Exponential stabilization of a viscoelastic wave equation with dynamic boundary conditions

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Abstract. In this paper, we study the stabilization of a semi-linear viscoelastic wave equation subject to semi-linear and dynamical boundary conditions. The kernels used are of strongly positive definite type. We prove that internal and boundary memory damping are strong enough, via transmission process $(u|_{\Gamma} = v)$, to stabilize the whole system.

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1. Introduction

Let us consider the following type of Wentzell system with dynamic boundary condition

$$\begin{cases} u'' + (1 - \Delta)u - g * (1 - \Delta)u + h_1(u) = f_1(t) & \text{in } \Omega \times \mathbb{R}_+, \\ u = v & \text{on } \Gamma \times \mathbb{R}_+, \\ v'' + \frac{\partial u}{\partial \nu} - g * \frac{\partial u}{\partial \nu} + (1 - \Delta_{\Gamma})v - g * (1 - \Delta_{\Gamma})v + h_2(v) = f_2(t) & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ (u(0), v(0)) = (u_0, v_0), (u'(0), v'(0)) = (u_1, v_1) & \text{in } \Omega \times \Gamma, \end{cases} \quad (1.1)$$

where Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) having a smooth boundary $\Gamma := \partial\Omega$. Let Γ_0 and Γ_1 be closed nonempty disjoint subsets of Γ with $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

We denote by $' = \frac{\partial}{\partial t}$, ∇_{Γ} the tangential-gradient on Γ , Δ_{Γ} the Laplacian operator on Γ and by $\frac{\partial u}{\partial \nu}$ the normal derivative where ν represents the unit outward normal to Γ .

We suppose that h_i are globally lipschitz, that is $h_i \in C^1(\mathbb{R})$, $h_i(0) = 0$, $i = 1, 2$ and there exist some constants $c_1, c_2 > 0$; $p > 1$; $(n - 2)p \leq n$ such

that

$$\begin{cases} \text{for every } s_1, s_2 \in \mathbb{R}, \\ |h_1(s_1) - h_1(s_2)| \leq c_1(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \\ |h_2(s_1) - h_2(s_2)| \leq c_2(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|. \end{cases} \quad (1.2)$$

We assume that

$$R(U) = \int_{\Omega} F_1(u(t))dx + \int_{\Gamma} F_2(v(t))d\Gamma \geq 0, \quad (1.3)$$

where

$$U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

and

$$F_1(u) = \int_0^u h_1(s)ds, \quad F_2(v) = \int_0^v h_2(s)ds.$$

This coupled system is a semilinear version (with memory terms) of the Wentzell's model. The model describes a vibrating of an elastic body with a thin body of high rigidity on its boundary. Recently, the above problem without terms of memory has been studied by Khemmoudj and Medjden [16]. In this paper the authors prove exponential stability by considering a localized linear feedback $a(x)u'_1$ acting in the first equation and $b(x)u'_2$ acting in the second equation, following the ideas firstly introduced by Zuazua [22] combined with new tools. More recently, the previous results, were extended by Cavalcanti et al. [8] by considering linear operators $\mathcal{A}$ and $\mathcal{A}_T$, depending on the spacial variable x , instead of $-\Delta$ and $-\Delta_{\Gamma}$, respectively, and in addition, supplemented by a nonlinear and localized dissipation $a(x)g_1(u')$ acting in the first equation and $b(x)g_2(v')$ acting in the second equation. This was done by combining new estimates of energy combined with Riemannian geometric methods due to Lasiecka et al. [17] see, for instance.

The polynomial stabilization of the following wave equation on the unit square of the plane with Wentzell boundary conditions has been showed in [18]

$$\begin{cases} u_{tt}(t) - \Delta u(t) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \nu} - \Delta_{\Gamma} u(t) + u_t(t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \end{cases} \quad (1.4)$$

where $\Omega = (0, 1)^2$ is the unit square with boundary Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, with a positive measure for Γ_0 and Γ_1 .

Many constitutive models for viscoelastic materials lead to equations of motion which have the form of a linear hyperbolic PDE perturbed by a dissipative integral term of Volterra type, having a nonnegative, decreasing convolution kernel, see, e.g., [10, 11]. We refer to [12] for more details concerning the physical phenomena which are modeled by differential equations with memory, as well as the problem of the modeling of materials with memory. In this direction, the existence and uniqueness of solutions is studied by Prüss [19].

In [4], the authors studied the stability for the following class of nonlinear integrodifferential equations

$$u_{tt}(t) + F(u(t), u_t(t)) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = f(t), \quad t \geq 0 \quad (1.5)$$

where A is a positive operator on a Hilbert space X , with dense domain $D(A)$ and F is a functional defined on $D(A^\theta)$, $0 < \theta \leq \frac{1}{2}$. Here, the kernel β is not assumed to be absolutely continuous but just integrable. Instead of higher regularity, the authors assumed that $k(t) := \int_0^\infty \beta(s)ds$ is a kernel of positive type satisfying $k(0) < 1$. They showed that the solutions decay exponentially at ∞ in the energy norm.

It is important to mention that in [3], Cannarsa and Sforza studied the following semilinear integro-differential equation

$$u_{tt}(t) + Au(t) - \int_0^t g(t-s)Au(s)ds = \nabla F(u(t)), \quad t \in (0, \infty), \quad (1.6)$$

where A is a positive operator on a Hilbert space X , with dense domain $D(A)$, ∇F denotes the gradient of a Gateaux differentiable functional $F : D(A^{1/2}) \rightarrow \mathbb{R}$ and g are some kernels that decay exponentially at infinity and possess strongly positive definite primitives. The exponential stability of weak solutions is obtained in the energy norm.

Alabau-Boussouira et al. [2] proved the exponential and polynomial decay of the energy for the problem (1.6) under minimal assumptions on g . They assumed that $g : [0, \infty) \rightarrow [0, \infty)$ is a locally absolutely continuous function satisfying, for some $p \in (2, \infty]$,

$$g(0) > 0, \quad \int_0^\infty g(t)dt < 1, \quad g' \leq -kg^{1+\frac{1}{p}}.$$

The decay results at infinity of the energy of any mild solution of (1.6), with sufficiently small initial data, are obtained with the same (exponential or polynomial) rate as g .

Cavalcanti and Oquendo [9] considered

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div} a(x)g(t-s)\nabla u(s)ds + b(x)h(u_t) + f(u) = 0,$$

in $\Omega \times \mathbb{R}_+$, where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) and $b : \Omega \rightarrow \mathbb{R}_+$, a function, which may be null on a part of the domain Ω . Under the conditions that $b(x) \geq b_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometric restrictions, $a(x) + b(x) \geq \rho > 0$ for all $x \in \Omega$ and g satisfying

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0.$$

They improved the result of [7] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. The problems, where the damping induced by the integral terms cooperating with another one acting on a part of the boundary or with a strong damping, were also discussed by Cavalcanti et al. [5, 6].

Finally, Aassila et al. [1] studied the wave equation subject to viscoelastic effects on the boundary and nonlinear boundary feedback.

In our work, we assume that the kernel g verifies the same assumptions that those of [3]. A natural question arises in the context of the viscoelastic Wentzell problem with dynamic boundary conditions: Is it possible to stabilize the system by considering some memory damping acting in the first and second equation? The main goal of this paper is exactly to give an answer to this question, namely: to prove that a memory damping is strong enough, via transmission process ($u|_{\Gamma} = v$), to assure the asymptotic stability of the whole system. This is a hard problem since we have dynamic boundary conditions, instead of static one. In other words: we have additional terms in the energy which come from a dynamic boundary so that we have to deal with these terms when we estimate the integral inequalities of energy. The strategy used is basically to make use of multipliers as in Cannarsa and Sforza [3] with new ingredients that will be clarified below.

Our paper is organized as follows: In Sect. 2 we give the notations used in our work, some definitions and preliminaries. Section 3 is devoted to the well-posedness and some energy estimates. In Sect. 4 we prove the exponential decay of the energy associated to the solution of the problem (1.1).

2. Definitions and preliminaries

2.1. Notations

In what follows, we prepare some notations and hypotheses which will be needed in our paper.

Let X be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. For $T \in (0, +\infty]$, we define the spaces $L^p(0, T; X)$, $1 \leq p < \infty$, as follow

$$L^p(0, T; X) = \left\{ v : (0, T) \longrightarrow X \text{ measurable, } \|v\|_{p,T}^p := \int_0^T \|v(t)\|^p dt < \infty \right\},$$

and

$$L^\infty(0, T; X) = \left\{ v : (0, T) \longrightarrow X \text{ measurable, } \|v\|_{\infty,T} := \operatorname{ess\,sup}_{0 \leq t \leq T} \|v(t)\| < \infty \right\},$$

For $1 \leq p \leq \infty$, we will use the abbreviation $\|v\|_p$ to denote $\|v\|_{p,\infty}$. We denote by $L_{loc}^p(0, \infty; X)$ the space of functions belonging to $L^p(0, T; X)$ for any $T \in (0, \infty)$. If $X = \mathbb{R}$, we will use the notation $L^p(0, T)$ to denote the space $L^p(0, T; \mathbb{R})$ and $L_{loc}^p(0, \infty)$ for $L_{loc}^p(0, \infty; \mathbb{R})$. We denote by $C^k([0, T]; X)$, $k = 0, 1, 2$, the space of continuous functions $v : [0, T] \longrightarrow X$, such that all their derivatives of order $\leq k$ are continuous functions on $[0, T]$. We will write $C([0, T]; X)$ for $C^0([0, T]; X)$.

We set

$$\begin{aligned} H_{\Gamma_0}^1(\Omega) &= \{u \in H^1(\Omega), \ u|_{\Gamma_0} = 0\}, \\ \mathbb{V} &= \{z = (u, v) \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega) \times H^2(\Gamma), \ u|_{\Gamma} = v\}, \\ \tilde{\mathbb{V}} &= \{z = (u, v) \in H_{\Gamma_0}^1(\Omega) \times H^1(\Gamma), \ u|_{\Gamma} = v\}, \\ \mathbb{H} &= L^2(\Omega) \times L^2(\Gamma), \end{aligned}$$

$\tilde{\mathbb{V}}$ and $\mathbb{H}$ are two Hilbert spaces and $\tilde{\mathbb{V}}$ is dense in $\mathbb{H}$ with continuous injection. $\tilde{\mathbb{V}}$ and $\mathbb{H}$ are equipped with inner products

$$\begin{aligned} \langle z_1, z_2 \rangle_{\mathbb{H}} &= \langle u_1, u_2 \rangle_{L^2(\Omega)} + \langle v_1, v_2 \rangle_{L^2(\Gamma)}, \\ \langle z_1, z_2 \rangle_{\tilde{\mathbb{V}}} &= \langle u_1, u_2 \rangle_{H^1(\Omega)} + \langle v_1, v_2 \rangle_{H^1(\Gamma)}, \end{aligned}$$

and canonical norms

$$\begin{aligned} \|z\|_{\mathbb{H}}^2 &= \|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Gamma)}^2, \\ \|z\|_{\tilde{\mathbb{V}}}^2 &= \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\Gamma} v\|_{L^2(\Gamma)}^2. \end{aligned}$$

For any $g \in L_{loc}^1(0, \infty)$ and any $y \in L_{loc}^1(0, \infty; \mathbb{H})$ we define

$$g * y(t) = (g * y_1(t), g * y_2(t)),$$

where

$$g * y_i(t) = \int_0^t g(t-s)y_i(s)ds, \quad i = 1, 2, \quad t \geq 0.$$

2.2. Basic properties of kernels

In this section we briefly summarize some properties of positive definite and strongly positive definite kernels which will play an essential role in our computations. The results present here are introduced in Cannarsa et al. [3], see also Prüss [19] and are repeated just for the reader's convenience.

A function $g \in L_{loc}^1(0, \infty)$ is said to be a positive definite kernel if

$$\int_0^t \langle g * y(s), y(s) \rangle_{\mathbb{H}} ds \geq 0, \quad t \geq 0, \quad (2.1)$$

for any $y \in L_{loc}^2(0, \infty; \mathbb{H})$.

A function g is called a strongly positive definite kernel if there exists a constant $\delta > 0$ such that $g(t) - \delta e^{-t}$ is positive definite, namely

$$\int_0^t \langle g * y(s), y(s) \rangle_{\mathbb{H}} ds \geq \delta \int_0^t \langle e * y(s), y(s) \rangle_{\mathbb{H}} ds, \quad t \geq 0, \quad (2.2)$$

for any $y \in L_{loc}^2(0, \infty; \mathbb{H})$, where $e(t) = e^{-t}$.

In what follows, we assume that g is a function such that $g \in L^1(0, \infty)$. Then we define a function G by

$$G(t) := \int_t^\infty g(s)ds.$$

The following Proposition which was studied and proved in [3], is useful to study exponential decay.

Proposition 2.1. *Let $g \in L^1(0, \infty)$ be a function such that $t \rightarrow \int_t^\infty g(s)ds$ is strongly positive definite. Then $t \rightarrow \int_t^\infty e^{-\sigma s} g(s)ds$, $\sigma > 0$, is strongly positive definite as well.*

In addition, if δ is the constant in (2.2) corresponding to $t \rightarrow \int_t^\infty g(s)ds$, then the analogous constant δ_σ for $t \rightarrow \int_t^\infty e^{-\sigma s} g(s)ds$ is given by

$$\delta_\sigma = \frac{\delta}{(\sigma + 1)^2}. \quad (2.3)$$

Now, adapting some estimates for positive definite kernels (see [14, 15, 20, 21]), we obtain the following three Lemmas.

Lemma 2.2. *Let $g \in C([0, \infty))$ be a positive definite kernel. Then $g(0) \geq 0$ and*

$$\|g * y(t)\|_{\mathbb{H}}^2 \leq 2g(0) \int_0^t \langle g * y(\tau), y(\tau) \rangle_{\mathbb{H}} d\tau, \quad t \geq 0, \quad (2.4)$$

for any $y \in L^1_{loc}(0, \infty; \mathbb{H})$.

Proof. It's analogous to the proof of Lemma 6.1 in [20]. $\square$

Note that if g is a strongly positive definite kernel and δ is the constant in (2.2), then by Lemma 2.2 we have $g(0) - \delta \geq 0$, that is

$$g(0) \geq \delta. \quad (2.5)$$

Lemma 2.3. *Let g be a strongly positive definite kernel satisfying $g, g' \in L^1(0, \infty)$. Then*

$$\int_0^t \|g * y(\tau)\|_{\mathbb{H}}^2 d\tau \leq \frac{1}{\delta} (\|g\|_1^2 + 4\|g'\|_1^2) \int_0^t \langle g * y(\tau), y(\tau) \rangle_{\mathbb{H}} d\tau, \quad t \geq 0, \quad (2.6)$$

for any $y \in L^1_{loc}(0, \infty; \mathbb{H})$, where δ is the constant in (2.2).

Proof. The proof is similar to that of Lemma 2.3 (see [15]). We omit the details here. $\square$

Lemma 2.4. *Let $g \in L^1_{loc}(0, \infty)$ be a strongly positive definite kernel. Then*

$$\begin{aligned} \int_0^t \|y(\tau)\|_{\mathbb{H}}^2 d\tau &\leq \|y(0)\|_{\mathbb{H}}^2 \\ &+ \frac{2}{\delta} \left\{ \int_0^t \langle g * y(\tau), y(\tau) \rangle_{\mathbb{H}} d\tau + \int_0^t \langle g * y'(\tau), y'(\tau) \rangle_{\mathbb{H}} d\tau \right\}, \end{aligned} \quad (2.7)$$

for any $t \geq 0$ and $y \in L^2_{loc}(0, \infty; \mathbb{H})$ with $y' \in L^1_{loc}(0, \infty; \mathbb{H})$, where δ is the constant in (2.2).

Proof. It follows the lines in the proof of Lemma 2.5 (see [14]) with some changes and necessary computations. $\square$

Classical results for integral equations (see, e.g., [13], Theorem 2.3.5) ensure that, for any kernel $g \in L^1_{loc}(0, \infty)$ and any $\psi \in L^1_{loc}(0, \infty; X)$, the problem

$$\varphi(t) - g * \varphi(t) = \psi(t), \quad t \geq 0, \quad (2.8)$$

admits a unique solution $\varphi \in L^1_{loc}(0, \infty; X)$. In particular, there is a unique solution $r \in L^1_{loc}(0, \infty)$ of

$$r(t) - g * r(t) = g(t), \quad t \geq 0. \quad (2.9)$$

Such a solution is called the resolvent kernel of g . Furthermore, the solution φ of (2.8) is given by the variation of constants formula

$$\varphi(t) = \psi(t) + r * \psi(t), \quad t \geq 0, \quad (2.10)$$

where r is the resolvent kernel of g .

The following corollary (see, e.g., [3] Corollary 2.12) provides uniform estimates for solutions of integral equations.

Corollary 2.5. *Let $g \in L^1(0, \infty)$ be such that $\int_0^\infty g(s)ds < 1$ and suppose that $G(t) = \int_t^\infty g(s)ds$ is a strongly positive definite kernel. Then,*

- (a) *the resolvent kernel r of g belongs to $L^1(0, \infty)$;*
- (b) *for any $\psi \in L^p(0, \infty; X)$, $1 \leq p \leq \infty$, the solution φ of equation*

$$\varphi(t) - g * \varphi(t) = \psi(t), \quad t \geq 0,$$

belongs to $L^p(0, \infty; X)$ and

$$\|\varphi\|_p \leq (1 + \|r\|_1)\|\psi\|_p. \quad (2.11)$$

To simplify the notations, the problem (1.1) can be formulated as

$$U''(t) + \mathbb{A}U(t) - \int_0^t g(t-s)\mathbb{A}U(s)ds + H(U(t)) = f(t), \quad t \geq 0, \quad (2.12)$$

with initial conditions

$$U(0) = U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad U'(0) = U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad (2.13)$$

where

$$\mathbb{A} = \begin{pmatrix} 1 - \Delta & 0 \\ \frac{\partial}{\partial \nu} & 1 - \Delta_\Gamma \end{pmatrix}, \quad (2.14)$$

and

$$U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad H(U(t)) = \begin{pmatrix} h_1(u(t)) \\ h_2(v(t)) \end{pmatrix}, \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

Later, we will denote $\|f_1\|_1 + \|f_2\|_1$ by $\|f\|_1$.

$\mathbb{A}$ is a self-adjoint linear operator on $\mathbb{H}$ with dense domain

$$D(\mathbb{A}) = \{(u, v) \in \mathbb{V}, \mathbb{A}(u, v) \in \mathbb{H}\}.$$

For the reader's convenience (see Prüss [19], Cannarsa and Sforza [3] for the case of the abstract equation), we recall the notion of resolvent for the equation

$$U''(t) + \mathbb{A}U(t) - \int_0^t g(t-s)\mathbb{A}U(s)ds = 0, \quad (2.15)$$

where $\mathbb{A}$ is the self-adjoint linear operator on $\mathbb{H}$, defined by (2.14) and $g \in L^1_{loc}(0, \infty)$.

A family $\{S(t)\}_{t \geq 0}$ of bounded linear operators in $\mathbb{H}$ is called a resolvent for equation (2.15) if the following conditions are satisfied:

(S1) $S(0) = I$ and $S(t)$ is strongly continuous on $[0, \infty)$, that is, for all $x \in \mathbb{H}$, $S(\cdot)x$ is continuous,

(S2) $S(t)$ commutes with $\mathbb{A}$, which means that $S(t)D(\mathbb{A}) \subset D(\mathbb{A})$ and

$$\mathbb{A}S(t)x = S(t)\mathbb{A}x, \quad x \in D(\mathbb{A}), \quad t \geq 0,$$

(S3) for any $x \in D(\mathbb{A})$, $S(\cdot)x$ is twice continuously differentiable in $\mathbb{H}$ on $[0, \infty)$ and $S'(0)x = 0$,

(S4) for any $x \in D(\mathbb{A})$ and any $t \geq 0$,

$$S''(t)x + \mathbb{A}S(t)x - \int_0^t g(t-\tau)\mathbb{A}S(\tau)x d\tau = 0.$$

3. Well-posedness and energy estimates

3.1. Local existence of mild and strong solutions

Our assumptions on the kernel g are the following:

Assumption (H1). Assume that $g \in L^1(0, \infty)$ such that

$$\int_0^\infty g(t)dt < 1, \quad (3.1)$$

and

$$t \mapsto \int_t^\infty g(s)ds \quad \text{is positive definite.}$$

Let $0 < T \leq \infty$ be given and $f \in L^1(0, T; \mathbb{H})$. To begin with, we recall some notions of solutions.

A function U is called a strong solution of (2.12) on $[0, T]$, $T > 0$, if

$$U \in C^2([0, T]; \mathbb{H}) \cap C([0, T]; D(\mathbb{A})),$$

and satisfies (2.12) for $t \in [0, T]$.

Let $U_0 \in \tilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$. A function $U \in C^1([0, T]; \mathbb{H}) \cap C([0, T]; \tilde{\mathbb{V}})$ is called a mild solution of (2.12) on $[0, T]$ with initial conditions (2.13) if

$$U(t) = S(t)U_0 + \int_0^t S(\tau)U_1 d\tau + \int_0^t 1 * S(t-\tau)(-H(U(\tau)) + f(\tau))d\tau, \quad (3.2)$$

where $S(t)$ is the resolvent for (2.15).

Notice that a strong solution is also a mild one.

A generalized solution (or a weak solution) of (2.12) is a function $U \in C^1([0, T]; \mathbb{H}) \cap C([0, T]; \tilde{\mathbb{V}})$ such that, for any $W \in \tilde{\mathbb{V}}$, $\langle U'(t), W(t) \rangle_{\mathbb{H}} \in C^1([0, T])$ and for any $t \in [0, T]$ one has

$$\begin{aligned}
& \frac{d}{dt} \langle U'(t), W(t) \rangle_{\mathbb{H}} + \langle \text{Grad } U(t), \text{Grad } W(t) \rangle_{\mathbb{H}} + \langle U(t), W(t) \rangle_{\mathbb{H}} \\
& - \int_0^t g(t-s) \{ \langle \text{Grad } U(s), \text{Grad } W(s) \rangle_{\mathbb{H}} \\
& + \langle U(s), W(s) \rangle_{\mathbb{H}} \} ds + \langle H(U(t)), W(t) \rangle_{\mathbb{H}} \\
& = \langle f(t), W(t) \rangle_{\mathbb{H}}.
\end{aligned} \tag{3.3}$$

were $\text{Grad } U(t) = (\nabla u(t), \nabla_{\Gamma} v(t))$.

3.1.1. Local well-posedness of mild solution. The next proposition ensures the local existence and uniqueness of mild solution to (2.12)–(2.13). For the proof, see ([4], Proposition 4.3 and Theorem 4.5).

Proposition 3.1. *Let $0 < T \leq \infty$ be given. Let $U_0 \in \widetilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$ and $f \in L^1(0, T; \mathbb{H})$. Then, Under assumption (H1), (1.2)–(1.3), there exists a positive number $T_0 \leq T$ such that the Cauchy problem (2.12)–(2.13) admits a unique mild solution U on $[0, T_0]$.*

In addition, for any $U_0, \widetilde{U}_0 \in \widetilde{\mathbb{V}}$, $U_1, \widetilde{U}_1 \in \mathbb{H}$ and $f, \widetilde{f} \in L^1(0, T; \mathbb{H})$, there exists a constant $C_T > 0$ (depending on T) such that, called U and $\widetilde{U}$ the mild solutions with data U_0, U_1, f and $\widetilde{U}_0, \widetilde{U}_1, \widetilde{f}$ respectively, we have

$$\begin{aligned}
& \| \text{Grad } (U(t) - \widetilde{U}(t)) \|_{\mathbb{H}} + \| U(t) - \widetilde{U}(t) \|_{\mathbb{H}} + \| U'(t) - \widetilde{U}'(t) \|_{\mathbb{H}} \\
& \leq C_T (\| \text{Grad } (U_0 - \widetilde{U}_0) \|_{\mathbb{H}} + \| U_0 - \widetilde{U}_0 \|_{\mathbb{H}} + \| U_1 - \widetilde{U}_1 \|_{\mathbb{H}} + \| f - \widetilde{f} \|_{1,T}),
\end{aligned} \tag{3.4}$$

for any $t \in [0, \min\{T_0, \widetilde{T}_0\}]$.

3.1.2. Local well-posedness of strong solution. Assuming more regular data and using standard argumentations, one can show that the mild solution is a strong one.

Proposition 3.2. *Let $U_0 \in D(\mathbb{A})$, $U_1 \in \widetilde{\mathbb{V}}$ and $f \in W^{1,1}(0, T; \mathbb{H})$. Then, the mild solution (given by Proposition 3.1) of the Cauchy problem (2.12)–(2.13) on $[0, T_0]$, $T_0 \in (0, T]$, is a strong solution.*

In addition, U belongs to $C^1([0, T_0]; \widetilde{\mathbb{V}})$.

Our global existence result is now stated as follows.

3.2. Global well-posedness of mild and strong solutions

In this section, we will investigate the existence in the large of the solution of (2.12)–(2.13).

Let us define the energy of a solution U of (2.12)–(2.13) by

$$E(t) = \frac{1}{2} \| U'(t) \|_{\mathbb{H}}^2 + \frac{1}{2} \left(1 - \int_0^\infty g(s) ds \right) (\| \text{Grad } U(t) \|_{\mathbb{H}}^2 + \| U(t) \|_{\mathbb{H}}^2) + R(U(t)), \tag{3.5}$$

where $R(U(t))$ is given by (1.3).

Now, we suppose that assumption (H1) is satisfied and set

$$G(t) := \int_t^\infty g(s) ds, \quad t \geq 0.$$

The following Theorems ensure the global existence and uniqueness of mild and strong solutions U of (2.12)–(2.13).

Theorem 3.3. *Under assumptions (H1), (1.2) and (1.3), for any $U_0 \in \tilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$, and any $f \in L^1(0, \infty; \mathbb{H})$ the problem (2.12)–(2.13) possesses a unique mild solution U on $[0, \infty)$.*

Moreover, $E(t)$ is positive and satisfies

$$E(t) \geq \frac{1}{2} \|U'(t)\|_{\mathbb{H}}^2 + \frac{1 - G(0)}{2} \{ \|U(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U(t)\|_{\mathbb{H}}^2 \}, \quad (3.6)$$

$$E(t) \leq \tilde{C}_1 \quad (3.7)$$

for any $t \geq 0$, where $\tilde{C}_1 = \tilde{C}_1(\|U_1\|, \|\text{Grad } U_0\|, \|f\|_1)$, is positive, increasing function in each variable.

Theorem 3.4. *Under assumptions (H1), (1.2) and (1.3), for any $U_0 \in D(\mathbb{A})$, $U_1 \in \tilde{\mathbb{V}}$ and $f \in W_{loc}^{1,1}(0, \infty; \mathbb{H}) \cap L^1(0, \infty; \mathbb{H})$ the problem (2.12)–(2.13) possesses a unique strong solution on $[0, \infty)$. In addition, U belongs to*

$$C^1([0, \infty[; \tilde{\mathbb{V}}),$$

and for any $t \geq 0$

$$E(t) + \int_0^t \{ \langle G * \text{Grad } U'(s), \text{Grad } U'(s) \rangle_{\mathbb{H}} + \langle G * U'(s), U'(s) \rangle_{\mathbb{H}} \} ds \leq \tilde{C}_2, \quad (3.8)$$

where $\tilde{C}_2 = \tilde{C}_2(\|U_1\|, \|\text{Grad } U_0\|, \|f\|_1)$ is a positive function, increasing in each variable.

The following lemma is useful for the derivation of some priori estimates.

Lemma 3.5. *For any $w \in C^1([0, T]; \mathbb{H})$, $T > 0$, the following holds true for any $t \in [0, T]$*

$$\begin{aligned} \int_0^t \langle g * w(s), w'(s) \rangle_{\mathbb{H}} ds &= - \int_0^t \langle G * w'(s), w'(s) \rangle_{\mathbb{H}} ds \\ &\quad + \frac{G(0)}{2} (\|w(t)\|_{\mathbb{H}}^2 + \|w(0)\|_{\mathbb{H}}^2) \\ &\quad - G(t) \langle w(0), w(t) \rangle_{\mathbb{H}} - \int_0^t g(s) \langle w(0), w(s) \rangle_{\mathbb{H}} ds, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \langle w(t), w(t) - g * w(t) \rangle_{\mathbb{H}} &= + G(t) \langle w(0), w(t) \rangle_{\mathbb{H}} + (1 - G(0)) \|w(t)\|_{\mathbb{H}}^2 \\ &\quad + \langle w(t), G * w'(t) \rangle_{\mathbb{H}}. \end{aligned} \quad (3.10)$$

In addition, if $G \in L^2(0, \infty)$, then we have for any $t \in [0, T]$

$$\begin{aligned} \int_0^t \|w(s) - g * w(s)\|_{\mathbb{H}}^2 ds &\leq 2(1 - G(0)) \int_0^t \langle w(s), w(s) - g * w(s) \rangle_{\mathbb{H}} ds \\ &\quad + 2 \|G\|_2^2 \|w(0)\|_{\mathbb{H}}^2 \\ &\quad + 2 \int_0^t \|G * w'(s)\|_{\mathbb{H}}^2 ds. \end{aligned} \quad (3.11)$$

The proof is similar to that of Lemma 3.4 in [3], with some changes and necessary computations.

Lemma 3.6. (i) If $U_0 \in D(\mathbb{A})$, $U_1 \in \widetilde{\mathbb{V}}$ and $f \in W^{1,1}(0, T; \mathbb{H})$, then the strong solution U of problem (2.12)–(2.13) on $[0, T_0]$, $T_0 \leq T$, satisfies

$$\begin{aligned} E(t) &+ \int_0^t \{ \langle G * \text{Grad } U'(s), \text{Grad } U'(s) \rangle_{\mathbb{H}} + \langle G * U'(s), U'(s) \rangle_{\mathbb{H}} \} ds \\ &= E(0) + G(0) \{ \| \text{Grad } U_0 \|_{\mathbb{H}}^2 + \| U_0 \|_{\mathbb{H}}^2 \} \\ &\quad - \{ \langle \text{Grad } U_0, G(t) \text{Grad } U(t) \rangle_{\mathbb{H}} + \langle U_0, G(t)U(t) \rangle_{\mathbb{H}} \} \\ &\quad - \int_0^t g(s) \{ \langle \text{Grad } U_0, \text{Grad } U(s) \rangle_{\mathbb{H}} + \langle U_0, U(s) \rangle_{\mathbb{H}} \} ds \\ &\quad + \int_0^t \langle f(s), U'(s) \rangle_{\mathbb{H}} ds, \end{aligned} \quad (3.12)$$

for any $t \in [0, T_0]$.

(ii) If $U_0 \in \widetilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$ and $f \in L^1(0, T; \mathbb{H})$, then the mild solution U of problem (2.12)–(2.13) on $[0, T_0]$ satisfies

$$\begin{aligned} E(t) &\leq E(0) + G(0) \{ \| \text{Grad } U_0 \|_{\mathbb{H}}^2 + \| U_0 \|_{\mathbb{H}}^2 \} \\ &\quad - \{ \langle \text{Grad } U_0, G(t) \text{Grad } U(t) \rangle_{\mathbb{H}} + \langle U_0, G(t)U(t) \rangle_{\mathbb{H}} \} \\ &\quad - \int_0^t g(s) \{ \langle \text{Grad } U_0, \text{Grad } U(s) \rangle_{\mathbb{H}} + \langle U_0, U(s) \rangle_{\mathbb{H}} \} ds \\ &\quad + \int_0^t \langle f(s), U'(s) \rangle_{\mathbb{H}} ds, \end{aligned} \quad (3.13)$$

for any $t \in [0, T_0]$.

Proof. (i) Firstly, Let us suppose that $U_0 \in D(\mathbb{A})$, $U_1 \in \widetilde{\mathbb{V}}$, and $f \in W^{1,1}(0, T; \mathbb{H})$; in particular, the strong solution U belongs to $C^1([0, T_0]; \widetilde{\mathbb{V}})$, see Proposition 3.2.

Taking the inner product of (2.12) with $U'(t)$, we have, for $t \geq 0$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \{ \| U(t) \|_{\mathbb{H}}^2 + \| U'(t) \|_{\mathbb{H}}^2 + \| \text{Grad } U(t) \|_{\mathbb{H}}^2 \} \\ &\quad - \{ \langle g * \text{Grad } U(t), \text{Grad } U'(t) \rangle_{\mathbb{H}} + \langle g * U(t), U'(t) \rangle_{\mathbb{H}} \} + \frac{d}{dt} R(U(t)) \\ &= \langle f(t), U'(t) \rangle_{\mathbb{H}}. \end{aligned}$$

Integrating over $[0, t]$ gives

$$\begin{aligned} &\frac{1}{2} \{ \| U(t) \|_{\mathbb{H}}^2 + \| U'(t) \|_{\mathbb{H}}^2 + \| \text{Grad } U(t) \|_{\mathbb{H}}^2 \} \\ &\quad - \int_0^t \{ \langle g * \text{Grad } U(\tau), \text{Grad } U'(\tau) \rangle_{\mathbb{H}} + \langle g * U(\tau), U'(\tau) \rangle_{\mathbb{H}} \} d\tau + R(U(t)) \\ &= \frac{1}{2} \{ \| U_0 \|_{\mathbb{H}}^2 + \| U_1 \|_{\mathbb{H}}^2 + \| \text{Grad } U_0 \|_{\mathbb{H}}^2 \} + R(U_0) + \int_0^t \langle f(\tau), U'(\tau) \rangle_{\mathbb{H}} ds. \end{aligned}$$

From the above equality, using (3.9) with $w = U$ and then with $w = \text{Grad}U$, we obtain

$$\begin{aligned}
 & \frac{1}{2} \|U'(t)\|_{\mathbb{H}}^2 + \frac{1 - G(0)}{2} \{ \|\text{Grad } U(t)\|_{\mathbb{H}}^2 + \|U(t)\|_{\mathbb{H}}^2 \} \\
 & \quad + \int_0^t \{ \langle G * \text{Grad } U'(\tau), \text{Grad } U'(\tau) \rangle_{\mathbb{H}} + \langle G * U'(\tau), U'(\tau) \rangle_{\mathbb{H}} \} d\tau + R(U(t)) \\
 & = \frac{1}{2} \|U_1\|_{\mathbb{H}}^2 + \frac{1 + G(0)}{2} \{ \|\text{Grad } U_0\|_{\mathbb{H}}^2 + \|U_0\|_{\mathbb{H}}^2 \} \\
 & \quad - \{ \langle \text{Grad } U_0, G(t) \text{Grad } U(t) \rangle_{\mathbb{H}} + \langle U_0, G(t)U(t) \rangle_{\mathbb{H}} \} \\
 & \quad - \int_0^t g(\tau) \{ \langle \text{Grad } U_0, \text{Grad } U(\tau) \rangle_{\mathbb{H}} + \langle U_0, U(\tau) \rangle_{\mathbb{H}} \} d\tau \\
 & \quad + R(U_0) + \int_0^t \langle f(\tau), U'(\tau) \rangle_{\mathbb{H}} d\tau,
 \end{aligned}$$

whence (3.12) follows.

(ii) Since G is a positive definite kernel, we have

$$\int_0^t \langle G * \text{Grad } U'(s), \text{Grad } U'(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle G * U'(s), U'(s) \rangle_{\mathbb{H}} ds \geq 0,$$

so, an approximation argument based on (3.4) suffices to prove (3.13) for mild solutions. $\square$

Proof of Theorem 3.3. For showing global well-posedness of the solution U of (2.12)–(2.13), it suffices to prove the boundedness of $\|U'(t)\|_{\mathbb{H}}$, $\|\text{Grad } U(t)\|_{\mathbb{H}}$ and $\|U(t)\|_{\mathbb{H}}$ by a constant independent of t on the maximal interval $[0, T)$ of existence.

From the assumption (1.3), we have $R(U(t)) \geq 0$. So, for any $t \in [0, T)$, we obtain

$$E(t) \geq \frac{1}{2} \|U'(t)\|_{\mathbb{H}}^2 + \frac{1 - G(0)}{2} \{ \|\text{Grad } U(t)\|_{\mathbb{H}}^2 + \|U(t)\|_{\mathbb{H}}^2 \}. \quad (3.14)$$

We know that

$$R(U(t)) = \int_{\Omega} F_1(u(t)) dx + \int_{\Gamma} F_2(v(t)) d\Gamma.$$

Since h_i is globally lipschitz, then $|F_i(s)| \leq c'|s|^2$, $i = 1, 2$, for every $s \in \mathbb{R}$. So, for all $t \geq 0$,

$$R(U(t)) \leq c \|U(t)\|_{\mathbb{H}}^2,$$

where c and c' are some positive constants.

Substituting the above inequality, for $t = 0$, into (3.13), we obtain

$$\begin{aligned} E(t) &\leq \frac{1}{2} \|U_1\|_{\mathbb{H}}^2 + \frac{1-G(0)}{2} \{ \|Grad U_0\|_{\mathbb{H}}^2 + \|U_0\|_{\mathbb{H}}^2 \} + c \|U_0\|_{\mathbb{H}}^2 \\ &\quad + G(0) \{ \|Grad U_0\|_{\mathbb{H}}^2 + \|U_0\|_{\mathbb{H}}^2 \} + \frac{\varepsilon}{2} \{ \|Grad U(t)\|_{\mathbb{H}}^2 + \|U(t)\|_{\mathbb{H}}^2 \} \\ &\quad \times \frac{1}{2\varepsilon} \|g\|_1 \{ \|g\|_1 + 1 \} \{ \|Grad U_0\|_{\mathbb{H}}^2 + \|U_0\|_{\mathbb{H}}^2 \} \\ &\quad + \frac{\varepsilon}{2} \int_0^t |g(s)| \{ \|Grad U(s)\|_{\mathbb{H}}^2 + \|U(s)\|_{\mathbb{H}}^2 \} ds \\ &\quad + \frac{1}{2} \|f\|_1 + \frac{1}{2} \int_0^t \{ |f_1(s)| + |f_2(s)| \} \|U'(s)\|_{\mathbb{H}}^2 ds, \end{aligned}$$

where $\varepsilon \in (0, \frac{1-G(0)}{2})$.

Using Poincaré's inequality and the continuous embedding:

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1), \quad 2 \leq q \leq \bar{q}$$

where

$$\bar{q} = \begin{cases} \frac{2(n-1)}{n-2}, & \text{if } n \geq 3, \\ +\infty, & \text{if } n = 1, 2. \end{cases}$$

we obtain

$$\begin{aligned} E(t) &\leq (\|U_1\|_{\mathbb{H}}^2 + \|Grad U_0\|_{\mathbb{H}}^2 + \|f\|_1) \left(3 + k_1 + \frac{k_2}{2\varepsilon} \|g\|_1 (\|g\|_1 + 1) \right) \\ &\quad + \frac{\varepsilon}{2} \{ \|Grad U(t)\|_{\mathbb{H}}^2 + \|U(t)\|_{\mathbb{H}}^2 \} \\ &\quad + \frac{1}{2} \int_0^t \{ |g(s)| + |f_1(s)| + |f_2(s)| \} \{ \varepsilon \|U(s)\|_{\mathbb{H}}^2 \\ &\quad + \|Grad U(s)\|_{\mathbb{H}}^2 + \|U'(s)\|_{\mathbb{H}}^2 \} ds, \end{aligned} \tag{3.15}$$

where k_1, k_2 are some positive constants.

From (3.15) and (3.14), we deduce

$$\begin{aligned} &\frac{1}{2} \|U'(t)\|_{\mathbb{H}}^2 + \frac{1-G(0)-\varepsilon}{2} \{ \|U(t)\|_{\mathbb{H}}^2 + \|Grad U(t)\|_{\mathbb{H}}^2 \} \\ &\leq (\|U_1\|_{\mathbb{H}}^2 + \|Grad U_0\|_{\mathbb{H}}^2 + \|f\|_1) \left(3 + k_1 + \frac{k_2}{2\varepsilon} \|g\|_1 (\|g\|_1 + 1) \right) \\ &\quad + \frac{1}{2} \int_0^t \{ |g(s)| + |f_1(s)| + |f_2(s)| \} \{ \varepsilon \|U(s)\|_{\mathbb{H}}^2 \\ &\quad + \|Grad U(s)\|_{\mathbb{H}}^2 + \|U'(s)\|_{\mathbb{H}}^2 \} ds. \end{aligned}$$

Since $\varepsilon \in (0, \frac{1-G(0)}{2})$, then $1 - G(0) - \varepsilon > \varepsilon$, so

$$\begin{aligned} & \|U'(t)\|_{\mathbb{H}}^2 + \varepsilon\{\|U(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U(t)\|_{\mathbb{H}}^2\} \\ & \leq (\|U_1\|_{\mathbb{H}}^2 + \|\text{Grad } U_0\|_{\mathbb{H}}^2 + \|f\|_1) \left(6 + 2k_1 + \frac{k_2}{\varepsilon} \|g\|_1 (\|g\|_1 + 1)\right) \\ & \quad + \int_0^t \{|g(s)| + |f_1(s)| + |f_2(s)|\} \{\varepsilon\|U(s)\|_{\mathbb{H}}^2 \\ & \quad + \|\text{Grad } U(s)\|_{\mathbb{H}}^2 + \|U'(s)\|_{\mathbb{H}}^2\} ds. \end{aligned}$$

Applying Gronwall's Lemma, we get for any $t \in [0, T]$

$$\begin{aligned} & \|U'(t)\|_{\mathbb{H}}^2 + \varepsilon\{\|U(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U(t)\|_{\mathbb{H}}^2\} \\ & \leq (\|U_1\|_{\mathbb{H}}^2 + \|\text{Grad } U_0\|_{\mathbb{H}}^2 + \|f\|_1) e^{\|g\|_1} \left(6 + 2k_1 + \frac{k_2}{\varepsilon} \|g\|_1 (\|g\|_1 + 1)\right) \\ & \quad \times e^{\|f\|_1}. \end{aligned} \tag{3.16}$$

To sum up, for any $U_0 \in \tilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$ and any $f \in L^1(0, \infty; \mathbb{H})$ we have

$$E(t) \geq \frac{1}{2} \|U'(t)\|_{\mathbb{H}}^2 + \varepsilon\{\|U(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U(t)\|_{\mathbb{H}}^2\},$$

and

$$\begin{aligned} & \|U'(t)\|_{\mathbb{H}}^2 + \varepsilon\{\|U(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U(t)\|_{\mathbb{H}}^2\} \\ & \leq (\|U_1\|_{\mathbb{H}}^2 + \|\text{Grad } U_0\|_{\mathbb{H}}^2 + \|f\|_1) e^{\|g\|_1 + \|f\|_1} \left(6 + 2k_1 + \frac{k_2}{\varepsilon} \|g\|_1 (\|g\|_1 + 1)\right). \end{aligned}$$

Moreover, putting the above estimate into (3.15), we obtain

$$E(t) \leq \tilde{C}_1, \quad \forall t \in [0, T],$$

where $\tilde{C}_1(\|U_1\|, \|\text{Grad } U_0\|, \|f\|_1)$ is a positive, increasing function such that $\tilde{C}_1(0) = 0$ and independent of T . We deduce that U is global and we obtain the estimates (3.6)–(3.7).

Proof of Theorem 3.4. To obtain (3.8) we follow the same steps of that of Theorem 3.3 with (3.12) instead of (3.13).

4. Stabilization

Our goal in this section is to show that the energy of the solution associated to (2.12)–(2.13) decays exponentially.

For any measurable function $g : (0, \infty) \longrightarrow X$ and $\alpha \in \mathbb{R}$ let us set

$$g_\alpha(t) := e^{\alpha t} g(t), \quad t > 0.$$

Now, we state the following hypothesis which will be assumed in this paper:

Assumption (H2). There exists $\alpha_0 > 0$ such that $g_{\alpha_0} \in L^1(0, \infty)$ and $t \longrightarrow \int_t^\infty g_{\alpha_0}(s) ds$ is a strongly positive definite kernel.

Our main result of stabilization reads as follows

Theorem 4.1. Assume that hypotheses (H1), (H2) with $\alpha_0 > 0$, and (1.2), (1.3) are satisfied. Then, there exist some positive constants K_i , $i = 0, 1$, such that, for any $U_0 \in \tilde{\mathbb{V}}$, $U_1 \in \mathbb{H}$ and any $f_{\eta_0} \in L^1(0, \infty; \mathbb{H})$, $\eta_0 > 0$, the energy $E(t)$ of the mild solution U of (2.12)–(2.13) decays to zero exponentially as

$$E(t) \leq K_0 e^{-2\alpha t}, \quad \forall t \geq 0, \quad (4.1)$$

and

$$\int_0^\infty e^{2\alpha s} E(s) ds \leq K_1, \quad (4.2)$$

for any $\alpha \in [0, \alpha^*]$, where $\alpha^* \in]0, \min\{\alpha_0, \eta_0\}[$ and $K_i = K_i(\|U_1\|, \|\text{Grad} U_0\|, \|f\|_1)$, are positive, increasing functions such that $K_i(0) = 0$, $i = 0, 1$.

Now, for the reader's convenience (see [3]), we recall the following remark, that will be used next.

Remark 4.2. 1. For any $\alpha \in [0, \alpha_0]$, $g_\alpha \in L^1(0, \infty)$ and

$$G_\alpha(t) = \int_t^\infty g_\alpha(s) ds$$

is a strongly positive definite kernel with

$$\delta_\alpha := \frac{\delta_{\alpha_0}}{(1 + \alpha_0 - \alpha)^2} \geq \frac{\delta_{\alpha_0}}{(1 + \alpha_0)^2}, \quad (4.3)$$

as one can see, noting that $G_\alpha(t) = \int_t^\infty e^{-(\alpha_0 - \alpha)s} g_{\alpha_0}(s) ds$ and applying Proposition 2.1 to g_{α_0} . Moreover, thanks to (2.5) and (4.3), we have

$$G_\alpha(0) \geq \delta_\alpha \geq \frac{\delta_{\alpha_0}}{(1 + \alpha_0)^2}. \quad (4.4)$$

2. Thanks to $g_{\alpha_0} \in L^1(0, \infty)$, the dominated convergence theorem and

$$G(0) < 1,$$

we have for any $\alpha \in [0, \alpha_0]$

$$1 - G_\alpha(0) \geq \frac{1 - G(0)}{2} > 0. \quad (4.5)$$

3. For any $\alpha \in [0, \alpha_0 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_0$, $G_\alpha \in L^1(0, \infty)$ and

$$\|G_\alpha\|_1 \leq \frac{1}{\varepsilon_0} \|g_{\alpha_0}\|_1, \quad (4.6)$$

because it suffices to note that

$$\begin{aligned} \int_0^\infty |G_\alpha(s)| ds &\leq \int_0^\infty s e^{\alpha s} |g(s)| ds \\ &= \int_0^\infty s e^{-(\alpha_0 - \alpha)s} e^{\alpha_0 s} |g(s)| ds \leq \frac{1}{\varepsilon_0} \int_0^\infty e^{\alpha_0 s} |g(s)| ds. \end{aligned} \quad (4.7)$$

4. Observe that for any $\alpha \in (0, \alpha_0 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_0$, $G_\alpha \in L^2(0, \infty)$ and also by (4.6) we have

$$\|G_\alpha\|_2^2 \leq \|G_\alpha\|_1 \|g_\alpha\|_1 \leq \frac{1}{\varepsilon_0} \|g_{\alpha_0}\|_1^2. \quad (4.8)$$

Proof of Theorem 4.1. The strategy used here is the same as that adopted by Cannarsa and Sforza [3].

4.1. The linear Problem

At first, let us consider the linear case $H = 0$.

$$\begin{cases} U''(t) + \mathbb{A}U(t) - \int_0^t g(t-s)\mathbb{A}U(s)ds = f(t), & t \geq 0, \\ U(0) = U_0, \quad U'(0) = U_1 \end{cases} \quad (4.9)$$

We assume that $U_0 \in D(\mathbb{A})$, $U_1 \in \widetilde{\mathbb{V}}$ and $f \in C^1([0, \infty); \mathbb{H})$. Then, the mild solution U is a strong one. Multiplying U by an exponential function $e^{\alpha t}$ for some $\alpha \geq 0$, we see that $U_\alpha(t) = e^{\alpha t}U(t)$ satisfies

$$\begin{cases} U_\alpha''(t) - 2\alpha U_\alpha'(t) + \alpha^2 U_\alpha(t) + \mathbb{A}U_\alpha(t) - g_\alpha * \mathbb{A}U_\alpha(t) = f_\alpha(t), \\ U_\alpha(0) = U_0, \quad U_\alpha'(0) = U_1 + \alpha U_0. \end{cases} \quad (4.10)$$

Let us define the energy of a solution U_α of (4.10) by

$$\begin{aligned} E_\alpha(t) &= \frac{1}{2} \|U_\alpha'(t)\|_{\mathbb{H}}^2 + \frac{1}{2} (1 - G_\alpha(0)) \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \\ &\quad + \frac{1}{2} (1 + \alpha^2 - G_\alpha(0)) \|U_\alpha(t)\|_{\mathbb{H}}^2. \end{aligned}$$

Note that

$$E(t) = E_0(t).$$

Below, we denote by C_i , $i \in \mathbb{N}$, positive constants depending only on data, $\|U_1\|$, $\|Grad U_0\|$ and $\|f\|_1$.

The proof of the exponential decay of the energy will be achieved with the help of a sequence of lemmas.

Lemma 4.3. *For all solution U_α of (4.10), any $\alpha \in [0, \alpha_0]$ and $t \geq 0$, we have*

$$\begin{aligned} &\frac{1 - G_\alpha(0)}{2} \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \\ &\quad + \int_0^t \langle G_\alpha * U_\alpha'(s), U_\alpha'(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle G_\alpha * Grad U_\alpha'(s), Grad U_\alpha'(s) \rangle_{\mathbb{H}} ds \\ &\leq C_1 + H_{\alpha,1}(t) + 2\alpha \int_0^t \|U_\alpha'(s)\|_{\mathbb{H}}^2 ds, \end{aligned} \quad (4.11)$$

where, $C_1 > 0$ and

$$\begin{aligned} H_{\alpha,1}(t) &= -G_\alpha(t) \{ \langle U_0, U_\alpha(t) \rangle_{\mathbb{H}} + \langle Grad U_0, Grad U_\alpha(t) \rangle_{\mathbb{H}} \} \\ &\quad - \int_0^t g_\alpha(s) \{ \langle U_0, U_\alpha(s) \rangle_{\mathbb{H}} + \langle Grad U_0, Grad U_\alpha(s) \rangle_{\mathbb{H}} \} ds \\ &\quad + \int_0^t \langle f_\alpha(s), U_\alpha'(s) \rangle_{\mathbb{H}} ds. \end{aligned} \quad (4.12)$$

Proof. Taking the inner product of (4.10) by $U'_\alpha(t)$ and integrating between 0 and t , we have, for $t \geq 0$,

$$\begin{aligned} & \frac{1}{2} \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \frac{1}{2} \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 + \frac{(\alpha^2 + 1)}{2} \|U_\alpha\|_{\mathbb{H}}^2 \\ & - \int_0^t \langle g_\alpha * U_\alpha(\tau), U'_\alpha(\tau) \rangle_{\mathbb{H}} d\tau - \int_0^t \langle g_\alpha * Grad U_\alpha(\tau), Grad U'_\alpha(\tau) \rangle_{\mathbb{H}} d\tau \\ & = \frac{1}{2} \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{1}{2} \|Grad U_\alpha(0)\|_{\mathbb{H}}^2 + \frac{(\alpha^2 + 1)}{2} \|U_\alpha(0)\|_{\mathbb{H}}^2 \\ & + 2\alpha \int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau + \int_0^t \langle f_\alpha(\tau), U'_\alpha(\tau) \rangle_{\mathbb{H}} d\tau. \end{aligned} \quad (4.13)$$

Therefore, applying (3.9) with $w = U_\alpha$ and then with $w = Grad U_\alpha$ to the integrals in the left-hand side of the above equality, one gets (4.11)–(4.12). $\square$

Lemma 4.4. *If U_α is a solution of (4.10), then there exists a number $\alpha_1 > 0$, such that for any $\alpha \in [0, \alpha_1 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_1$ and $t \geq 0$ the following two estimates hold true*

$$\begin{aligned} & \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds \\ & \leq \frac{\alpha_1}{\varepsilon_0} (C_2 + H_{\alpha,2}(t)) \\ & + \alpha \frac{\alpha_1}{\varepsilon_0} \left(\frac{(1 + \alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2}{1 - G(0)} \right) \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds, \end{aligned} \quad (4.14)$$

$$\begin{aligned} & \frac{1 - G_\alpha(0)}{2} \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \frac{1}{2} \{ \|Grad (U_\alpha - g_\alpha * U_\alpha)(t)\|_{\mathbb{H}}^2 + \|(U_\alpha - g_\alpha * U_\alpha)(t)\|_{\mathbb{H}}^2 \} \\ & + \alpha^2 \frac{1 - G_\alpha(0)}{4} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\ & \leq C_2 + H_{\alpha,2}(t) + \alpha \left(\frac{(1 + \alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2}{1 - G(0)} \right) \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds, \end{aligned} \quad (4.15)$$

where, $C_2 > 0$ and

$$\begin{aligned} H_{\alpha,2}(t) &= \frac{2\alpha^2}{1 - G_\alpha(0)} H_{\alpha,1}(t) \\ & - G_\alpha(t) \langle U'_\alpha(0), U'_\alpha(t) \rangle_{\mathbb{H}} - \int_0^t g_\alpha(s) \langle U'_\alpha(0), U'_\alpha(s) \rangle_{\mathbb{H}} ds \\ & - \int_0^t g_\alpha(s) \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds \\ & - \int_0^t g_\alpha(s) \langle Grad U_0, Grad (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds \\ & + \int_0^t \langle f_\alpha(s), U'_\alpha(s) - g_\alpha * U'_\alpha(s) \rangle_{\mathbb{H}} ds. \end{aligned} \quad (4.16)$$

Proof. Taking the inner product of (4.10) by

$$U'_\alpha(t) - g_\alpha * U'_\alpha(t) = \frac{d}{dt}(U_\alpha - g_\alpha * U_\alpha)(t) + g_\alpha(t)U_0, \quad (4.17)$$

and, since $U \in C^1([0, \infty); \mathbb{V})$ (see Theorem 3.4), we get

$$\begin{aligned} & \langle U''_\alpha(t), U'_\alpha(t) - g_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} - 2\alpha \langle U'_\alpha(t), U'_\alpha(t) - g_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} \\ & + \alpha^2 \langle U_\alpha(t), U'_\alpha(t) - g_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} \\ & + \frac{1}{2} \frac{d}{dt} \{ \| \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & + g_\alpha(t) \{ \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(t) \rangle_{\mathbb{H}} + \langle \text{Grad} U_0, \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \rangle_{\mathbb{H}} \} \\ & = \langle f_\alpha(t), U'_\alpha(t) - g_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}}. \end{aligned}$$

Now, integrating the above equality from 0 to t , we obtain

$$\begin{aligned} & \frac{1}{2} \| U'_\alpha(t) \|_{\mathbb{H}}^2 + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & - \int_0^t \langle U''_\alpha(s), g_\alpha * U'_\alpha(s) \rangle_{\mathbb{H}} ds \\ & \frac{1}{2} \{ \| U'_\alpha(0) \|_{\mathbb{H}}^2 + \| U_0 \|_{\mathbb{H}}^2 + \| \text{Grad} U_0 \|_{\mathbb{H}}^2 \} \\ & - \int_0^t g_\alpha(s) \{ \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \\ & + \langle \text{Grad} U_0, \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \} ds \\ & + \int_0^t \{ \langle f_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\ & + 2\alpha \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\ & - \alpha^2 \int_0^t \langle U_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds. \end{aligned}$$

To estimate the integral, in the left-hand side, we use (3.9) with $w = U'_\alpha$ to obtain

$$\begin{aligned} & \frac{1 - G_\alpha(0)}{2} \| U'_\alpha(t) \|_{\mathbb{H}}^2 + \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds \\ & + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & = \frac{1 + G_\alpha(0)}{2} \| U'_\alpha(0) \|_{\mathbb{H}}^2 + \frac{1}{2} \{ \| U_0 \|_{\mathbb{H}}^2 + \| \text{Grad} U_0 \|_{\mathbb{H}}^2 \} - G_\alpha(t) \langle U'_\alpha(0), U'_\alpha(t) \rangle_{\mathbb{H}} \\ & - \int_0^t g_\alpha(s) \langle U'_\alpha(0), U'_\alpha(s) \rangle_{\mathbb{H}} ds \\ & - \int_0^t g_\alpha(s) \{ \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \\ & + \langle \text{Grad} U_0, \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \} ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \{ \langle f_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& \quad + 2\alpha \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& \quad - \alpha^2 \int_0^t \langle U_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds.
\end{aligned} \tag{4.18}$$

We must evaluate the last two terms in the right-hand side. Applying (3.11) with $w = U'_\alpha$, gives

$$\begin{aligned}
& 2 \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& \leq \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds + \int_0^t \|(U'_\alpha - g_\alpha * U'_\alpha)(s)\|_{\mathbb{H}}^2 ds \\
& \leq \int_0^t \|U'_\alpha\|_{\mathbb{H}}^2 ds + 2(1 - G_\alpha(0)) \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& \quad + 2\|G_\alpha\|_2^2 \|U'_\alpha(0)\|_{\mathbb{H}}^2 + 2 \int_0^t \|G_\alpha * U''_\alpha(s)\|_{\mathbb{H}}^2 ds,
\end{aligned}$$

whence, thanks also to (2.6),

$$\begin{aligned}
& 2G_\alpha(0) \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& \leq 2\|G_\alpha\|_2^2 \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds + 2 \int_0^t \|G_\alpha * U''_\alpha(s)\|_{\mathbb{H}}^2 ds \\
& \leq 2\|G_\alpha\|_2^2 \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds \\
& \quad + \frac{2}{\delta_\alpha} (\|G_\alpha\|_1^2 + 4\|g_\alpha\|_1^2) \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds.
\end{aligned} \tag{4.19}$$

Next, using (4.17), yields

$$\begin{aligned}
& \int_0^t \langle U_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& = \langle U_\alpha, U_\alpha - g_\alpha * U_\alpha \rangle_{\mathbb{H}}|_0^t + \int_0^t g_\alpha(s) \langle U_0, U_\alpha(s) \rangle_{\mathbb{H}} ds \\
& \quad - \int_0^t \langle U'_\alpha(s), (U_\alpha - g_\alpha * U_\alpha(s)) \rangle_{\mathbb{H}} ds \\
& = \langle U_\alpha, U_\alpha - g_\alpha * U_\alpha \rangle_{\mathbb{H}}|_0^t + \int_0^t g_\alpha(s) \langle U_0, U_\alpha(s) \rangle_{\mathbb{H}} ds \\
& \quad + \frac{1}{2} \|U_0\|_{\mathbb{H}}^2 - \frac{1}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
& \quad + \int_0^t \langle U'_\alpha(s), g_\alpha * U_\alpha(s) \rangle_{\mathbb{H}} ds.
\end{aligned}$$

Now, we evaluate the last term in the right-hand side by means of (3.9) with $w = U_\alpha$ and obtain

$$\begin{aligned}
 & \int_0^t \langle U_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
 &= \langle U_\alpha, U - g_\alpha * U_\alpha \rangle_{\mathbb{H}}|_0^t + \frac{1 + G_\alpha(0)}{2} \|U_0\|_{\mathbb{H}}^2 \\
 &\quad - G_\alpha(t) \langle U_0, U_\alpha(t) \rangle_{\mathbb{H}} - \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
 &\quad - \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds
 \end{aligned} \tag{4.20}$$

From (3.10), we obtain

$$\begin{aligned}
 & \langle U_\alpha(t), (U_\alpha - g_\alpha * U_\alpha)(t) \rangle_{\mathbb{H}} - G_\alpha(t) \langle U_0, U_\alpha(t) \rangle_{\mathbb{H}} \\
 &= (1 - G_\alpha(0)) \|U_\alpha(t)\|_{\mathbb{H}}^2 + \langle U_\alpha(t), G_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}},
 \end{aligned}$$

so, (4.20) can be written in the following form

$$\begin{aligned}
 & \int_0^t \langle U_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
 &= \frac{G_\alpha(0) - 1}{2} \|U_0\|_{\mathbb{H}}^2 + \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
 &\quad + \langle U_\alpha(t), G_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} \\
 &\quad - \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds
 \end{aligned} \tag{4.21}$$

Plugging (4.19) and (4.21) into (4.18), we get

$$\begin{aligned}
 & \frac{1 - G_\alpha(0)}{2} \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds + \alpha^2 \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
 &\quad + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\
 &\leq \frac{1 + G_\alpha(0)}{2} \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{1}{2} \{ \|U_0\|_{\mathbb{H}}^2 + \| \text{Grad} U_0 \|_{\mathbb{H}}^2 \} \\
 &\quad + \alpha \frac{2 \|G_\alpha\|_2^2}{G_\alpha(0)} \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \alpha^2 \frac{1 - G_\alpha(0)}{2} \|U_0\|_{\mathbb{H}}^2 - G_\alpha(t) \langle U'_\alpha(0), U'_\alpha(t) \rangle_{\mathbb{H}} \\
 &\quad - \int_0^t g_\alpha(s) \langle U'_\alpha(0), U'_\alpha(s) \rangle_{\mathbb{H}} ds \\
 &\quad - \int_0^t g_\alpha(s) \{ \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \\
 &\quad \quad + \langle \text{Grad} U_0, \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \} ds \\
 &\quad + \int_0^t \langle f_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
 &\quad + 2\alpha \frac{\|G_\alpha\|_1^2 + 4\|g_\alpha\|_1^2}{G_\alpha(0)\delta_\alpha} \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha}{G_\alpha(0)} \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds - \alpha^2 \langle U_\alpha(t), G_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} \\
& + \alpha^2 \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds.
\end{aligned} \tag{4.22}$$

A combination of (4.6), (4.4) and (4.3), for $\alpha \leq \frac{\alpha_0}{2}$ yields

$$\frac{\|G_\alpha\|_1^2 + 4\|g_\alpha\|_1^2}{G_\alpha(0)\delta_\alpha} \leq \frac{4(1 + \alpha_0^2)(1 + \alpha_0)^4}{\alpha_0^2\delta_{\alpha_0}^2} \|g_{\alpha_0}\|_1^2 \leq \frac{1}{2\alpha_1},$$

where

$$\alpha_1 = \min \left\{ \frac{\alpha_0^2\delta_{\alpha_0}^2}{8(1 + \alpha_0^2)(1 + \alpha_0)^4\|g_{\alpha_0}\|_1^2}, \frac{\alpha_0}{2} \right\}, \tag{4.23}$$

and from (2.4)

$$\begin{aligned}
-\langle U_\alpha(t), G_\alpha * U'_\alpha(t) \rangle_{\mathbb{H}} & \leq \frac{1 - G_\alpha(0)}{4} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
& + \frac{2G_\alpha(0)}{1 - G_\alpha(0)} \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds,
\end{aligned}$$

so, in view also of (4.11), from (4.22) we obtain

$$\begin{aligned}
& \frac{1 - G_\alpha(0)}{2} \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds \\
& + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\
& + \alpha^2 \frac{1 - G_\alpha(0)}{4} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\
& \leq C_2 + \frac{2\alpha^2}{1 - G_\alpha(0)} H_{\alpha,1}(t) - G_\alpha(t) \langle U'_\alpha(0), U'_\alpha(t) \rangle_{\mathbb{H}} \\
& - \int_0^t g_\alpha(s) \langle U'_\alpha(0), U'_\alpha(0) \rangle_{\mathbb{H}} ds + \int_0^t \langle f_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds \\
& - \int_0^t g_\alpha(s) \{ \langle U_0, (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \\
& \quad + \langle \text{Grad} U_0, \text{Grad} (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} \} ds \\
& + \frac{\alpha}{\alpha_1} \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds \\
& + \alpha \left(\frac{1}{G_\alpha(0)} + \frac{4\alpha^2}{1 - G_\alpha(0)} \right) \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds,
\end{aligned}$$

for some $C_2 > 0$. Finally, we can absorb the term $\frac{\alpha}{\alpha_1} \int_0^t \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} ds$ by the similar term in the left-hand side and the estimates (4.14) and (4.15) hold true for $\alpha \leq \alpha_1 - \varepsilon_0$. $\square$

For all $t \geq 0$, applying (2.7) with $g = G_\alpha$ and $y = U'_\alpha$, we get

$$\begin{aligned} \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds &\leq \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{2}{\delta_\alpha} \int_0^t \{ \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} \\ &\quad + \langle G_\alpha * U''_\alpha(s), U''_\alpha(s) \rangle_{\mathbb{H}} \} ds. \end{aligned} \quad (4.24)$$

The following lemma gives us an estimate of $\int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds$.

Lemma 4.5. *If U_α is a solution of (4.10), then there exists a number $\alpha_2 > 0$, given by*

$$\alpha_2 = \min \left\{ \frac{\delta_{\alpha_0}}{4(1+\alpha_0)^2} \left(1 + \frac{(1+\alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2}{1-G(0)} \right)^{-1}, \frac{\alpha_1}{2}, \eta_0 \right\},$$

such that for any $\alpha \in [0, \alpha_2 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_2$ and $t \geq 0$, we have

$$\begin{aligned} \int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau &\leq \frac{\alpha_2}{\varepsilon_0} \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\varepsilon_0} (C_1 + 2C_2) \\ &\quad + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\varepsilon_0} (H_{\alpha,1}(t) + 2H_{\alpha,2}(t)), \end{aligned} \quad (4.25)$$

where η_0 has been introduced in statement of Theorem 4.1.

Proof. We estimate the term $\int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau$ using (4.24). Indeed, in view of (4.3), (4.11) and (4.14), we obtain, for $\alpha \leq \frac{\alpha_1}{2}$ and $t \geq 0$,

$$\begin{aligned} &\int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau \\ &\leq \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{2(1+\alpha_0)^2}{\delta_{\alpha_0}} (C_1 + 2C_2) + \frac{2(1+\alpha_0)^2}{\delta_{\alpha_0}} (H_{\alpha,1}(t) + 2H_{\alpha,2}(t)) \\ &\quad + \frac{\alpha}{\alpha_2} \int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau. \end{aligned}$$

For $\alpha \leq \alpha_2 - \varepsilon_0$, the inequality

$$\begin{aligned} \int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau &\leq \frac{\alpha_2}{\varepsilon_0} \|U'_\alpha(0)\|_{\mathbb{H}}^2 + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\varepsilon_0} (C_1 + 2C_2) \\ &\quad + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\varepsilon_0} (H_{\alpha,1}(t) + 2H_{\alpha,2}(t)), \end{aligned} \quad (4.26)$$

holds true, whence (4.25) follows. $\square$

Lemma 4.6. *Let U_α be a solution of (4.10), then there exists a number $\alpha_2 > 0$, such that for any $\alpha \in [0, \alpha_2 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_2$ and $t \geq 0$, we have*

$$\begin{aligned} &\frac{1-G_\alpha(0)}{2} \{ \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U_\alpha(t)\|_{\mathbb{H}}^2 \} \\ &\quad + \frac{1}{2} \{ \|(U_\alpha - g_\alpha * U_\alpha)(t)\|_{\mathbb{H}}^2 + \|\text{Grad } (U_\alpha - g_\alpha * U_\alpha)(t)\|_{\mathbb{H}}^2 \} \\ &\quad + \alpha^2 \frac{1-G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \leq C_3, \end{aligned} \quad (4.27)$$

$$\int_0^t \|U'_\alpha(\tau)\|_{\mathbb{H}}^2 d\tau \leq C_4, \quad (4.28)$$

where α_2 is given by Lemma 4.5 and $C_3, C_4 > 0$ are positive constants.

Proof. Putting (4.25) into (4.15) and taking into account (4.11), (4.12) and (4.16), we deduce

$$\begin{aligned} & \frac{1 - G_\alpha(0)}{2} \{ \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \} \\ & + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| Grad (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & + \alpha^2 \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\ & \leq \varepsilon \{ \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \alpha^2 \|U_\alpha(t)\|_{\mathbb{H}}^2 + \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \} \\ & + C_\varepsilon + C \int_0^t (\|f_{\eta_0}(s)\|_1 + |g_{\alpha_0}(s)|) \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds \\ & + C \int_0^t |g_{\alpha_0}(s)| \{ \| (U_\alpha - g_\alpha * U_\alpha)(s) \|_{\mathbb{H}}^2 + \| Grad (U_\alpha - g_\alpha * U_\alpha)(s) \|_{\mathbb{H}}^2 \} ds \\ & + C \int_0^t |g_{\alpha_0}(s)| \{ \|U_\alpha(s)\|_{\mathbb{H}}^2 + \|Grad U_\alpha(s)\|_{\mathbb{H}}^2 \} ds \\ & + C \int_0^t \|f_{\eta_0}(s)\|_1 \| (U'_\alpha - g_\alpha * U'_\alpha)(s) \|_{\mathbb{H}} ds, \end{aligned}$$

for any $\varepsilon > 0$, C_ε and C are positive constants.

Using Gronwall's Lemma, we obtain

$$\begin{aligned} & \frac{1 - G_\alpha(0)}{2} \{ \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \} \\ & + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| Grad (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & + \alpha^2 \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\ & \leq C + C' \int_0^t \|f_{\eta_0}(s)\|_1 \| (U'_\alpha - g_\alpha * U'_\alpha)(s) \|_{\mathbb{H}}^2 ds \end{aligned}$$

where C and C' are positive constants depending on data $\|U_1\|, \|Grad U_0\|, \|f\|_1$. Since for any $T > 0$,

$$(\|U'_\alpha - g_\alpha * U'_\alpha\|_{\mathbb{H}})_{\infty, T} \leq (1 + \|g_{\alpha_0}\|_1) (\|U'_\alpha\|_{\mathbb{H}})_{\infty, T},$$

we have

$$\begin{aligned} & \frac{1 - G_\alpha(0)}{2} \{ \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \|Grad U_\alpha(t)\|_{\mathbb{H}}^2 \} \\ & + \frac{1}{2} \{ \| (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 + \| Grad (U_\alpha - g_\alpha * U_\alpha)(t) \|_{\mathbb{H}}^2 \} \\ & + \alpha^2 \frac{1 - G_\alpha(0)}{2} \|U_\alpha(t)\|_{\mathbb{H}}^2 \\ & \leq \varepsilon (\|U'_\alpha\|_{\mathbb{H}}^2)_{\infty, T} + C_\varepsilon, \end{aligned}$$

for any $t \in [0, T]$, so we can control the term $(\|U'_\alpha\|_{\mathbb{H}}^2)_{\infty, T}$ by some positive constant independent of t , and hence (4.27) follows.

Finally, functions $H_{\alpha,1}(t)$ and $H_{\alpha,2}(t)$ in (4.25) can be bounded thanks to (4.27), so one obtains (4.28). $\square$

Lemma 4.7. *For all solution U_α of (4.10), any $\alpha \in [0, \alpha_2 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_2$, and $t \geq 0$ we have*

$$\int_0^t \{ \|(U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 + \|\text{Grad } (U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 \} ds \leq C_5, \quad (4.29)$$

where $C_5 > 0$.

Proof. Taking the inner product of (4.10) by $U_\alpha(t) - g_\alpha * U_\alpha(t)$ and integrating over $[0, t]$, we have

$$\begin{aligned} & \int_0^t \{ \|(U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 + \|\text{Grad } (U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 \} ds \\ &= \int_0^t \langle f_\alpha(s), (U_\alpha(s) - g_\alpha * U_\alpha(s)) \rangle_{\mathbb{H}} ds - \langle U'_\alpha, U_\alpha - g_\alpha * U_\alpha \rangle_{\mathbb{H}}|_0^t \\ &+ \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds - \int_0^t g_\alpha(s) \langle U'_\alpha(s), U_0 \rangle_{\mathbb{H}} ds \\ &+ 2\alpha \int_0^t \langle U'_\alpha(s), (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds \\ &- \alpha^2 \int_0^t \langle U_\alpha(s), (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds. \end{aligned} \quad (4.30)$$

In order to estimate the last term in the right-hand side of (4.30), we use (3.11) and get

$$\begin{aligned} & -(1 - G_\alpha(0)) \int_0^t \langle U_\alpha(s), (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds \\ & \leq -\frac{1}{2} \int_0^t \|(U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 ds + \|G_\alpha\|_2^2 \|U_0\|_{\mathbb{H}}^2 + \int_0^t \|G_\alpha * U'_\alpha(s)\|_{\mathbb{H}}^2 ds. \end{aligned}$$

Putting the above inequality into (4.30), we obtain

$$\begin{aligned} & \int_0^t \|\text{Grad } (U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 ds \\ &+ \left(\frac{\alpha^2}{2(1 - G_\alpha(0))} + 1 \right) \int_0^t \|(U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 ds \\ & \leq \int_0^t \|f_{\eta_0}(s)\|_1 \|U_\alpha(s) - g_\alpha * U_\alpha(s)\|_{\mathbb{H}} ds - \langle U'_\alpha, U_\alpha - g_\alpha * U_\alpha \rangle_{\mathbb{H}}|_0^t \\ &+ \int_0^t \langle U'_\alpha(s), (U'_\alpha - g_\alpha * U'_\alpha)(s) \rangle_{\mathbb{H}} ds - \int_0^t g_\alpha(s) \langle U'_\alpha(s), U_0 \rangle_{\mathbb{H}} ds \\ &+ \frac{\alpha^2 \|G_\alpha\|_2^2}{1 - G_\alpha(0)} \|U_0\|_{\mathbb{H}}^2 + \frac{\alpha^2}{1 - G_\alpha(0)} \int_0^t \|G_\alpha * U'_\alpha(s)\|_{\mathbb{H}}^2 ds \end{aligned}$$

$$\begin{aligned}
& +4(1-G_\alpha(0)) \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds \\
& + \frac{\alpha^2}{4(1-G_\alpha(0))} \int_0^t \|(U_\alpha - g_\alpha * U_\alpha)(s)\|_{\mathbb{H}}^2 ds.
\end{aligned}$$

Using Lemma 2.3, the integral $\int_0^t \|G_\alpha * U'_\alpha(s)\|_{\mathbb{H}}^2 ds$ in the right-hand side of the above inequality will be estimated by

$$\begin{aligned}
& \int_0^t \|G_\alpha * U'_\alpha(s)\|_{\mathbb{H}}^2 ds \\
& = \frac{1}{\delta_\alpha} \left\{ \|G_\alpha\|_I^2 + 4 \|g_\alpha\|_I^2 \right\} \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds.
\end{aligned}$$

Next, from Lemma 4.3, we deduce

$$\begin{aligned}
\int_0^t \|G_\alpha * U'_\alpha(s)\|_{\mathbb{H}}^2 ds & \leq \frac{1}{\delta_\alpha} \left\{ \|G_\alpha\|_I^2 + 4 \|g_\alpha\|_I^2 \right\} \int_0^t \langle G_\alpha * U'_\alpha(s), U'_\alpha(s) \rangle_{\mathbb{H}} ds \\
& \leq \frac{1}{\delta_\alpha} \left\{ \|G_\alpha\|_I^2 + 4 \|g_\alpha\|_I^2 \right\} \\
& \quad \times \left\{ C_1 + H_{\alpha,1}(t) + 2\alpha \int_0^t \|U'_\alpha(s)\|_{\mathbb{H}}^2 ds \right\}.
\end{aligned}$$

Finally, to conclude, in view of (4.27) and (4.28), the desired estimate (4.29) follows. $\square$

Proof of Theorem 4.1 (continued). From Lemma 4.6, we get, for any $t \geq 0$,

$$\|U_\alpha\|_{\mathbb{H}}^2 + \|U'_\alpha(t)\|_{\mathbb{H}}^2 + \|\text{Grad } U_\alpha(t)\|_{\mathbb{H}}^2 \leq K_0, \quad (4.31)$$

where $K_0 = K_0(\|\text{Grad } U_0\|, \|U_1\|, \|f_{\eta_0}\|_1)$ is a positive, increasing function such that $K_0(0) = 0$.

Since the resolvent kernel of g_α is given by $r_\alpha(t) = e^{\alpha t} r(t)$, we can apply Corollary 2.5 to get

$$(\|\text{Grad } U_\alpha\|_{\mathbb{H}})_\infty \leq (1 + \|r_{\alpha_0}\|_1)(\|\text{Grad } (U_\alpha - g_\alpha * U_\alpha)\|_{\mathbb{H}})_\infty, \quad (4.32)$$

$$(\|U_\alpha\|_{\mathbb{H}})_\infty \leq (1 + \|r_{\alpha_0}\|_1)(\|U_\alpha - g_\alpha * U_\alpha\|_{\mathbb{H}})_\infty. \quad (4.33)$$

Using (4.32), (4.33), (2.11) and (4.29), we can show a uniform estimate for

$$\int_0^\infty \|\text{Grad } U_\alpha(s)\|_{\mathbb{H}}^2 ds,$$

and

$$\int_0^\infty \|U_\alpha(s)\|_{\mathbb{H}}^2 ds,$$

as well. Therefore, thanks also to (4.28), we have

$$\int_0^\infty \{\|U_\alpha(s)\|_{\mathbb{H}}^2 + \|\text{Grad } U_\alpha(s)\|_{\mathbb{H}}^2 + \|U'_\alpha(s)\|_{\mathbb{H}}^2\} ds \leq K_1, \quad (4.34)$$

where $K_1 = K_1(\|\text{Grad } U_0\|, \|U_1\|, \|f_{\eta_0}\|_1)$ is a positive, increasing function such that $K_1(0) = 0$.

4.2. The semilinear problem

Here, let us consider the semi-linear problem, that is the case where $H \neq 0$.

$$U_\alpha''(t) - 2\alpha U_\alpha'(t) + \alpha^2 U_\alpha(t) + \mathbb{A}U_\alpha(t) - g_\alpha * \mathbb{A}u_\alpha(t) + e^{\alpha t} H(U(t)) = f_\alpha(t) \quad (4.35)$$

and let us define the energy of the solution U_α associated to the problem (4.35) by

$$E_\alpha(t) = \frac{1}{2} \|U_\alpha'(t)\|_{\mathbb{H}}^2 + \frac{1 - G_\alpha(0)}{2} \|\text{Grad} U_\alpha(t)\|_{\mathbb{H}}^2 + \left(\frac{\alpha^2 + 1 - G(0)}{2} \right) \|U_\alpha(t)\|_{\mathbb{H}}^2 + \int_0^t e^{\alpha s} \langle H(U(s)), U_\alpha'(s) \rangle_{\mathbb{H}} ds.$$

We have the additional following term

$$\int_0^t e^{\alpha s} \langle H(U(s)), U_\alpha'(s) \rangle_{\mathbb{H}} ds,$$

in the expression of energy.

Now, we follow the same strategy used in the linear case: as in Lemmas 4.3, 4.4 and 4.7, we multiply (4.35) by the three multipliers $U_\alpha'(t)$, $U_\alpha'(t) - g_\alpha * U_\alpha'(t)$, $U_\alpha(t) - g_\alpha * U_\alpha(t)$ and then integrate from 0 to t . For the first multiplier $U_\alpha'(t)$, the nonlinear term can be written as

$$\int_0^t e^{\alpha s} \langle H(U(s)), U_\alpha'(s) \rangle_{\mathbb{H}} ds. \quad (4.36)$$

Using Cauchy-Schwartz's, Young's inequalities and (1.2), (4.36) gives

$$\int_0^t e^{\alpha s} \langle H(U(s)), U_\alpha'(s) \rangle_{\mathbb{H}} ds \leq \tilde{c}_1 \int_0^t \{ \|U_\alpha'(s)\|_{\mathbb{H}}^2 + \|U_\alpha(s)\|_{\mathbb{H}}^2 \} ds,$$

arguing similarly as above, we obtain

$$\int_0^t e^{\alpha s} \langle H(U(s)), (U_\alpha' - g_\alpha * U_\alpha')(s) \rangle_{\mathbb{H}} ds \leq \tilde{c}_2 \int_0^t \{ \|U_\alpha'(s)\|_{\mathbb{H}}^2 + \|U_\alpha(s)\|_{\mathbb{H}}^2 \} ds,$$

and

$$\int_0^t e^{\alpha s} \langle H(U(s)), (U_\alpha - g_\alpha * U_\alpha)(s) \rangle_{\mathbb{H}} ds \leq \tilde{c}_3 \int_0^t \|U_\alpha(s)\|_{\mathbb{H}}^2 ds,$$

where $\tilde{c}_i$, $i = 1, 2, 3$ are positive constants.

Therefore, thanks to the above three estimates, (4.31) and (4.34) also hold in the nonlinear case.

Recalling that $U(t) = e^{-\alpha t} U_\alpha(t)$ and $U'(t) = e^{-\alpha t} U_\alpha'(t) - \alpha e^{-\alpha t} U_\alpha(t)$, we obtain the estimates (4.1) and (4.2) for strong solutions. Since none of the above constants depend on the regularity of U , an approximation argument suffices to extend such a conclusion to mild solutions.

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References

- [1] Aassila, M., Cavalcanti, M.M., Soriano, J.A.: Asymptotic stability and energy decay rates for solutions of the wave equation with memory in a starshaped domain. *SIAM J. Control Optim.* **38**(5), 1581–1602 (2000)
- [2] Alabau-Boussouira, F., Cannarsa, P., Sforza, D.: Decay estimates for second order evolution equations with memory. *J. Funct. Anal.* **254**, 1342–1372 (2008)
- [3] Cannarsa, P., Sforza, D.: integro-differential equations of hyperbolic type with positive definite kernels. *J. Differ. Equ.* **250**, 4289–4335 (2011)
- [4] Cannarsa, P., Sforza, D.: A stability result for a class of nonlinear integrodifferential equations with L^1 kernels. *Appl. Math. (Warsaw)* **35**, 395–430 (2008)
- [5] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ferreira, J.: Existence and uniform decay for nonlinear viscoelastic equation with strong damping. *Math. Methods Appl. Sci.* **24**, 1043–1053 (2001)
- [6] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Prates Filho, J.S., Soriano, J.A.: Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. *Differ. Integr. Equ.* **14**, 85–116 (2001)
- [7] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping. *Electron. J. Differ. Equ.* **44**, 1–14 (2002)
- [8] Cavalcanti, M.M., Khemmoudj, A., Medjden, M.: Uniform stabilization of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary condition. *J. Math. Anal. Appl* **328**(2), 900–930 (2007)
- [9] Cavalcanti, M.M., Oquendo, H.P.: Frictional versus viscoelastic damping in a semilinear wave equation. *SIAM J. Control Optim.* **42**(4), 1310–1324 (2003)
- [10] Dafermos, C.M.: Asymptotic stability in viscoelasticity. *Arch. Rational. Mech. Anal.* **37**, 297–308 (1970)
- [11] Dafermos, C.M.: On abstract Volterra equation with applications to linear viscoelasticity. *J. Differ. Equ.* **7**, 554–569 (1970)
- [12] Fabrizio, M., Giorgi, C., Pata, V.: A new approach to equations with memory. *Arch. Rational. Mech. Anal.* **198**, 189–232 (2010)
- [13] Gripenberg, G., Londen, S.O., Staffans, O.J.: Volterra Integral and Functional Equations. *Encyclopedia Math. Appl.*, vol. 34. Cambridge Univ. Press, Cambridge (1990)
- [14] Hrusa, W.J., Nohel, J.A.: The Cauchy problem in one-dimensional nonlinear viscoelasticity. *J. Differ. Equ.* **59**, 388–412 (1985)
- [15] Kawashima, S.: Global solutions to the equation of viscoelasticity with fading memory. *J. Differ. Equ.* **101**, 388–420 (1993)

- [16] Khemmoudj, A., Medjden, M.: Exponential decay for the semilinear damped Cauchy-Ventcel problem. *Bol. Soc. Parana. Mat.* **22**(2), 97–116 (2004)
- [17] Lasiecka, I., Triggiani, R., Yao, P.F.: Inverse/observability estimates for second-order hyperbolic equations with variable coefficients. *J. Math. Anal. Appl.* **235**(1), 13–57 (1999)
- [18] Nicaise, S., Laoubi, K.: Polynomial stabilization of the wave equation with Ventcel's boundary conditions. *Math. Nachr.* **283**, 10 (2010)
- [19] Prüss, J.: *Evolutionary Integral Equations and Applications*. Monogr. Math., vol. 87. Birkhäuser Verlag, Basel (1993)
- [20] Staffans, O.J.: Positive definite measures with applications to a Volterra equation. *Trans. Am. Math. Soc.* **218**, 219–237 (1976)
- [21] Staffans, O.J.: On a nonlinear hyperbolic Volterra equation. *SIAM J. Math. Anal.* **11**, 793–812 (1980)
- [22] Zuazua, E.: Exponential decay for the semilinear wave equation with locally distributed damping. *Commun. Partial Differ. Equ.* **15**(2), 205–235 (1990)

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