Bound states of a nonhomogeneous nonlinear Schrödinger equation with non symmetric potential

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Abstract. Bound state solutions are found via a linking theorem for a class of nonhomogeneous nonlinear Schrödinger equations with nonsymmetric potentials, using concentration compactness arguments and projections on a general Pohozaev type manifold.

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1. Introduction

In this work we intend to address a question posed by Azzollini and Pomponio in [3] on the existence of a solution for the problem

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^N, \ N \geq 3; \\ u > 0, \end{cases} \quad (P)$$

assuming very general hypotheses on the nonlinearity f as in [7], for instance that f is not superlinear at infinity.

The first part of [3] is dedicated to prove that (P) has at least a radially symmetric positive solution under the assumptions that $V(x) \geq 0$, $\lim_{|x| \rightarrow \infty} V(x) = 0$ and V is radially symmetric, among other natural hypotheses. Furthermore, in case V is not necessarily symmetric they obtained a

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nonexistence result of a ground state solution of problem (P) but in this more general setting, they did not show existence of a positive solution.

The main obstacle in finding a solution of (P) in this case is that the geometrical hypotheses on the potential V do not allow to use concentration compactness arguments as in [17] (see also [9]). There is a problem of lack of compactness due to the fact that the problem is studied in an unbounded domain in $\mathbb{R}^N$. In general, this difficulty is circumvented by assuming symmetry properties of V , for example that V is radially symmetric as in [3, 7] and [24] among many others. Our objective is to prove the existence of a positive solution of (P) with $V(x) \geq V_\infty$, $\lim_{|x| \rightarrow \infty} V(x) = V_\infty$, verifying suitable decay assumptions, but not requiring any symmetry properties.

Problem (P) has a variational structure: its solutions are critical points of the energy functional $I : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} F(u) dx, \quad (1.1)$$

where $F(u) = \int_0^u f(s) ds$. We will study problem (P) for f nonhomogeneous and asymptotically linear at infinity, not necessarily satisfying that $f(s)/s$ is an increasing function. This kind of nonlinearity imposes another difficulty to the problem. Projections on Nehari manifold of the functional associated with the equation (P) are not possible in general. Therefore, one is motivated to use the more suitable projections on the set of points which satisfy the Pohozaev identity [19], the so-called the Pohozaev manifold of (P) , instead. In recent years the use of Pohozaev manifold was shown very effective when treating nonlinearities which do not satisfy Ambrosetti–Rabinowitz superquadraticity condition [2] and the monotonicity condition $f(s)/s$ increasing [12]; this idea is performed in [3, 16, 18] and [23] and references therein.

In this scenario, the problem can be successfully treated by a minimization argument on the Pohozaev manifold [3]. Due to the asymptotically linear function f and the mountain pass geometry of the functional I associated with (P) , the use of a Cerami sequence is more appropriate in order to obtain levels of compactness of the functional I (see [8]).

Existence of solution for problem (P) with spherical symmetry and a superlinear f was obtained using constrained minimization for instance by Strauss [24], Berestycki and Lions [7]. They take advantage of the role played by radial symmetry in restoring compactness in $\mathbb{R}^N$.

Nonsymmetric problem (P) for a power like $f(u) = |u|^{p-1}u$, $1 < p < 2^* - 1$, and $0 < V_0 < V(x) \leq V_\infty$ was solved in [17] by constrained minimization and concentration-compactness arguments and in [20] when V is coercive. In case $V_\infty \leq V(x)$ and f is superlinear at infinity nonsymmetric problems cannot be solved by minimization [9]. In order to find a positive solution Benci and Cerami in [6] gave a description of the behavior of the Palais–Smale sequences

together with some energy estimates, which allow to locate some interval of values in which the compactness is obtained [9]. Under this type of nonlinearity and some decay assumption of $V(x)$ to limit value V_∞ , Bahri and Lions [4] were able to prove that (P) has a positive solution. On the other hand, if f is asymptotically linear at infinity and radial symmetry is assumed, the existence of a positive solution is exploited in [3, 7, 25] and their references.

The nonsymmetric asymptotically linear problem (P) under the hypothesis $0 < V_0 < V(x) \leq V_\infty$ was addressed by Jeanjean and Tanaka in [14] where they obtained a positive solution. Costa and Tehrani in [10] prove the existence of a positive solution for the nonautonomous asymptotically linear nonlinearity $f(x, u)$. They work with $V(x) = \lambda > 0$ constant and restricted to $f(x, s)/s$ a nondecreasing function of s on $[0, \infty)$. These problems model the propagation of a light beam when passing through a saturable medium, as well described in [26] and references therein. Problem (P) cannot be handled by the same tools when $V_\infty < V(x)$ and $f(x, u)$ is asymptotically linear at infinity. Moreover, if $f(x, s)/s$ is not increasing of s on $[0, \infty)$ then the projections on Nehari manifold are no longer suitable. Our purpose is to face this problem under this generality adapting the ideas developed in [16]. Indeed, this is the counterpart of the problem in [16] where the variable potential plays the role of the nonautonomous terms in there. Some of the arguments performed here are closely related to those of our previous paper [16]. So, in the proofs of some technical results we will omit the details and refer to where adaptations are needed. To our knowledge this general problem has not been solved before. We will consider the elliptic problem (P) , with $N \geq 3$ and these hypothesis on the functions V and f , given by

- (V_1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$;
- (V_2) $\lim_{|x| \rightarrow +\infty} V(x) = V_\infty < 1, \quad 0 < V_\infty \leq V(x)$;
- (V_3) $\langle \nabla V(x) | x \rangle \leq 0$, for all $x \in \mathbb{R}^N$, with the strict inequality holding on a subset of positive Lebesgue measure of $\mathbb{R}^N$;
- (V_4) $NV(x) + \langle \nabla V(x) | x \rangle \geq NV_\infty$ for all $x \in \mathbb{R}^N$;
- (V_5) $\langle \nabla V(x) | x \rangle + \frac{xH(x)x}{N} \leq 0$, for all $x \in \mathbb{R}^N$,

where H represents the Hessian matrix of the function V .

Later in this paper we will also assume yet another condition (V_6) , requiring that the supremum of $|V(x) - V_\infty|$ is not large (see Lemma 4.12).

We also assume the following conditions on the function f :

- (f_1) $f \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\lim_{s \rightarrow 0^+} \frac{f(s)}{s} = 0$;
- (f_2) $\lim_{s \rightarrow +\infty} \frac{f(s)}{s} = 1$;

(f_3) if $F(s) = \int_0^s f(t)dt$ and $Q(s) = \frac{1}{2}f(s)s - F(s)$, then there exists a constant $D \geq 1$ such that

$$0 < Q(s) \leq D Q(t), \quad \text{for all } 0 < s \leq t, \\ \lim_{s \rightarrow \infty} Q(s) = +\infty.$$

Under the previous hypotheses Azzollini and Pomponio in [3] have proven the following nonexistence result.

Theorem 1.1. *Assume $(V_1) - (V_4)$ and $(f_1) - (f_3)$. Then, $p = \inf_{u \in \mathcal{P}} I(u)$ is not a critical level for the functional I . In particular, the infimum p is not achieved.*

Now we can state our main existence result, which complements the results obtained in [3].

Theorem 1.2. *Assume $(V_1) - (V_6)$, $(f_1) - (f_3)$ and $f \in Lip(\mathbb{R}^+, \mathbb{R}^+)$ and problem*

$$-\Delta u + V_\infty u = f(u) \quad \text{in } \mathbb{R}^N, \quad |u(x)| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (1.2)$$

admits a unique positive solution. Then equation (P) has a positive solution $u \in H^1(\mathbb{R}^N)$.

Remark 1.3. By (f_1) and (V_2) it follows that $f'(0) = 0$ and since f' is continuous, there exists a $\delta > 0$ such that $g(s) = -V_\infty s + f(s)$, $g(0) = 0$ and g is non-increasing on the interval $0 \leq s < \delta$. Under these conditions the hypotheses of Theorem 1 in [22] are satisfied and we may conclude that a positive solution of (1.2) is radially symmetric about the origin in $\mathbb{R}^N$. Moreover, if we assume that for some $b > 0$ the function g satisfies:

(H_1) $g(s) \leq 0$ on $(0, b]$ and $g(s) > 0$ for $s > b$;

(H_2) $\frac{sg'(s)}{g(s)}$ non-increasing in (b, ∞) ,

then Theorem 1 in [21] applies to show that (1.2) has a unique radial positive solution.

Remark 1.4. Conditions (V_3) and (V_4) imply

$$\langle \nabla V(x) | x \rangle \rightarrow 0, \quad \text{if } |x| \rightarrow \infty.$$

Remark 1.5. Since we are looking for positive solutions, we take as usual $f(s)$ defined on all $s \in \mathbb{R}$, making $f(s) = 0$ if $s \leq 0$. Thus, as we know that the critical points of the functional I associated with the equation (P) are weak solutions of the equation, if u is a critical point of I then

$$0 = I'(u)u^- = \int_{\mathbb{R}^N} \nabla u \nabla u^- + V(x)uu^- dx - \int_{\mathbb{R}^N} f(u)u^- dx \\ = \int_{\mathbb{R}^N} |\nabla u^-|^2 + V(x)|u^-|^2 dx = \|u^-\|^2,$$

where $u^- := \min\{u, 0\}$. Thus, necessarily we have $u \geq 0$.

Conditions (f_1) and (f_2) imply that, given $\varepsilon > 0$ and $2 \leq p \leq 2^*$, there exists a positive constant $C = C(\varepsilon, p)$ such that for all s in $\mathbb{R}$

$$|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + C|s|^p. \quad (1.3)$$

In what follows we will denote $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx$ the norm in $H^1(\mathbb{R}^N)$, which is equivalent to the usual one, since (V_2) holds.

2. A Pohozaev type identity

We do not assume that $f(s)/s$ is nondecreasing and hence the path $\gamma(t) := I(tu)$ may not intersect the Nehari manifold $\mathcal{N} := \{u \neq 0; I'(u)u = 0\}$ for a unique t . Indeed, it may happen that it does not intersect the Nehari manifold at all or that it intersects the Nehari manifold in infinitely many points. This is the main reason why we are led to look for a different approach, using the Pohozaev identity and manifold. De Figueiredo, Lions and Nussbaum in [11], Lemma 1.1, (see also Proposition 2.1 in [16]) provide a Pohozaev type identity for problem (P) . We observe that from Lemma 1.1 in [11], any solution of equation (P) satisfies the Pohozaev identity

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V(x)u^2 + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x) | x \rangle u^2 = N \int_{\mathbb{R}^N} F(u). \quad (2.1)$$

In what follows we consider the set of points $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ that satisfy the Pohozaev identity associated with (P) :

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ such that } u \text{ satisfies (2.1)}\}. \quad (2.2)$$

Lemma 2.1. *Let the functional $J : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ be defined by*

$$J(u) = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V(x)u^2 + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x) | x \rangle u^2 - N \int_{\mathbb{R}^N} F(u).$$

Then, the following holds:

- (a) $\{u \equiv 0\}$ is an isolated point of $J^{-1}(\{0\})$;
- (b) $\mathcal{P} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid J(u) = 0\}$ is a closed set;
- (c) $\mathcal{P}$ is a C^1 manifold;
- (d) There exists $\sigma > 0$ such that $\|u\| > \sigma$, for all $u \in \mathcal{P}$.

Proof. (a) Indeed, using condition (V_4) it follows

$$\begin{aligned} J(u) &\geq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty u^2 dx - N \int_{\mathbb{R}^N} F(u) dx \\ &> \frac{C(N-2)}{2} \|u\|^2 - N \int_{\mathbb{R}^N} F(u) dx. \end{aligned}$$

By the Sobolev embedding and (1.3), it holds

$$\begin{aligned} J(u) &> \frac{N-2}{2} \|u\|^2 - \frac{N\varepsilon}{2V_\infty} \int_{\mathbb{R}^N} V_\infty |u|^2 dx - NC \int_{\mathbb{R}^N} |u|^p dx \\ &> \frac{1}{2} \left(C(N-2) - \frac{N\varepsilon}{V_\infty} \right) \|u\|^2 - NC \|u\|^p. \end{aligned}$$

If we take $\varepsilon > 0$ sufficiently small and $0 < \rho < 1$ such that $CV_\infty(N-2) - N\varepsilon > 0$ and $\rho^p < \frac{1}{4NC} \left(C(N-2) - \frac{N\varepsilon}{V_\infty} \right) \rho^2$, then if $\|u\| = \rho$, we obtain

$$J(u) > \frac{1}{4} \left(C(N-2) - \frac{N\varepsilon}{V_\infty} \right) \rho^2 > 0.$$

(b) J is a C^1 functional, thus $\mathcal{P} \cup \{0\} = J^{-1}(\{0\})$ is a closed subset. Moreover, $\{u \equiv 0\}$ is an isolated point in $J^{-1}(\{0\})$ and the claim in (b) follows.

(c) Consider the derivative of J at u and apply on u :

$$\begin{aligned} J'(u)u &= (N-2) \int_{\mathbb{R}^N} |\nabla u|^2 dx + N \int_{\mathbb{R}^N} V(x)u^2 dx + \int_{\mathbb{R}^N} \langle \nabla V(x)|x \rangle u^2 dx \\ &\quad - N \int_{\mathbb{R}^N} f(u)u dx. \end{aligned} \quad (2.3)$$

Since $u \in \mathcal{P}$, it follows that

$$\begin{aligned} (N-2) \int_{\mathbb{R}^N} |\nabla u|^2 dx &= -N \int_{\mathbb{R}^N} V(x)u^2 dx \\ &\quad - \int_{\mathbb{R}^N} \langle \nabla V(x)|x \rangle u^2 dx + 2N \int_{\mathbb{R}^N} F(u) dx. \end{aligned} \quad (2.4)$$

Using (2.4) in (2.3) and recalling (2.1) and (f_3) we obtain

$$\begin{aligned} J'(u)u &= -N \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} \langle \nabla V(x)|x \rangle u^2 dx + 2N \int_{\mathbb{R}^N} F(u) dx \\ &\quad + N \int_{\mathbb{R}^N} V(x)u^2 dx + \int_{\mathbb{R}^N} \langle \nabla V(x)|x \rangle u^2 dx - N \int_{\mathbb{R}^N} f(u)u dx \\ &= N \int_{\mathbb{R}^N} 2F(u) - f(u)u dx < 0. \end{aligned}$$

Therefore, if $u \in \mathcal{P}$, then $J'(u)u < 0$, which shows that $\mathcal{P}$ is a C^1 manifold.

(d) Consider $u \in \mathcal{P}$, then u satisfies $J(u) = 0$ and, by using (V_4) , we obtain

$$N \int_{\mathbb{R}^N} F(u) dx \geq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty u^2 dx \geq C \|u\|^2.$$

By (1.3) with $p > 2$

$$C \|u\|^2 \leq N \int_{\mathbb{R}^N} F(u) dx \leq N\varepsilon \int_{\mathbb{R}^N} u^2 dx + NC \int_{\mathbb{R}^N} |u|^p dx,$$

which implies

$$\left(C - \frac{N\varepsilon}{V_\infty}\right) \|u\|^2 < C\|u\|^p$$

and hence there is a constant $\sigma > 0$ such that if $u \in \mathcal{P}$, then $\|u\| > \sigma$. $\square$

Lemma 2.2. *Assume (V_1) , (V_5) and (f_1) , then $\mathcal{P}$ is a natural constraint of problem (P) .*

Proof. Let u be a critical point of the functional I restricted to the manifold $\mathcal{P}$. By the theorem of Lagrange multipliers, there exists a $\mu \in \mathbb{R}$ such that

$$I'(u) + \mu J'(u) = 0, \quad \mu \in \mathbb{R}.$$

We will show that $\mu = 0$. Evaluating the linear functional above at $u \in \mathcal{P}$, we obtain

$$I'(u)u + \mu J'(u)u = 0,$$

which is equivalent to

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx - \int_{\mathbb{R}^N} f(u)u dx \\ &\quad + \mu \left[(N-2) \int_{\mathbb{R}^N} |\nabla u|^2 dx + N \int_{\mathbb{R}^N} V(x)u^2 dx \right. \\ &\quad \left. + \int_{\mathbb{R}^N} \langle \nabla V(x) | x \rangle u^2 dx - N \int_{\mathbb{R}^N} f(u)u dx \right]. \end{aligned}$$

This expression may be associated with the equation

$$-\Delta u + V(x)u - f(u) + \mu [-(N-2)\Delta u + NV(x)u + \langle \nabla V(x) | x \rangle u - Nf(u)] = 0,$$

which can be rewritten as

$$-(1 + \mu(N-2))\Delta u + V(x)(1 + \mu N)u + \mu \langle \nabla V(x) | x \rangle u = (1 + \mu N)f(u). \quad (2.5)$$

Moreover, the solutions of this equation satisfy a Pohozaev identity and have an associated Pohozaev manifold defined by $\tilde{\mathcal{P}}^{-1}(\{0\})$, where

$$\begin{aligned} \tilde{\mathcal{P}}(u) &= (1 + \mu(N-2)) \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad - N \int_{\mathbb{R}^N} G^*(x, u) dx - \sum_{i=1}^N \int_{\mathbb{R}^N} G_{x_i}^*(x, u) x_i dx, \end{aligned}$$

with

$$G^*(x, u) = -\frac{V(x)}{2}(1 + \mu N)u^2 - \mu \langle \nabla V(x) | x \rangle \frac{u^2}{2} + (1 + \mu N)F(u),$$

and

$$\sum_{i=1}^N \int_{\mathbb{R}^N} G_{x_i}^*(x, u) x_i dx = \int_{\mathbb{R}^N} -\langle \nabla V(x) \mid x \rangle (1 + \mu N) \frac{u^2}{2} dx - \mu \int_{\mathbb{R}^N} x H(x) x \frac{u^2}{2} dx,$$

where $H(x)$ denotes the Hessian matrix of $V(x)$. Therefore, $\tilde{\mathcal{P}}$ may be seen as follows

$$\begin{aligned} \tilde{\mathcal{P}}(u) &= (1 + \mu(N - 2)) \left(\frac{N - 2}{2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad - N \int_{\mathbb{R}^N} \left[(1 + \mu N) F(u) - V(x) (1 + \mu N) \frac{u^2}{2} - \mu \langle \nabla V(x) \mid x \rangle \frac{u^2}{2} \right] dx \\ &\quad + \int_{\mathbb{R}^N} \left[\langle \nabla V(x) \mid x \rangle (1 + \mu N) \frac{u^2}{2} + \mu \langle \nabla V(x) \mid x \rangle \frac{u^2}{2} + \mu x H(x) x \frac{u^2}{2} \right] dx \\ &= (1 + \mu(N - 2)) \frac{(N - 2)}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - N(1 + \mu N) \int_{\mathbb{R}^N} F(u) dx \\ &\quad + (1 + \mu N) \int_{\mathbb{R}^N} (NV(x) + \langle \nabla V(x) \mid x \rangle) \frac{u^2}{2} dx + N\mu \int_{\mathbb{R}^N} \langle \nabla V(x) \mid x \rangle \frac{u^2}{2} dx \\ &\quad + \mu \int_{\mathbb{R}^N} x H(x) x \frac{u^2}{2} dx. \end{aligned}$$

But recalling that $u \in \mathcal{P}$ and replacing (2.4) in the previous equation, it follows that

$$\begin{aligned} \tilde{\mathcal{P}}(u) &= (1 + \mu(N - 2)) \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - (1 + \mu N) \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad + N\mu \int_{\mathbb{R}^N} \langle \nabla V(x) \mid x \rangle \frac{u^2}{2} dx + \mu \int_{\mathbb{R}^N} x H(x) x \frac{u^2}{2} dx \\ &= -\mu(N - 2) \int_{\mathbb{R}^N} |\nabla u|^2 dx + N\mu \int_{\mathbb{R}^N} \langle \nabla V(x) \mid x \rangle \frac{u^2}{2} dx \\ &\quad + \mu \int_{\mathbb{R}^N} x H(x) x \frac{u^2}{2} dx. \end{aligned}$$

On the other hand, u is a solution of the Eq. (2.5) and thus satisfies $\tilde{\mathcal{P}}(u) = 0$. This yields

$$\mu(N - 2) \int_{\mathbb{R}^N} |\nabla u|^2 dx = \mu N \int_{\mathbb{R}^N} \left[\langle \nabla V(x) \mid x \rangle \frac{u^2}{2} + \frac{x H(x) x}{N} \frac{u^2}{2} \right] dx.$$

By the hypothesis (V_5) , we have that the right hand side of the equation is always nonpositive, while the left hand side is always positive, if $\mu > 0$ (or vice-versa, if $\mu < 0$), unless $\mu = 0$. Therefore, in the equation

$$I'(u) + \mu J'(u) = 0,$$

it holds that $\mu = 0$ and it follows $I'(u) = 0$, so u is a critical point of I . $\square$

3. Projections on Pohozaev manifold

In order to do so we exploit the main relations between the Pohozaev manifold $\mathcal{P}$ associated with the nonautonomous problem (P) and the Pohozaev manifold $\mathcal{P}_\infty$ associated with the autonomous problem at infinity in $\mathbb{R}^N$ (1.2). The functional associated with the Eq. (1.2) is given by

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V_\infty u^2 dx - \int_{\mathbb{R}^N} F(u) dx,$$

and the Pohozaev manifold $\mathcal{P}_\infty$ is given by

$$\mathcal{P}_\infty = \{u \in H^1 \setminus \{0\} \mid J_\infty(u) = 0\},$$

with

$$J_\infty(u) = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - N \int_{\mathbb{R}^N} G_\infty(u) dx,$$

where $G_\infty(u) = F(u) - V_\infty \frac{u^2}{2}$. Also, we consider the set of paths

$$\Gamma_\infty = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, I_\infty(\gamma(1)) < 0\},$$

and define the min-max mountain pass level for I_∞ by

$$c_\infty := \min_{\gamma \in \Gamma_\infty} \max_{0 \leq t \leq 1} I_\infty(\gamma(t)).$$

Lemma 3.1. *Suppose that $\int_{\mathbb{R}^N} G_\infty(u) dx > 0$, then there exist unique $\theta_1(u) > 0$ and $\theta_2(u) > 0$ such that $u(\cdot/\theta_1) \in \mathcal{P}$ and $u(\cdot/\theta_2) \in \mathcal{P}_\infty$.*

Proof. The proof of existence of $\theta_1(u) > 0$ and $\theta_2(u) > 0$ is in [3] and the uniqueness can be proved as in [16]. $\square$

Lemma 3.2. *Let $\mathcal{O} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\}; \int_{\mathbb{R}^N} G_\infty(u) dx > 0\}$ then the function $\theta_1 : \mathcal{O} \rightarrow \mathbb{R}^+$ given by $u \mapsto \theta_1(u)$, such that $u(\cdot/\theta_1(u)) \in \mathcal{P}$, is continuous.*

Proof. The proof is analogous to the proof of Lemma 3.3 in [16]. $\square$

Lemma 3.3. *If $u \in \mathcal{P}_\infty$, then there exists $\theta > 0$ such that $u(\cdot/\theta) \in \mathcal{P}$ and $\theta > 1$.*

Proof. The proof is similar to that presented in [16]. $\square$

Lemma 3.4. (Lemma 4.6 in [3]) *If $u \in \mathcal{P}$, then there exists $\theta > 0$ such that $u(\cdot/\theta) \in \mathcal{P}_\infty$ and $\theta < 1$.*

Lemma 3.5. *If $u \in \mathcal{P}_\infty$, then $u(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^N$. Moreover, there exists $\theta_y > 0$ such that $u\left(\frac{\cdot - y}{\theta_y}\right) \in \mathcal{P}$ and*

$$\lim_{|y| \rightarrow \infty} \theta_y = 1.$$

Proof. The proof is similar to that of Lemma 3.7 in [16], with a function K defined by

$$K(\theta_{y_n}x + y_n) := V(\theta_{y_n}x + y_n) + \frac{\langle \nabla V(\theta_{y_n}x + y_n) | (\theta_{y_n}x + y_n) \rangle}{N}.$$

□

Lemma 3.6. $\sup_{y \in \mathbb{R}^N} \theta_y = \bar{\theta} < \infty$ and $\bar{\theta} > 1$.

Proof. See Lemma 3.8 in [16].

□

Lemma 3.7. *There exists a real number $\hat{\sigma} > 0$ such that $\inf_{u \in \mathcal{P}} \|\nabla u\|_2 \geq \hat{\sigma}$.*

Lemma 3.8. $p =: \inf_{u \in \mathcal{P}} I(u) > 0$.

For the proofs see [3] and [16].

Remark 3.9. We recall that Jeanjean and Tanaka have shown in [15] that

$$\inf_{u \in \mathcal{P}_\infty} I_\infty(u) = c_\infty.$$

Remark 3.10. If $u \in H^1(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} G_\infty(u) dx > 0$ and $\theta > 0$ is such that $u(\cdot/\theta) \in \mathcal{P}_\infty$, then we may write

$$I_\infty(u(x/\theta)) = \frac{\theta^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla u|^2 dx. \quad (3.1)$$

Lemma 3.11. $p = c_\infty$.

Proof. See Lemma 4.5 and Remark 4.7 in [3].

□

The proof of Theorem 1.1 in [3] is a consequence of the previous lemma.

4. A barycenter function and existence of a positive solution

This section is dedicated to prove the existence of a positive solution for problem (P). The previous results suggest that we should look for solutions in the range of energy levels above c_∞ . Ambrosetti, Cerami and Ruiz in [1] use a barycenter function together with the Nehari manifold associated with the problem in order to obtain a solution in the energy level above the ground state. Similarly to what was done in [16], we will use a combination of Pohozaev manifold and barycenter constraints and apply a version of the Linking Theorem.

Let c be the min-max mountain pass level for the functional I given by

$$c = \min_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad (4.1)$$

where

$$\Gamma := \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) | \gamma(0) = 0, I(\gamma(1)) < 0 \}.$$

Lemma 4.1. $c_\infty = c = p$.

The proof of this lemma can be found in [16], Lemma 4.1 and Lemma 4.2.

Now we observe the following property of the Pohozaev manifold $\mathcal{P}$ with respect to the paths in the Mountain Pass Theorem.

Lemma 4.2. *Let $\gamma \in \Gamma$, there exists $s \in (0, 1)$ such that $\gamma(s)$ intersects $\mathcal{P}$.*

Proof. From Lemma 2.1(a) there exists $\rho > 0$ such that, if $0 < \|u\| < \rho$, then $J(u) > 0$. Furthermore, observe that

$$J(u) = NI(u) - \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x) | x \rangle u^2 dx.$$

From condition (V_3) , it follows that

$$J(u) < NI(u).$$

Therefore, if $\gamma \in \Gamma$, on one hand $J(\gamma(0)) = 0$, and on the other hand $J(\gamma(1)) < NI(\gamma(1)) < 0$, since $I(\gamma(1)) < 0$, and we conclude that there exists $s \in (0, 1)$, for which $\|\gamma(s)\| > \rho$ and such that $J(\gamma(s)) = 0$. The function $\gamma(s)$ satisfies $\gamma(s) \in \mathcal{P}$, which shows that every path $\gamma \in \Gamma$ intersects $\mathcal{P}$. $\square$

A sequence (u_n) is said to be a Cerami sequence for the functional I at level d in $\mathbb{R}$, denoted by $(Ce)_d$, if $I(u_n) \rightarrow d$ and $\|I'(u_n)\|(1 + \|u_n\|) \rightarrow 0$. Now we show that, if $d > 0$, then any $(Ce)_d$ sequence for the functional I is bounded, up to a subsequence.

Lemma 4.3. *If (u_n) is a $(Ce)_d$ sequence with $d > 0$, then it has a bounded subsequence.*

Proof. Suppose, by contradiction, that $\|u_n\| \rightarrow \infty$. Let us define $\hat{u}_n := \frac{u_n}{\|u_n\|}$, then $\hat{u}_n$ is a bounded sequence and $\|\hat{u}_n\| = 1$, thus $\hat{u}_n \rightharpoonup \hat{u}$, up to a subsequence. Therefore, one of the two cases occur:

$$\text{Case 1 : } \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\hat{u}_n(x)|^2 dx > 0;$$

$$\text{Case 2 : } \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\hat{u}_n(x)|^2 dx = 0.$$

Suppose case 2, and take $L > 1$ which gives

$$I\left(\frac{L}{\|u_n\|} u_n\right) = \frac{1}{2} L^2 - \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|} u_n\right) dx.$$

Given $\varepsilon > 0$, (1.3) implies

$$\int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|} u_n\right) dx < \frac{\varepsilon}{2V_\infty} L^2 \int_{\mathbb{R}^N} V_\infty \hat{u}_n^2 dx + CL^p \int_{\mathbb{R}^N} \hat{u}_n^p dx.$$

By Lions' lemma [27],

$$\int_{\mathbb{R}^N} \hat{u}_n^p dx \rightarrow 0.$$

Moreover,

$$\int_{\mathbb{R}^N} V_\infty \hat{u}_n^2 dx \leq \int_{\mathbb{R}^N} V(x) \hat{u}_n^2 dx \leq \|u_n\|^2 \leq 1,$$

and hence

$$\int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|} u_n\right) dx < \frac{\varepsilon}{2V_\infty} L^2 + o_n(1).$$

With $\varepsilon = \frac{V_\infty}{2} > 0$, we have

$$I\left(\frac{L}{\|u_n\|} u_n\right) > \frac{L^2}{2} - \left(\frac{L^2}{4} + o_n(1)\right) = \frac{L^2}{4} - o_n(1).$$

The assumption $\|u_n\| \rightarrow \infty$, implies that $\frac{L}{\|u_n\|} \in (0, 1)$ for $n > 0$ sufficiently large, so

$$\max_{t \in [0,1]} I(tu_n) \geq I\left(\frac{L}{\|u_n\|} u_n\right) > \frac{L^2}{4} - o_n(1).$$

Let $t_n \in (0, 1)$ be such that $I(t_n u_n) = \max_{t \in [0,1]} I(tu_n)$. Then

$$I(t_n u_n) > \frac{L^2}{4} - o_n(1). \quad (4.2)$$

But since $t_n \leq 1$, using (f_3) we obtain

$$\begin{aligned} I(t_n u_n) &= I(t_n u_n) - \frac{1}{2} I'(t_n u_n)(t_n u_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(t_n u_n)(t_n u_n) - F(t_n u_n) \right) dx + o_n(1) \\ &\leq D \int_{\mathbb{R}^N} \left(\frac{1}{2} f(u_n) u_n - F(u_n) \right) dx + o_n(1) \\ &= D \left(I(u_n) - \frac{1}{2} I'(u_n) u_n \right) + o_n(1) \\ &= Dd + o_n(1). \end{aligned} \quad (4.3)$$

Together, (4.2) and (4.3) give

$$\frac{L^2}{4} - o_n(1) < I(t_n u_n) \leq Dd + o_n(1).$$

Taking $L > 0$ sufficiently large we now arrive at a contradiction.

On the other hand, suppose case 1 occurs, that is $\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\hat{u}_n|^2 dx = \delta > 0$. If (y_n) is a subsequence such that $|y_n| \rightarrow \infty$ and $\int_{B_1(y_n)} |\hat{u}_n|^2 dx > \delta/2$, recalling that $\hat{u}_n(x + y_n) \rightharpoonup \bar{u}(x)$, we obtain

$$\int_{B_1(0)} |\hat{u}_n(x + y_n)|^2 dx > \delta/2,$$

and hence

$$\int_{B_1(0)} |\bar{u}(x)|^2 dx > \delta/2,$$

i.e. $\bar{u} \neq 0$. Thus, there exists $\Omega \subset B_1(0)$, with $|\Omega| > 0$ such that

$$0 < |\bar{u}(x)| = \lim_{n \rightarrow \infty} |\hat{u}_n(x + y_n)| = \lim_{n \rightarrow \infty} \frac{|u_n(x + y_n)|}{\|u_n\|}, \quad \forall x \in \Omega.$$

Again using the hypothesis $\|u_n\| \rightarrow \infty$, it follows that

$$u_n(x + y_n) \rightarrow \infty, \quad \forall x \in \Omega \subset B_1(0).$$

Thus, by condition (f_3) and Fatou's lemma,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{1}{2} f(u_n(x + y_n)) u_n(x + y_n) - F(u_n(x + y_n)) dx \\ & \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{1}{2} f(u_n(x + y_n)) u_n(x + y_n) - F(u_n(x + y_n)) dx \\ & \geq \int_{\Omega} \liminf_{n \rightarrow \infty} \frac{1}{2} f(u_n(x + y_n)) u_n(x + y_n) - F(u_n(x + y_n)) dx \\ & = +\infty. \end{aligned}$$

On the other hand, we have that $|I'(u_n)u_n| \leq \|I'(u_n)\| \|u_n\| \rightarrow 0$ and hence $I'(u_n)u_n = o_n(1)$. Therefore,

$$\int_{\mathbb{R}^N} \frac{1}{2} f(u_n) u_n - F(u_n) dx = I(u_n) - \frac{1}{2} I'(u_n)u_n \leq d + 1,$$

which gives a contradiction.

If y_n is bounded, $|y_n| \leq R$, $R > 1$, we obtain

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\hat{u}_n(x + y_n)|^2 dx \leq \int_{B_{2R}(0)} |\hat{u}_n(x + y_n)|^2 dx,$$

and since $\hat{u}_n(x + y_n) \rightarrow \bar{u}$ in $B_{2R}(0)$, it follows that

$$\delta/2 \leq \int_{B_1(0)} |\bar{u}(x)|^2 dx.$$

Similarly to the previous case, there exists $\Omega \subset B_1(0)$, $|\Omega| > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{|u_n(x + y_n)|}{\|u_n\|} = \lim_{n \rightarrow \infty} |\hat{u}_n(x + y_n)| = |\bar{u}(x)| \neq 0, \quad \forall x \in \Omega.$$

The argument follows as above for the case where $|y_n| \rightarrow \infty$ and we arrive at a contradiction. Therefore, neither case 1 nor case 2 can occur and the lemma is proved. $\square$

Lemma 4.4. (Splitting) *Let $(u_n) \in H^1(\mathbb{R}^N)$ be a bounded sequence such that*

$$I(u_n) \rightarrow d > 0 \quad \text{and} \quad I'(u_n) \rightarrow 0 \quad \text{in } H^{-1}.$$

Replacing (u_n) by a subsequence, if necessary, there exists a solution $\bar{u}$ of (P) , a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, u^2, \dots, u^k$ and k sequences of points $(y_n^j) \in \mathbb{R}^N$, $1 \leq j \leq k$, satisfying:

- (a) $u_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^N)$ or
- (b) u^j are nontrivial solutions of (1.2);
- (c) $|y_n^j| \rightarrow \infty$ and $|y_n^j - y_n^i| \rightarrow \infty, i \neq j$;
- (d) $u_n - \sum_{i=1}^k u^i(x - y_n^i) \rightarrow \bar{u}$;
- (e) $I(u_n) \rightarrow I(\bar{u}) + \sum_{i=1}^k I_\infty(u^i)$.

Proof. See Lemma 2.9 in [18]. □

Corollary 4.5. *If $I(u_n) \rightarrow c_\infty$ and $\|I'(u_n)\|(1 + \|u_n\|) \rightarrow 0$, then either (u_n) is relatively compact or the splitting lemma holds with $k = 1$ and $\bar{u} = 0$.*

Lemma 4.6. *The functional I satisfies condition (Ce) at level $d \in (c_\infty, 2c_\infty)$.*

Proof. Take a sequence $(u_n) \in H^1(\mathbb{R}^N)$ such that $I(u_n) \rightarrow d$ and $\|I'(u_n)\|(1 + \|u_n\|) \rightarrow 0$. By Lemma 4.3, (u_n) is bounded. Applying Lemma 4.4, up to subsequences, we have

$$u_n - \sum_{j=1}^k u^j(x - y_n^j) \rightarrow \bar{u}, \quad \text{in } H^1(\mathbb{R}^N),$$

where u^j is a weak solution of the problem at infinity, $|y_n^j| \rightarrow \infty$ and $\bar{u}$ is a weak solution of equation (P) . Moreover,

$$I(u_n) = I(\bar{u}) + \sum_{j=1}^k I_\infty(u_j) + o_n(1).$$

Since $d < 2c_\infty$, then $k < 2$. If $k = 1$, we have two cases to distinguish:

- (1) $\bar{u} \neq 0$, which implies $I(\bar{u}) \geq c_\infty$ and hence $I(u_n) \geq 2c_\infty$.
- (2) $\bar{u} = 0$, which yields $I(u_n) \rightarrow I_\infty(u_1)$. Since the positive solution of (1.2) is unique, it follows that $I_\infty(u_1) = c_\infty$.

In both cases we arrive at a contradiction with the fact that $d \in (c_\infty, 2c_\infty)$. Therefore, we must have $k = 0$ and the convergence $u_n \rightarrow \bar{u}$. □

Lemma 4.7. *Let c be as in (4.1), then there exists a $(Ce)_c$ sequence $(u_n) \subset H^1(\mathbb{R}^N)$.*

Proof. The idea of the proof is to use the Ghoussoub–Preiss theorem (Theorem 6 in [13]), in the same way as done in [16], Lemma 4.5. In this way, we obtain a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that

$$\delta(u_n, \mathcal{P}) \rightarrow 0 \tag{4.4}$$

$$I(u_n) \rightarrow c \tag{4.5}$$

$$\|I'(u_n)\|(1 + \|u_n\|) \rightarrow 0, \tag{4.6}$$

where δ denotes the geodesic metric on $H^1(\mathbb{R}^N)$, defined by

$$\delta(u, v) = \inf \left\{ \int_0^1 \frac{\|h'(t)\|}{1 + \|h(t)\|} dt : h \in C([0, 1], H^1(\mathbb{R}^N)), h(0) = u \text{ and } h(1) = v \right\}.$$

□

Definition 4.8. Define the *barycenter* function of a given function $u \neq 0 \in H^1(\mathbb{R}^N)$ as follows: let

$$\mu(u)(x) = \frac{1}{|B_1|} \int_{B_1(x)} |u(y)| dy,$$

with $\mu(u) \in L^\infty(\mathbb{R}^N)$ and is a continuous function. Subsequently, take

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2} \max \mu(u) \right]^+.$$

It follows that $\hat{u} \in C_0(\mathbb{R}^N)$. Now define the *barycenter* of u by

$$\beta(u) = \frac{1}{\|\hat{u}\|_{L^1}} \int_{\mathbb{R}^N} x \hat{u}(x) dx \in \mathbb{R}^N.$$

Since $\hat{u}$ has compact support, by definition, $\beta(u)$ is well defined.

The function β satisfies the properties:

- (a) β is a continuous function in $H^1(\mathbb{R}^N) \setminus \{0\}$.
- (b) If u is radial, then $\beta(u) = 0$.
- (c) Given $y \in \mathbb{R}^N$ and defining $u_y(x) := u(x - y)$, then $\beta(u_y) = \beta(u) + y$.

Now we define

$$b := \inf \{I(u); u \in \mathcal{P} \text{ and } \beta(u) = 0\}. \quad (4.7)$$

It is clear that $b \geq c_\infty$. Moreover, the following result is true:

Lemma 4.9. $b > c_\infty$.

Proof. Suppose, by contradiction, that $b = c_\infty$. By the definition of b , there exists a (minimizing) sequence $(u_n) \subset \{u \in H^1(\mathbb{R}^N) \mid u \in \mathcal{P}, \beta(u) = 0\}$ such that

$$I(u_n) \rightarrow b > 0.$$

An argument similar to that of Lemma 4.10 in [16], yields that the sequence u_n is bounded. Since $b = p$ by Lemmas 4.1 from [16] and Lemma 4.1, then $\{u_n\}$ is also a minimizing sequence of I on $\mathcal{P}$. By Ekeland Variational Principle (Theorem 8.5 in [27]) there exists another sequence $\{\tilde{u}_n\} \subset \mathcal{P}$ such that:

- (i) $I(\tilde{u}_n) \rightarrow p$;
- (ii) $I'(\tilde{u}_n) \rightarrow 0$;
- (iii) $\|\tilde{u}_n - u_n\| \rightarrow 0$.

Pohozaev manifold is a natural constraint, hence in fact $I'(\tilde{u}_n) \rightarrow 0$. Indeed, suppose $I'(\tilde{u}_n)$ does not go to zero. That means there exists $\varepsilon_0 > 0$ and a subsequence $\{\tilde{u}_{n_j}\}$, with $n_j \rightarrow \infty$, such that

$$\|I'(\tilde{u}_{n_j})\| > \varepsilon_0.$$

Since f is Lipschitz continuous,

$$\begin{aligned} |(I'(\tilde{u}_{n_j}) - I'(v))\varphi| &= \left| \int (\nabla \tilde{u}_{n_j} - \nabla v)\varphi dx + \int V(x)(\tilde{u}_{n_j} - v)\varphi dx \right. \\ &\quad \left. - \int (f(\tilde{u}_{n_j}) - f(v))\varphi \right| \\ &\leq \|\tilde{u}_{n_j} - v\| \|\varphi\| + K \|\tilde{u}_{n_j} - v\| \|\varphi\| \\ &= (1 + K) \|\tilde{u}_{n_j} - v\| \|\varphi\|. \end{aligned}$$

Thus, if

$$\|\tilde{u}_{n_j} - v\| < \frac{\tilde{\delta}}{1 + K} := 3\delta$$

then

$$\|I'(\tilde{u}_{n_j}) - I'(v)\| < \tilde{\delta}.$$

This yields,

$$\varepsilon_0 - \tilde{\delta} < \|I'(\tilde{u}_{n_j})\| - \tilde{\delta} < \|I'(v)\|.$$

For $\tilde{\delta} > 0$ sufficiently small, we have $\lambda := \varepsilon_0 - \tilde{\delta} > 0$ and for all n_j ,

$$\text{for all } v \in B_{3\delta}(\tilde{u}_{n_j}), \quad \text{then } \|I'(v)\| > \lambda.$$

Let $\varepsilon := \min\{\frac{p}{2}, \frac{\lambda\delta}{8}\}$ and $S := \{\tilde{u}_{n_j}\}$. By Lemma 2.3 in [27], there is a deformation η on the level p , taking all the points of S_δ to the level $p - \varepsilon$.

$$I(\eta(1, u)) \leq I(u) \quad \text{for all } u \in H^1(\mathbb{R}^N)(\mathbb{R}^N).$$

Moreover, for n_j sufficiently large,

$$\max_{t>0} I(\eta(1, \tilde{u}_{n_j}(\frac{\cdot}{t}))) \leq p - \varepsilon$$

because $\{\tilde{u}_{n_j}\}$ is a minimizing sequence, $I(\tilde{u}_{n_j}) < p + \frac{\varepsilon}{2}$, for n_j sufficiently large, and since $\tilde{u}_{n_j} \in \mathcal{P}$,

$$\max_{t>0} I\left(\tilde{u}_{n_j}\left(\frac{\cdot}{t}\right)\right) = I(\tilde{u}_{n_j}) \rightarrow p.$$

On the other hand, $\gamma_0(t) = \eta(1, \tilde{u}_{n_j}(\frac{\cdot}{t}))$ is a path in Γ and hence

$$c \leq \max_{t>0} \gamma_0(t) \leq p - \varepsilon < p.$$

But this is a contradiction, because we have proved that $p = c$. Moreover, since f is Lipschitz continuous, $I'(\tilde{u}_n) \rightarrow 0$ and $\|\tilde{u}_n - u_n\| \rightarrow 0$ imply $I'(u_n) \rightarrow 0$, as $n \rightarrow \infty$. Therefore, the sequence $\{u_n\}$ satisfies the assumptions of Corollary 4.5

and since $p = c_\infty$ and is not attained, then the splitting lemma holds with $k = 1$. This yields

$$u_n(x) \rightarrow u^1(x - y_n), \quad (4.8)$$

where $y_n \in \mathbb{R}^N$, $|y_n| \rightarrow \infty$ and u^1 is a solution of the problem at infinity. By making a translation, we obtain

$$u_n(x + y_n) = u^1(x) + o_n(1).$$

Calculating the barycenter function on both sides, we have

$$\beta(u_n(x + y_n)) = \beta(u_n) - y_n,$$

with $\beta(u_n) = 0$ and

$$\beta(u^1(x) + o_n(1)) \rightarrow \beta(u^1(x)),$$

since β is a continuous function. On one side, $\beta(u^1(x)) = 0$ and, on the other, $|y_n| \rightarrow \infty$ so we arrive at a contradiction. Therefore, we must have $b > c_\infty$. $\square$

Let us consider again the positive, radially symmetric, ground state solution $w \in H^1(\mathbb{R}^N)$ of the autonomous problem at infinity. We define the operator $\Pi : \mathbb{R}^N \rightarrow \mathcal{P}$ by

$$\Pi[y](x) = w\left(\frac{x - y}{\theta_y}\right),$$

where θ_y is exactly the real number θ which projects $w(\cdot - y)$ onto the Pohozaev manifold $\mathcal{P}$. Π is a continuous function of y because θ_y is unique and $\theta_y(w(\cdot - y))$ is a continuous function of $w(\cdot - y)$.

We verify some properties of this operator Π . The first one, whose proof is found in [16], Lemma 4.12, is the following:

Lemma 4.10. $\beta(\Pi[y](x)) = y$.

Lemma 4.11. $I(\Pi[y]) \rightarrow c_\infty$, if $|y| \rightarrow \infty$.

Proof. In fact, $\Pi[y] \in \mathcal{P}$ and in $\mathcal{P}$ the functional I can be written as

$$I(\Pi[y]) = \frac{1}{N} \int_{\mathbb{R}^N} \left| \nabla w\left(\frac{x - y}{\theta_y}\right) \right|^2 dx + \frac{1}{2N} \int_{\mathbb{R}^N} \langle \nabla V(x) | x \rangle w^2\left(\frac{x - y}{\theta_y}\right) dx.$$

Moreover, since $w \in \mathcal{P}_\infty$, we have that $I_\infty(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^2 dx$ and we can perform a change of variables to obtain

$$\begin{aligned} I(\Pi[y]) &= \frac{\theta_y^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla w|^2 dx \\ &\quad + \frac{\theta_y^N}{N} \int_{\mathbb{R}^N} \langle \nabla V(\theta_y x + y) | (\theta_y x + y) \rangle w^2 dx \\ &= \theta_y^{N-2} I_\infty(w) + \frac{\theta_y^N}{N} \int_{\mathbb{R}^N} \langle \nabla V(\theta_y x + y) | (\theta_y x + y) \rangle w^2 dx. \end{aligned}$$

Using Lebesgue Dominated Convergence Theorem, (1.4) and the information that $\theta_y \rightarrow 1$ if $|y| \rightarrow \infty$, we get

$$\lim_{|y| \rightarrow \infty} \int_{\mathbb{R}^N} \langle \nabla V(\theta_y x + y) | (\theta_y x + y) \rangle w^2(x) dx = 0.$$

Therefore, $I(\Pi[y]) \rightarrow c_\infty$, if $|y| \rightarrow \infty$, and the lemma is proved. $\square$

Lemma 4.12. *Assume*

$$(V_6) \quad \sup_{\mathbb{R}^N} |V(x) - V_\infty| < \frac{c_\infty}{\bar{\theta}^N \|w\|_2^2},$$

where $\bar{\theta} = \sup_{y \in \mathbb{R}^N} \theta_y$. Then $I(\Pi[y]) < 2c_\infty$.

Proof. Noting that I_∞ is translation invariant, the maximum of $t \mapsto I_\infty(w(\cdot/t))$ is attained at $t = 1$ and that $\theta_y > 1$ and using (V_6) , we obtain

$$\begin{aligned} I(\Pi[y]) &= I_\infty(\Pi[y]) + I(\Pi[y]) - I_\infty(\Pi[y]) \\ &\leq I_\infty(w) + \int_{\mathbb{R}^N} (V(x) - V_\infty)(\Pi[y])^2 dx \\ &< c_\infty + \frac{c_\infty}{\bar{\theta}^N \|w\|_2^2} \int_{\mathbb{R}^N} w^2 \left(\frac{x - y}{\theta_y} \right) dx \\ &= c_\infty + \frac{c_\infty \theta_y^N}{\bar{\theta}^N \|w\|_2^2} \|w\|_2^2 \\ &\leq c_\infty + c_\infty. \end{aligned}$$

This yields $I(\Pi[y]) < 2c_\infty$. $\square$

We recall a version of the Linking Theorem with Cerami condition by Bartolo, Benci and Fortunato in [5] (see Theorem 2.3), which we state here for the sake of completeness.

Definition 4.13. Let S be a closed subset of a Banach space X , and Q a submanifold of X with relative boundary ∂Q . We say that S and ∂Q ‘link’ if:

- (1) $S \cap \partial Q = \emptyset$;
- (2) for any $h \in C^0(X, X)$ such that $h|_{\partial Q} = id$, then $h(Q) \cap S \neq \emptyset$.

Moreover, if S and Q are as above and B is a subset of $C^0(X, X)$, then S and ∂Q ‘link’ with respect to B if (1) and (2) hold for any $h \in B$.

Theorem 4.14. (Linking) *Suppose that $I \in C^1(X, \mathbb{R})$ is a functional satisfying (Ce) condition. Consider a closed subset $S \subset X$ and a submanifold $Q \subset X$ with relative boundary ∂Q ; suppose also that:*

- (a) S and ∂Q ‘link’;
- (b) $\alpha = \inf_{u \in S} I(u) > \sup_{u \in \partial Q} I(u) = \alpha_0$;
- (c) $\sup_{u \in Q} I(u) < +\infty$.

If $B = \{h \in C^0(X, X); h|_{\partial Q} = id\}$, then the real number $\tau = \inf_{h \in B} \sup_{u \in Q} I(h(u))$ defines a critical value of I , with $\tau \geq \alpha$.

We conclude this section proving the main existence result Theorem 1.2. The proof follows closely the ideas in [16]. We include the details to highlight the role played by a combination of the Pohozaev and barycenter constraints.

Proof. Condition (V_2) implies that $I_\infty(u) < I(u)$ for all $u \in H^1(\mathbb{R}^N) \setminus \{0\}$. In particular, $I_\infty(\Pi[y]) < I(\Pi[y])$, for any $y \in \mathbb{R}^N$. Since $b > c_\infty$, from Lemma 4.9, and $I(\Pi[y]) \rightarrow c_\infty$ if $|y| \rightarrow \infty$, from Lemma 4.11, then there exists $\bar{\rho} > 0$ such that for every $\rho \geq \bar{\rho}$,

$$c_\infty < \max_{|y|=\rho} I(\Pi[y]) < b. \quad (4.9)$$

In the setting of the linking theorem we define

$$Q := \Pi(\overline{B_{\bar{\rho}}(0)})$$

and

$$S := \{u \in H^1(\mathbb{R}^N) \mid u \in \mathcal{P}, \beta(u) = 0\},$$

and will show that ∂Q and S “link”. By Lemma 4.10 $\beta(\Pi[y]) = y$, we obtain that $\partial Q \cap S = \emptyset$, because if $u \in S$, then $\beta(u) = 0$, and if $u \in \partial Q$, then $\beta(u) = y \neq 0$, due to equality $|y| = \bar{\rho}$. Moreover, $h(Q) \cap S \neq \emptyset$, for any $h \in \mathcal{H}$, where

$$\mathcal{H} = \{h \in C(Q, \mathcal{P}); h|_{\partial Q} = id\}.$$

Indeed, given $h \in \mathcal{H}$, let us define $T : \overline{B_{\bar{\rho}}(0)} \rightarrow \mathbb{R}^N$ for $T(y) = \beta \circ h \circ \Pi[y]$. The function T is continuous, because it is the composition of continuous functions. For any $|y| = \bar{\rho}$, we have that $\Pi[y] \in \partial Q$, thus $h \circ \Pi[y] = \Pi[y]$, because $h|_{\partial Q} = id$, and hence from Lemma 4.10 $T(y) = y$. By the Fixed Point Theorem of Brouwer, we conclude that there exists $\tilde{y} \in B_{\bar{\rho}}(0)$ such that $T(\tilde{y}) = 0$, which implies $h(\Pi[\tilde{y}]) \in S$. Therefore $h(Q) \cap S \neq \emptyset$ and S and ∂Q “link”.

Furthermore, from the definitions of b and Q and the inequalities (4.9), we may write

$$b = \inf_S I > \max_{\partial Q} I.$$

Let us define

$$d = \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)).$$

Then we have $d \geq b$. Indeed, we have already proved that $h(Q) \cap S \neq \emptyset$, for all $h \in \mathcal{H}$. If h is fixed, then there exists $w \in S$ such that w also belongs to $h(Q)$, which means that $w = h(v)$ for some $v \in \Pi(\overline{B_{\bar{\rho}}(0)})$. Therefore,

$$I(w) \geq \inf_{u \in S} I(u) \quad \text{and} \quad \max_{u \in Q} I(h(u)) \geq I(h(v)).$$

This gives

$$\max_{u \in Q} I(h(u)) \geq I(h(v)) = I(w) \geq \inf_{u \in S} I(u) = b,$$

and hence

$$\inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)) \geq b.$$

In particular, it follows that $d > c_\infty$, because from Lemma 4.9 we know that $b > c_\infty$. Furthermore, if we take $h = id$, then

$$\inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)) < \max_{u \in Q} I(u) < 2c_\infty,$$

by Lemma 4.12. This implies $d < 2c_\infty$. The two inequalities give $c_\infty < d < 2c_\infty$, thus from Lemma 4.6 (Ce) condition is satisfied at level d . Therefore, we can apply the Linking Theorem and conclude that d is a critical level for the functional I and there exists a nontrivial solution $u \in H^1(\mathbb{R}^N)$ of the equation (P). Reasoning as usual, because of the hypotheses on f , and using the maximum principle we conclude that u is positive and the theorem is proved. $\square$

A final remark is to be made. The previous results can be obtained likewise in the case f is super quadratic and sub critical at infinity.

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