Large time behavior of solutions for a complex-valued quadratic heat equation

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Abstract. In this paper we study the existence and the asymptotic behavior of global solutions for a parabolic system related to the complex-valued heat equation with quadratic nonlinearity: $\partial_t z = \Delta z + z^2$, $t > 0$, $x \in \mathbb{R}^N$, with initial data $z_0 = u_0 + iv_0$. We show that if $u_0(x) \sim c|x|^{-2\alpha_1}$ and $v_0(x) \sim c|x|^{-2\alpha'_1}$, as $|x| \rightarrow \infty$ with $\alpha_1 \geq 1$, $2\alpha'_1 - \alpha_1 \geq 1$, $\frac{N}{2\alpha_1} > 1$, $\frac{N}{2\alpha'_1} > 1$ ($|c|$ is sufficiently small), then the solution is global and converges to a self-similar solution. We also establish the existence of four different self-similar behaviors. These behaviors depend on the values of α_1 and α'_1 . In particular, the real and the imaginary parts of the constructed solutions may have different behaviors in the L^∞ -norm for large time. Also, the real part may have different behaviors from those known for the real-valued quadratic heat equation.

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1. Introduction

In this paper we consider the complex-valued nonlinear heat equation:

$$\partial_t z = \Delta z + z^2, \quad (1.1)$$

$$z(0) = z_0, \quad (1.2)$$

where $z = z(t, x)$ is a complex valued function, $t > 0$, $x \in \mathbb{R}^N$, and the initial condition $z_0(x)$ is a complex valued function defined on $\mathbb{R}^N$. The aim of this paper is to prove the existence of global solutions for (1.1) and study their asymptotic behavior, as $t \rightarrow \infty$, for a class of initial data z_0 . In particular, we consider $z_0 = u_0 + iv_0$, where u_0, v_0 are given precisely in the space $L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ but not in the space $L^1(\mathbb{R}^N)$.

Equation (1.1) has a strong relation with the viscous Constantin–Lax–Majda equation. See [2] and reference therein. If we write $z(t, x) = u(t, x) +$

$iv(t, x)$, where $i^2 = -1$, then the above equation is rewritten as the parabolic system:

$$\begin{cases} \partial_t u = \Delta u + u^2 - v^2, \\ \partial_t v = \Delta v + 2uv, \end{cases} \quad (1.3)$$

$$u(0) = u_0, v(0) = v_0. \quad (1.4)$$

In the case where z is real-valued, that is $v \equiv 0$, then the previous system is reduced to the scalar quadratic heat equation

$$\partial_t u = \Delta u + u^2, \quad (1.5)$$

where $u = u(t, x)$ is a real valued function, $t > 0$, $x \in \mathbb{R}^N$.

In [1, 8], it is proved that if the initial value $u_0 \sim c|x|^{-2\alpha}$, as $|x| \rightarrow \infty$ with c is a small constant, $N > 2\alpha$ and $\alpha \geq 1$, then the solution u of (1.5) is global and is asymptotic, in some Lebesgue spaces, to a self-similar solution of (1.5) if $\alpha = 1$, and is asymptotic to $e^{t\Delta}(c|x|^{-2\alpha})$ if $\alpha > 1$, where $e^{t\Delta}$ is the heat semigroup. Moreover, in [8] it is proved that the asymptotic behavior holds in the L^∞ -norm. Furthermore, the resulting solution u satisfies, for large t , $\|u(t)\|_\infty \sim t^{-\alpha}$, $\alpha \geq 1$. The existence of self-similar solutions of (1.5) with initial data $u_0(x) = c|x|^{-2}$ is proved in [1]. See also [5]. We refer the reader to the paper [8] for more related results on the Eq. (1.5).

Many results have been established concerning existence of self-similar solutions and asymptotically self-similar solutions for other systems using the methods used in [1, 8]. In [6], the semi-linear parabolic system with nonlinear terms of the form $a_j|u|^{p_j-1}u + b_j|v|^{q_j-1}v$, $p_j, q_j > 1$, $a_j, b_j \in \mathbb{R}$, $j = 1, 2$, have been studied. The global existence and the existence of asymptotically self-similar solution are shown under some conditions on the parameters. In [7], the semi-linear parabolic system with product nonlinear terms of the form $a_j|u|^{p_j-1}u|v|^{q_j-1}v$, where $a_j \in \mathbb{R}$ and $p_j, q_j \geq 1$, $j = 1, 2$. If we treat the Eq. (1.1) as in [1, 8], we obtain similar results as in [1, 8] on the behavior of z . But, this will not show the difference on the behavior of the real and the imaginary part of the solution. On the other hand, the system (1.3) is not studied neither in [6] nor in [7]. In fact, the case where the sum and the product are in the same system is not considered. The aim of this paper is to study the large time behavior of global solutions for the system (1.3).

The Cauchy problem for the system (1.3) with an initial data

$$(u_0, v_0) \in (L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N))^2, \quad (1.6)$$

has a unique solution $(u, v) \in (C([0, T]; L^\infty(\mathbb{R}^N)))^2$, where $T = T(u_0, v_0) \in (0, \infty]$ denotes the maximal existence time of the solution. Moreover, we have either $T = \infty$, or

$$T < \infty \quad \text{and} \quad \limsup_{t \rightarrow T} \{\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty\} = \infty. \quad (1.7)$$

The system (1.3) has been studied in [3]. In particular, global existence results and large time behavior are established under some appropriate conditions on initial values. More precisely, the authors show that if the initial data (u_0, v_0) satisfy for all $x \in \mathbb{R}^N$

$$u_0(x) - Av_0(x) < 0, \quad (1.8)$$

with $A \in \mathbb{R}$ is a constant, then, the solution of (1.3) satisfying (1.6) exists globally in time. See [3, Theorem 1.1, p. 2]. Moreover, in [3] the authors study the large time behavior for the case where the initial data are positive and asymptotically constants. More precisely, they impose the following conditions on initial data:

$$u_0, v_0 \in C^1(\mathbb{R}^N), \quad 0 \leq u_0 \leq N_1, u_0 \not\equiv N_1, 0 \leq v_0 \leq L, L > 0, \quad (1.9)$$

$$\lim_{|x| \rightarrow \infty} u_0(x) = N_1, \quad \lim_{|x| \rightarrow \infty} v_0(x) = N_2, \quad N_1 > N_2 > 0, \quad (1.10)$$

and they prove that the solutions converges to the stationary solutions of (1.3). See [3, Theorem 1.4, p. 4].

In the present paper, a different class of initial values are considered. In particular, we do not impose a restriction on the sign of the initial data. Also, we do not consider the conditions (1.8) and (1.9). In our case (1.10) is satisfied for $N_1 = N_2 = 0$. We have obtained the following theorem.

Theorem 1.1. (Global existence) *Let N be an integer, α_1 and α'_1 be positive real numbers satisfying $\alpha_1 \geq 1$ and $2\alpha'_1 - \alpha_1 \geq 1$. Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Consider $(u_0, v_0) \in (L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N))^2$ be such that

$$u_0 \sim c_1(1 - \eta)|x|^{-2\alpha_1}, \quad v_0 \sim c_2(1 - \eta)|x|^{-2\alpha'_1}, \quad (1.11)$$

where η is an L^∞ cut-off function ($\eta \equiv 1$ near the origin and having a compact support). Then, there exists $\varepsilon > 0$ such that if $\max(|c_1|, |c_2|) < \varepsilon$, the solution of (1.3) with initial data (u_0, v_0) exists globally in time.

The case of $\alpha_1 = \alpha'_1$ is proved in [1, 8]. In fact, although in [1, 8] only the real valued nonlinear heat equation is considered, but the case $\alpha_1 = \alpha'_1$ can be proved using exactly the same calculations of [1, 8]. That is by working directly on the Eq. (1.1). Our result is new for $\alpha_1 \neq \alpha'_1$. For general formulation see Theorem 2.6 below. The method which we have employed is different from [3]. Our ideas are based on the technique used in [1, 6–8].

The existence of self-similar solutions for the system (1.3), [that is (1.1)], has been proved in [1]. It is proved there that if $u_0 = c|x|^{-2}$, $v_0 = c|x|^{-2}$, c sufficiently small, then there exists a self-similar solution of the system (1.3). Similar calculations, give the existence of a self-similar solution of the system (1.3) with initial data $(c_1|x|^{-2}, c_2|x|^{-2})$, with c_1, c_2 are two small constants. In order to study the different asymptotic behaviors of global solutions for the system (1.3), we construct self-similar solutions for the following asymptotic system:

$$\begin{cases} \partial_t w_1 = \Delta w_1 + w_1^2, \\ \partial_t w_2 = \Delta w_2 + 2w_1 w_2, \end{cases} \quad (1.12)$$

where $w_1(t, x)$ and $w_2(t, x)$ are real valued functions, $t > 0$, $x \in \mathbb{R}^N$. We thus have the following theorem.

FIGURE 1. Asymptotic behavior. Assume the condition on α_1, α'_1 : $\alpha_1 \geq 1$, $2\alpha'_1 - \alpha_1 \geq 1$, $\alpha_1, \alpha'_1 < N/2$. If $2\alpha'_1 - \alpha_1 > 1$ with $\alpha_1 > 1$, then, we have LB for z [case (i)]. If (α_1, α'_1) are on the line passing through the points $(0, 0.5)$ and $(1, 1)$ with $\alpha_1 > 1$, then, we have LNLB for $\text{Re}(z)$, LB for $\text{Im}(z)$ [case (ii)]. If (α_1, α'_1) are on the line passing through the points $(1, 0)$ and $(1, 1)$ with $\alpha'_1 > 1$, then, we have LB for $\text{Re}(z)$, NLB for $\text{Im}(z)$ [case (iii)]. Finally, if $(\alpha_1, \alpha'_1) = (1, 1)$, we have NLB for z [case (iv)]. Where LB linear behavior, LNLB linear and nonlocal behavior, NLB nonlinear behavior

Theorem 1.2. (Self-similar solutions) *Let N be an integer, α'_1 be positive real number satisfying*

$$\alpha'_1 > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Consider $u_0 = c_1|x|^{-2}$, $v_0 = c_2|x|^{-2\alpha'_1}$. Then, there exists $\varepsilon > 0$ such that if $\max(|c_1|, |c_2|) < \varepsilon$, there exists a global self-similar solution for the system

$$w_1(t) = e^{t\Delta}(c_1|x|^{-2}) + \int_0^t e^{(t-s)\Delta}(w_1^2(s)) ds, \quad (1.13)$$

$$w_2(t) = e^{t\Delta}(c_2|x|^{-2\alpha'_1}) + 2 \int_0^t e^{(t-s)\Delta}(w_1(s)w_2(s)) ds. \quad (1.14)$$

See Theorem 3.3 below for the general version of the previous result.

In this paper we also discuss all possible asymptotic self-similar behaviors for the system (1.3) which are described in the theorem below and illustrated by the Fig. 1.

Theorem 1.3. (Different asymptotic behavior) *Let N be an integer, α_1 and α'_1 be two real numbers satisfying*

$$\alpha_1 \geq 1, \quad 2\alpha'_1 - \alpha_1 \geq 1, \quad \frac{N}{2\alpha_1} > 1 \quad \text{and} \quad \frac{N}{2\alpha'_1} > 1.$$

Let $U = (u, v)$ be the global solution of (1.3) with initial value $(1 - \eta)(u_0, v_0)$ constructed by Theorem 1.1, $u_0 = c_1|x|^{-2\alpha_1}$, $v_0 = c_2|x|^{-2\alpha'_1}$ such that $\max(|c_1|, |c_2|) < \varepsilon$. Here η and ε are as in Theorem 1.1.

Then, for ε sufficiently small, there exists a real number $\delta_1 > 0$, such that for all δ satisfying $0 < \delta < \delta_1$, there exists a positive constant C_δ such that the following holds:

(i) If $\alpha_1 > 1$ and $2\alpha'_1 > \alpha_1 + 1$, we have

$$\|u(t) - e^{t\Delta}(c_1|x|^{-2\alpha_1})\|_\infty \leq C_\delta t^{-\alpha_1-\delta}, \quad \forall t > 0, \quad (1.15)$$

$$\|v(t) - e^{t\Delta}(c_2|x|^{-2\alpha'_1})\|_\infty \leq C_\delta t^{-\alpha'_1-\delta}, \quad \forall t > 0. \quad (1.16)$$

(ii) If $\alpha_1 > 1$ and $2\alpha'_1 = \alpha_1 + 1$, we have

$$\|u(t) - e^{t\Delta}(c_1|x|^{-2\alpha_1}) + g(t)\|_\infty \leq C_\delta t^{-\alpha_1-\delta}, \quad \forall t > 0, \quad (1.17)$$

$$\|v(t) - e^{t\Delta}(c_2|x|^{-2\alpha'_1})\|_\infty \leq C_\delta t^{-\alpha'_1-\delta}, \quad \forall t > 0, \quad (1.18)$$

where

$$g(t) = \int_0^t e^{(t-s)\Delta} [e^{s\Delta}(c_2|x|^{-2\alpha'_1})]^2 ds.$$

(iii) If $\alpha_1 = 1$ and $\alpha'_1 > 1$, we have

$$\|u(t) - w_1(t)\|_\infty \leq C_\delta t^{-1-\delta}, \quad \forall t > 0, \quad (1.19)$$

$$\|v(t) - w_2(t)\|_\infty \leq C_\delta t^{-\alpha'_1-\delta}, \quad \forall t > 0, \quad (1.20)$$

with w_1 and w_2 are given by (1.13)–(1.14).

Moreover, in all cases the resulting solution (u, v) satisfies, for large t

$$d_1 t^{-\alpha_1} \leq \|u(t)\|_\infty \leq d_2 t^{-\alpha_1}, \quad d_1 t^{-\alpha'_1} \leq \|v(t)\|_\infty \leq d_2 t^{-\alpha'_1}, \quad (1.21)$$

where d_1, d_2 are positive constants.

Remark 1.4. Using the method in [1, 8], we prove that, if $\alpha_1 = \alpha'_1 = 1$, we have

$$\|u(t) - w_1(t)\|_\infty \leq C_\delta t^{-1-\delta}, \quad \forall t > 0, \quad (1.22)$$

$$\|v(t) - w_2(t)\|_\infty \leq C_\delta t^{-1-\delta}, \quad \forall t > 0, \quad (1.23)$$

with w_1 and w_2 are the self-similar solution of system (1.3) with initial data $(c_1|x|^{-2}, c_2|x|^{-2})$.

It follows from the previous result that, the constructed solution (u, v) is asymptotically self-similar in the $L^\infty \times L^\infty$ -norm. See Theorem 4.2 below for general version of the previous Theorem. The case $\alpha_1 = \alpha'_1$ and $c_1 = c_2$ is proved in [1, 8]. Also, for the case $c_1 \neq c_2$ the result of [1, 8] can be extended with minor modification. The case $\alpha_1 \neq \alpha'_1$ is new. Note that the real part of the complex-valued heat equation and the real-valued heat equation behave differently. In particular, the behaviors (ii) and (iii) can not hold for the real-valued quadratic heat equation. This shows the influence of the imaginary part

FIGURE 2. Comparison between the L^∞ -norm of the real and the imaginary parts for large time. In the *shaded region*: $\alpha_1 > \alpha'_1$ and $2\alpha'_1 \geq \alpha_1 + 1$, we have $\|Re(z)(t)\|_\infty \sim \|Im(z)(t)\|_\infty$. In the *Hatched region*: $\alpha_1 < \alpha'_1$ and $2\alpha'_1 \geq 1$, we have $\|Im(z)(t)\|_\infty \ll \|Re(z)(t)\|_\infty$. On the line $\alpha_1 = \alpha'_1 \geq 1$, we have $\|Im(z)(t)\|_\infty \sim \|Re(z)(t)\|_\infty$.

on the large time behavior of the real part. For $\alpha_1 > 1$, we have that if $\alpha'_1 > \alpha_1$ the imaginary part v dominates the real part u , as $t \rightarrow \infty$ in the L^∞ -norm while if $\frac{1}{2} + \frac{\alpha_1}{2} \leq \alpha'_1 < \alpha_1$ the real part u dominates the imaginary part v (Fig. 2).

For blowing up solutions, in [4] a simultaneously blowing up solution for the system (1.3) is constructed such that the real part u and the imaginary part v of z blow up simultaneously at $x = 0$ and such that the real part dominates the imaginary part. Let us point also that the previous asymptotic behavior is not described in [3]. Moreover, the convergence results in [3] is proved under the assumptions (1.9), (1.10).

Remark 1.5. By the previous result we have the following comparison between the real part and the imaginary part of the solution:

- (i) If $\alpha_1 \geq 1$ and $\alpha'_1 = \alpha_1$, for large time t , we have $\|u(t)\|_\infty \sim t^{-\alpha_1}$ and $\|v(t)\|_\infty \sim t^{-\alpha_1}$, so, the real part $Re(z)$ and the imaginary part $Im(z)$ have the same behavior as $t \rightarrow \infty$ and we write $\|Re(z)(t)\|_\infty \sim \|Im(z)(t)\|_\infty$.
- (ii) If $\alpha_1 > 1$ and $\alpha'_1 > \alpha_1$, here we note that the real part $Re(z)$ dominates the imaginary part $Im(z)$ and we write $\|Im(z)(t)\|_\infty \ll \|Re(z)(t)\|_\infty$.
- (iii) If $\alpha_1 > 1$, $2\alpha'_1 - \alpha_1 \geq 1$ and $\alpha'_1 < \alpha_1$, here we note that the imaginary part $Im(z)$ dominates the real part $Re(z)$ and we write $\|Re(z)(t)\|_\infty \ll \|Im(z)(t)\|_\infty$.

In this work we study the global existence and the different asymptotic behaviors of mild global solutions for the following general parabolic system

$$\begin{cases} \partial_t u = \Delta u + a u^2 + b v^2, \\ \partial_t v = \Delta v + c uv, \end{cases} \quad (1.24)$$

with initial value

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad (1.25)$$

where $u(t, x)$ and $v(t, x)$ are real valued functions, $t > 0$, $x \in \mathbb{R}^N$, $a, b, c \in \mathbb{R}$. Note that system (1.24) is a generalization of the following equation

$$\partial_t z = \Delta z + a z^2, \quad (1.26)$$

for $t > 0$, $x \in \mathbb{R}^N$ and with initial data $z_0 = u_0 + iv_0$, $a \in \mathbb{R}$ and $b = -a$, $c = 2a$. By the well-known Duhamel formulation, we see that the problem can be further reduced into the following system of integral equations.

$$u(t) = e^{t\Delta} u_0 + a \int_0^t e^{(t-s)\Delta} (u^2(s)) ds + b \int_0^t e^{(t-s)\Delta} (v^2(s)) ds, \quad (1.27)$$

$$v(t) = e^{t\Delta} v_0 + c \int_0^t e^{(t-s)\Delta} (u(s)v(s)) ds, \quad (1.28)$$

where $e^{t\Delta}$ is the heat operator which can be regarded as the convolution with the heat kernel.

The method introduced by [1, 6–8] uses the integral formulation of the semi-linear equation or system under study. So, later on we will work on the above system of integral equations instead of (1.24). The first step is to prove the existence and uniqueness of global solutions for the integral formulation (1.27)–(1.28) in some functional spaces, and allowing to use a suitable contraction mapping argument on the whole time interval $(0, \infty)$. Second, we will construct self-similar solutions for the corresponding asymptotic system of (1.24):

$$\partial_t w_1 = \Delta w_1 + a \mu w_1^2 + b \nu w_2^2, \quad (1.29)$$

$$\partial_t w_2 = \Delta w_2 + c \mu w_1 w_2, \quad (1.30)$$

where $w_1(t, x)$ and $w_2(t, x)$ are real valued functions, $t > 0$, $x \in \mathbb{R}^N$, a, b, c are the same parameters as in system (1.24). The parameters μ and ν are defined by

$$\mu = \lim_{s \rightarrow \infty} s^{-2(\alpha_1 - 1)}, \quad \nu = \lim_{s \rightarrow \infty} s^{-2(2\alpha'_1 - \alpha_1 - 1)}. \quad (1.31)$$

In treating self-similar solutions, we shall use the standard method introduced in [6] (see also [1, 7, 8]). From the scaling principle, it is easy to see that if $u(t, x)$, $v(t, x)$ are a solutions of the Cauchy problem (1.24), then, the rescaled functions

$$u_\lambda(t, x) = \lambda^{2\alpha_1} u(\lambda^2 t, \lambda x), \quad v_\lambda(t, x) = \lambda^{2\alpha'_1} v(\lambda^2 t, \lambda x), \quad \forall \lambda > 0, \quad (1.32)$$

are also a solution of the system (1.24) with initial value $u_\lambda(0, x) = \lambda^{2\alpha_1} u_0(\lambda x)$, $v_\lambda(0, x) = \lambda^{2\alpha'_1} v_0(\lambda x)$. Then, a solution (u, v) is said to be self-similar when

$$(u(t, x), v(t, x)) = (\lambda^{2\alpha_1} u(\lambda^2 t, \lambda x), \lambda^{2\alpha'_1} v(\lambda^2 t, \lambda x)), \quad \forall \lambda > 0. \quad (1.33)$$

Finally, for the asymptotic behavior results, we prove that some of the global solutions of the system (1.27)–(1.28) are asymptotic, for large time, to self-similar solutions of integral system related to (1.29)–(1.30) with initial value $(c_1|x|^{-2\alpha_1}, c_2|x|^{-2\alpha'_1})$.

The rest of this paper is organized as follows. In the next section, we present some preliminary lemmas which will be needed in the proofs of the theorems. Furthermore, we state and we prove the main results concerning the global existence and continuous dependence. In Sect. 3, we prove the existence of global solutions and self-similar solutions for the asymptotic systems, also we demonstrate Theorems 1.1 and 1.2. Finally, in Sect. 4, we study the asymptotic behaviors for solutions of these systems with small initial values with respect to some norm and we prove Theorem 1.3. In this paper, we will denote by C a positive constant which can be different at different places and also denote it by C_δ to indicate that it depends on a real number δ . We sometimes denote $u(t, \cdot)$ by $u(t)$.

2. Global existence

Throughout this section, we give in the first subsection the following results which will be used later in the second subsection for the proof of the existence of global solution to (1.27)–(1.28).

2.1. Technical results

In this subsection, we shall state some basic facts and obtain auxiliary results. First of all, for $1 \leq r \leq \infty$, $L^r(\mathbb{R}^N)$ denotes the usual Lebesgue space on $\mathbb{R}^N$ with the norm $\|\cdot\|_r$. Let $e^{t\Delta}$ be the linear heat semigroup which is defined by

$$(e^{t\Delta}\varphi)(x) = (G(t) \star \varphi)(x),$$

where G is the heat kernel given by

$$G(t, x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, x \in \mathbb{R}^N, \quad (2.1)$$

and $\star$ stands for the convolution product. We consider α_1, α'_1 two real number such that

$$\alpha_1 \geq 1, \quad 2\alpha'_1 \geq \alpha_1 + 1, \quad (2.2)$$

and $\frac{N}{2\alpha_1} > 1, \frac{N}{2\alpha'_1} > 1$. One can choose a real number r_1 which satisfy

$$1 < \frac{\alpha_1}{1 + \alpha_1} r_1 < \frac{N}{2\alpha_1} < r_1 \quad \text{and} \quad \frac{\alpha'_1}{1 + \alpha'_1} r_1 < \frac{N}{2\alpha'_1} < r_1. \quad (2.3)$$

For r_1 satisfying (2.3), define the real numbers r_2, s_1 and s_2 by

$$r_2 = \frac{\alpha_1}{\alpha_2} r_1, \quad s_1 = \frac{\alpha_1}{\alpha'_1} r_1, \quad s_2 = \frac{\alpha_1}{\alpha'_2} r_1, \quad (2.4)$$

where α_2, α'_2 be two real number satisfying

$$\alpha_2 \leq 1, \quad 2\alpha'_2 \leq \alpha_2 + 1. \quad (2.5)$$

Now, for $\Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$, define $\mathcal{N}$ by

$$\mathcal{N}(\Phi) = \sup_{t>0} [t^{\beta_1} \|e^{t\Delta} \varphi_1\|_{r_1}, t^{\beta_2} \|e^{t\Delta} \varphi_1\|_{r_2}, t^{\gamma_1} \|e^{t\Delta} \varphi_2\|_{s_1}, t^{\gamma_2} \|e^{t\Delta} \varphi_2\|_{s_2}], \quad (2.6)$$

where $\beta_1, \beta_2, \gamma_1$ and γ_2 are given by

$$\beta_1 = \alpha_1 - \frac{N}{2r_1}, \quad \gamma_1 = \alpha'_1 - \frac{N}{2s_1}, \quad (2.7)$$

$$\beta_2 = \alpha_2 - \frac{N}{2r_2}, \quad \gamma_2 = \alpha'_2 - \frac{N}{2s_2}. \quad (2.8)$$

Now, we will give some preliminary lemmas which will be needed in the proofs of the main theorems. Lemma 2.1 is a direct consequence of the conditions (2.3)–(2.4) and (2.7)–(2.8). We will need some real interpolating number defined in the Lemma 2.3 which satisfy the conditions stated in Corollary 2.5. As well as, in the Corollary 2.5, the Part (1) represent compatibility conditions for the heat semigroup, that is $e^{t\Delta}$ maps between the appropriate Lebesgue spaces. The Part (2) is integrability conditions: to assure that the various integrals are convergent. Finally, Part (3) will allow the contraction mapping argument to be done on the time interval $(0, \infty)$ directly.

Lemma 2.1. *Let N be an integer. Let α_1, α'_1 be positive real numbers which satisfy (2.2). Let α_2, α'_2 be positive real numbers which satisfy (2.5). Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_1 be a real number satisfying (2.3) and let r_2, s_1 and s_2 be given by (2.4). Let $\beta_1, \beta_2, \gamma_1$ and γ_2 be given by (2.7) and (2.8). Then, we have the following:

- (i) $0 < \beta_2 < \beta_1, \quad 0 < \gamma_2 < \gamma_1,$
- (ii) $1 < \frac{\alpha_j}{1+\alpha_j} r_j < \frac{N}{2\alpha_j} < r_j, \quad 1 < \frac{\alpha'_j}{1+\alpha'_j} s_j < \frac{N}{2\alpha'_j} < s_j, \text{ for } j = 1, 2,$
- (iii) $\frac{1+\alpha_j}{\alpha_j} \beta_j < 1, \quad \frac{1+\alpha'_j}{\alpha'_j} \gamma_j < 1, \text{ for } j = 1, 2,$
- (iv) $\beta_j - \frac{N}{2\alpha_j r_j} - \frac{1+\alpha_j}{\alpha_j} \beta_j + 1 = 0, \quad \gamma_j - \frac{N}{2\alpha'_j s_j} - \frac{1+\alpha'_j}{\alpha'_j} \gamma_j + 1 = 0, \text{ for } j = 1, 2.$

Proof. The lemma follows from the expressions of r_1, r_2, s_1 and s_2 by a direct consequence of the conditions (2.3)–(2.4) and from the expressions of $\beta_1, \beta_2, \gamma_1$ and γ_2 concluded from the conditions (2.7)–(2.8). $\square$

Lemma 2.2. *Let $\alpha_1, \alpha'_1, \alpha_2$ and α'_2 be positive real numbers, then, they satisfy respectively (2.2) and (2.5) if and only if the following conditions hold:*

- (a) $0 < \alpha_2 \leq \alpha_1$ and $0 < \alpha'_2 \leq \alpha'_1,$
- (b) $\frac{\alpha_2}{\alpha_1} \leq \frac{\alpha_1+1}{2\alpha_1} \leq 1 \leq \frac{\alpha_2+1}{2\alpha_2} < \frac{\alpha_1}{\alpha_2}, \quad \frac{\alpha'_2}{\alpha'_1} \leq \frac{\alpha'_1+1}{2\alpha'_1} \leq 1 \leq \frac{\alpha'_2+1}{2\alpha'_2} < \frac{\alpha'_1}{\alpha'_2},$
- (c) $\frac{\alpha'_2}{\alpha_2} \leq \frac{\alpha_2+1}{2\alpha_2} \leq \frac{\alpha'_1}{\alpha_2}, \quad \frac{\alpha'_2}{\alpha_1} \leq \frac{\alpha_1+1}{2\alpha_1} \leq \frac{\alpha'_1}{\alpha_1}.$

Lemma 2.3. *Let N be an integer. Let α_1, α'_1 be positive real numbers which satisfy (2.2). Let α_2, α'_2 be positive real numbers which satisfy (2.5). Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_j, s_j, β_j and γ_j for $j = 1, 2$ be as in Lemma 2.1. Then, there exists the interpolating real numbers $r_{1j}, \beta_{1j}, \tau_{2j}$ and Γ_{2j} , $j = 1, 2$ which are defined as follow:

$$r_{1j} = \frac{2\alpha_j}{1 + \alpha_j} r_j; \quad \beta_{1j} = \frac{1 + \alpha_j}{2\alpha_j} \beta_j; \quad \tau_{2j} = \frac{\alpha'_j}{1 + \alpha'_j} s_j; \quad \Gamma_{2j} = \frac{1 + \alpha'_j}{\alpha'_j} \gamma_j.$$

Then, we have the following

$$r_{1j} \in [r_1, r_2] \cap [s_1, s_2], \quad \beta_{1j} \in [\beta_2, \beta_1] \cap [\gamma_2, \gamma_1], \quad \forall j = 1, 2.$$

Furthermore, there exists $r_{2j} \in [r_1, r_2]$, $s_{2j} \in [s_1, s_2]$, $\beta_{2j} \in [\beta_2, \beta_1]$ and $\gamma_{2j} \in [\gamma_2, \gamma_1]$ such that

$$\frac{1}{\tau_{2j}} = \frac{1}{r_{2j}} + \frac{1}{s_{2j}}, \quad \Gamma_{2j} = \beta_{2j} + \gamma_{2j}. \quad (2.9)$$

Also, there exists $0 \leq \theta_{1j}, \theta'_{1j} \leq 1$, for $j = 1, 2$ such that

$$\frac{1}{r_{1j}} = \frac{\theta_{1j}}{r_1} + \frac{1 - \theta_{1j}}{r_2}; \quad \beta_{1j} = \theta_{1j}\beta_1 + (1 - \theta_{1j})\beta_2, \quad (2.10)$$

$$\frac{1}{r_{1j}} = \frac{\theta'_{1j}}{s_1} + \frac{1 - \theta'_{1j}}{s_2}; \quad \beta_{1j} = \theta'_{1j}\gamma_1 + (1 - \theta'_{1j})\gamma_2. \quad (2.11)$$

Moreover, there exists $\theta_{2j}, \theta'_{2j} \in [0, 1]$, for $j = 1, 2$ such that

$$\begin{aligned} \frac{1}{r_{2j}} &= \frac{\theta_{2j}}{r_1} + \frac{1 - \theta_{2j}}{r_2}; & \beta_{2j} &= \theta_{2j}\beta_1 + (1 - \theta_{2j})\beta_2; \\ \frac{1}{s_{2j}} &= \frac{\theta'_{2j}}{s_1} + \frac{1 - \theta'_{2j}}{s_2}; & \gamma_{2j} &= \theta'_{2j}\gamma_1 + (1 - \theta'_{2j})\gamma_2. \end{aligned} \quad (2.12)$$

Proof. Owing to the conditions (2.3)–(2.4) and (2.7)–(2.8) and Lemma 2.1, the proof of the lemma is obvious and left to the reader. $\square$

Remark 2.4. As a choice, we can take, for $j = 1, 2$

$$r_{2j} = s_{2j} = \frac{2\alpha'_j}{1 + \alpha'_j} s_j; \quad \beta_{2j} = \gamma_{2j} = \frac{1 + \alpha'_j}{2\alpha'_j} \gamma_j.$$

As a corollary of the previous lemmas we have the following result.

Corollary 2.5. *Let N be an integer. Let α_1, α'_1 be positive real numbers which satisfy (2.2). Let α_2, α'_2 be positive real numbers which satisfy (2.5). Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Let $r_1, r_2, s_1, s_2, \beta_1, \beta_2, \gamma_1$ and γ_2 be as in Lemma 2.1. Let $r_{1j}, \tau_{2j}, \beta_{1j}$ and Γ_{2j} , $j = 1, 2$ be as in Lemma 2.3. Then, for $j = 1, 2$, we have:

- (1) $1 < \frac{r_{1j}}{2} < r_j$, $1 < \tau_{2j} < s_j$,
- (2) $2\beta_{1j} < 1$, $\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j}) < 1$, $\Gamma_{2j} < 1$, $\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j}) < 1$,
- (3) $1 + \beta_j - \frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j}) - 2\beta_{1j} = 0$, $1 + \gamma_j - \frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j}) - \Gamma_{2j} = 0$.

Proof. Owing to the conditions satisfied by r_1 , r_2 , s_1 , and s_2 in the Lemmas 2.1 and 2.3, and due to the expressions of β_1 , β_2 , γ_1 and γ_2 , the proof of Corollary 2.5 is simple and can be omitted. $\square$

2.2. Global existence

In this subsection, we give the Theorem and the proof of the global existence and continuous dependence of solutions for the system (1.27)–(1.28).

Theorem 2.6. (Global existence and continuous dependence) *Let N be an integer. Let α_1 , α'_1 , α_2 and α'_2 be positive real numbers satisfying*

$$\alpha_1 \geq 1, \quad 2\alpha'_1 \geq \alpha_1 + 1, \quad \alpha_2 \leq 1, \quad 2\alpha'_2 \leq \alpha_2 + 1.$$

Suppose that

$$\frac{N}{2\alpha_1} > 1 \quad \text{and} \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_1 be a real number satisfying (2.3) and let r_2 , s_1 and s_2 be defined in (2.4) β_1 , γ_1 and β_2 , γ_2 be defined by (2.7) and (2.8) respectively. Let $M > 0$ be such that

$$C := C(M) < 1, \tag{2.13}$$

where $C(M)$ is a positive constant given by (2.38) below. Choose $R > 0$ such that

$$R + CM \leq M. \tag{2.14}$$

If $\Phi = (\varphi_1, \varphi_2)$ be an element of $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ such that

$$\mathcal{N}(\Phi) \leq R, \tag{2.15}$$

where $\mathcal{N}$ is given by (2.6), then, there exists a unique global solution $U = (u, v)$ of the integral system (1.27)–(1.28) such that

$$\sup_{t>0} [t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|u(t)\|_{r_2}, t^{\gamma_1} \|v(t)\|_{s_1}, t^{\gamma_2} \|v(t)\|_{s_2}] \leq M. \tag{2.16}$$

Furthermore,

- (a) $u(t) - e^{t\Delta} \varphi_1 \in C([0, \infty), L^\tau(\mathbb{R}^N))$, for $\frac{\alpha_1 r_1}{1+\alpha_1} < \tau < \frac{N}{2\alpha_1}$.
- (b) $v(t) - e^{t\Delta} \varphi_2 \in C([0, \infty), L^{\tau'}(\mathbb{R}^N))$, for $\frac{\alpha'_1 s_1}{1+\alpha'_1} < \tau' < \frac{N}{2\alpha'_1}$.
- (c) $u(t) - e^{t\Delta} \varphi_1 \in L^\infty([0, \infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N))$, $v(t) - e^{t\Delta} \varphi_2 \in L^\infty([0, \infty), L^{\frac{N}{2\alpha'_1}}(\mathbb{R}^N))$.
- (d) $\lim_{t \searrow 0} u(t) = \varphi_1$ and $\lim_{t \searrow 0} v(t) = \varphi_2$ in the sense of tempered distributions.

Moreover, the global solution $U = (u, v)$ satisfies

$$\sup_{t>0} \left[t^{\alpha_1 - \frac{N}{2r}} \|u(t)\|_r, t^{\alpha_2 - \frac{N}{2r'}} \|u(t)\|_{r'}, t^{\alpha'_1 - \frac{N}{2s}} \|v(t)\|_s, t^{\alpha'_2 - \frac{N}{2s'}} \|v(t)\|_{s'} \right] < \infty, \tag{2.17}$$

for all $r \in [r_1, \infty]$, $r' \in [r_2, \infty]$, $s \in [s_1, \infty]$ and $s' \in [s_2, \infty]$.

In addition, if $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be respectively the solutions of (1.27)–(1.28) satisfying (2.16) with initial values $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ which verify (2.15), then, for all $r \in [r_1, \infty]$, $r' \in [r_2, \infty]$, $s \in [s_1, \infty]$ and $s' \in [s_2, \infty]$, we have

$$\sup_{t>0} [t^{\beta_1} \|u_1(t) - u_2(t)\|_r, t^{\beta_2} \|u_1(t) - u_2(t)\|_{r'}, \\ t^{\gamma_1} \|v_1(t) - v_2(t)\|_s, t^{\gamma_2} \|v_1(t) - v_2(t)\|_{s'}] \leq (1 - C)^{-1} \mathcal{N}(\Phi - \Psi), \quad (2.18)$$

where C is given in Theorem 2.6 by (2.38) below.

Proof. We look for global mild solutions of the system (1.27)–(1.28) with initial data $\Phi = (\varphi_1, \varphi_2)$ via a fixed point argument. Let X be the set of Bochner measurable functions

$$U : (0, \infty) \longrightarrow (L^{r_1}(\mathbb{R}^N) \cap L^{r_2}(\mathbb{R}^N)) \times (L^{s_1}(\mathbb{R}^N) \cap L^{s_2}(\mathbb{R}^N)) \\ t \longmapsto U(t) := (u(t), v(t))$$

such that

$$\|U\|_X = \sup_{t>0} [t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|u(t)\|_{r_2}, t^{\gamma_1} \|v(t)\|_{s_1}, t^{\gamma_2} \|v(t)\|_{s_2}] < \infty. \quad (2.19)$$

Let M be a positive real number. We denote by X_M the set of $u \in X$ such that $\|U\|_X \leq M$. Endowed with the metric $d(U_1, U_2) = \|U_1 - U_2\|_X$, X_M is a complete metric space. Define the mapping $\mathcal{F}_\Phi : X_M \longrightarrow X_M$ by

$$\mathcal{F}_\Phi(u, v) = (F_\Phi(u, v), G_\Phi(u, v)), \quad (2.20)$$

with

$$F_\Phi(u, v)(t) = e^{t\Delta} \varphi_1 + a \int_0^t e^{(t-s)\Delta} (u^2(s)) ds + b \int_0^t e^{(t-s)\Delta} (v^2(s)) ds, \quad (2.21)$$

$$G_\Phi(u, v)(t) = e^{t\Delta} \varphi_2 + c \int_0^t e^{(t-s)\Delta} (u(s)v(s)) ds, \quad (2.22)$$

where $\Phi = (\varphi_1, \varphi_2)$ belongs to $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ satisfies (2.15). We will show that $\mathcal{F}_\Phi$ given by (2.20) is a strict contraction mapping on X_M for suitable real numbers M and R . Let $\Phi = (\varphi_1, \varphi_2)$, $\Psi = (\psi_1, \psi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ satisfies (2.15) and $U_1 = (u_1, v_1)$, $U_2 = (u_2, v_2)$ be two elements of X_M . Then, we estimate $t^{\gamma_j} \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{s_j}$ and $t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j}$, for $j = 1, 2$. Thus, by using the smoothing effect

$$\|e^{t\Delta} \varphi\|_q \leq Ct^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_p, \quad (2.23)$$

for all $t > 0$, $\varphi \in L^p(\mathbb{R}^N)$, C is a positive constant and $1 \leq p \leq q \leq \infty$ and due to Corollary 2.5 part (1), we apply (2.23) for $(p, q) = (\tau_{2j}, s_j)$, $j = 1, 2$ and $\varphi = (u_1(s)v_1(s) - u_2(s)v_2(s))$, we write

$$t^{\gamma_j} \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{s_j} \leq t^{\gamma_j} \|e^{t\Delta} (\varphi_2 - \psi_2)\|_{s_j} + 2C|c|t^{\gamma_j} \\ \times \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j})} \\ \times \|u_1(s)v_1(s) - u_2(s)v_2(s)\|_{\tau_{2j}} ds, \quad (2.24)$$

where τ_{2j} are given by (2.9) for $j = 1, 2$. Now, we remark that

$$\|u_1 v_1 - u_2 v_2\|_{\tau_{2j}} \leq \|u_1(v_1 - v_2)\|_{\tau_{2j}} + \|v_2(u_1 - u_2)\|_{\tau_{2j}}.$$

Due to (2.9), we use the following Hölder inequality

$$\|fg\|_{\tau_{2j}} \leq \|f\|_{r_{2j}} \|g\|_{s_{2j}}, \quad (2.25)$$

for all $f \in L^{r_{2j}}(\mathbb{R}^N)$ and $g \in L^{s_{2j}}(\mathbb{R}^N)$, to obtain

$$\|u_1(v_1 - v_2)\|_{\tau_{2j}} + \|v_2(u_1 - u_2)\|_{\tau_{2j}} \leq \|u_1\|_{r_{2j}} \|v_1 - v_2\|_{s_{2j}} + \|v_2\|_{s_{2j}} \|u_1 - u_2\|_{r_{2j}}. \quad (2.26)$$

Injecting (2.26) in (2.24), we have

$$\begin{aligned} t^{\gamma_j} \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{s_j} &\leq t^{\gamma_j} \|e^{t\Delta}(\varphi_2 - \psi_2)\|_{s_j} + 2C|c|t^{\gamma_j} \\ &\quad \times \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j})} \left(\|u_1(s)\|_{r_{2j}} \|v_1(s) \right. \\ &\quad \left. - v_2(s)\|_{s_{2j}} + \|v_2(s)\|_{s_{2j}} \|u_1(s) - u_2(s)\|_{r_{2j}} \right) ds. \end{aligned} \quad (2.27)$$

By Lemma 2.3, we apply on the second term of the right-hand side of the last inequality the interpolation inequality

$$\|f\|_s \leq \|f\|_r^\theta \|f\|_\tau^{1-\theta}, \quad f \in L^r(\mathbb{R}^N) \cap L^\tau(\mathbb{R}^N), \quad (2.28)$$

where

$$1 \leq r \leq \tau \leq \infty, \quad \frac{1}{s} = \frac{\theta}{r} + \frac{1-\theta}{\tau}, \quad \theta \in [0, 1],$$

once with $(s, r, \tau) = (r_{2j}, r_1, r_2)$, $\theta = \theta_{2j}$, $f = u_1$ and $(s, r, \tau) = (s_{2j}, s_1, s_2)$, $\theta = \theta'_{2j}$, $f = v_1 - v_2$, for $j = 1, 2$ and once with the same triplet but with $f = u_1 - u_2$ for the first and $f = v_2$ for the second, with the fact that U_1 and U_2 belongs to X_M , we obtain

$$\begin{aligned} &t^{\gamma_j} \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{s_j} \\ &\leq t^{\gamma_j} \|e^{t\Delta}(\varphi_2 - \psi_2)\|_{s_j} + 2MC|c|t^{\gamma_j} \\ &\quad \times \left(\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j})} s^{-\Gamma_{2j}} ds \right) \|U_1 - U_2\|_X \\ &\leq \mathcal{N}(\Phi - \Psi) + 2MC|c|t^{1+\gamma_j - \frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j}) - \Gamma_{2j}} \\ &\quad \times \left(\int_0^1 (1-s)^{-\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j})} s^{-\Gamma_{2j}} ds \right) \\ &\quad \times \|U_1 - U_2\|_X. \end{aligned} \quad (2.29)$$

From properties (2) and (3) of Corollary 2.5, we deduce

$$t^{\gamma_j} \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{s_j} \leq \mathcal{N}(\Phi - \Psi) + C_{2j} \|U_1 - U_2\|_X, \quad (2.30)$$

where

$$C_{2j} = 2|c|MC \int_0^1 (1-s)^{-\frac{N}{2}(\frac{1}{\tau_{2j}} - \frac{1}{s_j})} s^{-(\beta_{2j} + \gamma_{2j})} ds, \quad (2.31)$$

which is a finite positive constant from property (2) of Corollary 2.5.

We now estimate $t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j}$. In fact, we have for $U_1, U_2 \in X_M$

$$\begin{aligned} t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j} &\leq t^{\beta_j} \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_j} \\ &\quad + |a| t^{\beta_j} \int_0^t \left\| e^{(t-s)\Delta} \left(u_1^2(s) - u_2^2(s) \right) \right\|_{r_j} ds \\ &\quad + |b| t^{\beta_j} \int_0^t \left\| e^{(t-s)\Delta} \left(v_1^2(s) - v_2^2(s) \right) \right\|_{r_j} ds. \end{aligned}$$

We compute, using the smoothing properties of the heat semigroup (2.23) with $(p, q) = (\frac{r_{1j}}{2}, r_j)$ on the second and the third term of the right-hand side of the last inequality and part (1) of Corollary 2.5, for $j = 1, 2$, we have

$$\begin{aligned} t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j} &\leq t^{\beta_j} \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_j} \\ &\quad + |a| C t^{\beta_j} \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} \|u_1^2(s) - u_2^2(s)\|_{r_{1j}/2} ds \\ &\quad + |b| C t^{\beta_j} \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} \|v_1^2(s) - v_2^2(s)\|_{r_{1j}/2} ds. \end{aligned} \quad (2.32)$$

It follows by the Hölder inequality, that for $j = 1, 2$

$$\begin{aligned} t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j} &\leq t^{\beta_j} \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_j} + |a| C t^{\beta_j} \\ &\quad \times \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} (\|u_1(s)\|_{r_{1j}} + \|u_2(s)\|_{r_{1j}}) \|u_1(s) - u_2(s)\|_{r_{1j}} ds \\ &\quad + |b| C t^{\beta_j} \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} (\|v_1(s)\|_{r_{1j}} + \|v_2(s)\|_{r_{1j}}) \\ &\quad \times \|v_1(s) - v_2(s)\|_{r_{1j}} ds. \end{aligned} \quad (2.33)$$

Using Lemma 2.3 and the interpolation inequality (2.28) with $(s, r, \tau) = (r_{1j}, r_1, r_2)$, $\theta = \theta_{1j}$ for $f \in \{u_1, u_2, u_1 - u_2\}$ on the second term of the right-hand side of the last inequality and with $(s, r, \tau) = (r_{1j}, s_1, s_2)$, $\theta = \theta'_{1j}$ for $f \in \{v_1, v_2, v_1 - v_2\}$ on the third term, and the fact that U_1 and U_2 belongs to X_M , we see that inequality (2.33) becomes

$$\begin{aligned} t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j} &\leq \mathcal{N}(\Phi - \Psi) + 2CM \\ &\quad \times (|a| + |b|) t^{1+\beta_j - \frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j}) - 2\beta_{1j}} \\ &\quad \times \left(\int_0^1 (1-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} s^{-2\beta_{1j}} ds \right) \|U_1 - U_2\|_X. \end{aligned} \quad (2.34)$$

We deduce, by using the expressions of r_{1j} and β_{1j} introduced in Lemma 2.3, and their properties given in (2) and (3) of Corollary 2.5 that

$$t^{\beta_j} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_j} \leq \mathcal{N}(\Phi - \Psi) + C_{1j} \|U_1 - U_2\|_X, \quad (2.35)$$

where

$$C_{1j} = 2M C(|a| + |b|) \int_0^1 (1-s)^{-\frac{N}{2}(\frac{2}{r_{1j}} - \frac{1}{r_j})} s^{-2\beta_{1j}} ds, \quad (2.36)$$

it follows by property (2) of Corollary 2.5 that C_{1j} , for $j = 1, 2$ is a finite positive constant.

So, inequalities (2.30) and (2.35) with $j = 1, 2$ lead to

$$\|\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Psi(U_2)\|_X \leq \mathcal{N}(\Phi - \Psi) + C\|U_1 - U_2\|_X, \quad (2.37)$$

where

$$C := C(M) = \max\{C_{1j}, C_{2j}, j = 1, 2\}. \quad (2.38)$$

Now if we take $\Psi = 0$ and $U_2 = 0$, the inequality (2.37) becomes

$$\|\mathcal{F}_\Phi(U_1)\|_X \leq \mathcal{N}(\Phi) + C\|U_1\|_X. \quad (2.39)$$

If we choose M and R such that (2.13) and (2.14) are satisfied, then, by (2.39), $\mathcal{F}_\Phi$ maps X_M into itself. For $\Phi = \Psi$, (2.37) becomes

$$\|\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Psi(U_2)\|_X \leq C\|U_1 - U_2\|_X,$$

hence inequality (2.13) gives that $\mathcal{F}_\Phi$ is a strict contraction mapping from X_M into itself. So $\mathcal{F}_\Phi$ has a unique fixed point $U = (u, v)$ in X_M which is the solution of the integral system (1.27)–(1.28). This terminates the proof of the existence of a unique global solution of (1.27)–(1.28) in X_M .

Now, we will prove the statements (a)–(d). Let τ be a positive real number satisfying

$$\frac{\alpha_1}{1 + \alpha_1} r_1 < \tau < \frac{N}{2\alpha_1}, \quad (2.40)$$

we have

$$\|u(t) - e^{t\Delta} \varphi_1\|_\tau \leq |a|C \int_0^t \|e^{(t-s)\Delta}(u(s)^2)\|_\tau ds + |b|C \int_0^t \|e^{(t-s)\Delta}(v(s)^2)\|_\tau ds.$$

The smoothing properties of the heat semigroup (2.23), where $(p, q) = (\frac{r_{11}}{2}, \tau)$ on the first and the second term of the right-hand side of the last inequality and (1) with $j = 1$ of Corollary 2.5, give

$$\begin{aligned} \|u(t) - e^{t\Delta} \varphi_1\|_\tau &\leq |a|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{11}} - \frac{1}{\tau})} \|u(s)\|_{r_{11}}^2 ds \\ &\quad + |b|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{11}} - \frac{1}{\tau})} \|v(s)\|_{r_{11}}^2 ds. \end{aligned}$$

By using the interpolation inequality (2.28) with $(s, r, \tau) = (r_{11}, r_1, r_2)$, $\theta = \theta_{11}$, $f = u$ on the first term of the right-hand side of the last inequality and for the second term we take $(s, r, \tau) = (r_{11}, s_1, s_2)$, $\theta = \theta'_{11}$, $f = v$ and the fact that U_1 and U_2 belongs to X_M , we obtain the following estimates

$$\begin{aligned} \|u(t) - e^{t\Delta} \varphi_1\|_\tau &\leq |a|CM^2 \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{11}} - \frac{1}{\tau})} s^{-2\beta_{11}} ds \\ &\quad + |b|CM^2 \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_{11}} - \frac{1}{\tau})} s^{-2\beta_{11}} ds. \end{aligned}$$

Thanks to the expressions of r_{11} and β_{11} given in Lemma 2.3 and due to (2.7), we deduce

$$\|u(t) - e^{t\Delta}\varphi_1\|_\tau \leq CM^2(|a| + |b|)t^{\frac{N}{2\tau} - \alpha_1} \int_0^1 (1-s)^{-\frac{N}{2}(\frac{2}{\tau_{11}} - \frac{1}{\tau})} s^{-2\beta_{11}} ds. \quad (2.41)$$

Using (2) with $j = 1$ of Corollary 2.5 and the fact that $\tau < r_1$, the latter inequality gives

$$\|u(t) - e^{t\Delta}\varphi_1\|_\tau \leq C_1 t^{\frac{N}{2\tau} - \alpha_1}, \quad (2.42)$$

where C_1 is a finite positive constant.

On the other hand, we set τ' be a positive real number satisfying

$$\frac{\alpha'_1}{1 + \alpha'_1} r_1 < \tau' < \frac{N}{2\alpha'_1}. \quad (2.43)$$

We aim to estimate $\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'}$, writing

$$\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'} \leq |c|C \int_0^t \|e^{(t-s)\Delta}(u(s)v(s))\|_{\tau_{21}} ds,$$

then, by using the smoothing effect, where $(p, q) = (\tau_{21}, \tau')$

$$\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'} \leq |c|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{21}} - \frac{1}{\tau'})} \|u(s)v(s)\|_{\tau_{21}} ds.$$

Then, we apply Hölder inequality (2.25), for $j = 1$, we find

$$\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'} \leq |c|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{21}} - \frac{1}{\tau'})} \|u(s)\|_{r_{21}} \|v(s)\|_{s_{21}} ds.$$

Thus, by the interpolation inequality (2.28), with $(s, r, \tau) = (r_{21}, r_1, r_2)$, $\theta = \theta_{21}$ for $f = u$ and for $f = v$, we take $(s, r, \tau) = (s_{21}, s_1, s_2)$, $\theta = \theta'_{21}$, on the term in the right-hand side of the last inequality and the fact that U_1 and U_2 belongs to X_M , we obtain

$$\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'} \leq |c|CM^2 \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{21}} - \frac{1}{\tau'})} s^{-\Gamma_{21}} ds. \quad (2.44)$$

Thanks to the expressions of τ_{21} and Γ_{21} given in Lemma 2.3 and due to (2.7), we deduce

$$\|v(t) - e^{t\Delta}\varphi_2\|_\tau \leq |c|CM^2 t^{\frac{N}{2\tau'} - \alpha'_1} \int_0^1 (1-s)^{-\frac{N}{2}(\frac{1}{\tau_{21}} - \frac{1}{\tau'})} s^{-\Gamma_{21}} ds. \quad (2.45)$$

Using (2) with $j = 1$ of Corollary 2.5 and the fact that $\tau' < s_1$, the latter inequality gives

$$\|v(t) - e^{t\Delta}\varphi_2\|_{\tau'} \leq C_2 t^{\frac{N}{2\tau'} - \alpha'_1}, \quad (2.46)$$

where C_2 is a finite positive constant.

Owing to the condition (2.40) and (2.43), the right-hand sides of (2.42) and (2.46) converge to zero as $t \searrow 0$. This proves statements (a), (b) and (d) of Theorem 2.6. Also, it is clear that (2.42) and (2.46) still hold if $\tau = \frac{N}{2\alpha_1}$ and $\tau' = \frac{N}{2\alpha'_1}$, this proves statement (c).

Finally, we shall demonstrate (2.17). The proof of the $L^\infty \times L^\infty$ estimates is based on an iterative argument as in [8]. Let us denote the real numbers

r_1, r_2, s_1 and s_2 respectively by $\tau_0, \tau'_0, \sigma_0$ and σ'_0 . Choose real numbers $\tau_1, \tau'_1, \sigma_1$ and σ'_1 such that, for $j = 1, 2$, we have

$$\frac{1 + \alpha_j}{\alpha_j} < \tau_0 < \tau_1, \quad \frac{N}{2} \left(\frac{1 + \alpha_j}{\alpha_j} \frac{1}{\tau_0} - \frac{1}{\tau_1} \right) < 1, \quad (2.47)$$

$$\frac{1 + \alpha'_j}{\alpha_j} < \tau_0 < \tau_1, \quad \frac{N}{2} \frac{\alpha'_j}{\alpha_j} \left(\frac{1 + \alpha'_j}{\alpha'_j} \frac{1}{\tau_0} - \frac{1}{\tau_1} \right) < 1, \quad (2.48)$$

and

$$\tau'_1 = \frac{\alpha_1}{\alpha_2} \tau_1, \quad \sigma_1 = \frac{\alpha_1}{\alpha'_1} \tau_1, \quad \sigma'_1 = \frac{\alpha_1}{\alpha'_2} \tau_1. \quad (2.49)$$

Remark that a such choice is possible thanks to (2.3). We will concentrate first on the equation satisfied by v which is written as follows:

$$v(t) = e^{(t/2)\Delta} v(t/2) + c \int_{t/2}^t e^{(t-s)\Delta} (u(s)v(s)) ds.$$

Therefore, the conditions (2.48), for $j = 1$ and (2.49) allows us to use the smoothing effect of the heat semigroup (2.23), with $(p, q) = (\sigma_0, \sigma_1)$ on the first term and $(p, q) = (\tau_{21}, \sigma_1)$ on the second term of the right-hand side of the inequality, and by applying Hölder inequality (2.25), for $j = 1$, we obtain

$$\begin{aligned} & t^{\alpha'_1 - \frac{N}{2\sigma_1}} \|v(t)\|_{\sigma_1} \\ & \leq C(t/2)^{\alpha'_1 - \frac{N}{2\sigma_0}} \|v(t/2)\|_{\sigma_0} + |c| C t^{\alpha'_1 - \frac{N}{2\sigma_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2} \left(\frac{1}{\tau_{21}} - \frac{1}{\sigma_1} \right)} \\ & \quad \times \|u(s)v(s)\|_{\tau_{21}} ds, \\ & \leq C(t/2)^{\alpha'_1 - \frac{N}{2\sigma_0}} \|v(t/2)\|_{\sigma_0} + |c| C t^{\alpha'_1 - \frac{N}{2\sigma_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2} \left(\frac{1}{\tau_{21}} - \frac{1}{\sigma_1} \right)} \|u(s)\|_{\tau_{21}} \\ & \quad \times \|v(s)\|_{s_{21}} ds. \end{aligned}$$

Using the interpolation inequality (2.28), with $(s, r, \tau) = (r_{21}, \tau_0, \tau'_0)$, $\theta = \theta_{21}$ and $(s, r, \tau) = (s_{21}, \sigma_0, \sigma'_0)$, $\theta = \theta'_{21}$ respectively for $f = u$ and $f = v$, and the fact that $U \in X_M$ we obtain

$$t^{\alpha'_1 - \frac{N}{2\sigma_1}} \|v(t)\|_{\sigma_1} \leq C D_0 + |c| C D_0^2 \int_{1/2}^1 (1-s)^{-\frac{N}{2} \left(\frac{1}{\tau_{21}} - \frac{1}{\sigma_1} \right)} s^{-\Gamma_{21}} ds, \quad (2.50)$$

where

$$\begin{aligned} D_0 = \sup_{t>0} & \left[t^{\alpha_1 - \frac{N}{2\tau_0}} \|u(t)\|_{\tau_0}, t^{\alpha_2 - \frac{N}{2\tau'_0}} \|u(t)\|_{\tau'_0}, t^{\alpha'_1 - \frac{N}{2\sigma_0}} \|v(t)\|_{\sigma_0}, t^{\alpha'_2 - \frac{N}{2\sigma'_0}} \|v(t)\|_{\sigma'_0} \right] \\ & < \infty. \end{aligned} \quad (2.51)$$

With the fact that (2.48) is satisfied, for $j = 1$ and (2.49), we deduce that the inequality (2.50) leads to

$$\sup_{t>0} \left[t^{\alpha'_1 - \frac{N}{2\sigma_1}} \|v(t)\|_{\sigma_1} \right] \leq C(D_0) < \infty. \quad (2.52)$$

Now, we estimate $t^{\alpha'_2 - \frac{N}{2\sigma'_1}} \|v(t)\|_{\sigma'_1}$. Indeed, we have

$$\begin{aligned} & t^{\alpha'_2 - \frac{N}{2\sigma'_1}} \|v(t)\|_{\sigma'_1} \\ & \leq C(t/2)^{\alpha'_2 - \frac{N}{2\sigma'_0}} \|v(t/2)\|_{\sigma'_0} + |c|Ct^{\alpha'_2 - \frac{N}{2\sigma'_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2}\left(\frac{1}{\tau_{22}} - \frac{1}{\sigma'_1}\right)} \\ & \quad \times \|u(s)v(s)\|_{\tau_{22}} ds \\ & \leq C(t/2)^{\alpha'_2 - \frac{N}{2\sigma'_0}} \|v(t/2)\|_{\sigma'_0} + |c|Ct^{\alpha'_2 - \frac{N}{2\sigma'_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2}\left(\frac{1}{\tau_{22}} - \frac{1}{\sigma'_1}\right)} \|u(s)\|_{r_{22}} \\ & \quad \times \|v(s)\|_{s_{22}} ds, \end{aligned}$$

where we have used (2.48), for $j = 2$, (2.49) and the smoothing effect of the heat semigroup (2.23), with $(p, q) = (\sigma'_0, \sigma'_1)$ on the first term and $(p, q) = (\tau_{22}, \sigma'_1)$ on the second term in the first estimate and Hölder inequality (2.25), for $j = 2$ in the last one.

Now, applying the interpolation inequality (2.28), with $(s, r, \tau) = (r_{22}, \tau_0, \tau'_0)$, $\theta = \theta_{22}$ and $(s, r, \tau) = (s_{22}, \sigma_0, \sigma'_0)$, $\theta = \theta'_{22}$ respectively for $f = u$ and $f = v$, and due to $U \in X_M$, we obtain

$$t^{\alpha'_2 - \frac{N}{2\sigma'_1}} \|v(t)\|_{\sigma'_1} \leq CD_0 + |c|CD_0^2 \int_{1/2}^1 (1-s)^{-\frac{N}{2}\left(\frac{1}{\tau_{22}} - \frac{1}{\sigma'_1}\right)} s^{-\Gamma_{21}} ds,$$

where D_0 is given in (2.51). By using (2.48), with $j = 2$, the latter inequality leads to

$$\sup_{t>0} \left[t^{\alpha'_2 - \frac{N}{2\sigma'_1}} \|v(t)\|_{\sigma'_1} \right] \leq C(D_0) < \infty. \quad (2.53)$$

Also, we write

$$u(t) = e^{(t/2)\Delta} u(t/2) + a \int_{t/2}^t e^{(t-s)\Delta} (u^2(s)) ds + b \int_{t/2}^t e^{(t-s)\Delta} (v^2(s)) ds.$$

Then, due to (2.47), for $j = 1$ we use the smoothing properties of the heat semigroup (2.23) with $(p, q) \in \{(\tau_0, \tau_1), (\frac{\tau_{11}}{2}, \tau_1)\}$ to obtain

$$\begin{aligned} t^{\alpha_1 - \frac{N}{2\tau_1}} \|u(t)\|_{\tau_1} & \leq C(t/2)^{\alpha_1 - \frac{N}{2\tau_0}} \|u(t/2)\|_{\tau_0} \\ & \quad + |a|Ct^{\alpha_1 - \frac{N}{2\tau_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2}\left(\frac{2}{r_{11}} - \frac{1}{\tau_1}\right)} \|u(s)\|_{r_{11}}^2 ds \\ & \quad + |b|Ct^{\alpha_1 - \frac{N}{2\tau_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2}\left(\frac{2}{r_{11}} - \frac{1}{\tau_1}\right)} \|v(s)\|_{r_{11}}^2 ds. \end{aligned}$$

Using the interpolation inequality (2.28) by taking $(s, r, \tau) = (r_{11}, \tau_0, \tau'_0)$, $\theta = \theta_{11}$ for $f = u$ and $(s, r, \tau) = (r_{11}, \sigma_0, \sigma'_0)$, $\theta = \theta'_{11}$ for $f = v$, we obtain

$$t^{\alpha_1 - \frac{N}{2\tau_1}} \|u(t)\|_{\tau_1} \leq CD_0 + C(|a| + |b|)D_0^2 \int_{1/2}^1 (1-s)^{-\frac{N}{2}\left(\frac{2}{r_{11}} - \frac{1}{\tau_1}\right)} s^{-2\beta_{11}} ds,$$

where D_0 is given in (2.51). Then, by (2.47), for $j = 1$, we have

$$\sup_{t>0} \left[t^{\alpha_1 - \frac{N}{2\tau_1}} \|u(t)\|_{\tau_1} \right] \leq C(D_0) < \infty. \quad (2.54)$$

Then, to estimate $t^{\alpha_2 - \frac{N}{2\tau'_1}} \|u(t)\|_{\tau'_1}$, we use the smoothing properties of the heat semigroup (2.23) under the assumption (2.47), for $j = 2$ and (2.49), with $(p, q, \varphi) = (\tau'_0, \tau'_1, u)$ on the first term, $(p, q, \varphi) = (\frac{r_{12}}{2}, \tau'_1, u^2)$ on the second term and $(p, q, \varphi) = (\frac{r_{12}}{2}, \tau'_1, v^2)$ on the third term, we obtain

$$\begin{aligned} t^{\alpha_2 - \frac{N}{2\tau'_1}} \|u(t)\|_{\tau'_1} &\leq C(t/2)^{\alpha_2 - \frac{N}{2\tau'_0}} \|u(t/2)\|_{\tau'_0} \\ &\quad + |a| C t^{\alpha_2 - \frac{N}{2\tau'_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2} \left(\frac{2}{r_{12}} - \frac{1}{\tau'_1} \right)} \|u(s)\|_{\tau_{12}}^2 ds \\ &\quad + |b| C t^{\alpha_2 - \frac{N}{2\tau'_1}} \int_{t/2}^t (t-s)^{-\frac{N}{2} \left(\frac{2}{r_{12}} - \frac{1}{\tau'_1} \right)} \|v(s)\|_{\tau_{12}}^2 ds. \end{aligned}$$

We consider $(s, r, \tau) = (r_{12}, \tau'_0, \tau'_1)$, $\theta = \theta_{12}$ and $(s, r, \tau) = (r_{12}, \sigma'_0, \sigma'_1)$, $\theta = \theta'_{12}$ in the interpolation inequality (2.28) to apply it in the last inequality respectively for $f = u$ and $f = v$ and so, we obtain

$$t^{\alpha_2 - \frac{N}{2\tau'_1}} \|u(t)\|_{\tau'_1} \leq C D_0 + C(|a| + |b|) D_0^2 \int_{1/2}^1 (1-s)^{-\frac{N}{2} \left(\frac{2}{r_{12}} - \frac{1}{\tau'_1} \right)} s^{-2\beta_{12}} ds,$$

where D_0 is given in (2.51). Then, by (2.47), for $j = 2$ and (2.49), we have

$$\sup_{t>0} \left[t^{\alpha_2 - \frac{N}{2\tau'_1}} \|u(t)\|_{\tau'_1} \right] \leq C(D_0) < \infty. \quad (2.55)$$

The inequalities (2.52), (2.53), (2.54) and (2.55) imply that

$$D_1 = \sup_{t>0} \left[t^{\alpha_1 - \frac{N}{2\tau_1}} \|u(t)\|_{\tau_1}, t^{\alpha_2 - \frac{N}{2\tau'_1}} \|u(t)\|_{\tau'_1}, t^{\alpha'_1 - \frac{N}{2\sigma_1}} \|v(t)\|_{\sigma_1}, t^{\alpha'_2 - \frac{N}{2\sigma'_1}} \|v(t)\|_{\sigma'_1} \right] < \infty.$$

We iterate this procedure. For the next step, we consider $\tau_2, \tau'_2, \sigma_2$ and σ'_2 verifying, for $j = 1, 2$

$$\frac{1 + \alpha_j}{\alpha_j} < \tau_1 < \tau_2, \quad \frac{N}{2} \left(\frac{1 + \alpha_j}{\alpha_j} \frac{1}{\tau_1} - \frac{1}{\tau_2} \right) < 1, \quad (2.56)$$

$$\frac{1 + \alpha'_j}{\alpha'_j} < \tau_1 < \tau_2, \quad \frac{N}{2} \frac{\alpha'_j}{\alpha_j} \left(\frac{1 + \alpha'_j}{\alpha'_j} \frac{1}{\tau_1} - \frac{1}{\tau_2} \right) < 1, \quad (2.57)$$

and

$$\tau'_2 = \frac{\alpha_1}{\alpha_2} \tau_2, \quad \sigma_2 = \frac{\alpha_1}{\alpha'_1} \tau_2, \quad \sigma'_2 = \frac{\alpha_1}{\alpha'_2} \tau_2. \quad (2.58)$$

Using similar estimates as the previous step with the following interpolating numbers

$$\frac{1}{r_{1,kj}} = \frac{\theta_{kj}}{\tau_1} + \frac{1 - \theta_{kj}}{\tau'_1}, \quad \forall k, j = 1, 2,$$

and

$$\frac{1}{r_{1,1j}} = \frac{\theta'_{1j}}{\sigma_1} + \frac{1 - \theta'_{1j}}{\sigma'_1}; \quad \frac{1}{s_{1,2j}} = \frac{\theta'_{2j}}{\sigma_1} + \frac{1 - \theta'_{2j}}{\sigma'_1}, \quad \forall j = 1, 2,$$

with θ_{kj} and θ'_{kj} are as in Lemma 2.3 for $k, j = 1, 2$ and $\frac{1}{\tau_{1,2j}} = \frac{1}{r_{1,2j}} + \frac{1}{s_{1,2j}}$. Remark that we set $r_{0,1j} = r_{1j}$ and $\frac{1}{\tau_{0,2j}} = \frac{1}{r_{0,2j}} + \frac{1}{s_{0,2j}}$, where $r_{0,2j} = r_{2j}, s_{0,2j} = s_{2j}$, for $j = 1, 2$ for the first step. We obtain

$$D_2 = \sup_{t>0} \left[t^{\alpha_1 - \frac{N}{2\tau_2}} \|u(t)\|_{\tau_2}, t^{\alpha_2 - \frac{N}{2\tau'_2}} \|u(t)\|_{\tau'_2}, t^{\alpha'_1 - \frac{N}{2\sigma_2}} \|u(t)\|_{\sigma_2}, t^{\alpha'_2 - \frac{N}{2\sigma'_2}} \|u(t)\|_{\sigma'_2} \right] \leq C(M) < \infty.$$

Therefore, we construct the sequences $\{\tau_n\}_n$ which satisfies for $j = 1, 2$ the following conditions

$$\begin{aligned} \frac{1 + \alpha_j}{\alpha_j} < \tau_n < \tau_{n+1}, \quad \frac{N}{2} \left(\frac{1 + \alpha_j}{\alpha_j} \frac{1}{\tau_n} - \frac{1}{\tau_{n+1}} \right) < 1, \\ \frac{1 + \alpha'_j}{\alpha'_j} < \tau_n < \tau_{n+1}, \quad \frac{N}{2} \frac{\alpha'_j}{\alpha'_j} \left(\frac{1 + \alpha'_j}{\alpha'_j} \frac{1}{\tau_n} - \frac{1}{\tau_{n+1}} \right) < 1, \end{aligned}$$

for all integer $n \geq 0$. Then, we set the sequences $\{\tau'_n\}_n$, $\{\sigma_n\}_n$ and $\{\sigma'_n\}_n$ such that for all $n = 0, 1, 2, \dots$

$$\tau'_n = \frac{\alpha_1}{\alpha_2} \tau_n, \quad \sigma_n = \frac{\alpha_1}{\alpha'_1} \tau_n, \quad \sigma'_n = \frac{\alpha_1}{\alpha'_2} \tau_n, \quad (2.59)$$

One can check that we can construct a suitable sequence $\{\tau_n\}_n$ such that it can reach ∞ , for some finite n_0 , that is $\tau_{n_0} = \infty$, this prove (2.17), for $r = s = \infty$. The inequality (2.17) follows by an argument interpolation.

The continuous dependence inequality (2.18), for $(r, s) = (r_1, s_1)$ of the solution on the initial data results by considering $\mathcal{F}_\Phi(U_1) = U_1$ and $\mathcal{F}_\Phi(U_2) = U_2$ in the estimate (2.37). Formula (2.18), for $r = s = \infty$ can be proved starting with (2.18), for $(r, s) = (r_1, s_1)$ and using an iterative procedure similar to the proof of (2.17), for $r = s = \infty$. By Lemma of interpolation, we conclude (2.18). This achieves the proof of Theorem 2.6. $\square$

3. Self-similar solutions

This section is concerned with the mild solutions of the system (1.29)–(1.30) which have a self-similar structure with power $(2\alpha_1, 2\alpha'_1)$. The integral system related to (1.29)–(1.30) is given by

$$w_1(t) = e^{t\Delta} \varphi_1 + a\mu \int_0^t e^{(t-s)\Delta} (w_1^2(s)) ds + b\nu \int_0^t e^{(t-s)\Delta} (w_2^2(s)) ds, \quad (3.1)$$

$$w_2(t) = e^{t\Delta} \varphi_2 + c\mu \int_0^t e^{(t-s)\Delta} (w_1(s)w_2(s)) ds, \quad (3.2)$$

where a, b, c are the same parameters appearing in system (1.24) and μ, ν are given by (1.31). We first establish the global existence only with the norm $\mathcal{N}_s$. This is needed to prove the existence of self-similar solutions in Theorem 3.3.

In this section, we will use α_1 and α'_1 satisfying (2.2), r_1 satisfying (2.3) and s_1 be given by (2.4), β_1 and γ_1 be given by (2.7).

For $\Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$, we define $\mathcal{N}_s$ by

$$\mathcal{N}_s(\Phi) := \sup_{t>0} [t^{\beta_1} \|e^{t\Delta} \varphi_1\|_{r_1}, t^{\gamma_1} \|e^{t\Delta} \varphi_2\|_{s_1}]. \quad (3.3)$$

For $0 < \delta < \delta_0$, where δ_0 is given by

$$\delta_0 = \frac{N}{2r_1} \max \left[\frac{1 + \alpha_1}{\alpha_1}, \frac{1 + \alpha'_1}{\alpha_1} \right] - \max(\alpha_1, \alpha'_1), \quad (3.4)$$

we define $\mathcal{N}_\delta$ by

$$\mathcal{N}_\delta(\Phi) := \sup_{t>0} [t^{\beta_1+\delta} \|e^{t\Delta} \varphi\|_{r_1}, t^{\gamma_1+\delta} \|e^{t\Delta} \varphi\|_{s_1}]. \quad (3.5)$$

We first give examples of homogeneous initial values with finite norm $\mathcal{N}$ defined by (2.6). We first recall the following result due to [1, 8].

Proposition 3.1. [1, 8] *Let N be an integer. Let α_1, α_2 be two real numbers such that*

$$\alpha_1 \geq \alpha_2, \quad \frac{N}{2\alpha_1} > 1.$$

Let r_1, r_2 be two real numbers such that

$$r_2 = \frac{\alpha_1}{\alpha_2} r_1, \quad r_1 > \frac{N}{2\alpha_1}.$$

Let

$$\beta_1 = \alpha_1 - \frac{N}{2r_1}, \quad \beta_2 = \alpha_2 - \frac{N}{2r_2}.$$

Let

$$\varphi(x) = \omega \left(\frac{x}{|x|} \right) |x|^{-2\alpha_1},$$

where $\omega \in L^{r_2}(S^{N-1})$. Then, for any cut-off function η satisfying

$$\eta \equiv 1 \quad \text{near the origin and having a compact support,} \quad (3.6)$$

we have the following:

- (i) $\sup_{t>0} t^{\beta_1} \|e^{t\Delta}(\varphi)\|_{r_1} < \infty$,
- (ii) $\sup_{t>0} t^{\beta_1+\delta} \|e^{t\Delta}(\eta\varphi)\|_{r_1} < \infty$, for $0 < \delta < \frac{N}{2} - \alpha_1$,
- (iii) $\sup_{t>0} t^{\beta_1} \|e^{t\Delta}((1-\eta)\varphi)\|_{r_1} < \infty$;
- (iv) $\sup_{t>0} t^{\beta_2} \|e^{t\Delta}((1-\eta)\varphi)\|_{r_2} < \infty$.

Also, we give examples of initial values with finite norm $\mathcal{N}_s$ given by (3.3). These initial values are used in Theorems 3.3 and 4.2 below.

Proposition 3.2. [6, Proposition 2.3] *Let N be an integer. Let α_1, α'_1 be positive real numbers which satisfy (2.2). Let α_2, α'_2 be positive real numbers which satisfy (2.5). Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_1 be a real number satisfying (2.3) and r_2, s_1 and s_2 be given by (2.4). Let β_1, γ_1 and β_2, γ_2 be given by (2.7) and (2.8) respectively. Let φ and ψ be two tempered distributions homogeneous of degree $-2\alpha_1, -2\alpha'_1$ respectively. If

$$\varphi(x) = \omega_1(x)|x|^{-2\alpha_1} \quad \text{and} \quad \psi(x) = \omega_2(x)|x|^{-2\alpha'_1}, \quad (3.7)$$

where $\omega_1 \in L^{r_2}(S^{N-1})$ and $\omega_2 \in L^{s_2}(S^{N-1})$ are homogeneous of degree 0, then,

$$\mathcal{N}_s((\varphi, \psi)) < \infty, \quad (3.8)$$

where $\mathcal{N}_s$ is given by (3.3). Also, for any cut-off function η satisfying

$$\eta \equiv 1 \quad \text{near the origin and having a compact support}, \quad (3.9)$$

we have the following:

- (i) $\sup_{t>0} t^{\beta_1+\delta} \|e^{t\Delta}(\eta\varphi)\|_{r_1} < \infty$, for $0 < \delta < \frac{N}{2} - \alpha_1$;
 $\sup_{t>0} t^{\gamma_1+\delta} \|e^{t\Delta}(\eta\psi)\|_{s_1} < \infty$, for $0 < \delta < \frac{N}{2} - \alpha'_1$;
- (ii) $\sup_{t>0} t^{\beta_2} \|e^{t\Delta}((1-\eta)\varphi)\|_{r_2} < \infty$; $\sup_{t>0} t^{\gamma_2} \|e^{t\Delta}((1-\eta)\psi)\|_{s_2} < \infty$;
- (iii) $\mathcal{N}((1-\eta)\varphi, (1-\eta)\psi) < \infty$, where $\mathcal{N}$ is given by (2.6).

We establish the global existence with the norm $\mathcal{N}_s$ which is given by (3.3) and with the norm $\mathcal{N}_\delta$ is given by (3.5). This is needed to prove the existence of self-similar solutions.

Theorem 3.3. *Let N be an integer. Let α_1 and α'_1 be positive real numbers satisfying (2.2). Suppose that*

$$\frac{N}{2\alpha_1} > 1 \quad \text{and} \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_1 be a real number satisfying (2.3) and s_1 be given in (2.4). Let β_1 and γ_1 be given by (2.7) and define $\beta_1(r)$ and $\gamma_1(s)$ by

$$\beta_1(r) = \alpha_1 - \frac{N}{2r}, \quad r \in [r_1, \infty], \quad (3.10)$$

$$\gamma_1(s) = \alpha'_1 - \frac{N}{2s}, \quad s \in [s_1, \infty]. \quad (3.11)$$

Let $M > 0$ be such that $C := C(M) < 1$, where C is given by (3.23) and (3.34) below. Choose $R > 0$ such that $R + CM \leq M$. If $\Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ such that

$$\mathcal{N}_s(\Phi) \leq R, \quad (3.12)$$

where $\mathcal{N}_s$ is given by (3.3), then, the integral system (3.1)–(3.2) has a unique global solution $W = (w_1, w_2)$ such that

$$\sup_{t>0} \left[t^{\beta_1(r)} \|w_1(t)\|_r, t^{\gamma_1(s)} \|w_2(t)\|_s \right] \leq M, \quad r \in [r_1, \infty], s \in [s_1, \infty]. \quad (3.13)$$

In addition, if $W = (w_1, w_2)$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2)$ respectively are the solutions of the system (3.1)–(3.2) with initial values $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ which satisfy (3.12), then, we have

$$\sup_{t>0} \left[t^{\beta_1} \|w_1(t) - \tilde{w}_1(t)\|_{r_1}, t^{\gamma_1} \|w_2(t) - \tilde{w}_2(t)\|_{s_1} \right] \leq (1 - C)^{-1} \mathcal{N}_s(\Phi - \Psi), \quad (3.14)$$

where $\mathcal{N}_s$ is given by (3.3). Also, for some $0 < \delta < \delta_0$, where δ_0 is given by (3.4), we have

$$\begin{aligned} & \sup_{t>0} [t^{\beta_1+\delta} \|w_1(t) - \tilde{w}_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t) - \tilde{w}_2(t)\|_{s_1}] \\ & \leq (1 - C_\delta)^{-1} \mathcal{N}_\delta(\Phi - \Psi), \end{aligned} \quad (3.15)$$

where $\mathcal{N}_\delta$ is given by (3.5), C_δ is given by (3.28) and (3.36) below. The positive constant M is chosen small enough so that $C_\delta < 1$.

Furthermore, If $\Phi = (\varphi_1, \varphi_2)$ and φ_1, φ_2 be as in the Proposition 3.2, then, the resulting solution of (3.1)–(3.2) with initial data Φ is self-similar, denote it by $W' = (w'_1, w'_2)$. Let $W = (w_1, w_2)$ be the solution of (3.1)–(3.2) with initial data $(1 - \eta)\Phi$, [η is a cut-off function satisfying (3.9)]. Then, we have

$$\|w_1(t) - w'_1(t)\|_r \leq C_\delta t^{-\beta_1(r)-\delta}, \quad \forall t > 0, r \in [r_1, \infty], \quad (3.16)$$

$$\|w_2(t) - w'_2(t)\|_s \leq C_\delta t^{-\gamma_1(s)-\delta}, \quad \forall t > 0, s \in [s_1, \infty]. \quad (3.17)$$

The real number M is smaller. At this stage, we conclude that the solution W is asymptotically self-similar in the $L^\infty \times L^\infty$ -norm.

Remark 3.4. 1. The statements (a)–(d) of Theorem 2.6 are also satisfied by global solution $W = (w_1, w_2)$ given by Theorem 3.3.

2. The solution W' given by Theorem 3.3 satisfy

$$W'(t, x) = (w'_1(t, x), w'_2(t, x)) = \left(t^{-\alpha_1} w'_1(1, x/\sqrt{t}), t^{-\alpha'_1} w'_2(1, x/\sqrt{t}) \right). \quad (3.18)$$

By using the simple dilatation argument, the results (3.16)–(3.17) is given as follow:

$$\|t^{\alpha_1} w_1(t, \cdot\sqrt{t}) - w'_1(1, \cdot)\|_r \leq C_\delta t^{-\delta}, \quad \forall t > 0, r \in [r_1, \infty], \quad (3.19)$$

$$\|t^{\alpha'_1} w_2(t, \cdot\sqrt{t}) - w'_2(1, \cdot)\|_s \leq C_\delta t^{-\delta}, \quad \forall t > 0, s \in [s_1, \infty]. \quad (3.20)$$

Proof of Theorem 3.3 Let X be the set of Bochner measurable functions:

$$\begin{aligned} W : (0, \infty) & \longrightarrow L^{r_1}(\mathbb{R}^N) \times L^{s_1}(\mathbb{R}^N) \\ t & \longrightarrow W(t) := (w_1(t), w_2(t)) \end{aligned}$$

such that

$$\|W\|_X := \sup_{t>0} [t^{\beta_1} \|w_1(t)\|_{r_1}, t^{\gamma_1} \|w_2(t)\|_{s_1}] < \infty, \quad (3.21)$$

where r_1 is a real number satisfying (2.3), s_1 is given by (2.4), β_1 and γ_1 are given by (2.7).

Let M be a positive real number. Define $X_M = \{W \in X, \|W\|_X \leq M\}$. In this proof, we do not need to treat all the cases described by μ and ν . In fact,

- (i) if $\mu = \nu = 0$, then the asymptotic system (3.1)–(3.2) is the linear heat system: $\partial_t w_1 = \Delta w_1, \partial_t w_2 = \Delta w_2$. This case can be treated as in [6],
- (ii) if $\mu = \nu = 1$, then the asymptotic system (3.1)–(3.2) is the nonlinear heat system (1.27)–(1.28). This case is studied in [8].

So, we are interested only by the cases $\mu = 1, \nu = 0$ and $\mu = 0, \nu = 1$.

Case 1: $\mu = 1$ and $\nu = 0$.

We have, by Lemma 2.3, $r_{11} = r_1, \beta_{11} = \beta_1, \frac{1}{\tau_{21}} = \frac{1}{r_1} + \frac{1}{s_1}$ and $\Gamma_{21} = \beta_1 + \gamma_1$. Let us consider the mapping $\mathcal{F}_\Phi^s$ defined by $\mathcal{F}_\Phi^s(W) = (F_\Phi^s(W), G_\Phi^s(W))$, where $W = (w_1, w_2) \in X_M, \Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ and

$$\begin{aligned} F_\Phi^s(W)(t) &= e^{t\Delta} \varphi_1 + a \int_0^t e^{(t-s)\Delta} (w_1^2(s)) ds, \\ G_\Phi^s(W)(t) &= e^{t\Delta} \varphi_2 + c \int_0^t e^{(t-s)\Delta} (w_1(s)w_2(s)) ds. \end{aligned}$$

Through calculations similar to the proof of Theorem 2.6, we have for $W, \tilde{W} \in X_M$

$$\|\mathcal{F}_\Phi^s(W) - \mathcal{F}_\Phi^s(\tilde{W})\|_{X_M} \leq \mathcal{N}_s(\Phi - \Psi) + C\|W - \tilde{W}\|_{X_M}, \quad (3.22)$$

where $\mathcal{N}_s$ is defined by (3.3), C is a finite positive constant given by

$$C = \max(C_1, C_2), \quad (3.23)$$

such that

$$C_1 = 2M|a|C \int_0^1 (1-\sigma)^{-\frac{N}{2r_1}} \sigma^{-2\beta_1} d\sigma, \quad (3.24)$$

$$C_2 = 2M|c|C \int_0^1 (1-\sigma)^{-\frac{N}{2r_1}} \sigma^{-\gamma_1-\beta_1} d\sigma. \quad (3.25)$$

So, the mapping $\mathcal{F}_\Phi^s$ has a unique fixed point in X_M and (3.14) follows similarly as in the proof of Theorem 2.6. Now, let $\Phi, \Psi, W = (w_1, w_2)$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2)$. We estimate the difference $W - \tilde{W}$ with the norm

$$\|W\|_{X,\delta} := \sup_{t>0} [t^{\beta_1+\delta} \|w_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t)\|_{s_1}] < \infty, \quad (3.26)$$

to obtain

$$\sup_{t>0} [t^{\beta_1+\delta} \|w_1(t) - \tilde{w}_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t) - \tilde{w}_2(t)\|_{s_1}] \leq (1 - C_\delta)^{-1} \mathcal{N}_\delta(\Phi - \Psi), \quad (3.27)$$

with

$$C_\delta = \max(C_{1\delta}, C_{2\delta}), \quad (3.28)$$

where

$$\begin{aligned} C_{1\delta} &= 2M|a|C \int_0^1 (1-\sigma)^{-\frac{N}{2r_1}} \sigma^{-2\beta_1-\delta} d\sigma, \\ C_{2\delta} &= 2M|c|C \int_0^1 (1-\sigma)^{-\frac{N}{2r_1}} \sigma^{-\gamma_1-\beta_1-\delta} d\sigma. \end{aligned}$$

Since $0 < \delta < \delta_0$, where δ_0 is given by (3.4) one can see that $2\beta_1 + \delta < 1, \beta_1 + \gamma_1 + \delta < 1$ and C_δ is a finite positive constant. This proves (3.15).

Let $\Phi = (\varphi_1, \varphi_2)$ be as in Proposition 3.2 and $\Psi = (1 - \eta)\Phi$. It follows from formula (3.15), that

$$\begin{aligned} \sup_{t>0} [t^{\beta_1+\delta} \|w_1(t) - w'_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t) - w'_2(t)\|_{s_1}] &\leq C\mathcal{N}_\delta(\Phi - (1 - \eta)\Phi) \\ &= C\mathcal{N}_\delta(\eta\Phi). \end{aligned} \quad (3.29)$$

Using part (i) of Proposition 3.2, we have that $\mathcal{N}_\delta(\eta\Phi) < \infty$, for $0 < \delta < \delta_0$. Thus, we get (3.16) and (3.17), for $(r, s) = (r_1, s_1)$.

The $L^\infty \times L^\infty$ - asymptotic self-similar estimates:

Here we write

$$\begin{aligned} w_2(t) - w'_2(t) &= e^{\frac{t}{2}\Delta} (w_2(t/2) - w'_2(t/2)) \\ &\quad + c \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (w_1(s)w_2(s) - w'_1(s)w'_2(s)) \, ds, \end{aligned}$$

thus, by the smoothing properties of the heat semigroup (2.23), for $(p, q) = (s_1, \infty)$ and $\varphi = w_2(t/2) - w'_2(t/2)$, we obtain

$$\begin{aligned} t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty &\leq C(t/2)^{\alpha'_1+\delta-\frac{N}{2s_1}} \|w_2(t/2) - w'_2(t/2)\|_{s_1} \\ &\quad + |c|Ct^{\alpha'_1+\delta} \left(\int_{t/2}^t \|w_1(s)w_2(s) - w'_1(s)w'_2(s)\|_\infty \, ds \right). \end{aligned}$$

Now, we compute that, for arbitrary $T > 0$

$$\begin{aligned} t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty &\leq C(t/2)^{\gamma_1+\delta} \|w_2(t/2) - w'_2(t/2)\|_{s_1} \\ &\quad + 2MC|c|t^{-\alpha_1+1} \left(\int_{1/2}^1 s^{-\alpha_1-\alpha'_1-\delta} \, ds \right) \\ &\quad \times \sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty, \right. \\ &\quad \left. t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right), \end{aligned}$$

where M is as in Theorem 2.6. By (3.17), for $s = s_1$ and the fact that $\alpha_1 = 1$, the last inequality becomes

$$\begin{aligned} \sup_{(0,T)} \left(t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right) &\leq C_\delta + |c|C(M, \delta) \sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) \right. \\ &\quad \left. - w'_1(t)\|_\infty, t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right). \end{aligned} \quad (3.30)$$

Similarly, we write

$$w_1(t) - w'_1(t) = e^{\frac{t}{2}\Delta} (w_1(t/2) - w'_1(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (w_1^2(s) - w_1'^2(s)) \, ds.$$

Thus, by the smoothing properties of the heat semigroup (2.23), for $(p, q) = (r_1, \infty)$ and $\varphi = w_1 - w'_1$, for arbitrary $T > 0$ and $0 < t \leq T$, we compute that

$$\begin{aligned} t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty &\leq C (t/2)^{\alpha_1+\delta-\frac{N}{2r_1}} \|w_1(t/2) - w'_1(t/2)\|_{r_1} \\ &\quad + 2M C |a| t^{-\alpha_1+1} \left(\int_{1/2}^1 s^{-2\alpha_1-\delta} ds \right) \\ &\quad \times \left(\sup_{(0,T)} t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty \right), \end{aligned}$$

where M is as in Theorem 2.6. By (3.16), for $r = r_1$ and the fact that $\alpha_1 = 1$, the last inequality becomes

$$\begin{aligned} \sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty \right) &\leq C_\delta + |a| C(M, \delta) \\ &\quad \times \sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty, t^{\alpha'_1+\delta} \right. \\ &\quad \left. \times \|w_2(t) - w'_2(t)\|_\infty \right). \end{aligned} \quad (3.31)$$

Since $C(M, \delta) \searrow 0$ if $M \searrow 0$, it is clear that if M is sufficiently small then (3.30) and (3.31) imply that

$$\sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty, t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right) \leq C_\delta. \quad (3.32)$$

Since the last constant C_δ does not depend on the arbitrary real number $T > 0$, the estimates (3.16) and (3.17), for $r = s = \infty$ are valid for all $t > 0$. The general case for (3.16) and (3.17) follows by interpolation.

Case 2: $\mu = 0$ and $\nu = 1$.

We have, by Lemma 2.3, $r_{11} = s_1$ and $\beta_{11} = \gamma_1$. Let us consider the mapping $\mathcal{F}_\Phi^s$ defined by $\mathcal{F}_\Phi^s(W) = (F_\Phi^s(W), G_\Phi^s(W))$, where $W = (w_1, w_2) \in X_M$, $\Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ and

$$\begin{aligned} F_\Phi^s(W)(t) &= e^{t\Delta} \varphi_1 + b \int_0^t e^{(t-s)\Delta} (w_2^2(s)) ds, \\ G_\Phi^s(W)(t) &= e^{t\Delta} \varphi_2. \end{aligned}$$

Through calculations similar to the proof of Theorem 2.6, we have for $W, \tilde{W} \in X_M$

$$\left\| \mathcal{F}_\Phi^s(W) - \mathcal{F}_\Phi^s(\tilde{W}) \right\|_{X_M} \leq \mathcal{N}_s(\Phi - \Psi) + C \|W - \tilde{W}\|_{X_M}, \quad (3.33)$$

where $\mathcal{N}_s$ is defined by (3.3), C is a finite positive constant given by

$$C = 2M|b|C \int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{2}{s_1}-\frac{1}{r_1}\right)} \sigma^{-2\gamma_1} d\sigma. \quad (3.34)$$

We conclude that the mapping $\mathcal{F}_\Phi^s$ has a unique fixed point in X_M and (3.14) follows as in Theorem 2.6.

Similarly, We estimate the difference $W - \tilde{W}$ with the norm defined by (3.26) and we set $\Phi, \Psi, W = (w_1, w_2)$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2)$. We find

$$\sup_{t>0} [t^{\beta_1+\delta} \|w_1(t) - \tilde{w}_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t) - \tilde{w}_2(t)\|_{s_1}] \leq (1 - C_\delta)^{-1} \mathcal{N}_\delta(\Phi - \Psi), \quad (3.35)$$

with

$$C_\delta = 2M|b|C \int_0^1 (1 - \sigma)^{-\frac{N}{2}(\frac{2}{s_1} - \frac{1}{r_1})} \sigma^{-2\gamma_1-\delta} d\sigma. \quad (3.36)$$

Since $0 < \delta < \delta_0$, where δ_0 is given by (3.4), note that $2\gamma_1 + \delta < 1$, C_δ is a finite positive constant and (3.15) follows.

Let $\Phi = (\varphi_1, \varphi_2)$ be as in Proposition 3.2 and $\Psi = (1 - \eta)\Phi$. It follows from the formula (3.15), that

$$\begin{aligned} \sup_{t>0} [t^{\beta_1+\delta} \|w_1(t) - w'_1(t)\|_{r_1}, t^{\gamma_1+\delta} \|w_2(t) - w'_2(t)\|_{s_1}] &\leq C\mathcal{N}_\delta(\Phi - (1 - \eta)\Phi) \\ &= C\mathcal{N}_\delta(\eta\Phi). \end{aligned} \quad (3.37)$$

Using part (i) of Proposition 3.2, we have that $\mathcal{N}_\delta(\eta\Phi) < \infty$, for $0 < \delta < \delta_0$. Thus, we obtain (3.16) and (3.17), for $(r, s) = (r_1, s_1)$.

The $L^\infty \times L^\infty$ -asymptotic self-similar estimates: here we write

$$w_2(t) - w'_2(t) = e^{\frac{t}{2}\Delta} (w_2(t/2) - w'_2(t/2)).$$

We compute, by the smoothing properties of the heat semigroup (2.23) and (3.17) for $s = s_1$

$$t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \leq C (t/2)^{\gamma_1+\delta} \|w_2(t/2) - w'_2(t/2)\|_{s_1} \leq C_\delta. \quad (3.38)$$

Now, we estimate $t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty$. We have

$$w_1(t) - w'_1(t) = e^{\frac{t}{2}\Delta} (w_1(t/2) - w'_1(t/2)) + b \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (w_2^2(s) - w_2'^2(s)) ds.$$

Thus, by the smoothing properties of the heat semigroup (2.23), for $(p, q) = (r_1, \infty)$ on the second and the third term of the right-hand side of the last inequality, for arbitrary $T > 0$ and $0 < t \leq T$, we compute that

$$\begin{aligned} t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty &\leq C (t/2)^{\alpha_1+\delta-\frac{N}{2r_1}} \|w_1(t/2) - w'_1(t/2)\|_{r_1} \\ &\quad + 2M C |b| t^{\alpha_1-2\alpha'_1+1} \left(\int_{1/2}^1 s^{-2\alpha'_1-\delta} ds \right) \\ &\quad \times \left(\sup_{(0,T)} t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right), \end{aligned} \quad (3.39)$$

where M is as in Theorem 2.6. By (3.16), for $r = r_1$, and the fact that $2\alpha'_1 = \alpha_1 + 1$, the last inequality becomes

$$\begin{aligned}
\sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty \right) &\leq C_\delta + |b|C(M, \delta) \\
&\quad \times \sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty, t^{\alpha'_1+\delta} \right. \\
&\quad \left. \times \|w_2(t) - w'_2(t)\|_\infty \right). \tag{3.40}
\end{aligned}$$

Since $C(M, \delta) \searrow 0$ if $M \searrow 0$, it is clear that if M is sufficiently small then (3.38) and (3.40) imply that

$$\sup_{(0,T)} \left(t^{\alpha_1+\delta} \|w_1(t) - w'_1(t)\|_\infty, t^{\alpha'_1+\delta} \|w_2(t) - w'_2(t)\|_\infty \right) \leq C_\delta. \tag{3.41}$$

Since the last constant C_δ does not depend on the arbitrary real number $T > 0$, the estimates (3.16) and (3.17) for $r = s = \infty$ are valid for all $t > 0$. The general case for (3.16) and (3.17) will be deduced from interpolation.

Proof of Theorem 1.1 Let (u_0, v_0) be as in the assumptions of the Theorem 1.1. Then, by Part (i) of Proposition 3.2, we have $\mathcal{N}((u_0, v_0)) < \infty$. Let $\varepsilon > 0$ be such that $\max(|c_1|, |c_2|) < \varepsilon$ and (u_0, v_0) satisfies (2.15). Then, by Theorem 2.6, there exists (u, v) global mild solution of (1.24). Since $(u_0, v_0) \in (L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N))^2$, then, (u, v) is the solution of (1.24) and is global.

Proof of Theorem 1.2 Let (u_0, v_0) be as in the assumptions of the Theorem 1.2. Then by (3.8), we have $\mathcal{N}((u_0, v_0)) < \infty$. Let $\varepsilon > 0$ be such that $\max(|c_1|, |c_2|) < \varepsilon$ and (u_0, v_0) satisfies (3.12). Then by Theorem 3.3, there exists (w_1, w_2) global mild solution of (1.12). Since $(u_0, v_0) \in (L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N))^2$, then, (w_1, w_2) is the solution of (1.12) and is global. Using the Theorem 3.3, we conclude that the resulting solution (w_1, w_2) is self-similar.

4. Asymptotically self-similar solutions

In this section, we are mainly concerned with the proof of the asymptotic behavior results for some global solutions of the system (1.27)–(1.28) with small initial data with respect to the norm $\mathcal{N}$. These allowed initial data (φ_1, φ_2) decrease in space as $(|x|^{-2\alpha_1}, |x|^{-2\alpha'_1})$, where α_1, α'_1 are any real numbers such that they satisfy (2.2) and $\alpha_1, \alpha'_1 < \frac{N}{2}$.

We fulfil the proof of Theorem 3.3 through the following lemma.

Lemma 4.1. *Let N be an integer. Let α_1, α'_1 and α_2, α'_2 be positive real numbers which satisfy respectively (2.2) and (2.5). Suppose that*

$$\frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_j, s_j, β_j and γ_j , for $j = 1, 2$ be as in Lemma 2.1. Let μ, ν be given by (1.31) and $\delta > 0$. Let r_{11} and β_{11} be as in Lemma 2.3 given also by (2.10). We define the interpolating numbers r'_{11} and β'_{11} as follow:

$$\frac{1}{r'_{11}} = \frac{1}{r_{11}} \left[1 + \frac{(1-\mu)\delta}{1+\alpha_1} \right], \quad \beta'_{11} = \beta_{11} \left[1 + \frac{(1-\mu)\delta}{1+\alpha_1} \right].$$

Moreover, by the formula (2.11) of r_{11} and β_{11} , we set the interpolating numbers s'_{11} and γ'_{11} which be written as follow:

$$\frac{1}{s'_{11}} = \frac{1}{r_{11}} \left[1 + \frac{(1-\nu)\delta}{1+\alpha_1} \right], \quad \gamma'_{11} = \beta_{11} \left[1 + \frac{(1-\nu)\delta}{1+\alpha_1} \right].$$

Furthermore, from the numbers τ_{21} and Γ_{21} cited in Lemma 2.1, we introduce the interpolating numbers τ'_{21} and Γ'_{21} as follow:

$$\frac{1}{\tau'_{21}} = \frac{1}{\tau_{21}} \left[1 + \frac{(1-\mu)\delta}{1+\alpha'_1} \right], \quad \Gamma'_{21} = \Gamma_{21} \left[1 + \frac{(1-\mu)\delta}{1+\alpha'_1} \right].$$

Also, there exists $r'_{21} \in [r_1, r_2]$, $s'_{21} \in [s_1, s_2]$, $\beta'_{21} \in [\beta_2, \beta_{21}]$ and $\gamma'_{21} \in [\gamma_2, \gamma_{21}]$ such that

$$\frac{1}{\tau'_{21}} = \frac{1}{r'_{21}} + \frac{1}{s'_{21}}, \quad \Gamma'_{21} = \beta'_{21} + \gamma'_{21}. \quad (4.1)$$

Then, there exists a real number $\delta_0 > 1$, which can be computed, such that for all $0 < \delta < \delta_0$ the numbers defined recently verify the following assumptions:

(i) There exists $0 \leq \zeta_{kj}, \zeta'_{kj} \leq 1$, for $(k, j) \in \{(1, 1), (2, 1)\}$ such that

$$\frac{1}{r'_{kj}} = \frac{\zeta_{kj}}{r_1} + \frac{1-\zeta_{kj}}{r_2}, \quad \frac{1}{s'_{kj}} = \frac{\zeta'_{kj}}{s_1} + \frac{1-\zeta'_{kj}}{s_2},$$

and

$$\beta'_{11} = \zeta_{11}\beta_1 + (1-\zeta_{11})\beta_2, \quad \gamma'_{11} = \zeta'_{11}\gamma_1 + (1-\zeta'_{11})\gamma_2,$$

$$\beta'_{21} = \zeta_{21}\beta_1 + (1-\zeta_{21})\beta_2, \quad \gamma'_{21} = \zeta'_{21}\gamma_1 + (1-\zeta'_{21})\gamma_2,$$

- (ii) $1 < \frac{r'_{11}}{2} < r_1$, $1 < \frac{s'_{11}}{2} < r_1$, $1 < \tau'_{21} < s_1$,
- (iii) $2\beta'_{11} + \mu\delta < 1$, $2\gamma'_{11} + \nu\delta < 1$, $\Gamma'_{21} + \mu\delta < 1$,
- (iv) $\frac{N}{2} \left(\frac{2}{r'_{11}} - \frac{1}{r_1} \right) < 1$, $\frac{N}{2} \left(\frac{2}{s'_{11}} - \frac{1}{r_1} \right) < 1$, $\frac{N}{2} \left(\frac{1}{\tau'_{21}} - \frac{1}{s_1} \right) < 1$,
- (v) $1 + \beta_1 - \frac{N}{2} \left(\frac{2}{r'_{11}} - \frac{1}{r_1} \right) - 2\beta'_{11} + (1-\mu)\delta = 0$,
 $1 + \beta_1 - \frac{N}{2} \left(\frac{2}{s'_{11}} - \frac{1}{r_1} \right) - 2\gamma'_{11} + (1-\nu)\delta = 0$,
 $1 + \gamma_1 - \frac{N}{2} \left(\frac{1}{\tau'_{21}} - \frac{1}{s_1} \right) - \Gamma'_{21} + (1-\mu)\delta = 0$.

Proof. Owing to the conditions satisfied by $r_1, r_2, s_1, s_2, \beta_1, \beta_2, \gamma_1$ and γ_2 in the Lemmas 2.1 and 2.3 and in the Corollary 2.5, the proof of Lemma 4.1 is simple and can be omitted. $\square$

Theorem 4.2. Let N be an integer. Let $\alpha_1, \alpha'_1, \alpha_2$ and α'_2 be positive real numbers which satisfy

$$\alpha_1 \geq 1, \quad 2\alpha'_1 \geq \alpha_1 + 1 \quad \alpha_2 \leq 1, \quad 2\alpha'_2 \leq \alpha_2 + 1.$$

Suppose that

$$\frac{N}{2\alpha_1} > 1 \quad \text{and} \quad \frac{N}{2\alpha'_1} > 1.$$

Let r_1 be a real number satisfying (2.3) and s_1 defined in (2.4). Let β_1 and γ_1 be given by (2.7). Let Φ and η be as in Proposition 3.2. Assume that (2.13)–(2.14) are satisfied.

Let W' given by (3.18) be the self-similar solution of (1.29)–(1.30) with initial data Φ given by Theorem 3.3 and let U be the global solution of (1.27)–(1.28) with initial data $(1 - \eta)\Phi$ constructed by Theorem 2.6 (we multiply Φ by a small constant so that (2.15) and then (3.12) are satisfied with both Φ and $(1 - \eta)\Phi$). Then, there exists a real number $\delta_1 > 0$, which can be computed, such that for all δ satisfying $0 < \delta < \delta_1$ there exists a positive constant C_δ such that for $r \in [r_1, \infty]$, $s \in [s_1, \infty]$,

$$\|u(t) - w'_1(t)\|_r \leq C_\delta t^{-\beta_1(r) - \delta}, \quad \forall t > 0, \quad (4.2)$$

$$\|v(t) - w'_2(t)\|_s \leq C_\delta t^{-\gamma_1(s) - \delta}, \quad \forall t > 0, \quad (4.3)$$

where $\beta_1(r)$ and $\gamma_1(s)$ are given respectively by (3.10) and (3.11).

Hence $U = (u, v)$ is asymptotically self-similar in the $L^\infty \times L^\infty$ -norm. Furthermore, there exists two positive constants d_1 and d_2 such that, for large time, we have

$$d_1 t^{-\alpha_1} \leq \|u(t)\|_\infty \leq d_2 t^{-\alpha_1}, \quad d_1 t^{-\alpha'_1} \leq \|v(t)\|_\infty \leq d_2 t^{-\alpha'_1}. \quad (4.4)$$

We require that the real number M appearing in Theorem 2.6 must be smaller so that $C_\delta < 1$ where C_δ is given by (3.28) and (3.36).

Remark 4.3. Remark that, by the dilatation argument, we can prove the following inequalities:

$$\|t^{\alpha_1} u(t, \sqrt{t}) - w'_1(1, \cdot)\|_r \leq C_\delta t^{-\delta}, \quad \forall t > 0, \quad r \in [r_1, \infty], \quad (4.5)$$

$$\|t^{\alpha'_1} v(t, \sqrt{t}) - w'_2(1, \cdot)\|_s \leq C_\delta t^{-\delta}, \quad \forall t > 0, \quad s \in [s_1, \infty]. \quad (4.6)$$

Proof. Over the proof, we use the notations established in Lemma 4.1. The system (1.27)–(1.28) coincides with the system (3.1)–(3.2) in the particular case where $\mu = \nu = 1$. As well as the estimates (4.2) and (4.3) follow from those established in the Theorem 3.3. We prove the results of Theorem 4.2 only for two cases $\mu = 1, \nu = 0$ and $\mu = 0, \nu = 1$, since the cases $\mu = \nu = 0$ and $\mu = \nu = 1$ are treated as well as in [6, 8] respectively.

Case 1: $\mu = 1$ and $\nu = 0$.

Step 1: $L^{r_1} \times L^{s_1}$ -asymptotic results. Let $\Psi = (\psi_1, \psi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ satisfying the condition (2.15). Let $U = (u, v)$ be the solution of (1.27)–(1.28) with initial data Ψ constructed by Theorem 2.6. Let $W = (w_1, w_2)$ be the solution of the system:

$$w_1(t) = e^{t\Delta} \psi_1 + a \int_0^t e^{(t-s)\Delta} (w_1^2(s)) ds, \quad (4.7)$$

$$w_2(t) = e^{t\Delta} \psi_2 + c \int_0^t e^{(t-s)\Delta} (w_1(s)w_2(s)) ds, \quad (4.8)$$

which given by Theorem 3.3. We have

$$\begin{aligned} \|u(t) - w_1(t)\|_{r_1} &\leq |a| \int_0^t \|e^{(t-s)\Delta} (u^2(s) - w_1^2(s))\|_{r_1} ds \\ &\quad + |b| \int_0^t \|e^{(t-s)\Delta} (v^2(s))\|_{r_1} ds, \end{aligned} \quad (4.9)$$

$$\|v(t) - w_2(t)\|_{s_1} \leq |c| \int_0^t \|e^{(t-s)\Delta}(u(s)v(s) - w_1(s)w_2(s))\|_{s_1} ds. \quad (4.10)$$

Let δ be such that $0 < \delta < \delta_1$, where δ_1 is a small positive real number. We first deal with the inequality (4.10). By Lemma 4.1 and the smoothing properties of the heat semigroup (2.23) with $(p, q) = (\tau_{21}, s_1)$ and $\varphi = u(s)v(s) - w_1(s)w_2(s)$, we have

$$\|v(t) - w_2(t)\|_{s_1} \leq C|c| \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{\tau_{21}} - \frac{1}{s_1})} \|u(s)v(s) - w_1(s)w_2(s)\|_{\tau_{21}} ds, \quad (4.11)$$

where τ_{21} is given in Lemma 2.3. In fact, since $\mu = 1$ we have $\alpha_1 = 1$ and we write $\frac{1}{\tau_{21}} = \frac{1+\alpha'_1}{r_1} = \frac{1}{r_1} + \frac{1}{s_1}$. So, by Hölder inequality, we get

$$\|uv - w_1w_2\|_{\tau_{21}} \leq \|v\|_{s_1}\|u - w_1\|_{r_1} + \|w_1\|_{r_1}\|v - w_2\|_{s_1}. \quad (4.12)$$

Let T be an arbitrary positive number. Injecting (4.12) in (4.11), we obtain, for $0 < t \leq T$

$$t^{\gamma_1+\delta}\|v(t) - w_2(t)\|_{s_1} \leq C'_{1\delta} t^{\gamma_1+\delta-\frac{N}{2r_1}-\beta_1-\gamma_1-\delta+1} \times \left[\sup_{t \in (0, T]} \left(t^{\beta_1+\delta}\|u(t) - w_1(t)\|_{r_1}, t^{\gamma_1+\delta}\|v(t) - w_2(t)\|_{s_1} \right) \right],$$

where

$$C'_{1\delta} = 2CM|c| \int_0^1 (1-s)^{-\frac{N}{2r_1}} s^{-\beta_1-\gamma_1-\delta} ds,$$

the positive constants $C'_{1\delta}$ is finite. Note that, in this case we have $\alpha_1 = 1$, so

$$\gamma_1 + \delta - \frac{N}{2r_1} - \beta_1 - \gamma_1 - \delta + 1 = 0.$$

Then, we obtain

$$t^{\gamma_1+\delta}\|v(t) - w_2(t)\|_{s_1} \leq C'_{1\delta} \sup_{t \in (0, T]} \left(t^{\beta_1+\delta}\|u(t) - w_1(t)\|_{r_1}, t^{\gamma_1+\delta}\|v(t) - w_2(t)\|_{s_1} \right). \quad (4.13)$$

Now we estimate the inequality (4.9). By Lemma 4.1 and the smoothing properties of the heat semigroup (2.23) with $(p, q, \varphi) = (\frac{r_1}{2}, r_1, u^2 - w_1^2)$ on the first term and with $(p, q, \varphi) = (\frac{s'_{11}}{2}, r_1, v^2)$ on the second term of the right-hand side of the inequality (4.9), we have

$$\begin{aligned} \|u(t) - w_1(t)\|_{r_1} &\leq C|a| \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{r_1} - \frac{1}{r_1})} \|u^2(s) - w_1^2(s)\|_{r_1/2} ds \\ &\quad + C|b| \int_0^t (t-s)^{-\frac{N}{2}(\frac{2}{s'_{11}} - \frac{1}{r_1})} \|v^2(s)\|_{s'_{11}/2} ds. \end{aligned} \quad (4.14)$$

Let $T > 0$ be an arbitrary real number. Using Lemma 4.1, Eq. (2.28) with $(s, r, \tau, \theta) = (s'_{11}, s_1, s_2, \zeta'_{11})$ on the second term of the right-hand side of the last inequality, we get, for $0 < t \leq T$

$$t^{\beta_1+\delta}\|u(t) - w_1(t)\|_{r_1} \leq C_{1\delta} \sup_{t \in (0, T]} (t^{\beta_1+\delta}\|u(t) - w_1(t)\|_{r_1}) + C_{2\delta},$$

where

$$C_{1\delta} = 2MC|a| \int_0^1 (1-s)^{-\frac{N}{2r_1}} s^{-2\beta_1-\delta} ds,$$

$$C_{2\delta} = M^2C|b| \int_0^1 (1-s)^{-\frac{N}{2}\left(\frac{2}{s_{11}^2}-\frac{1}{r_1}\right)} s^{-2\gamma'_{11}} ds.$$

By Lemma 4.1, we find that the positive constants $C_{1\delta}, C_{2\delta}$ are finite. Thus, we obtain

$$t^{\beta_1+\delta} \|u(t) - w_1(t)\|_{r_1} \leq (1 - C_{1\delta})^{-1} C_{2\delta}, \quad (4.15)$$

for a given constant C_δ , since it does not depend on T the pervious inequalities hold on the whole interval $(0, \infty)$. It follows that, for all δ satisfying, $0 < \delta < \delta_1$, there exists a positive constant C_δ , such that

$$\|u(t) - w_1(t)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta}, \quad \forall t > 0, \quad (4.16)$$

$$\|v(t) - w_2(t)\|_{s_1} \leq C_\delta t^{-\gamma_1-\delta}, \quad \forall t > 0. \quad (4.17)$$

Step 2: $L^\infty \times L^\infty$ -asymptotic result. Let $\Phi = (\varphi_1, \varphi_2)$ be as in Proposition 3.2, then using the results (3.16)–(3.17), for $(r, s) = (r_1, s_1)$ of Theorem 3.3, one obtains that, for all $0 < \delta < \delta_0$, there exists a positive constant $C_\delta > 0$, such that

$$\|w_1(t) - w'_1(t)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta}, \quad \forall t > 0, \quad (4.18)$$

$$\|w_2(t) - w'_2(t)\|_{s_1} \leq C_\delta t^{-\gamma_1-\delta}, \quad \forall t > 0, \quad (4.19)$$

where δ_0 is given by (3.4). Hence, $W = (w_1, w_2)$ is solution of (4.7)–(4.8) with initial data $(1 - \eta)\Phi$, where η is a cut-off function satisfying (3.9). Then by Proposition 3.2 (iii) we have that $\mathcal{N}((1 - \eta)\Phi)$ is finite. Thus, the inequalities (4.2) and (4.3) follow by writing $U - W' = U - W + W - W'$ and using (4.16)–(4.19).

Let $\delta > 0$ be sufficiently small and recall the following numbers given in (i) of Lemma 4.1,

$$\zeta_{11} = \theta_{11}; \quad \zeta'_{11} = \theta'_{11} + \delta \left[\frac{1}{2(\alpha'_1 - \alpha'_2)} \right]. \quad (4.20)$$

In particular, δ is chosen small enough so that ζ_{11} and ζ'_{11} belong to $[0, 1]$ and $0 < \delta < \delta_1$. Write now

$$v(t) - w_2(t) = e^{\frac{t}{2}\Delta} (v(t/2) - w_2(t/2))$$

$$+ c \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (u(s)v(s) - w_1(s)w_2(s)) ds.$$

Then, by (2.23), (4.12) and (4.17), we have

$$\|v(t) - w_2(t)\|_\infty \leq \left\| e^{\frac{t}{2}\Delta} (v(t/2) - w_2(t/2)) \right\|_\infty$$

$$+ |c| \int_{\frac{t}{2}}^t \left\| e^{(t-s)\Delta} (u(s)v(s) - w_1(s)w_2(s)) \right\|_\infty ds.$$

That is

$$\begin{aligned}
 t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty &\leq C (t/2)^{\gamma_1+\delta} \|(v(t/2) - w_2(t/2))\|_{s_1} \\
 &\quad + |c| C t^{\alpha'_1+\delta} \int_{\frac{t}{2}}^t \|(u(s)v(s) - w_1(s)w_2(s))\|_\infty ds \\
 &\leq C_\delta + C |c| t^{\alpha'_1+\delta} \int_{t/2}^t \left(\|v(s)\|_\infty \|u(s) - w_1(s)\|_\infty \right. \\
 &\quad \left. + \|w_1(s)\|_\infty \|v(s) - w_2(s)\|_\infty \right) ds.
 \end{aligned}$$

Using (2.17), we get for arbitrary $T > 0$, we obtain

$$\begin{aligned}
 t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty &\leq C_\delta + C(M) t^{\alpha'_1+\delta} \left(\int_{t/2}^t s^{-\alpha_1-\alpha'_1-\delta} ds \right) \\
 &\quad \times \sup_{t \in (0, T]} \left(t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty, t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \right) \\
 &\leq C_\delta + C(M) t^{\alpha'_1+\delta-\alpha_1-\alpha'_1-\delta+1} \left(\int_{1/2}^1 s^{-\alpha_1-\alpha'_1-\delta} ds \right) \\
 &\quad \times \sup_{t \in (0, T]} \left(t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty, t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \right).
 \end{aligned}$$

Thus, we obtain, from $\alpha_1 = 1$, that

$$\begin{aligned}
 &t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \\
 &\leq C_\delta + C(M, \delta) \sup_{t \in (0, T]} \left(t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty, t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \right). \quad (4.21)
 \end{aligned}$$

Let us write now

$$\begin{aligned}
 u(t) - w_1(t) &= e^{\frac{t}{2}\Delta} (u(t/2) - w_1(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (u^2(s) - w_1^2(s)) ds \\
 &\quad + b \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (v^2(s)) ds.
 \end{aligned}$$

We have

$$\begin{aligned}
 t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty &\leq t^{\alpha_1+\delta} \|e^{\frac{t}{2}\Delta} (u(t/2) - w_1(t/2))\|_\infty \\
 &\quad + |a| t^{\alpha_1+\delta} \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (u^2(s) - w_1^2(s))\|_\infty ds \\
 &\quad + |b| t^{\alpha_1+\delta} \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (v^2(s))\|_\infty ds.
 \end{aligned}$$

Therefore by (2.23) and (4.16), we obtain

$$\begin{aligned}
 t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty &\leq C (t/2)^{\beta_1+\delta} \|(u(t/2) - w_1(t/2))\|_{r_1} + C |a| t^{\alpha_1+\delta} \\
 &\quad \times \int_{t/2}^t \|u^2(s) - w_1^2(s)\|_\infty ds
 \end{aligned}$$

$$\begin{aligned}
& +C|b|t^{\alpha_1+\delta}\int_{t/2}^t\|v^2(s)\|_\infty ds \\
& \leq C_\delta+C|a|t^{\alpha_1+\delta}\int_{t/2}^t(\|u(s)\|_\infty+\|w_1(s)\|_\infty) \\
& \quad \times\|u(s)-w_1(s)\|_\infty ds \\
& +C|b|t^{\alpha_1+\delta}\int_{t/2}^t\|v(s)\|_\infty^2 ds.
\end{aligned}$$

Due to (2.17), and an interpolation argument, we get for arbitrary $T > 0$,

$$\begin{aligned}
t^{\alpha_1+\delta}\|u(t)-w_1(t)\|_\infty & \leq C_\delta+2MC|a|t^{\alpha_1+\delta}\left(\int_{t/2}^t s^{-2[\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2]-\delta} ds\right) \\
& \times\sup_{(0,T]}\left(t^{\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2+\delta}\|u(t)-w_1(t)\|_\infty\right) \\
& +CM^2|b|t^{\alpha_1+\delta}\left(\int_{t/2}^t s^{-2[\alpha'_1\zeta'_{11}+(1-\zeta'_{11})\alpha'_2]} ds\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
t^{\alpha_1+\delta}\|u(t)-w_1(t)\|_\infty & \leq C_\delta+2MC|a|t^{1+\alpha_1-2\alpha_1\zeta_{11}-2(1-\zeta_{11})\alpha_2} \\
& \times\left(\int_{1/2}^1 s^{-2[\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2]-\delta} ds\right) \\
& \times\sup_{(0,T]}\left(t^{\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2+\delta}\|u(t)-w_1(t)\|_\infty\right) \\
& +M^2C|b|t^{1+\alpha_1-2(\alpha'_1\zeta'_{11}+(1-\zeta'_{11})\alpha'_2)+\delta} \\
& \times\left(\int_{1/2}^1 s^{-2[\alpha'_1\zeta'_{11}+(1-\zeta'_{11})\alpha'_2]} ds\right).
\end{aligned}$$

From the expressions of ζ_{11} , ζ'_{11} given by (4.20), we obtain

$$\begin{aligned}
t^{\alpha_1+\delta}\|u(t)-w_1(t)\|_\infty & \leq C_\delta+C(M,\delta)+C(M^2) \\
& \times\sup_{(0,T]}\left(t^{\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2+\delta}\|u(t)-w_1(t)\|_\infty\right). \quad (4.22)
\end{aligned}$$

We substitute $\zeta_{11} = \theta_{11} = 1$ and obtain

$$\sup_{(0,T]}\left(t^{\alpha_1\zeta_{11}+(1-\zeta_{11})\alpha_2+\delta}\|u(t)-w_1(t)\|_\infty\right)=\sup_{(0,T]}\left(t^{\alpha_1+\delta}\|u(t)-w_1(t)\|_\infty\right). \quad (4.23)$$

Case 2: $\mu = 0$ and $\nu = 1$.

Step 1: $L^{r_1} \times L^{s_1}$ -asymptotic results. Let $\Psi = (\psi_1, \psi_2) \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ satisfying the condition (2.15). Let $U = (u, v)$ be the solution of (1.27)–(1.28) with initial data Ψ constructed by Theorem 2.6. Let $W = (w_1, w_2)$ be the

solution of the system:

$$w_1(t) = e^{t\Delta}\psi_1 + b \int_0^t e^{(t-s)\Delta} w_2^2(s) ds, \quad (4.24)$$

$$w_2(t) = e^{t\Delta}\psi_2, \quad (4.25)$$

which is given by Theorem 3.3. We have

$$\begin{aligned} \|u(t) - w_1(t)\|_{r_1} &\leq |b| \int_0^t \|e^{(t-s)\Delta}(v^2(s) - w_2^2(s))\|_{r_1} ds \\ &\quad + |a| \int_0^t \|e^{(t-s)\Delta}(u^2(s))\|_{r_1} ds, \end{aligned} \quad (4.26)$$

$$\|v(t) - w_2(t)\|_{s_1} \leq |c| \int_0^t \|e^{(t-s)\Delta}(u(s)v(s))\|_{s_1} ds. \quad (4.27)$$

Let δ be such that $0 < \delta < \delta_1$, where δ_1 is a small positive real number. We first deal with the inequality (4.27). By Lemma 4.1 and the smoothing properties of the heat semigroup (2.23) with $(p, q) = (\tau'_{21}, s_1)$ and $\varphi = u(s)v(s)$, we have

$$\|v(t) - w_2(t)\|_{s_1} \leq C|c| \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{1}{\tau'_{21}} - \frac{1}{s_1}\right)} \|u(s)v(s)\|_{\tau'_{21}} ds. \quad (4.28)$$

where τ'_{21} is given by (4.1). Using (4.1), Hölder inequality, we get

$$\|uv\|_{\tau'_{21}} \leq \|v\|_{s'_{21}} \|u\|_{r'_{21}}. \quad (4.29)$$

Let T be an arbitrary positive number. Injecting (4.29) in (4.28) and applying (2.28) with $(s, r, \tau, \theta) = (r'_{21}, r_1, r_2, \zeta_{21})$ for $f = u$ and $(s, r, \tau, \theta) = (s'_{21}, s_1, s_2, \zeta'_{21})$ for $f = v$ and using Lemma 4.1, we obtain, for $0 < t \leq T$

$$t^{\gamma_1+\delta} \|v(t) - w_2(t)\|_{s_1} \leq C'_{1\delta}, \quad (4.30)$$

where

$$C'_{1\delta} = CM^2|c| \int_0^1 (1-s)^{-\frac{N}{2}\left(\frac{1}{\tau'_{21}} - \frac{1}{s_1}\right)} s^{-\Gamma'_{21}} ds.$$

By parts (iii), (iv) of Lemma 4.1 and by the fact $0 < \delta < \delta_1$, the positive constants $C'_{1\delta}$ is finite.

Now we estimate the inequality (4.26). By Lemma 4.1 and the smoothing properties of the heat semigroup (2.23) with $(p, q, \varphi) = (\frac{s_1}{2}, r_1, v^2 - w_2^2)$ and $(p, q, \varphi) = (\frac{r'_{11}}{2}, r_1, u^2)$ respectively on the first and the second term of the right-hand side of the inequality (4.26), we have

$$\begin{aligned} \|u(t) - w_1(t)\|_{r_1} &\leq C|a| \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{2}{r'_{11}} - \frac{1}{r_1}\right)} \|u^2(s)\|_{r'_{11}/2} ds \\ &\quad + C|b| \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{2}{s_1} - \frac{1}{r_1}\right)} \|v^2(s) - w_2^2(s)\|_{s_1/2} ds. \end{aligned} \quad (4.31)$$

Let $T > 0$ be an arbitrary real number. Using Lemma 4.1, Eq. (2.28) with $(s, r, \tau, \theta) = (r'_{11}, r_1, r_2, \zeta_{11})$ on the first term of the right-hand side of the last

inequality, we get, for $0 < t \leq T$

$$t^{\beta_1+\delta} \|u(t) - w_1(t)\|_{r_1} \leq C_{1\delta} + C_{2\delta} t^{\beta_1+\delta} \|v(t) - w_2(t)\|_{s_1},$$

where

$$C_{1\delta} = M^2 C |a| \int_0^1 (1-s)^{-\frac{N}{2} \left(\frac{2}{r_{11}'} - \frac{1}{r_1} \right)} s^{-2\beta_{11}'} ds,$$

$$C_{2\delta} = 2MC |b| \int_0^1 (1-s)^{-\frac{N}{2s_1}} s^{-2\gamma_1-\delta} ds.$$

By Lemma 4.1, we find that the positive constants $C_{1\delta}, C_{2\delta}$ are finite. Thus, we obtain for a given constant C_δ , since it does not depend on T the pervious inequalities hold on the whole interval $(0, \infty)$. It follows that, for all δ satisfying, $0 < \delta < \delta_1$, there exists a positive constant C_δ , such that (4.16) and (4.17) hold.

Step 2: $L^\infty \times L^\infty$ -asymptotic result. Let $\Phi = (\varphi_1, \varphi_2)$ be as in Proposition 3.2, then using the inequalities (3.16)–(3.17), for $(r, s) = (r_1, s_1)$ of Theorem 3.3, one obtains that, for all $0 < \delta < \delta_0$, there exists a positive constant $C_\delta > 0$, such that (4.18) and (4.19) hold. Hence, $W = (w_1, w_2)$ is solution of (4.24)–(4.25) with initial data $(1 - \eta)\Phi$, where η is a cut-off function satisfying (3.9). Then, by Proposition 3.2 (iii), we have that $\mathcal{N}((1 - \eta)\Phi)$ is finite. Thus, the inequalities (4.2) and (4.3) follow by writing $U - W' = U - W + W - W'$ and using (4.16)–(4.19).

Let $\delta > 0$ be sufficiently small and recall the following numbers given in (i) of Lemma 4.1,

$$\zeta_{11} = \theta_{11} + \delta \left[\frac{1}{2(\alpha_1 - \alpha_2)} \right]; \quad \zeta'_{11} = \theta'_{11}, \quad (4.32)$$

$$\zeta_{21}(\alpha_1 - \alpha_2) + \zeta'_{21}(\alpha'_1 - \alpha'_2) = \theta_{21}(\alpha_1 - \alpha_2) + \theta'_{21}(\alpha'_1 - \alpha'_2) + \delta. \quad (4.33)$$

In particular, δ is chosen small enough so that $\zeta_{11}, \zeta'_{11}, \zeta_{21}$ and ζ'_{21} belong to $[0, 1]$ and $0 < \delta < \delta_1$. Write now

$$v(t) - w_2(t) = e^{\frac{t}{2}\Delta} (v(t/2) - w_2(t/2)) + c \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (u(s)v(s)) ds.$$

We have

$$t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \leq t^{\alpha'_1+\delta} \|e^{\frac{t}{2}\Delta} (v(t/2) - w_2(t/2))\|_\infty$$

$$+ |c| t^{\alpha'_1+\delta} \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (u(s)v(s))\|_\infty ds.$$

That is, then by (2.23) we obtain

$$t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \leq C (t/2)^{\gamma_1+\delta} \|(v(t/2) - w_2(t/2))\|_{s_1}$$

$$+ |c| C t^{\alpha'_1+\delta} \int_{\frac{t}{2}}^t \|u(s)\|_\infty \|v(s)\|_\infty ds.$$

Using (2.17), (4.17) and an interpolation argument, we get for arbitrary $T > 0$, we obtain

$$\begin{aligned}
& t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \\
& \leq C_\delta + C|c|M^2 t^{\alpha'_1+\delta} \left(\int_{1/2}^1 s^{-[\alpha_1 \zeta_{21} + (1-\zeta_{21})\alpha_2 + \alpha'_1 \zeta'_{21} + (1-\zeta'_{21})\alpha'_2]} ds \right) \\
& \leq C_\delta + C|c|M^2 t^{1+\alpha'_1 - (\alpha_1 \zeta_{21} + (1-\zeta_{21})\alpha_2 + \alpha'_1 \zeta'_{21} + (1-\zeta'_{21})\alpha'_2)} \\
& \quad \times \left(\int_{1/2}^1 s^{-[\alpha_1 \zeta_{21} + (1-\zeta_{21})\alpha_2 + \alpha'_1 \zeta'_{21} + (1-\zeta'_{21})\alpha'_2] - \delta} ds \right).
\end{aligned}$$

Now thanks to the relation between ζ_{21} and ζ'_{21} given by (4.33), we obtain

$$t^{\alpha'_1+\delta} \|v(t) - w_2(t)\|_\infty \leq C_\delta + C|c|M^2. \quad (4.34)$$

Let us write now

$$\begin{aligned}
u(t) - w_1(t) &= e^{\frac{t}{2}\Delta} (u(t/2) - w_1(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (u^2(s)) ds \\
&\quad + b \int_{\frac{t}{2}}^t e^{(t-s)\Delta} (v^2(s) - w_2^2(s)) ds.
\end{aligned}$$

We have

$$\begin{aligned}
t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty &\leq t^{\alpha_1+\delta} \|e^{\frac{t}{2}\Delta} (u(t/2) - w_1(t/2))\|_\infty \\
&\quad + |a| \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (u^2(s))\|_\infty ds \\
&\quad + |b| \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (v^2(s) - w_2^2(s))\|_\infty ds.
\end{aligned}$$

Therefore by (2.23) and (4.16), we obtain

$$\begin{aligned}
t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty &\leq C (t/2)^{\beta_1+\delta} \|u(t/2) - w_1(t/2)\|_{r_1} \\
&\quad + C |a| t^{\alpha_1+\delta} \int_{t/2}^t \|u^2(s)\|_\infty ds \\
&\quad + C |b| t^{\alpha_1+\delta} \int_{t/2}^t \|v^2(s) - w_2^2(s)\|_\infty ds \\
&\leq C_\delta + C |a| t^{\alpha_1+\delta} \int_{t/2}^t (\|u(s)\|_\infty^2 ds \\
&\quad + C |b| t^{\alpha_1+\delta} \int_{t/2}^t (\|v(s)\|_\infty + \|w_2(s) \\
&\quad \times \|v(s) - w_2(s)\|_\infty ds.
\end{aligned}$$

Due to (2.17) and an interpolation argument, we get for arbitrary $T > 0$,

$$\begin{aligned}
t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty &\leq C_\delta + C t^{\alpha_1+\delta} M^2 |a| \\
&\quad \times \left(\int_{t/2}^t s^{-2[\alpha_1 \zeta_{11} + (1-\zeta_{11})\alpha_2]} ds \right)
\end{aligned}$$

$$\begin{aligned}
& + 2Ct^{\alpha_1+\delta}M|b| \left(\int_{t/2}^t s^{-2} [\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2]^{-\delta} ds \right) \\
& \times \sup_{(0,T]} \left(t^{\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2 + \delta} \|v(t) - w_2(t)\|_\infty \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty \\
& \leq C_\delta + M^2 C|a| t^{1+\alpha_1-2(\alpha_1 \zeta_{11} + (1-\zeta_{11})\alpha_2)} \left(\int_{1/2}^1 s^{-2} [\alpha_1 \zeta_{11} + (1-\zeta_{11})\alpha_2] ds \right) \\
& + 2MC|b| t^{1+\alpha_1-2(\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2)} \left(\int_{1/2}^1 s^{-2} [\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2]^{-\delta} ds \right) \\
& \times \sup_{(0,T]} \left(t^{\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2 + \delta} \|v(t) - w_2(t)\|_\infty \right).
\end{aligned}$$

From the expressions of ζ_{11} , ζ'_{11} given by (4.32), we obtain

$$\begin{aligned}
t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty & \leq C_\delta + C(M^2) + C(M, \delta) \\
& \times \sup_{(0,T]} \left(t^{\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2 + \delta} \|v(t) - w_2(t)\|_\infty \right). \quad (4.35)
\end{aligned}$$

We substitute $\zeta'_{11} = \theta'_{11} = 1$ and obtain

$$\sup_{(0,T]} \left(t^{\alpha'_1 \zeta'_{11} + (1-\zeta'_{11})\alpha'_2 + \delta} \|v(t) - w_2(t)\|_\infty \right) = \sup_{(0,T]} \left(t^{\alpha'_1 + \delta} \|v(t) - w_2(t)\|_\infty \right). \quad (4.36)$$

By combining (4.35)–(4.36), we arrive to

$$\begin{aligned}
t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty & \leq C_\delta + C(M^2) + C(M, \delta) \\
& \times \sup_{(0,T]} \left(t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty, t^{\alpha'_1 + \delta} \|v(t) - w_2(t)\|_\infty \right). \quad (4.37)
\end{aligned}$$

It follows, for M sufficiently small and the fact that the constants appearing in (4.34) and (4.37) do not depend on T , that

$$\sup_{(0,T]} \left(t^{\alpha_1+\delta} \|u(t) - w_1(t)\|_\infty, t^{\alpha'_1 + \delta} \|v(t) - w_2(t)\|_\infty \right) \leq C_\delta. \quad (4.38)$$

Finally, the conclusion follows by writing $\|u(t) - w'_1(t)\|_\infty \leq \|u(t) - w_1(t)\|_\infty + \|w_1(t) - w'_1(t)\|_\infty$ and $\|v(t) - w'_2(t)\|_\infty \leq \|v(t) - w_2(t)\|_\infty + \|w_2(t) - w'_2(t)\|_\infty$ and using (3.16), (3.17) for $r = s = \infty$ and (4.38). We finish the proof of Theorem 4.2 by using an interpolation argument to get the general case. $\square$

Proof of Theorem 1.3 Let u_0 and v_0 be as in the assumptions of the Theorem 1.2. Let $W = (w_1, w_2)$ be the self-similar of (1.12) with initial data (u_0, v_0) given by Theorem 1.2 and let $U = (u, v)$ the global solution of (1.24) with initial data $(1 - \eta)(u_0, v_0)$ constructed by Theorem 1.1. Then, by the

inequalities (4.2) and (4.3), for $r = s = \infty$ in Theorem 4.2, we find (1.15)–(1.23).

Remark 4.4. Our results concern also the more general Eq. (1.26) with $a \in \mathbb{C}$. In fact, the same technique and results hold if we take $a = a_1 + ia_2$, where $a_1, a_2 \in \mathbb{R}$. Let us consider a more general equation

$$\begin{cases} \partial_t z = \Delta z + a z^2, \\ z(0) = u_0 + iv_0, \end{cases} \quad (4.39)$$

with $a = a_1 + ia_2$, $a_1, a_2 \in \mathbb{R}$. The parabolic system derived from (4.39) is written as follows:

$$\begin{cases} \partial_t u = \Delta u + a_1 u^2 - 2a_2 uv - a_1 v^2, \\ \partial_t v = \Delta v + a_2 u^2 + 2a_1 uv - a_2 v^2, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x). \end{cases} \quad (4.40)$$

Define the asymptotic system of (4.40):

$$\partial_t w_1 = \Delta w_1 + a_1 \mu w_1^2 - 2a_2 \mu_1 w_1 w_2 - a_1 \nu w_2^2, \quad (4.41)$$

$$\partial_t w_2 = \Delta w_2 + a_2 \nu_1 w_1^2 + 2a_1 \mu w_1 w_2 - a_2 \mu_1 w_2^2, \quad (4.42)$$

The parameters μ, ν are given by (1.31) and μ_1, ν_1 are defined by

$$\mu_1 = \lim_{s \rightarrow \infty} s^{-2(\alpha'_1 - 1)}, \quad \nu_1 = \lim_{s \rightarrow \infty} s^{-2(2\alpha_1 - \alpha'_1 - 1)}, \quad (4.43)$$

where α_1, α'_1 be positive real numbers satisfying (2.2) and also the following conditions

$$\alpha'_1 \geq 1, \quad 2\alpha_1 - \alpha'_1 \geq 1, \quad \frac{N}{2\alpha_1} > 1, \quad \frac{N}{2\alpha'_1} > 1.$$

Consider $(u_0, v_0) \in (L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N))^2$ be such that

$$u_0 \sim c_1(1 - \eta)|x|^{-2\alpha_1}, \quad v_0 \sim c_2(1 - \eta)|x|^{-2\alpha'_1}, \quad (4.44)$$

where η is an L^∞ cut-off function ($\eta \equiv 1$ near the origin and having a compact support). There exists $\varepsilon > 0$ such that if $\max(|c_1|, |c_2|) < \varepsilon$, the solution of (4.40) with initial data (u_0, v_0) exists globally in time and describes the different asymptotic behavior. In fact, for ε sufficiently small, there exists a real number $\delta_1 > 0$, such that for all δ satisfying $0 < \delta < \delta_1$ there exists a positive constant C_δ such that the following holds:

(i) If $\alpha_1 > 1$, $2\alpha'_1 > \alpha_1 + 1$ and $2\alpha_1 > \alpha'_1 + 1$, we have

$$\|u(t) - e^{t\Delta}(c_1|x|^{-2\alpha_1})\|_\infty \leq C_\delta t^{-\alpha_1 - \delta}, \quad \forall t > 0, \quad (4.45)$$

$$\|v(t) - e^{t\Delta}(c_2|x|^{-2\alpha'_1})\|_\infty \leq C_\delta t^{-\alpha'_1 - \delta}, \quad \forall t > 0. \quad (4.46)$$

(ii) If $\alpha_1 > 1$ and $2\alpha'_1 = \alpha_1 + 1$, we have

$$\|u(t) - e^{t\Delta}(c_1|x|^{-2\alpha_1}) + f(t)\|_\infty \leq C_\delta t^{-\alpha_1 - \delta}, \quad \forall t > 0, \quad (4.47)$$

$$\|v(t) - e^{t\Delta}(c_2|x|^{-2\alpha'_1})\|_\infty \leq C_\delta t^{-\alpha'_1 - \delta}, \quad \forall t > 0, \quad (4.48)$$

where

$$f(t) = a_1 \int_0^t e^{(t-s)\Delta} [e^{s\Delta} (c_2 |x|^{-2\alpha'_1})]^2 ds.$$

(iii) If $\alpha'_1 > 1$ and $2\alpha_1 = \alpha'_1 + 1$, we have

$$\|u(t) - e^{t\Delta}(c_1 |x|^{-2\alpha_1})\|_\infty \leq C_\delta t^{-\alpha_1 - \delta}, \quad \forall t > 0, \quad (4.49)$$

$$\|v(t) - e^{t\Delta}(c_1 |x|^{-2\alpha'_1}) - h(t)\|_\infty \leq C_\delta t^{-\alpha'_1 - \delta}, \quad \forall t > 0, \quad (4.50)$$

where

$$h(t) = a_2 \int_0^t e^{(t-s)\Delta} [e^{s\Delta} (c_2 |x|^{-2\alpha_1})]^2 ds.$$

(iv) If $\alpha_1 = \alpha'_1 = 1$, we have

$$\|u(t) - w_1(t)\|_\infty \leq C_\delta t^{-1-\delta}, \quad \forall t > 0, \quad (4.51)$$

$$\|v(t) - w_2(t)\|_\infty \leq C_\delta t^{-1-\delta}, \quad \forall t > 0, \quad (4.52)$$

with w_1 and w_2 are the self-similar solution of system (4.40) with initial data $(c_1 |x|^{-2}, c_2 |x|^{-2})$.

In particular, the terms $2a_1 uv$, $2a_2 uv$ are negligible on the large time behavior unlike the case where $a \in \mathbb{R}$.

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