domain, $\gamma \in \mathbb{R}$, $f \in L^2(\Omega)$, n_i^ε is the unit outward normal to Ω_i^ε, for $i = 1, 2$.

The two components are open disjoint subsets of the open bounded set $\Omega \in \mathbb{R}^n$, separated by a common boundary Γ^ε. The component Ω_2^ε is the (non connected) intersection of Ω with the union of the ε-periodic translated sets of εY_2, where $\overline{Y_2}$ is contained in the reference periodicity cell Y. The other one $\Omega_1^\varepsilon = \Omega \backslash \overline{\Omega}_2^\varepsilon$ is connected (see Fig. 1 in the next section).

We assume that the thermal conductivity of material is given by $A^\varepsilon(x) = A(x/\varepsilon)$, the matrix A being Y-periodic, uniformly elliptic and bounded. The nonlinear terms h_i^ε in the equation are defined by $h_i^\varepsilon(x, s) = h_i(x/\varepsilon, s)$ for $i = 1, 2$, where h_1 and h_2 are Carathéodory functions satisfying some natural growth conditions.

In the boundary conditions in (1.1) the nonlinear term h^ε is given by $h^\varepsilon(x, s) = h(x/\varepsilon, s/\varepsilon)$, so that the heat flux through the interface is continuous, and proportional via a nonlinear function h to the jump of the temperature. This means that the rate of heat transfer across the interfacial barrier is proportional to the temperature difference by a nonlinear rule. Actually, according to experimental results, in most cases, the flux on the interface is not a linear function of the difference of temperature between two components (see for instance [10]). This nonlinear condition generalizes the linear one of the well-known Barenblatt's model (see [5]).

Let us point out that in this work we allow the holes to meet the boundary of the domain Ω and then Ω_1^ε, Ω_2^ε, Γ^ε are somewhat different from those considered in the linear case [26], where Ω_2^ε only contains the holes inside Ω.

Let us also mention that nonlinear interface conditions also arise in some evolution problem modeling the electrical conduction in biological tissues, in the presence of an interfacial resistance property of the cell membranes, see [2].

In the present work, we apply the periodic unfolding method to get the asymptotic behavior of the nonlinear problem (1.1) as ε tends to zero. In [26] the authors have recently adapted this method to the case of two-component domains by considering two unfolding operators: the first one, denoted $\mathcal{T}_1^\varepsilon$, acts on functions defined in Ω_1^ε and was originally denoted $\mathcal{T}_\varepsilon^*$ in the literature for perforated domains [13,15,16]; the second one, appropriate to case of two-component domains and denoted $\mathcal{T}_2^\varepsilon$, concerns functions defined in Ω_2^ε. They maps functions defined on the oscillating domains Ω_1^ε, Ω_2^ε into functions defined

on the fixed ones $\Omega \times Y_1$ and $\Omega \times Y_2$, respectively. In particular, the relationship between these two operators and the properties of their traces on the common boundaries was also studied. These results were applied in order to recover and improve the homogenization and corrector results obtained by S. Monsurrò and P. Donato in [20, 25, 36].

The periodic unfolding method was introduced for the first time by Cioranescu et al. [11]. It gives an elementary proof for the classical periodic homogenization problem originally considered in [6] (see also [14, 40]) and corrector results under natural assumptions (see [12] for a comprehensive presentation). It was extended to the case of perforated domains in [13, 15, 16] and allows to avoid using the extension operators, so that no condition for the existence of such extensions is required as traditionally done in the literature (see [17, 18]).

Let us mention that if $\{v_n\}$ is a bounded sequence of $L^p(\Omega)$, then the weak L^p-convergence of the unfolded sequence $\{\mathcal{T}_\varepsilon(v_n)\}$ is equivalent to the two-scale convergence of $\{v_n\}$. We recall that the notion of two-scale convergence has been introduced by Nguetseng in [38] and further developed by Allaire in [1] with applications to periodic homogenization.

In the first part of the present work, we show an existence and uniqueness result of the weak solution of problem (1.1), using the classical Minty–Browder theorem. Then, we complete the study done in [26] with some new results needed here for our nonlinear problem, including some convergence results concerning the limit of the unfolded nonlinear terms in the variational formulation of the problem (see Proposition 4.7 for more details). In the second part, we study the homogenization of problem (1.1) for the different values of $\gamma \leq 1$.

Let us briefly describe the difficulties due to the presence of the nonlinear jump and the nonlinear function h_2. The main difficulty is related to the nonlinearity in the boundary condition, which gives rise to a term of the form

$$\varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon\big(x, u_1^\varepsilon - u_2^\varepsilon\big)(v_1 - v_2)\, d\sigma_x = \varepsilon^\gamma \frac{1}{|Y|} \int_{\Omega \times \Gamma} h\left(y, \frac{\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)}{\varepsilon}\right) \\ \times (\mathcal{T}_1^\varepsilon(v_1) - \mathcal{T}_2^\varepsilon(v_2))\, dx\, d\sigma_y.$$

In view of the results in [26], we prove that this term goes to zero for $\gamma > -1$, while for $\gamma < -1$ the choice of test functions without jump, which makes the term vanish, is enough to pass to the limit. This implies that the homogenized matrix A_γ^0 is the same as that of the linear case with jump studied before in the literature.

The case $\gamma = -1$ presents a major difficulty since we have only the weak convergence of $\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)/\varepsilon$ in $L^2(\Omega \times \Gamma)$ so that we need to identify the limit of the unfolded boundary term. To do that, we use the Minty method for monotone operators together with a suitable choice of sequence of test functions and we obtain a boundary term in the limit problem. The homogenized matrix is described in a more complicated way, since it is expressed via a non-

linear function involving the correctors, as shown by Corollary 5.5. Moreover, in this case, the uniqueness of the limit problem needs to be proved.

Concerning the nonlinear function h_2, it leads to pass to the limit in terms of the form

$$\int_{\Omega_2^\varepsilon} h_2^\varepsilon (x, u_2^\varepsilon) \, v_2 \; dx = \frac{1}{|Y|} \int_{\Omega \times Y_2} h_2 \left(y, \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\right) \mathcal{T}_2^\varepsilon (v_2) \; dx \, dy,$$

where the weak convergence of $\mathcal{T}_2^\varepsilon (u_2^\varepsilon)$ in $L^2 \left(\Omega, H^1 (Y_2)\right)$ proved in [26] is not sufficient to pass to the limit. To overcome this difficulty we improve here the previous results proving that this convergence is actually strong for $\gamma < 1$ (see Theorem 4.5). For the case $\gamma = 1$, the assumption that $h_2 \equiv 0$ is required since we cannot improve the weak convergence result of $\mathcal{T}_2^\varepsilon (u_2^\varepsilon)$. Some results for this case have been announced in [41] where also Ω_2^ε is supposed connected.

The pioneer paper on this subject was proposed by Auriault and Ene in [4] for the linear case, using the multiple scale method. In [20,21,25,36,37], the linear case ($h_1 = h_2 = 0$ and $h(y, s) = s\tilde{h}(y)$, where $\tilde{h}(y)$ is an Y-periodic positive bounded function) was studied using the Tartar method and assuming that the holes do not meet the boundary.

For similar homogenization problems of elliptic type we refer the reader to [9,27,28,30,33,34,39] and for related parabolic problems [22–24,29,31], together with the references therein. For the homogenization of linear and quasilinear elliptic problems with a nonlinear Robin condition containing a nonlinear term with the same growth as that in the boundary condition considered in the present paper, we refer to [8,16], respectively, where the periodic unfolding method was used.

The paper is organized as follows:
In Sect. 2, we introduce the problem together with the assumptions concerning the functions h, h_1 and h_2.
Section 3 is devoted to the existence and uniqueness of a weak solution of problem (1.1).
In Section 4, we recall some properties of the unfolding operators introduced in [13,15,26] and give some new convergence results.
Finally, in Sect. 5 we state and prove the homogenization results for the different case of the parameter, $\gamma < -1$, $\gamma = -1$ and $\gamma \in \;]-1, 1]$.

2. Statement of the problem

2.1. Notations

In this section we introduce the main notations and the two-component domain.

In $\mathbb{R}^n$ ($n \geq 2$), let Ω be an open bounded set with a Lipschitz continuous boundary $\partial \Omega$ and $Y := \prod_{i=1}^n [0, l_i[$ be a reference cell, with $l_i > 0$, $i = 1, \dots, n$. We assume that Y_1 and Y_2 are two disjoint connected open subsets of Y such that $Y_2 \neq \varnothing$, $\overline{Y}_2 \subset Y$ and $Y = Y_1 \cup \overline{Y}_2$, with a common boundary $\Gamma = \partial Y_2$ Lipschitz continuous (see Fig. 1).
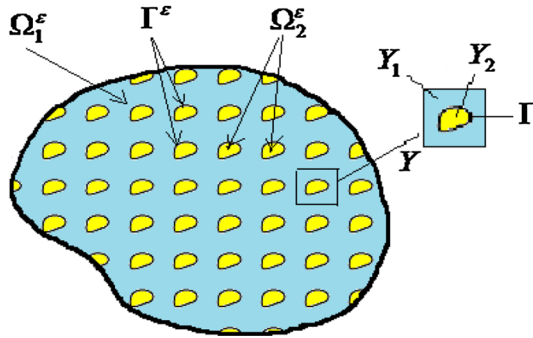
FIGURE 1. The two-component composite

Remark 2.1. For the sake of simplicity, in this paper we assume that Y_2 is connected. Actually the results given here can be extended to the case where Y_2 has a finite number of connected components, in the spirit of Theorems 2.21 and 3.11 in [26].

Throughout this paper, ε will take its values in a positive real sequence which tends to zero and c is a constant independent of ε.

For almost every $z \in \mathbb{R}^n$, we use $[z]_Y$ to denote its integer part $(k_1 l_1, \ldots, k_n l_n)$, $k_i \in \mathbb{Z}$ for $i = 1, \ldots, n$, such that $z - [z]_Y \in Y$ and set $\{z\}_Y = z - [z]_Y$.

Then, we have

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Let us introduce the two-component domain as follows (see Fig. 1):

- for any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $k_l = (k_1 l_1, \ldots, k_n l_n)$ and

$$Y^k = k_l + Y, \quad Y_i^k = k_l + Y_i, \quad i = 1, 2,$$

- $K_\varepsilon = \left\{ k \in \mathbb{Z}^n \mid \varepsilon Y_i^k \cap \Omega \neq \varnothing, \ i = 1, 2 \right\}$ and

$$\Omega_1^\varepsilon = \Omega \setminus \bigcup_{k \in K_\varepsilon} \varepsilon \overline{Y_2^k}, \quad \Gamma^\varepsilon = \partial \Omega_1^\varepsilon \cap \Omega, \quad \Omega_2^\varepsilon = \Omega \setminus \overline{\Omega_1^\varepsilon}. \tag{2.1}$$

We allow here the holes to meet the boundary of the domain Ω. This makes the domains Ω_1^ε, Ω_2^ε and Γ^ε somewhat different from themselves in our last paper [26] for the linear case, where Ω_2^ε only contains the holes inside Ω. Here the domain Ω_2^ε is the intersection of Ω and the union of the εY-periodic translated sets of the hole εY_2. Consequently, the interface Γ^ε is bigger than the corresponding in [26].

The following notations are also used in the sequel:

- $\theta_i = \dfrac{|Y_i|}{|Y|}$, $i = 1, 2$,
- $\widetilde{u}$: the zero extension to the whole Ω of a function u defined on Ω_1^ε or Ω_2^ε,
- χ_ω: the characteristic function of each open set ω of $\mathbb{R}^n$,

- $\mathcal{M}_\omega(f) := \dfrac{1}{|\omega|} \int_\omega f \, dx$, for any open set ω of $\mathbb{R}^n$ and for any $f \in L^1(\omega)$,

- $M(\alpha, \beta, \mathcal{O})$: the set of the $n \times n$ matrix-valued functions A in $(L^\infty(\mathcal{O}))^{n^2}$ such that, for any $\lambda \in \mathbb{R}^n$,

$$\begin{cases} (A(x)\lambda, \lambda) \geq \alpha \, |\lambda|^2 & \text{a.e. in } \mathcal{O}, \\ |A(x)\lambda| \leq \beta \, |\lambda| & \text{a.e. in } \mathcal{O}, \end{cases}$$

for $\alpha, \ \beta \in \mathbb{R}, \ \ 0 < \alpha \leq \beta$.

2.2. The problem

Our goal is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u_1^\varepsilon) + h_1^\varepsilon(x, u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon \nabla u_2^\varepsilon) + h_2^\varepsilon(x, u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = -A^\varepsilon \nabla u_2^\varepsilon . n_2^\varepsilon & \text{on } \Gamma^\varepsilon, \\ -A^\varepsilon \nabla u_1^\varepsilon . n_1^\varepsilon = \varepsilon^{\gamma+1} h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u_i^\varepsilon = 0 & \text{on } \partial\Omega \cap \partial\Omega_i^\varepsilon, \ \ i = 1, 2, \end{cases} \qquad (2.2)$$

where the coefficient matrix A^ε and the nonlinear terms h^ε, h_i^ε are given by

$$\begin{cases} A^\varepsilon(x) = A\left(\dfrac{x}{\varepsilon}\right), \\ h^\varepsilon(x, s) = h\left(\dfrac{x}{\varepsilon}, \dfrac{s}{\varepsilon}\right), \\ h_i^\varepsilon(x, s) = h_i\left(\dfrac{x}{\varepsilon}, s\right) \text{ for } i = 1, 2, \end{cases} \qquad (2.3)$$

with A, h_1 and h_2 satisfying assumptions (2.4)–(2.5) below.

The linear case, where $h_1 = h_2 = 0$ and h is of the form $h(y, s) = s\tilde{h}(y)$, was studied, under the assumption that the holes do not intersect the exterior boundary, by Tartar's method in [25,36], by two-scale method in [28] and recently by unfolding in [26].

Here we investigate a more general situation, where a nonlinear function of the jump of the solution appears on the rapidly oscillating interface together with the presence of the two nonlinear zero-order terms h_1 and h_2.

Throughout the paper, we assume that

$$\begin{cases} \gamma \leq 1, \ \ f \in L^2(\Omega), \\ A \in M(\alpha, \beta, Y), \quad A \text{ is } Y\text{-periodic}, \end{cases} \qquad (2.4)$$

and

$$\begin{cases} h \text{ satisfies } (\mathcal{H}_1) - (\mathcal{H}_3), \\ h_1 \text{ satisfies } (\mathcal{H}_1) \text{ and } (\mathcal{H}_4), \\ h_2 \text{ satisfies } (\mathcal{H}_1) \text{ and } (\mathcal{H}_4'), \end{cases} \qquad (2.5)$$

for some $\alpha, \beta \in \mathbb{R}, 0 < \alpha \leq \beta$ and assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ shown below.

Furthermore, the homogenization results obtained in Sect. 5 depend on the value of γ. Then, for the reader's convenience, we list the different cases and the corresponding assumptions as follows:

$$
\begin{cases}
h \text{ or } h_2 \text{ is strictly increasing} & \text{if } \gamma \neq \pm 1, \\
h \text{ is strictly increasing} & \text{if } \gamma = -1, \\
h \text{ is strictly increasing and } h_2 \equiv 0 & \text{if } \gamma = 1.
\end{cases}
\tag{2.6}
$$

Let us introduce the hypotheses related to the two-variable functions h, h_1 and h_2.

Assumption $\mathcal{H}_1$: The function $g(y,s) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_1)$ iff

1. g is a Carathéodory function,
2. $g(\cdot, s)$ is a (measurable) Y-periodic function for all $s \in \mathbb{R}$,
3. $g(y, \cdot)$ is an increasing function in $C^1(\mathbb{R})$ s.t. $g(y, 0) = 0$ for all $y \in \mathbb{R}^n$.

Assumption $\mathcal{H}_2$: The function $g(y,s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_2)$ iff there exists a constant $c > 0$ and an exponent q, with

$$
1 \leq q < 2 \text{ if } n = 2, 3 \quad \text{and} \quad 1 \leq q < \frac{n}{n-2} \text{ if } n > 3,
$$

such that

$$
\left| \frac{\partial g}{\partial s}(y, s) \right| \leq c \left(1 + |s|^{q-1} \right) \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.
$$

Assumption $\mathcal{H}_3$: The function $g(y,s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_3)$ iff there exists a constant $c > 0$ such that

$$
sg(y, s) \geq c|s|^2 \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.
$$

Assumption $\mathcal{H}_4$: The function $g(y,s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_4)$ (resp. $(\mathcal{H}_4')$) iff there exists a constant $c > 0$ and an exponent p_1 (resp. p_2), with

$$
1 \leq p_1 < +\infty \text{ if } n = 2 \quad \text{and} \quad 1 \leq p_1 \leq \frac{n+2}{n-2} \text{ if } n > 2,
$$

(resp. $1 \leq p_2 \leq \min\left\{2, \frac{n+2}{n-2}\right\}$) such that

$$
\left| \frac{\partial g}{\partial s}(y, s) \right| \leq c \left(1 + |s|^{p_i - 1} \right) \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R}.
$$

Remark 2.2. It is easy to check that

(i) the function $g(y, s) = \xi(y)\left(s + |s|^{q-1} s\right)$ (resp. $\xi(y)\left(s + |s|^{p_1 - 1} s\right)$) satisfies assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ (resp. $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$) where $\xi(y)$ is any positive bounded measurable Y-periodic function,

(ii) if $g(y, s)$ satisfies assumption $(\mathcal{H}_1)$, then $sg(y, s) \geq 0$ for a.e. $y \in Y$ and for all $s \in \mathbb{R}$,

(iii) if $g(y, s)$ satisfies assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, then for some $c > 0$,

$$
|g(y, s)| \leq c(1 + |s|^q) \quad \text{for a.e. } y \in Y \text{ and for all } s \in \mathbb{R},
$$

since, due to the fact that $g(y, 0) = 0$ for all $y \in \mathbb{R}^n$, we have

$$
|g(y, s)| = \left| \int_0^s \frac{\partial g}{\partial \tau}(y, \tau) d\tau \right| \leq c \left| \int_0^s \left(1 + |\tau|^{q-1} \right) d\tau \right| \leq c(|s| + |s|^q),
$$

and

- if $|s| \leq 1$, $|g(y, s)| \leq c(1 + |s|^q)$,
- if $|s| > 1$, $|g(y, s)| \leq 2c |s|^q < 2c(1 + |s|^q)$.

We now introduce the functional spaces adapted to our problem.

Definition 2.3. For every $\gamma \in \mathbb{R}$,

$$H_\gamma^\varepsilon \doteq \left\{ u = (u_1, u_2) \mid u_1 \in V_1^\varepsilon, \ u_2 \in V_2^\varepsilon \right\} \tag{2.7}$$

equipped with the norm

$$\|u\|_{H_\gamma^\varepsilon}^2 = \|\nabla u_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla u_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^\gamma \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2, \tag{2.8}$$

where $V_i^\varepsilon \doteq \left\{ v \in H^1(\Omega_i^\varepsilon) \mid v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i^\varepsilon \right\}$, $i = 1, 2$, endowed respectively with the norms

$$\|v\|_{V_1^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}, \quad \|v\|_{V_2^\varepsilon} = \left(\|v\|_{L^2(\Omega_2^\varepsilon)}^2 + \|\nabla v\|_{L^2(\Omega_2^\varepsilon)}^2 \right)^{1/2}.$$

Remark 2.4. (i) As observed in [13], the Lipschitz assumption of $\partial\Omega$ implies that for every open subset Ω_0 of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega_0$ and $\partial\Omega \cap \partial\Omega_1^\varepsilon = \partial\Omega \cap \Omega_0$,

$$V_1^\varepsilon = \left\{ v \in H^1(\Omega_1^\varepsilon) \mid \exists v' \in H^1(\Omega_0^\varepsilon), \ v' = 0 \text{ in } \Omega_0^\varepsilon \backslash \overline{\Omega_1^\varepsilon} \text{ and } v = v'|_{\Omega_1^\varepsilon} \right\},$$

where $\Omega_0^\varepsilon = \Omega_0 \backslash \bigcup_{k \in \mathbb{Z}^n} \varepsilon \overline{Y_2^k}$.

(ii) The norm $\|\cdot\|_{V_1^\varepsilon}$ is equivalent to $\|\cdot\|_{H^1(\Omega_1^\varepsilon)}$ by constants independent of ε since the Poincaré inequality holds in the space V_1^ε with a constant c independent of ε, that is

$$\|v\|_{L^2(\Omega_1^\varepsilon)} \leq c \|\nabla v\|_{L^2(\Omega_1^\varepsilon)} \quad \forall v \in V_1^\varepsilon,$$

(see Theorem 2.9 in [13] for more details).

The space H_γ^ε defined above is not exactly the one introduced in [24–26,36], since as mentioned above, here the holes can meet the boundary of Ω and then Γ_ε is somewhat different. Nevertheless, it is easy to check that the properties of the space H_γ^ε proved in the papers quoted above are still true for this situation. In particular, the following proposition is still true.

Proposition 2.5. [24] *There exist some positive constants c, c_1 and c_2, independent of ε, such that*

$$\|u\|_{H_\gamma^\varepsilon}^2 \leq c \left(1 + \varepsilon^{\gamma-1} \right) \|u\|_{V_1^\varepsilon \times V_2^\varepsilon}^2 \quad \forall \gamma \in \mathbb{R}, \ \forall u \in H_\gamma^\varepsilon.$$

Moreover, if $\gamma \leq 1$

$$c_1 \|u\|_{V_1^\varepsilon \times V_2^\varepsilon}^2 \leq \|u\|_{H_\gamma^\varepsilon}^2 \leq c_2 \left(1 + \varepsilon^{\gamma-1} \right) \|u\|_{V_1^\varepsilon \times V_2^\varepsilon}^2 \quad \forall u \in H_\gamma^\varepsilon. \tag{2.9}$$

Let us now turn back to problem (2.2), whose variational formulation is

$$\begin{cases} \text{Find } u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in H_\gamma^\varepsilon \text{ such that} \\[2mm] \displaystyle\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla v_1 \ dx + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \nabla v_2 \ dx + \int_{\Omega_1^\varepsilon} v_1 h_1^\varepsilon \left(x, u_1^\varepsilon\right) \ dx \\[2mm] + \displaystyle\int_{\Omega_2^\varepsilon} v_2 h_2^\varepsilon \left(x, u_2^\varepsilon\right) \ dx + \varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} \left(v_1 - v_2\right) h^\varepsilon \left(x, u_1^\varepsilon - u_2^\varepsilon\right) \ d\sigma \\[2mm] = \displaystyle\int_{\Omega_1^\varepsilon} f v_1 \ dx + \int_{\Omega_2^\varepsilon} f v_2 \ dx \ \forall \left(v_1, v_2\right) \in H_\gamma^\varepsilon. \end{cases} \qquad (2.10)$$

We state below an existence and uniqueness result for this problem, which is a particular case of Theorem 3.1 proved in the next section.

Theorem 2.6. *Let the function h satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and h_1, h_2 satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. Suppose further that h or h_2 fulfills assumption $(\mathcal{H}_3)$. Then, for every fixed ε, problem (2.10) has a solution $u^\varepsilon \in H_\gamma^\varepsilon$.*

If moreover h or h_2 is strictly increasing, then the solution is unique.

We end this section by some a priori estimates for problem (2.10).

Proposition 2.7. (a priori estimates) *Let $\gamma \leq 1$ and suppose that the function h satisfies assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ and h_1, h_2 satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a weak solution of problem (2.2), then there exists a positive constant c, independent of ε, such that*

$$\begin{cases} \|(\nabla u_1^\varepsilon, \nabla u_2^\varepsilon)\|_{L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon)} \leq c, \\[1mm] \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq c \ \varepsilon^{-\gamma/2}. \end{cases} \qquad (2.11)$$

Proof. With the assumptions given as above, problem (2.2) has a weak solution due to Theorem 2.6. Arguing as in the linear case [26], together with the coerciveness of h, the proof of (2.11) is straightforward. $\qquad\square$

Remark 2.8. For $\gamma = 1$, if the coerciveness assumption on h is replaced by that on h_2, then we still deduce the a priori estimates in (2.11) by using (2.9).

3. An existence and uniqueness result

Let $\mathcal{O}$ be a Lipschitz bounded open set of $\mathbb{R}^n$. We take $\mathcal{O}_1$, $\mathcal{O}_2$ as the open disjoint subsets of $\mathcal{O}$ such that $\mathcal{O}_1$ is connected and $\mathcal{O}_2$ has a finite number of connected components. We also assume that the interface S between $\mathcal{O}_1$, $\mathcal{O}_2$ is Lipschitz continuous and set $S_1 = \partial\mathcal{O} \cap \partial\mathcal{O}_1$, $S_2 = \partial\mathcal{O} \cap \partial\mathcal{O}_2$.

The main result of this section concerns the existence and the uniqueness of a weak solution of the following more general problem:

$$\begin{cases} -\mathrm{div}\left(B\nabla u_1\right) + h_1\left(x, u_1\right) = f & \text{in } \mathcal{O}_1, \\ -\mathrm{div}\left(B\nabla u_2\right) + h_2\left(x, u_2\right) = f & \text{in } \mathcal{O}_2, \\ B\nabla u_1.n_1 = -B\nabla u_2.n_2 & \text{on } S, \\ -B\nabla u_1.n_1 = h\left(x, \delta(u_1 - u_2)\right) & \text{on } S, \\ u_1 = 0 & \text{on } S_1, \\ u_2 = 0 & \text{on } S_2, \end{cases} \qquad (3.1)$$

for a given $\delta \in \mathbb{R}$.

The existence and uniqueness result stated in Theorem 2.6 will follow as a particular case.

Let us define the functional space H as follows

$$H \doteq \{u = (u_1, u_2) | \ u_1 \in V_1, \ u_2 \in V_2\},$$

endowed with the norm

$$\|u\|_H^2 = \|\nabla u_1\|_{L^2(\mathcal{O}_1)}^2 + \|\nabla u_2\|_{L^2(\mathcal{O}_2)}^2 + \|u_1 - u_2\|_{L^2(S)}^2 \quad \forall u = (u_1, u_2) \in H,$$

where

$$V_i = \{v \in H^1(\mathcal{O}_i) : v = 0 \text{ on } S_i\}, \quad i = 1, 2,$$

with the norms $\|v\|_{V_1} = \|\nabla v\|_{L^2(\mathcal{O}_1)}$ and $\|v\|_{V_2} = \|v\|_{H^1(\mathcal{O}_2)}$.
N.B. In this section, we suppose that all assumptions $\mathcal{H}$ introduced in Sect. 2 are written with $\mathcal{O}$ instead of Y and without any periodicity assumptions.

Now we introduce an assumption more general than $(\mathcal{H}_2)$, which is enough to obtain the results for the existence and the uniqueness.

Assumption $\mathcal{H}_2'$: The function $g(y, s) : Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies assumption $(\mathcal{H}_2')$ iff there exists a constant $c > 0$ and an exponent q, with

$$1 \leq q < +\infty \text{ if } n = 2 \quad \text{and} \quad 1 \leq q \leq \frac{n}{n-2} \text{ if } n > 2$$

such that

$$\left| \frac{\partial g}{\partial s}(y, s) \right| \leq c \left(1 + |s|^{q-1}\right) \quad \forall (y, s) \in Y \times \mathbb{R}.$$

Theorem 3.1. *Let $f \in L^2(\mathcal{O})$ and $B \in M(\alpha, \beta, \mathcal{O})$. Suppose that the function h satisfies assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2')$ and h_1, h_2 satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. Assume further that h or h_2 satisfies assumption $(\mathcal{H}_3)$. Then, there exists a solution $u = (u_1, u_2) \in H$ of the following variational formulation of problem (3.1):*

$$\begin{cases} \text{Find } u = (u_1, u_2) \in H \text{ such that} \\ \displaystyle\int_{\mathcal{O}_1} B\nabla u_1 \nabla v_1 \ dx + \int_{\mathcal{O}_2} B\nabla u_2 \nabla v_2 \ dx + \int_{\mathcal{O}_1} v_1 h_1(x, u_1) \ dx \\ \quad + \displaystyle\int_{\mathcal{O}_2} v_2 h_2(x, u_2) \ dx + \int_S (v_1 - v_2) h(x, \delta(u_1 - u_2)) \ d\sigma \\ \quad = \displaystyle\int_{\mathcal{O}_1} f v_1 \ dx + \int_{\mathcal{O}_2} f v_2 \ dx \quad \forall (v_1, v_2) \in H. \end{cases} \quad (3.2)$$

Furthermore, the solution is unique if h or h_2 is strictly increasing.

This theorem is proved at the end of this section, after some preliminary results. The next proposition shows in particular that all the terms in (3.2) make sense:

Proposition 3.2. *Let the function h satisfy assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2')$ and the functions h_1, h_2 satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_4)$. Suppose that r, r', s, s' are such that*

(i) *if $n = 2$, r, $s \in [1, +\infty[$ and $r \geq qr'$, $s \geq ps'$,*

(ii) *if $n > 2$, $r = \dfrac{2(n-1)}{n-2}$, $s = 2^* = \dfrac{2n}{n-2}$ and $\dfrac{1}{r} + \dfrac{1}{r'} = \dfrac{1}{s} + \dfrac{1}{s'} = 1$.*

Then, the following three maps are bounded and continuous:

$$J : u = (u_1, u_2) \in H \longmapsto J(u) \in L^{r'}(S),$$

$$J_i : u_i \in H^1(\mathcal{O}_i) \longmapsto J_i(u_i) \in L^{s'}(\mathcal{O}_i), \quad i = 1, 2,$$

where $J(u)(x) = h(x, \delta(u_1(x) - u_2(x)))$, $J_i(u_i)(x) = h_i(x, u_i(x))$ and for every $u_i, v_i \in H^1(\mathcal{O}_i)$, $i = 1, 2$,

$$\begin{cases} (v_1 - v_2)\, h(x, \delta(u_1 - u_2)) \in L^1(S), \\ v_i h_i(x, u_i) \in L^1(\mathcal{O}_i). \end{cases} \tag{3.3}$$

Moreover, there exists a positive constant c such that

$$\|J(u)\|_{L^{r'}(S)} \le c\left(1 + \|u_1 - u_2\|_{L^r(S)}^{r/r'}\right). \tag{3.4}$$

Proof. Let us first note that $J = g_1 \circ g_2$ where

$$g_1 \colon v \in L^r(S) \longmapsto g_1(v) \in L^{r'}(S)$$

is defined by $g_1(v)(x) = h(x, v(x))$ and

$$g_2 \colon u = (u_1, u_2) \in H \longmapsto u_1 - u_2 \in L^r(S).$$

Taking into account that h satisfies $(\mathcal{H}_1)$ and $(\mathcal{H}'_2)$, by the classical result in [32], g_1 is continuous and bounded. On the other hand, arguing as in [7,16], the linear continuity of the inclusions $H^1(\mathcal{O}_i) \subset L^r(S)$ for $i = 1, 2$, gives that g_2 is bounded and continuous. Hence, the operator J is bounded and continuous from H into $L^{r'}(S)$, which provides $(3.3)_1$.

We now turn our attention to the boundedness and the continuity of the two maps J_1, J_2. If $n > 2$, the embeddings $H^1(\mathcal{O}_i) \subset L^{2^*}(\mathcal{O}_i)$ for $i = 1, 2$, are continuous.

Then, taking $v_i \in H^1(\mathcal{O}_i)$, since the functions h_1, h_2 satisfy $(\mathcal{H}_4)$ and $p_1 \le \dfrac{n+2}{n-2} = \dfrac{2^*}{(2^*)'}$, we have

$$|h_i(x, v_i)| \le c(1 + |v_i|^{p_1}) \le 2c\left(1 + |v_i|^{\frac{2^*}{(2^*)'}}\right), \quad \text{for } i = 1, 2.$$

Consequently, the functions J_i, $i = 1, 2$, are bounded and continuous from $H^1(\mathcal{O}_i)$ into $L^{(2^*)'}(\mathcal{O}_i)$, which implies $(3.3)_2$.

For $n = 2$ we argue in a similar way as above, using the continuous embedding $H^1(\mathcal{O}_i) \subset L^s(\mathcal{O}_i)$, $i = 1, 2$, $\forall s \in [1, +\infty[$.

Finally, inequality (3.4) comes from similar arguments as in [7]. $\qquad \square$

Let us now recall a result from [36] that we shall need in the sequel.

Lemma 3.3. [36] *There exists a constant $c > 0$ such that*

$$\|u\|_{L^2(\mathcal{O}_1) \times L^2(\mathcal{O}_2)} \le c\, \|u\|_H,$$

for every $u \in H$.

Proof of Theorem 3.1. We first prove the existence of the solution by using the Minty–Browder theorem for the operator $G : u = (u_1, u_2) \in H \mapsto G(u) \in H'$ where

$$\langle G(u), v \rangle_{H',H} = \sum_{i=1,2} \int_{\mathcal{O}_i} B \nabla u_i \nabla v_i \, dx + \sum_{i=1,2} \int_{\mathcal{O}_i} v_i h_i(x, u_i) \, dx$$

$$+ \int_S (v_1 - v_2) h(x, \delta(u_1 - u_2)) \, d\sigma - \sum_{i=1,2} \int_{\mathcal{O}_i} f v_i \, dx.$$

Clearly, the continuity and the boundedness of G follow from Proposition 3.2. Moreover, the fact that the functions h, h_1 and h_2 satisfy assumption $(\mathcal{H}_1)$ provides the monotonicity of G.

Let us turn our attention into the coerciveness of G. If h satisfies assumption $(\mathcal{H}_3)$, taking into account Remark 2.2 and Lemma 3.3, we have

$$\langle G(u), u \rangle_{H',H} \geq \alpha \|\nabla u_1\|^2_{L^2(\mathcal{O}_1)} + \alpha \|\nabla u_2\|^2_{L^2(\mathcal{O}_2)} + c \|u_1 - u_2\|^2_{L^2(S)}$$

$$- \|f\|_{L^2(\mathcal{O})} \left(\|u_1\|_{L^2(\mathcal{O}_1)} + \|u_2\|_{L^2(\mathcal{O}_2)} \right)$$

$$\geq c_1 \|u\|^2_H - c_2 \|u\|_H, \quad \forall u \in H,$$

If the coerciveness assumption of h is replaced by that of h_2, then

$$\langle G(u), u \rangle_{H',H} \geq \alpha \|\nabla u_1\|^2_{L^2(\mathcal{O}_1)} + \alpha \|\nabla u_2\|^2_{L^2(\mathcal{O}_2)} + c \|u_2\|^2_{L^2(\mathcal{O}_2)}$$

$$- \|f\|_{L^2(\mathcal{O})} \left(\|u_1\|_{L^2(\mathcal{O}_1)} + \|u_2\|_{L^2(\mathcal{O}_2)} \right)$$

$$\geq c \left(\|u_1\|^2_{H^1(\mathcal{O}_1)} + \|u_2\|^2_{H^1(\mathcal{O}_2)} \right) - c_1 \|u\|_H$$

$$\geq c_2 \|u\|^2_H - c_1 \|u\|_H, \quad \forall u \in H.$$

Obviously, the above inequalities imply that G is coercive.

Hence, by the Minty–Browder theorem, the operator G is surjective so that there exists $u \in H$ such that $G(u) = 0$, that is a solution $u \in H$ of problem (3.2).

Let us now check the uniqueness of the solution.

Suppose that problem (3.1) has two solutions $u = (u_1, u_2)$, $w = (w_1, w_2) \in H$. Choosing as test function $v = u - w$ in (3.2), we have

$$\sum_{i=1,2} \int_{\mathcal{O}_i} B \left(\nabla u_i - \nabla w_i \right) \left(\nabla u_i - \nabla w_i \right) dx$$

$$+ \sum_{i=1,2} \int_{\mathcal{O}_i} (u_i - w_i) \left[h_i(x, u_i) - h_i(x, w_i) \right] dx$$

$$+ \int_S \left[h(x, \delta(u_1 - u_2)) - h(x, \delta(w_1 - w_2)) \right] \left[(u_1 - u_2) - (w_1 - w_2) \right] d\sigma = 0.$$

$$(3.5)$$

By virtue of the ellipticity of B, the monotonicity of h, h_1, h_2 and the fact that h (or h_2) is strictly increasing, equality (3.5) gives

$$\begin{cases} \nabla (u_i - w_i) = 0 & \text{in } \mathcal{O}_i \quad \text{for } i = 1, 2, \\ u_1 - u_2 = w_1 - w_2 & \text{on } S \text{ (or } u_2 = w_2 \text{ in } \mathcal{O}_2). \end{cases}$$

Then, using the Poincaré inequality in V_1 we obtain $u \equiv w$, which ends the proof. $\qquad\square$

4. Periodic unfolding method for two-component domains

In this section, we first recall the unfolding operators $\mathcal{T}_1^\varepsilon$, $\mathcal{T}_2^\varepsilon$ for the two-component domain and the boundary unfolding operator $\mathcal{T}_b^\varepsilon$ together with their main properties.

The operators $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_b^\varepsilon$ have been introduced in [13, 15, 26] and $\mathcal{T}_2^\varepsilon$ in [26], where the relationship between the traces of $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$ is also studied. Although as mentioned in Section 2 the domain Ω_2^ε defined in [26] is somehow different, all the results therein concerning the unfolding operators are still true for this case. Indeed, the main difference concerns $\mathcal{T}_1^\varepsilon$, due to the fact that u_1^ε does not belong to $V^\varepsilon \doteq \left\{ v \in H^1 (\Omega_1^\varepsilon) \middle| \ v = 0 \text{ on } \partial\Omega \right\}$ as in the linear case studied in [26], but belongs to V_1^ε (see Definition 2.3). Also, the fact that the domain Ω_2^ε contains some more holes (or part of them) does not make any relevant difference. Consequently, it is easy to check, taking into account [13, Theorem 2.13], that the convergence results recalled in Theorem 4.3 below are still true in this case.

In the second part of this section, we state some results concerning the composed operators $\mathcal{T}_b^\varepsilon \circ h$ and $\mathcal{T}_i^\varepsilon \circ h_i$, $i = 1, 2$ where h, h_1 and h_2 are the nonlinear functions in problem (2.2). Moreover, we prove (see Theorem 4.5 below) a new strong convergence result for $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$, which is needed to pass to the limit in the unfolded nonlinear term concerning h_2 in the variational formulation (2.10).

In order to define the periodic unfolding operators, let us introduce the following sets as in [13] (see Fig. 2).

- $\widehat{K}_\varepsilon = \left\{ k \in \mathbb{Z}^n \middle| \ \varepsilon Y^k \subset \Omega \right\}$, $\widehat{\Omega}_\varepsilon = int \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon \left(k_l + \overline{Y} \right)$, $\Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon$,

- $\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k$, $\qquad \Lambda_i^\varepsilon = \Omega_i^\varepsilon \backslash \widehat{\Omega}_i^\varepsilon$, $\quad i = 1, 2$.

4.1. The periodic unfolding operators

Definition 4.1. For any function ϕ Lebesgue-measurable on Ω_i^ε, the periodic unfolding operators $\mathcal{T}_i^\varepsilon$, $i = 1, 2$ are defined by the formula

$$\mathcal{T}_i^\varepsilon (\phi) (x, y) = \begin{cases} \phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{a.e. } (x, y) \in \Lambda_\varepsilon \times Y_i. \end{cases}$$
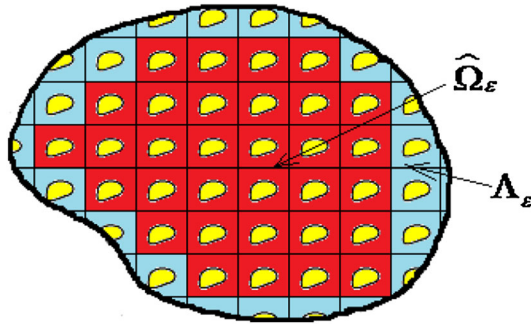
FIGURE 2. The sets $\widehat{\Omega}_\varepsilon$ and Λ_ε

For any function ϕ Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by

$$\mathcal{T}_b^\varepsilon\left(\phi\right)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & a.e.\ (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & a.e.\ (x,y) \in \Lambda_\varepsilon \times \Gamma. \end{cases}$$

For the readers' convenience, we recall the main properties of $\mathcal{T}_i^\varepsilon$ for $i = 1, 2$ as follows:

Proposition 4.2. [13,15,26] *For $p \in [1, +\infty[$, the operators $\mathcal{T}_i^\varepsilon$ are linear and continuous from $L^p\left(\Omega_i^\varepsilon\right)$ to $L^p\left(\Omega \times Y\right)$, $i = 1, 2$; moreover,*

(i) *$\mathcal{T}_i^\varepsilon\left(\varphi\psi\right) = \mathcal{T}_i^\varepsilon\left(\varphi\right)\mathcal{T}_i^\varepsilon\left(\psi\right)$, for every φ, ψ Lebesgue-measurable on Ω_i^ε.*

(ii) *For every $\varphi \in L^1\left(\Omega_i^\varepsilon\right)$, one has*

$$\frac{1}{|Y|}\int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon\left(\varphi\right)(x,y)\ dx\ dy = \int_{\widehat{\Omega}_i^\varepsilon} \varphi\left(x\right)\ dx = \int_{\Omega_i^\varepsilon} \varphi\left(x\right)\ dx - \int_{\Lambda_i^\varepsilon} \varphi\left(x\right)\ dx.$$

(iii) *For every $\varphi \in L^p\left(\Omega_i^\varepsilon\right)$,*

$$\left\|\mathcal{T}_i^\varepsilon\left(\varphi\right)\right\|_{L^p(\Omega \times Y_i)} \leq |Y|^{1/p}\left\|\varphi\right\|_{L^p(\Omega_i^\varepsilon)}.$$

(iv) *For $\varphi \in L^p\left(\Omega\right)$,*

$$\mathcal{T}_i^\varepsilon\left(\varphi\right) \longrightarrow \varphi \quad \text{strongly in } L^p\left(\Omega \times Y_i\right).$$

(v) *Let $\{\varphi_\varepsilon\}$ be a sequence in $L^p\left(\Omega\right)$ such that $\varphi_\varepsilon \longrightarrow \varphi$ strongly in $L^p\left(\Omega\right)$. Then,*

$$\mathcal{T}_i^\varepsilon\left(\varphi^\varepsilon\right) \longrightarrow \varphi \quad \text{strongly in } L^p\left(\Omega \times Y_i\right).$$

(vi) *Let $\varphi \in L^p\left(Y_i\right)$ be a Y-periodic function and set $\varphi^\varepsilon\left(x\right) = \varphi(x/\varepsilon)$. Then*

$$\mathcal{T}_i^\varepsilon\left(\varphi^\varepsilon\right) \longrightarrow \varphi \quad \text{strongly in } L^p\left(\Omega \times Y_i\right).$$

(vii) *Let $\varphi_\varepsilon \in L^p\left(\Omega_i^\varepsilon\right)$ satisfy $\left\|\varphi_\varepsilon\right\|_{L^p(\Omega_i^\varepsilon)} \leq C$. If $\mathcal{T}_i^\varepsilon\left(\varphi_\varepsilon\right) \rightharpoonup \widehat{\varphi}$ weakly in $L^p\left(\Omega \times Y_i\right)$, then*

$$\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}\left(\widehat{\varphi}\right) \quad \text{weakly in } L^p\left(\Omega\right).$$

(viii) *If $\varphi \in W^{1,p}\left(\Omega_i^\varepsilon\right)$, then $\nabla_y\left[\mathcal{T}_i^\varepsilon\left(\varphi\right)\right] = \varepsilon\mathcal{T}_i^\varepsilon\left(\nabla\varphi\right)$ and $\mathcal{T}_i^\varepsilon\left(\varphi\right) \in L^p\left(\Omega, W^{1,p}\left(Y_i\right)\right)$.*

(ix) *If $\varphi \in L^s\left(\Gamma^\varepsilon\right)$ for $s \in [1, +\infty[$, then*

$$\left\|\mathcal{T}_b^\varepsilon\left(\varphi\right)\right\|_{L^s(\Omega \times \Gamma)} \leq |Y|^{1/s}\, \varepsilon^{1/s}\, \|\varphi\|_{L^s(\Gamma^\varepsilon)}.$$

Let us now recall the main convergence results which have been proved in [13, 26].

Theorem 4.3. [13, 26] *Let $\gamma \in \mathbb{R}$ and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H_γ^ε. Then,*

$$\left\|\mathcal{T}_1^\varepsilon\left(u_1^\varepsilon\right) - \mathcal{T}_2^\varepsilon\left(u_2^\varepsilon\right)\right\|_{L^2(\Omega \times \Gamma)} \leq C\varepsilon^{\frac{1-\gamma}{2}},$$

and there exists a subsequence (still denoted ε), $u_1 \in H_0^1(\Omega)$ and $\widehat{u}_1 \in L^2\left(\Omega, H_{per}^1(Y_1)\right)$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ a.e. $x \in \Omega$ such that

$$\begin{cases} \mathcal{T}_1^\varepsilon\left(u_1^\varepsilon\right) \longrightarrow u_1 & \text{strongly in } L^2\left(\Omega, H^1(Y_1)\right), \\ \mathcal{T}_1^\varepsilon\left(\nabla u_1^\varepsilon\right) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 & \text{weakly in } L^2\left(\Omega \times Y_1\right). \end{cases} \tag{4.1}$$

Moreover, if $\gamma \leq 1$, there exists a subsequence (still denoted ε), $u_2 \in L^2(\Omega)$ and $\overline{u}_2 \in L^2\left(\Omega, H^1(Y_2)\right)$ with $\mathcal{M}_\Gamma(\overline{u}_2) = 0$ a.e. $x \in \Omega$ such that

$$\begin{cases} \mathcal{T}_2^\varepsilon\left(u_2^\varepsilon\right) \rightharpoonup u_2 & \text{weakly in } L^2\left(\Omega, H^1(Y_2)\right), \\ \mathcal{T}_2^\varepsilon\left(\nabla u_2^\varepsilon\right) \rightharpoonup \nabla_y \overline{u}_2 & \text{weakly in } L^2\left(\Omega \times Y_2\right). \end{cases} \tag{4.2}$$

Furthermore, if $\gamma < 1$, then $u_1 = u_2$ and

(i) *if $\gamma < -1$, we have*

$$\widehat{u}_1 = \overline{u}_2 - y_\Gamma \nabla u_1 \quad \text{on } \Omega \times \Gamma, \tag{4.3}$$

where $y_\Gamma = y - \mathcal{M}_\Gamma(y)$.

(ii) *if $\gamma = -1$, there exists $\xi_\Gamma \in L^2(\Omega)$ such that*

$$\frac{\mathcal{T}_1^\varepsilon\left(u_1^\varepsilon\right) - \mathcal{T}_2^\varepsilon\left(u_2^\varepsilon\right)}{\varepsilon} \rightharpoonup \widehat{u}_1 - \overline{u}_2 + y_\Gamma \nabla u_1 + \xi_\Gamma \quad \text{weakly in } L^2\left(\Omega \times \Gamma\right). \tag{4.4}$$

Remark 4.4. From now on, we set

$$\widehat{u}_2 = \overline{u}_2 - y_\Gamma \nabla u_1 - \xi_\Gamma \in L^2(\Omega, H^1(Y_2)). \tag{4.5}$$

With this notation, in $(4.2)_2$ we can replace $\nabla_y \overline{u}_2$ by $\nabla u_1 + \nabla_y \widehat{u}_2$ and (4.3)-(4.4) can be rewritten respectively as follows:

$$\widehat{u}_1 = \widehat{u}_2 + \xi_\Gamma \quad \text{on } \Omega \times \Gamma \quad \text{if } \gamma < -1, \tag{4.6}$$

$$\mathcal{T}_1^\varepsilon\left(u_1^\varepsilon\right) - \mathcal{T}_2^\varepsilon\left(u_2^\varepsilon\right)/\varepsilon \rightharpoonup \widehat{u}_1 - \widehat{u}_2 \quad \text{weakly in } L^2\left(\Omega \times \Gamma\right) \quad \text{if } \gamma = -1. \tag{4.7}$$

Let us mention that, when comparing to [26], the notations of $\widehat{u}_2$ and $\overline{u}_2$ used here are interchanged, in order to simplify the presentation.

4.2. Some convergence results

The main difficulty when passing to the limit in the unfolded version of problem (2.2) is clearly given by the nonlinear terms. Concerning $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$, its weak convergence in $L^2\left(\Omega, H^1(Y_2)\right)$ provided by Theorem 4.3 is not sufficient.

This is why we first need to prove that the above convergence is actually strong. We do that in the next theorem, then we obtain the convergences of the unfolded nonlinear terms.

Theorem 4.5. *Let $\gamma < 1$ and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H_γ^ε. Then, there exists a subsequence (still denoted ε) and $u_1 \in H_0^1(\Omega)$ such that*

$$\begin{cases} \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \longrightarrow u_1 & \text{strongly in } L^2\left(\Omega, H^1(Y_2)\right), \\ \mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \longrightarrow 0 & \text{strongly in } L^2\left(\Omega, L^r(\Gamma)\right), \end{cases} \tag{4.8}$$

where r is defined in Proposition 3.2.

Proof. From the proof of Theorem 2.18 in [26], we have

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \longrightarrow u_1 \quad \text{strongly in } L^2(\Omega \times \Gamma). \tag{4.9}$$

Moreover, since Y_2 is bounded with a Lipschitz boundary Γ, by the Friedrichs inequality we have for some positive constant c,

$$\|v\|_{L^2(Y_2)}^2 \le c\left(\|v\|_{L^2(\Gamma)}^2 + \|\nabla v\|_{L^2(Y_2)}^2\right) \quad \forall v \in H^1(Y_2),$$

which applied to the function $\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - u_1$ gives

$$\|\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - u_1\|_{L^2(\Omega \times Y_2)}^2 \le c\left(\|\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - u_1\|_{L^2(\Omega \times \Gamma)}^2 + \|\nabla_y \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times Y_2)}^2\right), \tag{4.10}$$

since u_1 is independent of y. Note that

$$\|\nabla_y \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega \times Y_2)} = \varepsilon\|\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\|_{L^2(\Omega \times Y_2)} \le c\,\varepsilon, \tag{4.11}$$

in view of Proposition 4.2(iii), (viii) and the fact that u^ε is a bounded sequence in H_γ^ε. Taking into account (4.9)–(4.11), we deduce (4.8)$_1$.

For the second convergence in (4.8) one has

$$\int_\Omega \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^r(\Gamma)}^2 \, dx$$

$$\le 2\int_\Omega \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - u_1\|_{L^r(\Gamma)}^2 \, dx + 2\int_\Omega \|\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - u_1\|_{L^r(\Gamma)}^2 \, dx$$

$$\le c\left(\int_\Omega \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - u_1\|_{H^1(Y_1)}^2 \, dx + \int_\Omega \|\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - u_1\|_{H^1(Y_2)}^2 \, dx\right) \longrightarrow 0$$

where we use (4.1)$_1$, (4.8)$_1$ and the continuous embeddings $H^1(Y_i) \subset L^r(\Gamma)$, $i = 1, 2$ (see Proposition 3.2). This completes the proof. □

The following lemma is needed to prove the next proposition.

Lemma 4.6. *There exists a positive constant c, independent of ε, such that, for any $v \in V_1^\varepsilon$,*

$$\|v\|_{L^s(\Omega_1^\varepsilon)} \le c\|v\|_{V_1^\varepsilon}, \tag{4.12}$$

for every $s \in [1, 2^*]$ *if* $n > 2$ *and* $s \in [1, +\infty[$ *if* $n = 2$.

Proof. Let Ω_0 be an open subset of $\mathbb{R}^n$ such that $\Omega \subset \Omega_0$ and $d(\partial\Omega_0, \Omega) > diam(Y)$. Set $\Omega_0^\varepsilon = \Omega_0 \backslash \bigcup_{k \in \mathbb{Z}^n} \varepsilon \overline{Y_2^k}$. Observe that, for ε small enough, $\partial\Omega_0$ does not meet the boundary of the holes intersecting $\partial\Omega$. Consequently, if $\widetilde{v}$ is the zero extension to Ω_0^ε of a function $v \in V_1^\varepsilon$, then according to Remark 2.4(i), $\widetilde{v} \in H^1(\Omega_0^\varepsilon)$ and $\widetilde{v} = 0$ on $\partial\Omega_0^\varepsilon$.

Hence, from classical results in [17, 18], there exists an extension operator P^ε from $H^1(\Omega_0^\varepsilon)$ to $H_0^1(\Omega_0)$ such that,

$$\|P^\varepsilon \widetilde{v}\|_{H_0^1(\Omega_0)} \le c \|\widetilde{v}\|_{H^1(\Omega_0^\varepsilon)},$$

where the constant c is independent of ε.

As a result, by using the Sobolev embedding $H_0^1(\Omega_0) \subset L^s(\Omega_0)$, we obtain

$$\|v\|_{L^s(\Omega_1^\varepsilon)} = \|\widetilde{v}\|_{L^s(\Omega_0^\varepsilon)} \le \|P^\varepsilon \widetilde{v}\|_{L^s(\Omega_0)} \le c_1 \|P^\varepsilon \widetilde{v}\|_{H_0^1(\Omega_0)} \le c_2 \|v\|_{V_1^\varepsilon},$$

where the constant c_2 does not depend on ε, which concludes the proof. $\square$

Proposition 4.7. *Let the function h_1 satisfy assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_4)$ *and h_2 satisfy assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_4')$. *If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a bounded sequence in H_γ^ε, then there exists a subsequence (still denoted ε) and $u_1 \in H_0^1(\Omega)$ such that*

(i) $\mathcal{T}_1^\varepsilon(h_1^\varepsilon(x, u_1^\varepsilon)) \longrightarrow h_1(y, u_1)$ *strongly in* $L^{t/p_1}(\Omega \times Y_1)$,
(ii) $\mathcal{T}_2^\varepsilon(h_2^\varepsilon(x, u_2^\varepsilon)) \longrightarrow h_2(y, u_1)$ *strongly in* $L^{2/p_2}(\Omega \times Y_2)$ *if $\gamma < 1$,*

where

$$\begin{cases} t = max\{2, p_1\} & if \ n = 2, \\ t \in \left[max\left\{2, \frac{n+2}{n-2}\right\}, 2^* \right[& if \ n > 2. \end{cases}$$

Remark 4.8. In order to treat the nonlinear term by unfolding, in the sequel we use the fact that, from the definition of the unfolding operators and assumption $(\mathcal{H}_1)$ on h, h_1 and h_2,

$$\begin{cases} \mathcal{T}_i^\varepsilon(h_i^\varepsilon(\cdot, v))(x, y) = h_i(y, \mathcal{T}_i^\varepsilon(v)(x, y)), & \text{for } i = 1, 2, \\ \mathcal{T}_b^\varepsilon(h^\varepsilon(\cdot, w))(x, y) = h(y, \mathcal{T}_b^\varepsilon(w)(x, y)), \end{cases}$$

for any function v measurable on Ω_i^ε and w measurable on Γ^ϵ.

Proof of Proposition 4.7. From Proposition 4.2(iii) and Lemma 4.6, one has

$$\|\mathcal{T}_1^\varepsilon(u_1^\varepsilon)\|_{L^s(\Omega \times Y_1)} \le |Y|^{1/s} \|u_1^\varepsilon\|_{L^s(\Omega_1^\varepsilon)} \le c \|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \le c_1, \qquad (4.13)$$

for all $s \in [2, 2^*]$ if $n > 2$ and $s \in [2, +\infty[$ if $n = 2$.

Then, choosing $s = 2^*$ if $n > 2$ and $s > max\{2, p_1\}$ if $n = 2$, we obtain by the interpolation inequality,

$$\|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - u_1\|_{L^t(\Omega \times Y_1)} \le C \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon)$$
$$- u_1\|_{L^s(\Omega \times Y_1)}^{1-a} \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - u_1\|_{L^2(\Omega \times Y_1)}^a \quad \forall t \in [2, s[, \qquad (4.14)$$

with $a = \frac{2(s-t)}{t(s-2)} \in]0, 1]$. Hence, combining $(4.1)_1$, (4.13) and (4.14), we get, up to a subsequence,

$$\mathcal{T}_1^\varepsilon (u_1^\varepsilon) \longrightarrow u_1 \quad \text{strongly in } L^t (\Omega \times Y_1) \quad \forall t \in [2, s[, \qquad (4.15)$$

for the above choice of s.

Now, in (4.15) we choose $t = max\{2, p_1\}$ if $n = 2$ and $t \in \left[max\left\{2, \frac{n+2}{n-2}\right\}, 2^* \right[$ if $n > 2$. This means that $p_1 \leq t$. Since h_1 satisfies assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$, from the classical results in [32] and convergence (4.15), we obtain

$$h_1 (y, \mathcal{T}_1^\varepsilon (u_1^\varepsilon)) \longrightarrow h_1 (y, u_1) \quad \text{strongly in } L^{t/p_1} (\Omega \times Y_1), \qquad (4.16)$$

which implies (i) by Remark 4.8.

Finally, when $\gamma < 1$, convergence (ii) is a direct consequence of Theorem 4.5 and the growth condition $(\mathcal{H}_4')$ on h_2. $\qquad \square$

Remark 4.9. Due to the disconnectedness of the set Ω_2^ε, we cannot obtain for $v \in V_2^\varepsilon$ a similar result as that in Lemma 4.6. This is shown by the following counter-example, suggested by Mardare [35]:

Let us fix $k_0 \in \widehat{K}_\varepsilon$ and set

$$v = \begin{cases} \varepsilon^{-n/2} & \text{in } \varepsilon Y_2^{k_0}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, if $s > 2$,

$$\|v\|_{L^s(\Omega_2^\varepsilon)} = \varepsilon^{-\frac{n}{2}} \left| \varepsilon Y_2^{k_0} \right|^{1/s} = \varepsilon^{\frac{n}{s} - \frac{n}{2}} \left| Y_2^{k_0} \right|^{1/s} \to +\infty, \quad \text{as } \varepsilon \to 0,$$

while

$$\|v\|_{H^1(\Omega_2^\varepsilon)} = \|v\|_{L^2(\Omega_2^\varepsilon)} = \left| Y_2^{k_0} \right|^{1/s}.$$

5. Homogenization results

Here, we prove in details the result for $\gamma = -1$, which is more complicated. For the other cases we only clarify the points where the proof is different.

5.1. The case $\gamma = -1$

In this case, although the strong convergence of $\mathcal{T}_2^\varepsilon (u_2^\varepsilon)$ proved in Theorem 4.5 allows to pass to the limit in the nonlinear term h_2, we meet a main difficulty concerning the nonlinear function h of the jump in the equation. Indeed, from (4.4) we have only weak convergence of the term $(\mathcal{T}_1 (u_1^\varepsilon) - \mathcal{T}_2 (u_2^\varepsilon))/\varepsilon$ in $L^2 (\Omega \times \Gamma)$ and this does not allow to pass straightforward to the limit in the interface term, as done in [26] for a linear h.

Nevertheless, in the spirit of the Minty method for monotone operators, by choosing a sequence of suitable test functions, we are able to compute the limit of the unfolded term $h(y, \mathcal{T}_b^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)/\varepsilon)$, although we cannot improve the weak convergence of $\mathcal{T}_b^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)/\varepsilon$.

Theorem 5.1. *For $\gamma = -1$, let (2.4), (2.5) and (2.6)$_2$ hold true. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.2), then there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2\left(\Omega, H_{per}^1(Y_1)\right)$ and $\widehat{u}_2 \in L^2\left(\Omega, H^1(Y_2)\right)$ such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \longrightarrow u_1 & strongly\ in\ L^2\left(\Omega, H^1(Y_1)\right), \\ \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 & weakly\ in\ L^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \longrightarrow u_1 & strongly\ in\ L^2\left(\Omega, H^1(Y_2)\right), \\ \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_2 & weakly\ in\ L^2(\Omega \times Y_2). \end{cases} \tag{5.1}$$

The triplet $(u_1, \widehat{u}_1, \widehat{u}_2)$ is the unique solution of the following problem:

$$\begin{cases} Find\ u_1 \in H_0^1(\Omega),\ \widehat{u}_1 \in L^2\left(\Omega, H_{per}^1(Y_1)\right)\ with\ \mathcal{M}_\Gamma(\widehat{u}_1) = 0 \\ \quad and\ \widehat{u}_2 \in L^2\left(\Omega, H^1(Y_2)\right)\ such\ that \\ \dfrac{1}{|Y|} \displaystyle\sum_{i=1,2} \int_{\Omega \times Y_i} A(y)(\nabla u_1 + \nabla_y \widehat{u}_i)(\nabla \varphi + \nabla_y \Phi_i)\, dx\, dy + \dfrac{1}{|Y|} \sum_{i=1,2} \\ \quad \displaystyle\int_{\Omega \times Y_i} \varphi h_i(y, u_1)\, dx\, dy + \dfrac{1}{|Y|} \int_{\Omega \times \Gamma} h(y, \widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2)\ dx\, d\sigma_y \\ \quad = \displaystyle\int_\Omega f(x)\varphi(x)\ dx \quad \forall \varphi \in H_0^1(\Omega),\ \Phi_1 \in L^2\left(\Omega, H_{per}^1(Y_1)\right), \\ \quad \Phi_2 \in L^2\left(\Omega, H^1(Y_2)\right). \end{cases} \tag{5.2}$$

Remark 5.2. Observe that in view of Theorem 2.6, assumption (2.5) and the strict monotonicity (2.6)$_2$ of h imply that the variational problem (2.10) admits a unique solution $u^\varepsilon \in H_\gamma^\varepsilon$. Besides, Proposition 2.7 gives the boundedness of the norm of u^ε in H_γ^ε.

In the proof of Theorem 5.1, we will need the following result:

Lemma 5.3. *Let the spaces $W_{per}(Y_1)$ and $\mathcal{B}$ defined by*

$$W_{per}(Y_1) = \left\{ g \in H_{per}^1(Y_1) | \mathcal{M}_\Gamma(g) = 0 \right\},$$
$$\mathcal{B} = H_0^1(\Omega) \times L^2(\Omega, W_{per}(Y_1)) \times L^2(\Omega, H^1(Y_2)).$$

Then, $\|\cdot\|_{\mathcal{B}}$ given by the formula

$$\|v\|_{\mathcal{B}}^2 = \|\nabla v_1 + \nabla_y \widehat{v}_1\|_{L^2(\Omega \times Y_1)}^2 + \|\nabla v_1 + \nabla_y \widehat{v}_2\|_{L^2(\Omega \times Y_2)}^2$$
$$+ \|\widehat{v}_1 - \widehat{v}_2\|_{L^2(\Omega \times \Gamma)}^2,\ v = (v_1, \widehat{v}_1, \widehat{v}_2) \in \mathcal{B}$$

is a norm on $\mathcal{B}$.

Proof. Let us prove that if $\|v\|_{\mathcal{B}} = 0$ then $v \equiv 0$, the remaining properties of a norm being trivial. Suppose then that $\|v\|_{\mathcal{B}} = 0$. We have

$$\begin{cases} \widehat{v}_1 = \widehat{v}_2\ on\ \Omega \times \Gamma, \\ M = \|\nabla v_1 + \nabla_y \widehat{v}_1\|_{L^2(\Omega \times Y_1)}^2 + \|\nabla v_1 + \nabla_y \widehat{v}_2\|_{L^2(\Omega \times Y_2)}^2 = 0. \end{cases} \tag{5.3}$$

It is easy to see that

$$M = |Y| \, \|\nabla v_1\|_{L^2(\Omega)}^2 + \|\nabla_y \widehat{v}_1\|_{L^2(\Omega \times Y_1)}^2 + \|\nabla_y \widehat{v}_2\|_{L^2(\Omega \times Y_2)}^2$$

$$+ 2 \int_{\Omega \times Y_1} \nabla v_1 \nabla_y \widehat{v}_1 \, dx \, dy + 2 \int_{\Omega \times Y_2} \nabla v_1 \nabla_y \widehat{v}_2 \, dx \, dy. \qquad (5.4)$$

Applying Green's formula, we deduce from $(5.3)_1$ and the Y-periodicity of $\widehat{v}_1$,

$$\int_{\Omega \times Y_1} \nabla v_1 \nabla_y \widehat{v}_1 \, dx \, dy + \int_{\Omega \times Y_2} \nabla v_1 \nabla_y \widehat{v}_2 \, dx \, dy$$

$$= \sum_{i=1}^{n} \int_{\Omega} \frac{\partial v_1}{\partial x_i} \left(\int_{Y_1} \frac{\partial \widehat{v}_1}{\partial y_i} \, dy \right) dx + \sum_{i=1}^{n} \int_{\Omega} \frac{\partial v_1}{\partial x_i} \left(\int_{Y_2} \frac{\partial \widehat{v}_2}{\partial y_i} \, dy \right) dx$$

$$= \int_{\Omega \times \Gamma} \widehat{v}_1 \nabla v_1 . n_1 \, (y) \, dx \, d\sigma_y + \int_{\Omega \times \Gamma} \widehat{v}_2 \nabla v_1 . n_2 \, (y) \, dx \, d\sigma_y \qquad (5.5)$$

$$= \int_{\Omega \times \Gamma} (\widehat{v}_1 - \widehat{v}_2) \, \nabla v_1 . n_1 \, (y) \, dx \, d\sigma_y = 0,$$

since ∇v_1 is independent of y and $n_2 = -n_1$.

Combining (5.3)–(5.5) implies

$$\begin{cases} \nabla v_1 = 0 & \text{in } \Omega, \\ \nabla_y \widehat{v}_1 = 0 & \text{in } \Omega \times Y_1, \\ \nabla_y \widehat{v}_2 = 0 & \text{in } \Omega \times Y_2, \\ \widehat{v}_1 = \widehat{v}_2 & \text{on } \Omega \times \Gamma. \end{cases} \qquad (5.6)$$

This means that

$$\begin{cases} v_1 = 0 & \text{in } \Omega, \\ \widehat{v}_1 = 0 & \text{in } \Omega \times Y_1, \\ \widehat{v}_2 = C(x) & \text{in every connected component of } \Omega \times Y_2, \\ \widehat{v}_1 = \widehat{v}_2 & \text{on } \Omega \times \Gamma, \end{cases} \qquad (5.7)$$

where we make use of the Poincaré inequality on $H_0^1(\Omega)$ and the Poincaré–Wirtinger inequality on $W_{per}(Y_1)$. This yields $v \equiv 0$ and the proof is complete. $\qquad \square$

Proof of Theorem 5.1. The proof consists of three steps.
Step 1: *Passing to the limit.*

From Theorems 4.3, 4.5 and Remark 4.4, convergences (5.1) follow for a subsequence. In order to get the limit problem, we take (v_1, v_2) as test functions in (2.10), where

$$v_i = \varphi(x) + \varepsilon \omega_i(x) \, \psi_i^\varepsilon(x) \quad \text{with } \psi_i^\varepsilon(x) = \psi_i(x/\varepsilon), \ i = 1, 2,$$

for $\varphi, \omega_i \in \mathcal{D}(\Omega)$, $\psi_1 \in H_{per}^1(Y_1)$ and $\psi_2 \in H^1(Y_2)$ (extended by Y-periodicity to $\mathbb{R}^n$).

Proposition 4.2(v) gives

$$\mathcal{T}_i^\varepsilon(v_i) \longrightarrow \varphi \quad \text{strongly in } L^2(\Omega \times Y_i), \qquad (5.8)$$

since v_i strongly converges to φ in $L^2(\Omega)$. Let us also note that

$$\nabla v_i(x) = \nabla\varphi + \varepsilon\psi_i(x/\varepsilon)\nabla\omega_i(x) + \omega_i(x)\nabla_y\psi_i(x/\varepsilon).$$

Then, if we set $\Phi_i(x,y) = \omega_i(x)\psi_i(y)$, $i = 1,2$, we have

$$\mathcal{T}_i^\varepsilon(\nabla v_i) = \mathcal{T}_i^\varepsilon(\nabla\varphi) + \varepsilon\,\psi_i\,\mathcal{T}_i^\varepsilon(\nabla\omega_i) + \nabla_y\psi_i\,\mathcal{T}_i^\varepsilon(\omega_i)$$
$$\longrightarrow \nabla\varphi + \nabla_y\Phi_i \quad \text{strongly in } L^2(\Omega\times Y_i). \tag{5.9}$$

Hence, from convergences $(5.1)_{2,4}$, (5.8), (5.9) and Proposition 4.7, we obtain by unfolding,

$$\int_{\Omega_i^\varepsilon} A^\varepsilon\nabla u_i^\varepsilon\nabla v_i\ dx = \frac{1}{|Y|}\int_{\Omega\times Y_i} A(y)\,\mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon)\,\mathcal{T}_i^\varepsilon(\nabla v_i)\ dx\ dy$$
$$\longrightarrow \frac{1}{|Y|}\int_{\Omega\times Y_i} A(y)\,(\nabla u_1 + \nabla_y\widehat{u}_i)\,(\nabla\varphi + \nabla_y\Phi_i)\ dx\ dy, \tag{5.10}$$

$$\int_{\Omega_i^\varepsilon} h_i^\varepsilon(x, u_i^\varepsilon)\,v_i\ dx = \frac{1}{|Y|}\int_{\Omega\times Y_i} h_i(y, \mathcal{T}_i^\varepsilon(u_i^\varepsilon))\,\mathcal{T}_i^\varepsilon(v_i)\ dx\ dy$$
$$\longrightarrow \frac{1}{|Y|}\int_{\Omega\times Y_i} h_i(y, u_1)\,\varphi\ dx\ dy, \tag{5.11}$$

for $i = 1,2$ and

$$\int_{\Omega_1^\varepsilon} f v_1\ dx + \int_{\Omega_2^\varepsilon} f v_2\ dx \longrightarrow \int_\Omega \theta_1 f\varphi\ dx + \int_\Omega \theta_2 f\varphi\ dx = \int_\Omega f\varphi\ dx. \tag{5.12}$$

On the other hand, in view of Remark 4.8,

$$\int_{\Gamma^\varepsilon} h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon)\,(v_1 - v_2)\ d\sigma_x = \varepsilon\int_{\Gamma^\varepsilon} h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon)\,(\omega_1\psi_1^\varepsilon - \omega_2\psi_2^\varepsilon)\ d\sigma_x$$
$$= \frac{1}{|Y|}\int_{\Omega\times\Gamma} h\left(y, \frac{\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)}{\varepsilon}\right)(\psi_1(y)\mathcal{T}_1^\varepsilon(\omega_1) - \psi_2(y)\mathcal{T}_2^\varepsilon(\omega_2))\ dx\ d\sigma_y. \tag{5.13}$$

Note that Theorem 4.3 implies

$$\left\|\frac{\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)}{\varepsilon}\right\|_{L^2(\Omega\times\Gamma)} \leq c.$$

Thus, in view of assumption $(\mathcal{H}_2)$ on h and the fact that $q < 2$, there exists $\zeta \in L^{2/q}(\Omega\times\Gamma)$ such that, up to a subsequence of ε,

$$h\left(y, \frac{\mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)}{\varepsilon}\right) \rightharpoonup \zeta \quad \text{weakly in } L^{2/q}(\Omega\times\Gamma). \tag{5.14}$$

Then, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Gamma^\varepsilon} h^\varepsilon(x, u_1^\varepsilon - u_2^\varepsilon)\,(v_1 - v_2)\ d\sigma_x = \frac{1}{|Y|}\int_{\Omega\times\Gamma} \zeta\,(\Phi_1 - \Phi_2)\ dx\ d\sigma_y. \tag{5.15}$$

Collecting (5.10)–(5.12) and (5.15), passing to the limit as $\varepsilon \to 0$ in the variational formulation (2.10) for $\gamma = -1$ and (v_1, v_2) chosen as above, by density we deduce the following limit problem:

$$
\begin{cases}
\dfrac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A\left(y\right) \left(\nabla u_1 + \nabla_y \widehat{u}_i\right) \left(\nabla \varphi + \nabla_y \Phi_i\right) dx \, dy \\[2mm]
+ \dfrac{1}{|Y|} \int_{\Omega \times \Gamma} \zeta \left(\Phi_1 - \Phi_2\right) dx \, d\sigma_y + \dfrac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} \varphi h_i \left(y, u_1\right) dx \, dy \\[2mm]
= \displaystyle\int_\Omega f\left(x\right) \varphi\left(x\right) dx \quad \forall \varphi \in H_0^1\left(\Omega\right), \ \Phi_1 \in L^2\left(\Omega, H_{per}^1\left(Y_1\right)\right), \\[2mm]
\Phi_2 \in L^2\left(\Omega, H^1\left(Y_2\right)\right).
\end{cases}
\tag{5.16}
$$

Step 2: *Identification of the limit ζ in* (5.16).

We now show that

$$
\zeta = h\left(y, \widehat{u}_1 - \widehat{u}_2\right).
\tag{5.17}
$$

In order to do that, we use the Minty method for monotone operators, as done for instance in [19] for Leray–Lions operators by the unfolding method or in [3] for a parabolic problem with a jump of solution by the two-scale method.

This leads to use some test functions adapted to our nonlinear problem. Consider the sequences

$$
\varpi_i^\varepsilon\left(x\right) = \lambda_0\left(x\right) + \varepsilon \lambda_{i1}\left(x, \frac{x}{\varepsilon}\right) + k\varepsilon \lambda_{i2}\left(x, \frac{x}{\varepsilon}\right), \quad \text{for } i = 1, 2,
$$

where

$$
\begin{cases}
\lambda_0 \in \mathcal{D}\left(\Omega\right), \ \lambda_{1j} \in D\left(\Omega, C_{per}^\infty\left(Y_1\right)\right), \\
\lambda_{2j} \in D\left(\Omega, C^\infty\left(Y_2\right)\right) \text{ extended by Y-periodicity to } \mathbb{R}^n, \ j = 1, 2.
\end{cases}
\tag{5.18}
$$

First, taking $(u_1^\varepsilon - \varpi_1^\varepsilon, u_2^\varepsilon - \varpi_2^\varepsilon)$ as a test function in (2.10) written for $\gamma = -1$ implies

$$
\sum_{i=1,2} \int_{\Omega_i^\varepsilon} A^\varepsilon \nabla u_i^\varepsilon \left(\nabla u_i^\varepsilon - \nabla \varpi_i^\varepsilon\right) \, dx + \sum_{i=1,2} \int_{\Omega_i^\varepsilon} \left(u_i^\varepsilon - \varpi_i^\varepsilon\right) h_i^\varepsilon\left(x, u_i^\varepsilon\right) \, dx
$$

$$
+ \int_{\Gamma^\varepsilon} \left(u_1^\varepsilon - \varpi_1^\varepsilon - u_2^\varepsilon + \varpi_2^\varepsilon\right) h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) \, d\sigma_x = \sum_{i=1,2} \int_{\Omega_i^\varepsilon} f\left(u_i^\varepsilon - \varpi_i^\varepsilon\right) \, dx.
\tag{5.19}
$$

Moreover, from the monotonicity of h, h_1, h_2 and the assumption $A \in M\left(\alpha, \beta, Y\right)$ we have

$$
\sum_{i=1,2} \int_{\Omega_i^\varepsilon} A^\varepsilon\left(\nabla u_i^\varepsilon - \nabla \varpi_i^\varepsilon\right)\left(\nabla u_i^\varepsilon - \nabla \varpi_i^\varepsilon\right) \, dx
$$

$$
+ \sum_{i=1,2} \int_{\Omega_i^\varepsilon} \left(u_i^\varepsilon - \varpi_i^\varepsilon\right)\left[h_i^\varepsilon\left(x, u_i^\varepsilon\right) - h_i^\varepsilon\left(x, \varpi_i^\varepsilon\right)\right] \, dx
$$

$$
+ \int_{\Gamma^\varepsilon} \left(u_1^\varepsilon - u_2^\varepsilon - \varpi_1^\varepsilon + \varpi_2^\varepsilon\right)\left[h^\varepsilon\left(x, u_1^\varepsilon - u_2^\varepsilon\right) - h^\varepsilon\left(x, \varpi_1^\varepsilon - \varpi_2^\varepsilon\right)\right] \, d\sigma_x \geq 0.
\tag{5.20}
$$

Thus, combining (5.19) with (5.20) yields

$$\sum_{i=1,2} \int_{\Omega_i^\varepsilon} A^\varepsilon \nabla \varpi_i^\varepsilon \left(\nabla u_i^\varepsilon - \nabla \varpi_i^\varepsilon \right) \, dx + \sum_{i=1,2} \int_{\Omega_i^\varepsilon} \left(u_i^\varepsilon - \varpi_i^\varepsilon \right) h_i^\varepsilon \left(x, \varpi_i^\varepsilon \right) \, dx$$

$$+ \int_{\Gamma^\varepsilon} \left(u_1^\varepsilon - u_2^\varepsilon - \varpi_1^\varepsilon + \varpi_2^\varepsilon \right) h^\varepsilon \left(x, \varpi_1^\varepsilon - \varpi_2^\varepsilon \right) \, d\sigma_x \leq \sum_{i=1,2} \int_{\Omega_i^\varepsilon} f \left(u_i^\varepsilon - \varpi_i^\varepsilon \right) \, dx,$$

(5.21)

which gives, by unfolding, for ε small enough

$$\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \, \mathcal{T}_i^\varepsilon \left(\nabla \varpi_i^\varepsilon \right) \left[\mathcal{T}_i^\varepsilon \left(\nabla u_i^\varepsilon \right) - \mathcal{T}_i^\varepsilon \left(\nabla \varpi_i^\varepsilon \right) \right] \, dx \, dy$$

$$+ \frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon \left(u_i^\varepsilon - \varpi_i^\varepsilon \right) h_i \left(y, \mathcal{T}_i^\varepsilon \left(\varpi_i^\varepsilon \right) \right) \, dx \, dy$$

$$+ \frac{1}{|Y|} \int_{\Omega \times \Gamma} \left[\frac{\mathcal{T}_b^\varepsilon \left(u_1^\varepsilon - u_2^\varepsilon \right)}{\varepsilon} - \frac{\mathcal{T}_b^\varepsilon \left(\varpi_1^\varepsilon - \varpi_2^\varepsilon \right)}{\varepsilon} \right] h \left(y, \frac{\mathcal{T}_b^\varepsilon \left(\varpi_1^\varepsilon - \varpi_2^\varepsilon \right)}{\varepsilon} \right) \, dx \, d\sigma_y$$

$$\leq \frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon (f) \, \mathcal{T}_i^\varepsilon \left(u_i^\varepsilon - \varpi_i^\varepsilon \right) \, dx \, dy.$$

(5.22)

On the other hand, it is easily seen that, as $\varepsilon \to 0$

$$\begin{cases} \mathcal{T}_i^\varepsilon \left(\varpi_i^\varepsilon \right) \longrightarrow \lambda_0 \text{ strongly in } L^2 \left(\Omega \times Y_i \right), \\ \frac{1}{\varepsilon} \mathcal{T}_i^\varepsilon \left(\varepsilon \lambda_{i1} \left(x, \frac{x}{\varepsilon} \right) + k \varepsilon \lambda_{i2} \left(x, \frac{x}{\varepsilon} \right) \right) \longrightarrow \lambda_{i1}(x, y) + k \lambda_{i2}(x, y) \\ \text{strongly in } L^2(\Omega, H^1(Y_i)) \text{ and } L^2(\Omega \times \Gamma), \text{ for } i = 1, 2. \end{cases}$$

(5.23)

Consequently,

$$\frac{\mathcal{T}_b^\varepsilon \left(\varpi_1^\varepsilon - \varpi_2^\varepsilon \right)}{\varepsilon} \longrightarrow \lambda_{11} - \lambda_{21} + k \left(\lambda_{12} - \lambda_{22} \right) \quad \text{strongly in } L^2 \left(\Omega \times \Gamma \right).$$ (5.24)

Moreover, for $i = 1, 2$, since

$$\nabla \varpi_i^\varepsilon (x) = \nabla \lambda_0 (x) + \varepsilon \nabla \lambda_{i1} \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \lambda_{i1} \left(x, \frac{x}{\varepsilon} \right)$$

$$+ k \varepsilon \nabla \lambda_{i2} \left(x, \frac{x}{\varepsilon} \right) + k \nabla_y \lambda_{i2} \left(x, \frac{x}{\varepsilon} \right),$$

we obtain

$$\mathcal{T}_i^\varepsilon \left(\nabla \varpi_i^\varepsilon \right) \longrightarrow \nabla \lambda_0 (x) + \nabla_y \lambda_{i1} (x, y) + k \nabla_y \lambda_{i2} (x, y) \quad \text{strongly in } L^2 \left(\Omega \times Y_i \right).$$

Thus, passing to the limit in (5.22) as $\varepsilon \to 0$, one deduces

$$\sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \left(\nabla \lambda_0 + \nabla_y \lambda_{i1} + k \nabla_y \lambda_{i2} \right) \left[\nabla u_1 + \nabla_y \widehat{u}_i \right.$$

$$\left. - \left(\nabla \lambda_0 + \nabla_y \lambda_{i1} + k \nabla_y \lambda_{i2} \right) \right] \, dx \, dy$$

$$+ \int_{\Omega \times \Gamma} \left[\widehat{u}_1 - \widehat{u}_2 - \lambda_{11} + \lambda_{21} - k \left(\lambda_{12} - \lambda_{22} \right) \right] h \left(y, \lambda_{11} - \lambda_{21} + k \left(\lambda_{12} - \lambda_{22} \right) \right) dx \ d\sigma_y$$

$$+ \sum_{i=1,2} \int_{\Omega \times Y_i} \left(u_1 - \lambda_0 \right) h_i \left(y, \lambda_0 \right) \ dx \ dy \le |Y| \int_\Omega f \left(u_1 - \lambda_0 \right) \ dx \ dy, \qquad (5.25)$$

where we use (5.23), (5.24) and the convergence results in Theorem 4.3.

Now let us choose $\lambda_0 = \lambda_0^n$ and $\lambda_{i1} = \lambda_{i1}^n$, $i = 1, 2$ such that, as $n \to +\infty$

$$\begin{cases} \lambda_0^n \longrightarrow u_1 & \text{strongly in } H_0^1 \left(\Omega \right), \\ \lambda_{11}^n \longrightarrow \widehat{u}_1 & \text{strongly in } L^2 \left(\Omega, H_{per}^1 \left(Y_1 \right) \right), \\ \lambda_{21}^n \longrightarrow \widehat{u}_2 & \text{strongly in } L^2 \left(\Omega, H^1 \left(Y_2 \right) \right), \end{cases} \qquad (5.26)$$

and pass to the limit as $n \to +\infty$ in (5.25), we obtain

$$- k \sum_{i=1,2} \int_{\Omega \times Y_i} A \left(y \right) \left(\nabla u_1 + \nabla_y \widehat{u}_i + k \nabla_y \lambda_{i2} \right) \nabla_y \lambda_{i2} \ dx \ dy$$

$$- k \int_{\Omega \times \Gamma} \left(\lambda_{12} - \lambda_{22} \right) h \left(y, \widehat{u}_1 - \widehat{u}_2 + k \left(\lambda_{12} - \lambda_{22} \right) \right) \ dx \ d\sigma_y \le 0. \qquad (5.27)$$

Since (5.27) holds for every $k \in \mathbb{R}$, we deduce the following equality:

$$\sum_{i=1,2} \int_{\Omega \times Y_i} A \left(y \right) \left(\nabla u_1 + \nabla_y \widehat{u}_i + k \nabla_y \lambda_{i2} \right) \nabla_y \lambda_{i2} \ dx \ dy$$

$$+ \int_{\Omega \times \Gamma} \left(\lambda_{12} - \lambda_{22} \right) h \left(y, \widehat{u}_1 - \widehat{u}_2 + k \left(\lambda_{12} - \lambda_{22} \right) \right) \ dx \ d\sigma_y = 0. \qquad (5.28)$$

On the other hand, choosing $\varphi \equiv 0$, $\Phi_1 = \lambda_{12}$, $\Phi_2 = \lambda_{22}$ as the test functions in (5.16) implies

$$\sum_{i=1,2} \int_{\Omega \times Y_i} A \left(y \right) \left(\nabla u_1 + \nabla_y \widehat{u}_i \right) \nabla_y \lambda_{i2} \ dx \ dy + \int_{\Omega \times \Gamma} \zeta \left(\lambda_{12} - \lambda_{22} \right) \ dx \ d\sigma_y = 0. \qquad (5.29)$$

Combining (5.28) with (5.29), one gets

$$k \sum_{i=1,2} \int_{\Omega \times Y_i} A \left(y \right) \nabla_y \lambda_{i2} \nabla_y \lambda_{i2} \ dx \ dy - \int_{\Omega \times \Gamma} \zeta \left(\lambda_{12} - \lambda_{22} \right) \ dx \ d\sigma_y$$

$$+ \int_{\Omega \times \Gamma} \left(\lambda_{12} - \lambda_{22} \right) h \left(y, \widehat{u}_1 - \widehat{u}_2 + k \left(\lambda_{12} - \lambda_{22} \right) \right) \ dx \ d\sigma_y = 0. \qquad (5.30)$$

Thus, letting $k \to 0$ in (5.30) provides

$$\int_{\Omega \times \Gamma} \left(\lambda_{12} - \lambda_{22} \right) h \left(y, \widehat{u}_1 - \widehat{u}_2 \right) \ dx \ d\sigma_y = \int_{\Omega \times \Gamma} \zeta \left(\lambda_{12} - \lambda_{22} \right) \ dx \ d\sigma_y$$

which gives (5.17).

Step 3: *Existence and uniqueness of the solution of the limit problem* (5.2).

Let us consider operator $\mathcal{P} : u = (u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{B} \longmapsto \mathcal{P}(u) \in \mathcal{B}'$ defined by

$$\langle \mathcal{P}(u), v \rangle_{\mathcal{B}', \mathcal{B}} = \sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \left(\nabla u_1 + \nabla_y \widehat{u}_i \right) \left(\nabla v_1 + \nabla_y \widehat{v}_i \right) dx dy$$

$$+ \sum_{i=1,2} \int_{\Omega \times Y_i} v_1 h_i (y, u_1) \, dx dy$$

$$+ \int_{\Omega \times \Gamma} h(y, \widehat{u}_1 - \widehat{u}_2)(\widehat{v}_1 - \widehat{v}_2) \; dx \; d\sigma_y - |Y| \int_\Omega f v_1 \; dx \quad \forall v = (v_1, \widehat{v}_1, \widehat{v}_2) \in \mathcal{B},$$

where the space $\mathcal{B}$ is given in Lemma 5.3.

It is easily seen that $\mathcal{P}$ is bounded continuous, coercive and monotone. Thus, from the Minty–Browder theorem, $\mathcal{P}$ is surjective, which implies that Eq. (5.2) has a solution $u = (u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{B}$. Let us check the uniqueness of u. If problem (5.2) has two solutions $u = (u_1, \widehat{u}_1, \widehat{u}_2)$ and $\omega = (\omega_1, \widehat{\omega}_1, \widehat{\omega}_2)$, we have

$$\langle \mathcal{P}(u) - \mathcal{P}(\omega), v \rangle_{\mathcal{B}', \mathcal{B}} = 0 \quad \forall v = (v_1, \widehat{v}_1, \widehat{v}_2) \in \mathcal{B}.$$

Choosing $v = u - \omega$ yields

$$\langle \mathcal{P}(u) - \mathcal{P}(\omega), u - \omega \rangle_{\mathcal{B}', \mathcal{B}}$$

$$= \sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \left(\nabla u_1 - \nabla \omega_1 + \nabla_y \widehat{u}_i - \nabla_y \widehat{\omega}_i \right)$$

$$\times \left(\nabla u_1 - \nabla \omega_1 + \nabla_y \widehat{u}_i - \nabla_y \widehat{\omega}_i \right) dx \; dy$$

$$+ \sum_{i=1,2} \int_{\Omega \times Y_i} \left(h_i(y, u_1) - h_i(y, \omega_1) \right) (u_1 - \omega_1) \, dx \; dy \qquad (5.31)$$

$$+ \int_{\Omega \times \Gamma} \left(h(y, \widehat{u}_1 - \widehat{u}_2) - h(y, \widehat{\omega}_1 - \widehat{\omega}_2) \right) (\widehat{u}_1 - \widehat{u}_2 - \widehat{\omega}_1 + \widehat{\omega}_2) \, dx \; d\sigma_y = 0.$$

Using the assumption $A \in M(\alpha, \beta, Y)$ and the strictly monotonicity of h, we obtain

$$\begin{cases} \nabla(u_1 - \omega_1) + \nabla_y (\widehat{u}_i - \widehat{\omega}_i) = 0 & \text{in } \Omega \times Y_i, \; i = 1, 2, \\ \widehat{u}_1 - \widehat{\omega}_1 - (\widehat{u}_2 - \widehat{\omega}_2) = 0 & \text{on } \Omega \times \Gamma, \end{cases}$$

which by Lemma 5.3 gives $\|u - \omega\|_{\mathcal{B}} = 0$, so that $u \equiv \omega$. Hence, convergences (5.1) hold for the whole sequence. This completes the proof. $\qquad \square$

Remark 5.4. The fact that we cannot directly pass to the limit in the unfolded nonlinear term $h(y, \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)/\varepsilon)$ makes the situation here different from the case of nonlinear Robin boundary conditions treated in [7,16], where some weak convergence of the Nemytskii operator associated to the nonlinear function h was proved.

In order to describe the homogenized problem solved by u_1, we introduce for every $z \in \mathbb{R}^n$, the following problem:

$$
\begin{cases}
\text{Find } \left(\chi_1(\cdot, z), \; \chi_2(\cdot, z) \right) \in W_{per}(Y_1) \times H^1(Y_2) \text{ such that} \\[2mm]
\displaystyle\sum_{i=1,2} \int_{Y_i} A(y) \, \nabla_y \chi_i(y, z) \nabla v_i \; dy + \int_{\Gamma} h(y, \chi_1(y, z) - \chi_2(y, z))(v_1 - v_2) \, d\sigma \\[4mm]
\displaystyle = - \int_{Y_1} A(y) \, z \nabla v_1 \; dy - \int_{Y_2} A(y) \, z \nabla v_2 \; dy, \quad \forall v_1 \in H^1_{per}(Y_1), \; v_2 \in H^1(Y_2),
\end{cases}
\tag{5.32}
$$

where the space $W_{per}(Y_1)$ is defined in Lemma 5.3.

The existence and the uniqueness of the solution of the above problem can be proved using the Minty–Browder theorem as in the proof of Proposition 3.2, for the functional space

$$
W \doteq \left\{ v = (v_1, v_2) : \; v_1 \in W_{per}(Y_1), \; v_2 \in H^1(Y_2) \right\},
$$

endowed with the norm

$$
\|v\|_W^2 = \|\nabla v_1\|_{L^2(Y_1)}^2 + \|\nabla v_2\|_{L^2(Y_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2 \quad \forall v = (v_1, v_2) \in W.
$$

Corollary 5.5. *For $\gamma = -1$, under the hypotheses of Theorem 5.1, if $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.2), then there exists $u_1 \in H_0^1(\Omega)$ such that, for $i = 1, 2$,*

$$
\begin{cases}
\widetilde{u_i^\varepsilon} \rightharpoonup \theta_i u_1 & \text{weakly in } L^2(\Omega), \\[2mm]
A^\varepsilon \widetilde{\nabla u_i^\varepsilon} \rightharpoonup \theta_i \mathcal{M}_{Y_i}(A(y)[\nabla u_1 + \nabla_y \chi_i(y, \nabla u_1)]) & \text{weakly in } \left(L^2(\Omega) \right)^n,
\end{cases}
\tag{5.33}
$$

and u_1 is the unique solution of the following nonlinear problem:

$$
\begin{cases}
- div\left(D(\nabla u_1) \right) + \theta_1 \mathcal{M}_{Y_1}\left(h_1(\cdot, u_1) \right) + \theta_2 \mathcal{M}_{Y_2}\left(h_2(\cdot, u_1) \right) = f & \text{in } \Omega, \\[2mm]
u_1 = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{5.34}
$$

where

$$
D(z) = \sum_{i=1}^{2} \theta_i \mathcal{M}_{Y_i} \left[A(y)(z + \nabla_y \chi_i(y, z)) \right] = \mathcal{M}_Y(A)z
$$

$$
+ \sum_{i=1}^{2} \theta_i \mathcal{M}_{Y_i} \left(A(y) \nabla_y \chi_i(y, z) \right)
\tag{5.35}
$$

and for every $z \in \mathbb{R}^n$, the pair $(\chi_1(\cdot, z), \chi_2(\cdot, z))$ is the unique solution of problem (5.32).

Proof. Choosing $\varphi \equiv 0$ in (5.2) yields

$$
\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A(y) \left(\nabla u_1 + \nabla_y \widehat{u}_i \right) \nabla_y \Phi_i \; dx \, dy
$$

$$
+ \frac{1}{|Y|} \int_{\Omega \times \Gamma} h\left(y, \widehat{u}_1 - \widehat{u}_2 \right) \left(\Phi_1 - \Phi_2 \right) \; dx \, d\sigma_y = 0.
\tag{5.36}
$$

This implies that

$$
\begin{cases}
-\mathrm{div}_y \left(A \left(y \right) \nabla_y \widehat{u}_1 \left(x, y \right) \right) = \mathrm{div}_y \left(A \left(y \right) \nabla u_1 \left(x \right) \right) & \text{a.e. in } \Omega \times Y_1, \\
-\mathrm{div}_y \left(A \left(y \right) \nabla_y \widehat{u}_2 \left(x, y \right) \right) = \mathrm{div}_y \left(A \left(y \right) \nabla u_1 \left(x \right) \right) & \text{a.e. in } \Omega \times Y_2, \\
A \left(y \right) \left[\nabla_y \widehat{u}_1 \left(x, y \right) + \nabla u_1 \left(x \right) \right] n_1 \\
\quad = -A \left(y \right) \left[\nabla_y \widehat{u}_2 \left(x, y \right) + \nabla u_1 \left(x \right) \right] n_2 & \text{a.e. in } \Omega \times \Gamma, \\
-A \left(y \right) \left[\nabla_y \widehat{u}_1 \left(x, y \right) + \nabla u_1 \left(x \right) \right] n_1 = h \left(y, \widehat{u}_1 - \widehat{u}_2 \right) & \text{a.e. in } \Omega \times \Gamma, \\
\widehat{u}_1 \left(x, \cdot \right) \ Y\text{-periodic.}
\end{cases}
\tag{5.37}
$$

Clearly, (5.37) provides $(\widehat{u}_1, \widehat{u}_2)$ under the following forms:

$$
\begin{cases}
\widehat{u}_1 \left(x, y \right) = \chi_1(y, \nabla u_1), \\
\widehat{u}_2 \left(x, y \right) = \chi_2(y, \nabla u_1),
\end{cases}
\tag{5.38}
$$

where, for every $z \in \mathbb{R}^n$, the pair $(\chi_1(\cdot, z),\ \chi_2(\cdot, z)) \in W_{per}(Y_1) \times H^1(Y_2)$ is the solution of problem (5.32).

On the other hand, choosing $\Phi_1 = \Phi_2 \equiv 0$ in (5.2) gives

$$
\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A \left(y \right) \left(\nabla u_1 + \nabla_y \widehat{u}_i \right) \nabla \varphi \ dx \ dy + \frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} h_i \left(y, u_1 \right) \varphi \ dx \ dy
$$
$$
= \int_\Omega f \left(x \right) \varphi \left(x \right) \ dx,
\tag{5.39}
$$

that is

$$
\begin{cases}
\dfrac{1}{|Y|} \sum_{i=1,2} \mathrm{div}_x \int_{Y_i} A \left(y \right) \left[\nabla u_1 \left(x \right) + \nabla_y \chi_i (y, \nabla u_1) \right] dy \\
\quad + \dfrac{1}{|Y|} \sum_{i=1,2} \int_{Y_i} h_i \left(y, u_1 \left(x \right) \right) dy = f & \text{in } \Omega, \\
u_1 = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{5.40}
$$

where we replaced (5.38) in (5.39). This, using (5.35) gives the homogenized problem (5.34).

Let us now check the uniqueness of solution u_1 of problem (5.34). Suppose that u_1^* is a solution of (5.34) and set $\widehat{u}_1^* = \chi_1(y, \nabla u_1^*)$ and $\widehat{u}_2^* = \chi_2(y, \nabla u_1^*)$. Since $(\chi_1(\cdot, z), \chi_2(\cdot, z))$ is the unique solution of problem (5.32) for every $z \in \mathbb{R}^n$, $(u_1^*, \widehat{u}_1^*, \widehat{u}_2^*)$ satisfies (5.36) and (5.39). Summing these two equalities, one has that $(u_1^*, \widehat{u}_1^*, \widehat{u}_2^*)$ is a solution of problem (5.2) which by Theorem 5.1 has a unique solution. Hence, $u_1^* \equiv u_1$ and problem (5.34) admits a unique solution.

Finally, convergences (5.33) follow from Proposition 4.2(i), (vi), (vii) together with (5.1) and (5.38). $\qquad\square$

5.2. The case $\gamma < -1$

In this case, as done in [26], thanks to the relationship between the traces of $\widehat{u}_1$ and $\widehat{u}_2$ on Γ provided by Theorem 4.3, we can overcome the difficulty given by the presence of the nonlinear jump on the boundary.

Theorem 5.6. *For $\gamma < -1$, under assumptions* (2.4), (2.5) *and* $(2.6)_1$ *let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be the solution of problem* (2.2). *Then, there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$ and $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that convergences* (5.1) *hold. If $\widehat{u}$ (extended by periodicity) is given by*

$$\widehat{u} = \begin{cases} \widehat{u}_1 & in \ \Omega \times Y_1, \\ \widehat{u}_2 + \xi_\Gamma & in \ \Omega \times Y_2, \end{cases} \tag{5.41}$$

with ξ_Γ as in Theorem 4.3, *then the pair $(u_1, \widehat{u})$ is the unique solution of the following problem:*

$$\begin{cases} Find \ u_1 \in H_0^1(\Omega), \widehat{u} \in L^2\left(\Omega, H_{per}^1(Y)\right) \ with \ \mathcal{M}_\Gamma(\widehat{u}) = 0 \ a.e. \ in \ \Omega \\ \quad such \ that \\ \dfrac{1}{|Y|} \displaystyle\int_{\Omega \times Y} A(y)(\nabla u_1 + \nabla_y \widehat{u})(\nabla\varphi + \nabla_y \Phi) \ dx \ dy \\ \quad + \dfrac{1}{|Y|} \displaystyle\sum_{i=1,2} \int_{\Omega \times Y_i} h_i(y, u_1) \varphi \ dx \ dy \\ \quad = \displaystyle\int_\Omega f(x)\varphi(x) \ dx \quad \forall\varphi \in H_0^1(\Omega), \ \forall\Phi \in L^2\left(\Omega, H_{per}^1(Y)\right). \end{cases} \tag{5.42}$$

Proof. Let us choose

$$v_1 = v_2 = \varphi(x) + \varepsilon\omega(x)\psi^\varepsilon(x)$$

as the test functions in (2.10) where $\varphi, \omega \in \mathcal{D}(\Omega)$, $\psi \in H_{per}^1(Y)$ and $\psi^\varepsilon(x) = \psi(x/\varepsilon)$. This makes the integral on the interface vanish, so that we only have to treat the volume integrals.

By similar arguments as in the proof of Theorem 5.1, we get (5.8)–(5.12) where Φ_1 and Φ_2 are replaced by $\Phi(x, y) = \omega(x)\psi(y)$.

Hence, passing to the limit as $\varepsilon \to 0$ in the variational formulation (2.10) with the above choice of (v_1, v_2), by density we deduce

$$\frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} A(y)(\nabla u_1 + \nabla_y \widehat{u}_i)(\nabla\varphi + \nabla_y \Phi) \ dx \ dy$$

$$+ \frac{1}{|Y|} \sum_{i=1,2} \int_{\Omega \times Y_i} h_i(y, u_1) \ \varphi \ dx \ dy$$

$$= \int_\Omega f(x)\varphi(x) \ dx \quad \forall\varphi \in H_0^1(\Omega), \ \forall\Phi \in L^2\left(\Omega, H_{per}^1(Y)\right). \tag{5.43}$$

Let us now recall that from Remark 4.4 the relationship between the traces of $\widehat{u}_1$ and $\overline{u}_2$ on Γ is given by $\widehat{u}_1 = \widehat{u}_2 + \xi_\Gamma$ on $\Omega \times \Gamma$. Then, the extension by periodicity of the function $\widehat{u}$ defined by (5.41) is a function (still denoted by $\widehat{u}$) which belongs to $L^2\left(\Omega, H_{per}^1(Y)\right)$ and satisfies

$$\nabla_y \widehat{u}|_{\Omega \times Y_1} = \nabla_y \widehat{u}_1, \quad \nabla_y \widehat{u}|_{\Omega \times Y_2} = \nabla_y \widehat{u}_2.$$

Consequently, (5.43) gives the limit problem (5.42). □

Finally, the uniqueness of $(u_1, \widehat{u}_1, \widehat{u}_2)$, which implies the convergences of the whole sequence in (5.1), comes from the next corollary. $\qquad\square$

Corollary 5.7. *For $\gamma < -1$, under the hypotheses of Theorem 5.6, let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be the solution of problem (2.2). Then, there exists $u_1 \in H_0^1(\Omega)$ such that, for $i = 1, 2$*

$$\begin{cases} \widetilde{u}_i^\varepsilon \rightharpoonup \theta_i u_1 & \text{weakly in } L^2(\Omega), \\ A^\varepsilon \widetilde{\nabla u_i^\varepsilon} \rightharpoonup A_\gamma^i \nabla u_1 & \text{weakly in } \left(L^2(\Omega)\right)^n, \end{cases} \tag{5.44}$$

A_γ^1 and A_γ^2 being the constant matrices defined by

$$A_\gamma^1 e_j = \theta_1 \mathcal{M}_{Y_1}(Ae_j - A\nabla \overline{\chi}_j), \quad A_\gamma^2 e_j = \theta_2 \mathcal{M}_{Y_2}(Ae_j - A\nabla \overline{\chi}_j)$$

where $\{e_j\}_n$ is the canonical basis of $\mathbb{R}^n$ and the correctors $\overline{\chi}_j$, $j = 1, \dots, n$, are the unique solutions of the cell problems

$$\begin{cases} -\operatorname{div}(A(y)\nabla(\overline{\chi}_j - y_j)) = 0 & \text{in } Y, \\ \overline{\chi}_j \ Y\text{-periodic}, \ \mathcal{M}_\Gamma(\overline{\chi}_j) = 0. \end{cases} \tag{5.45}$$

The function u_1 is the unique solution of the following problem:

$$\begin{cases} -\operatorname{div}\left(A_\gamma^0 \nabla u_1\right) + \theta_1 \mathcal{M}_{Y_1}(h_1(\cdot, u_1)) + \theta_2 \mathcal{M}_{Y_2}(h_2(\cdot, u_1)) = f & \text{in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.46}$$

where the homogenized matrix A_γ^0 is given by

$$A_\gamma^0 e_j = \mathcal{M}_Y(Ae_j - A\nabla \overline{\chi}_j) = A_\gamma^1 e_j + A_\gamma^2 e_j. \tag{5.47}$$

Proof. Convergences (5.44) can be proved similarly as in [26].

Choosing $\varphi \equiv 0$ in (5.42), we get

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y)(\nabla u_1 + \nabla_y \widehat{u}) \nabla_y \Phi \, dx \, dy = 0,$$

which implies that

$$\begin{cases} \operatorname{div}_y [A(y)(\nabla_y \widehat{u}(x, y) + \nabla u_1(x))] = 0 \text{ a.e. in } \Omega \times Y, \\ \widehat{u}(x, \cdot) \quad Y\text{-periodic}. \end{cases}$$

Consequently by standard arguments, one obtains

$$\widehat{u}(x, y) = -\sum_{j=1}^n \frac{\partial u_1}{\partial x_j}(x) \overline{\chi}_j(y), \tag{5.48}$$

where $\overline{\chi}_j$, $j = 1, \dots, n$, are the solutions of the cell problems (5.45).

Now we choose $\Phi \equiv 0$ in (5.42),

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y)(\nabla u_1 + \nabla_y \widehat{u}) \nabla \varphi \, dx \, dy + \frac{1}{|Y|} \int_{\Omega \times Y_1} h_1(y, u_1) \varphi \, dx \, dy$$

$$+ \frac{1}{|Y|} \int_{\Omega \times Y_2} h_2(y, u_1) \varphi \, dx \, dy = \int_\Omega f(x) \varphi(x) \, dx,$$

that gives

$$\begin{cases} -\dfrac{1}{|Y|}\operatorname{div}\left(\int_Y A\left(y\right)\left[\nabla u_1\left(x\right)+\nabla_y\widehat{u}\left(x,y\right)\right]\,dy\right) \\ +\dfrac{1}{|Y|}\int_{Y_1} h_1\left(y,u_1\left(x\right)\right)dy+\dfrac{1}{|Y|}\int_{Y_2} h_2\left(y,u_1\left(x\right)\right)dy=f \quad \text{in } \Omega, \\ u_1=0 \text{ on } \partial\Omega. \end{cases} \quad (5.49)$$

Replacing (5.48) yields (5.46). The uniqueness of the solution u_1 of problem (5.46) is straightforward. $\qquad\square$

Remark 5.8. It is easy to check from the proof that for this case, we only need assumption $(\mathcal{H}_2')$ defined in Sect 3, instead of $(\mathcal{H}_2)$, to obtain the homogenization result.

5.3. The case $\gamma \in \,]-1,1]$

Let us turn to the last case, where we state the results for $\gamma \in \,]-1,1[$ and $\gamma = 1$ separately.

Theorem 5.9. *Assume that $\gamma \in \,]-1,1[$ and that (2.4)–$(2.6)_1$ hold. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.2), then there exist $u_1 \in H_0^1\left(\Omega\right)$ and $\widehat{u}_1 \in L^2\left(\Omega, H_{per}^1\left(Y_1\right)\right)$ such that*

$$\begin{cases} \mathcal{T}_1^\varepsilon\left(u_1^\varepsilon\right)\longrightarrow u_1 & \text{strongly in } L^2\left(\Omega, H^1\left(Y_1\right)\right), \\ \mathcal{T}_2^\varepsilon\left(u_2^\varepsilon\right)\longrightarrow u_1 & \text{strongly in } L^2\left(\Omega, H^1\left(Y_2\right)\right), \\ \mathcal{T}_1^\varepsilon\left(\nabla u_1^\varepsilon\right)\rightharpoonup \nabla u_1+\nabla_y\widehat{u}_1 & \text{weakly in } L^2\left(\Omega\times Y_1\right), \\ \mathcal{T}_2^\varepsilon\left(\nabla u_2^\varepsilon\right)\rightharpoonup 0 & \text{weakly in } L^2\left(\Omega\times Y_2\right), \end{cases} \quad (5.50)$$

and the pair $(u_1,\widehat{u}_1)$ is the unique solution of the problem

$$\begin{cases} \text{Find } u_1 \in H_0^1\left(\Omega\right), \widehat{u}_1 \in L^2\left(\Omega, W_{per}\left(Y_1\right)\right) \text{ such that,} \\ \dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y_1} A\left(y\right)\left(\nabla u_1+\nabla_y\widehat{u}_1\right)\left(\nabla\varphi+\nabla_y\Phi_1\right)\,dx\,dy \\ \qquad +\dfrac{1}{|Y|}\displaystyle\sum_{i=1,2}\int_{\Omega\times Y_i} h_i\left(y,u_1\right)\varphi\,dx\,dy \\ =\displaystyle\int_\Omega f\left(x\right)\varphi\left(x\right)\,dx, \quad \forall\varphi\in H_0^1\left(\Omega\right),\ \forall\Phi_1\in L^2\left(\Omega, H_{per}^1\left(Y_1\right)\right). \end{cases} \quad (5.51)$$

Proof. Theorems 4.3 and 4.5 imply convergences $(5.50)_{1-3}$ and

$$\mathcal{T}_2^\varepsilon\left(\nabla u_2^\varepsilon\right)\rightharpoonup \nabla_y\overline{u}_2 \quad \text{weakly in } L^2\left(\Omega\times Y_2\right), \quad (5.52)$$

for some $\overline{u}_2 \in L^2\left(\Omega, H^1\left(Y_2\right)\right)$, up to a subsequence.

We now choose as test functions in (2.10) the functions $v_i\left(x\right)=\varphi\left(x\right)+\varepsilon\omega_i\left(x\right)\psi_i^\varepsilon\left(x\right)$, $i=1,2$, where φ, $\omega_i\in\mathcal{D}\left(\Omega\right)$ and $\psi_i^\varepsilon\left(x\right)=\psi_i\left(\dfrac{x}{\varepsilon}\right)$, with $\psi_1\in H_{per}^1\left(Y_1\right)$ and $\psi_2\in H^1\left(Y_2\right)$ (extended by Y-periodicity to $\mathbb{R}^n$). Arguing as in the proof of Theorem 5.1 gives convergences (5.10)–(5.12).

Now we treat the nonlinear jump on the interface. Taking into account Remark 4.8, Proposition 4.2(ix) and the fact that the function h satisfies assumption $(\mathcal{H}_2)$, we have

$$\varepsilon^{\gamma+1} \int_{\Gamma^\varepsilon} h^\varepsilon \left(x, u_1^\varepsilon - u_2^\varepsilon\right) (v_1 - v_2) \ d\sigma_x$$

$$= \frac{\varepsilon^{\gamma+1}}{|Y|} \int_{\Omega \times \Gamma} h\left(y, \frac{\mathcal{T}_b^\varepsilon \left(u_1^\varepsilon - u_2^\varepsilon\right)}{\varepsilon}\right) \left(\psi_1(y) \mathcal{T}_1^\varepsilon(\omega_1) - \psi_2(y) \mathcal{T}_2^\varepsilon(\omega_2)\right) \ dx \ d\sigma_y$$

$$\leq c \, \varepsilon^{\gamma+1} \left\| h\left(y, \frac{\mathcal{T}_b^\varepsilon \left(u_1^\varepsilon - u_2^\varepsilon\right)}{\varepsilon}\right) \right\|_{L^{\frac{2}{q}}(\Omega \times \Gamma)} \left\| \psi_1(y) \mathcal{T}_1^\varepsilon(\varphi_1) \right.$$

$$\left. - \psi_2(y) \mathcal{T}_2^\varepsilon(\varphi_2) \right\|_{L^{\left(\frac{2}{q}\right)'}(\Omega \times \Gamma)}$$

$$\leq c \, \varepsilon^{\gamma+1} \left(1 + \varepsilon^{-q} \| \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)\|_{L^2(\Omega \times \Gamma)}^q\right)$$

$$\leq c \, \varepsilon^{\gamma+1} \left(1 + \varepsilon^{-q/2} \| u_1^\varepsilon - u_2^\varepsilon \|_{L^2(\Gamma^\varepsilon)}^q\right)$$

$$\leq c \, \varepsilon^{\gamma+1} \left(1 + \varepsilon^{-(1+\gamma)q/2}\right) \leq c \left(\varepsilon^{\gamma+1} + \varepsilon^{\frac{1}{2}(\gamma+1)(2-q)}\right) \longrightarrow 0, \qquad (5.53)$$

since $\gamma > -1$ and $q < 2$.

From (5.10) to (5.12) and (5.53), passing to the limit as $\varepsilon \to 0$ in the variational formulation (2.10) for (v_1, v_2) given as above, we get

$$\begin{cases} \forall \varphi \in H_0^1(\Omega), \ \forall (\Phi_1, \Phi_2) \in L^2\left(\Omega, H_{per}^1(Y_1)\right) \times L^2\left(\Omega, H^1(Y_2)\right), \\[2mm] \dfrac{1}{|Y|} \displaystyle\int_{\Omega \times Y_1} A(y) \left(\nabla u_1 + \nabla_y \widehat{u}_1\right) \left(\nabla \varphi + \nabla_y \Phi_1\right) \ dx \ dy \\[4mm] \quad + \dfrac{1}{|Y|} \displaystyle\sum_{i=1,2} \int_{\Omega \times Y_i} h_i(y, u_1) \varphi \, dx \, dy \\[4mm] \quad + \dfrac{1}{|Y|} \displaystyle\int_{\Omega \times Y_2} A(y) \nabla_y \overline{u}_2 \left(\nabla \varphi + \nabla_y \Phi_2\right) \ dx \ dy = \int_\Omega f(x) \varphi(x) \ dx. \end{cases}$$

As in the proof of Theorem 3.3 in [26], choosing $\varphi = \Phi_1 \equiv 0$ and $\Phi_2 \equiv 0$ gives that $\nabla_y \overline{u}_2 = 0$ in $\Omega \times Y_2$. Hence, convergence $(5.50)_4$ follows from (5.52). Finally, the uniqueness of $(u_1, \widehat{u}_1)$ which implies the convergences of the whole sequence in (5.50) comes from the next corollary. $\qquad \square$

By similar arguments to those used in Theorem 5.7, we can prove the following:

Corollary 5.10. *For $\gamma \in \,]\!-\!1, 1[$, assume that the hypotheses of Theorem 5.9 are satisfied and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.2). Then there exists $u_1 \in H_0^1(\Omega)$ such that*

$$\begin{cases} \widetilde{u}_i^\varepsilon \rightharpoonup \theta_i u_1 & \text{weakly in } L^2(\Omega), i = 1, 2, \\[1mm] A^\varepsilon \widetilde{\nabla u_1^\varepsilon} \rightharpoonup A_\gamma^0 \nabla u_1 & \text{weakly in } \left(L^2(\Omega)\right)^n, \\[1mm] A^\varepsilon \widetilde{\nabla u_2^\varepsilon} \rightharpoonup 0 & \text{weakly in } \left(L^2(\Omega)\right)^n, \end{cases} \qquad (5.54)$$

and u_1 is the unique solution of problem (5.46), where the homogenized matrix A_γ^0 is the constant matrix defined by

$$A_\gamma^0 e_j = \theta_1 \mathcal{M}_{Y_1} (A e_j - A \nabla \chi_j). \tag{5.55}$$

The correctors χ_j, $j = 1, \ldots, n$, are the unique solutions of the cell problems

$$\begin{cases} -div(A(y) \nabla(\chi_j - y_j)) = 0 & in \ Y_1, \\ A(y) \nabla(\chi_j - y_j).n_1 = 0 & on \ \Gamma, \\ \chi_j \ Y\text{-periodic}, \ \mathcal{M}_\Gamma(\chi_j) = 0. \end{cases} \tag{5.56}$$

Let us now state the homogenization result for the case $\gamma = 1$, where we make use of the same test functions and similar arguments in the proof of Theorem 5.9.

Theorem 5.11. *Let $\gamma = 1$ and the assumptions (2.4)–(2.6)$_3$ hold. If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the solution of problem (2.2), then there exist $u_1 \in H_0^1(\Omega)$ and $\widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1))$ such that convergences (5.50)$_{1,3,4}$ hold and $(u_1, \widehat{u}_1)$ uniquely satisfies*

$$\begin{cases} u_1 \in H_0^1(\Omega), \ \widehat{u}_1 \in L^2(\Omega, H_{per}^1(Y_1)) \ \ with \ \mathcal{M}_\Gamma(\widehat{u}_1) = 0 \ a.e. \ x \in \Omega, \\ \dfrac{1}{|Y|} \int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \nabla_y \Phi_1) \ dx \ dy \\ \quad + \dfrac{1}{|Y|} \int_{\Omega \times Y_1} h_1(y, u_1) \varphi \ dx \ dy \\ \quad = \displaystyle\int_\Omega f(x) \varphi(x) \ dx, \ \forall \varphi \in H_0^1(\Omega), \ \forall \Phi_1 \in L^2(\Omega, H_{per}^1(Y_1)). \end{cases}$$

Moreover, the homogenized problem is the following:

$$\begin{cases} -div\left(A_\gamma^0 \nabla u_1\right) + \theta_1 \mathcal{M}_{Y_1}(h_1(\cdot, u_1)) = f & in \ \Omega, \\ u_1 = 0 \ on \ \partial\Omega, \end{cases} \tag{5.57}$$

where A_γ^0 is defined by (5.55).

Remark 5.12. When $\gamma = 1$, Theorem 4.3 only provides the weak convergence of $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$ to the function u_2 in $L^2(\Omega; H^1(Y_2))$, so that we are not able to characterize u_2, except when h is linear as in [26]. In that case, this is done by choosing $v_1 = 0$ and $v_2 \in \mathcal{D}(\Omega)$ as test functions in the variational formulation and passing to the limit as $\varepsilon \to 0$. To do that, the weak convergence above is sufficient since $\varepsilon h(y, \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)/\varepsilon)$ is of the form $\widetilde{h}(y) \mathcal{T}_b^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)$. This is not the case anymore when h is not linear then here we cannot prove the uniqueness of u_2. Consequently, the convergence of $\mathcal{T}_2^\varepsilon(u_2^\varepsilon)$ in (4.2)$_1$ holds only for a subsequence.

Observe also that the assumption $h_2 \equiv 0$ is motivated by the fact that Theorem 4.5 does not apply for $\gamma = 1$, so that one cannot pass to the limit in the unfolded term concerning h_2. The same assumption has been used by

Timofte in [41] for the case where the domain Ω_2^ε is also connected. The case $h_2 \not\equiv 0$ remains an open problem.

Acknowledgements

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