Nonlinear Differ. Equ. Appl. 22 (2015), 513–542

1021-9722/15/040513-30
*published online* October 28, 2014
DOI 10.1007/s00030-014-0294-x

**Nonlinear Differential Equations and Applications NoDEA**

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# A control problem in phase transition modeling

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**Abstract.** We consider a nonlinear partial differential control system describing phase transitions taking account of hysteresis effects. The control constraint is given by a multivalued mapping with nonconvex closed bounded values in a finite dimensional space depending on the phase variables. Existence of solutions and topological properties of the set of admissible "trajectory-control" pairs are discussed in detail.

**Mathematics Subject Classification.** Primary 49J20; Secondary 49J30.

**Keywords.** Nonlinear evolution control systems, Subdifferential, Nonconvex constraints, Extreme points, Hysteresis.

## 1. Introduction

In the space-time cylinder $\Omega_T := [0,T] \times \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitzian domain and $T > 0$ is a fixed final time, we consider the system

$$w_t + \partial I_{K(v)}(w) \ni g(v,w) - \psi(v,w) - F(w), \tag{1.1}$$

$$cw_t + dv_t - \Delta v = a(v,w) + b(v,w)u, \tag{1.2}$$

$$v(x,0) = v_0(x),\ w(x,0) = w_0(x), \tag{1.3}$$

with boundary condition

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } [0,T] \times \partial\Omega, \tag{1.4}$$

subject to the control constraint

$$u \in U(t,x,v,w) \quad \text{on } \Omega_T, \tag{1.5}$$

where $K(v) = [f_*(v), f^*(v)]$ is a possibly degenerate interval in $\mathbb{R}$, the symbol $\partial I_{K(v)}$ denotes the subdifferential of its indicator function, $c,d$ are given positive constants, $f_*, f^*, g, a, b, \psi, F$ are given functions satisfying Hypothesis 1

The research of PK was supported by GAČR Grant P201/10/2315 and RVO: 67985840, the research of AT and ST was supported by RFBR Grant no. 13-01-00287. The support of the Russian Academy of Sciences during the stay of PK in Irkutsk is gratefully acknowledged.

below, $(v_0, w_0)$ are given initial conditions, and $U$ is a multivalued mapping with closed bounded values in $\mathbb{R}$ satisfying Hypothesis 2 below. Note that the differential inclusion (1.1) is related to the so-called *generalized play*, see [1].

We consider also alternative control constraints to (1.5) in the form

$$u \in \operatorname{co} U(t, x, v, w) \quad \text{on } \Omega_T, \tag{1.6}$$

$$u \in \operatorname{ext} \operatorname{co} U(t, x, v, w) \quad \text{on } \Omega_T, \tag{1.7}$$

where the symbol "co" stands for the convex hull of a set, and "ext" is the collection of all extreme points of a set. Since $U(t, x, v, w)$ is a closed bounded set in $\mathbb{R}$, the set $\operatorname{co} U(t, x, v, w)$ is a closed interval, and $\operatorname{ext} \operatorname{co} U(t, x, v, w)$ contains two points.

A system similar to (1.1)–(1.4) without control $u$ has been studied by many authors. We refer the reader to the references [2,3] for the physical interpretation and an extensive bibliography on the phenomena described by this system. When $f_*(v) = -1$, $f^*(v) = +1$, for instance, a system of the form

$$\left.\begin{aligned} \gamma(v, w) w_t + \partial I_{[-1,1]}(w) &\ni L(v - v_c) \\ L w_t + c v_t - \Delta v &= h(v, w, x, t) \end{aligned}\right\} \text{ in } \Omega_T, \tag{1.8}$$

with unknown state variables $v$ (the absolute temperature) and $w$ (the order parameter, or phase variable) can be derived from general thermodynamic principles as a model for phase transitions as proposed by Visintin in [4,5] under the name *relaxed Stefan problem*. In this model, $L$ (latent heat), $c$ (specific heat capacity), and $v_c$ (critical temperature) are positive material constants, $\gamma(v, w) > 0$ is the relaxation time characterizing the phase transition speed in the phase dynamics equation for different values of the state variables, and $h(v, w, x, t)$ is a variable heat source density in the energy balance equation.

The hysteresis character of phase transition processes has been repeatedly pointed out in the literature starting from [6], and the potential of specific hysteresis techniques has been exploited for solving more complex phase transition systems, e.g. with mechanical interaction as in [7]. Problem (1.1)–(1.2) can also be interpreted in this framework. The right hand side of Eq. (1.1) is obtained by dividing the first equation of (1.8) by the relaxation time under suitable (mild) assumptions on the nonlinearity. A possible optimization of the phase transition process can be achieved in a natural way by a partial control of the heat sources in the energy balance equation. This is why the control variable $u$ appears only on the right hand side of Eq. (1.2).

Note that differential inclusions of the type (1.1) are not just particular examples of hysteresis relations. It was proved in [8, Theorem 2.7.7] that *every scalar return point memory hysteresis operator* can be represented by first order differential inclusions with a one-parameter family of indicator functions.

The majority of works on systems with hysteresis have been mainly focused on existence, uniqueness and long time behavior of solutions. Along with the articles [2,3] of this type we also mention the works [9–11]. In all these works it was assumed that $\psi(v, w) \equiv 0$, $F(w) \equiv 0$, $b(v, w) \equiv 0$, the function $a(v, w)$ is bounded, the functions $f_*$ and $f^*$ describing the hysteresis region are also bounded. The only exception is the work [12] in which the system is given by Eqs. (1.1), (1.8) with $\psi(v,w) \equiv 0$, $F(w) \equiv 0$, and the functions $f_*$, $f^*$ have the form

$$f_*(r) = \alpha r + \beta_*,\ f^*(r) = \alpha r + \beta^*, \quad \alpha > 0,\ \beta_* < \beta^*,\ r \in \mathbb{R}. \tag{1.9}$$

Note however that the hysteresis region in this case though unbounded is of a very special type.

When it comes to control systems with hysteresis their study is at an initial stage. An interesting from the authors' point of view work [13] in this direction deals with the system described by Eqs. (1.1), (1.8) with $\psi(v,w) \equiv 0$, $F(w) \equiv 0$, the functions $f_*$, $f^*$ given by (1.9) and the control $h(t,x)$. In this work the problem of minimization of the integral over $T$ of the sum of squares of norms of the functions $v - v_g$, $w - w_g$ and $h$ with values in the space $L^2(\Omega)$, where $v_g$, $w_g$ are given functions, was considered. An approximating system was constructed by approximating the subdifferential $\partial I_v(w)$ by smooth functions and relations between solutions of the initial and approximating systems were studied. Necessary optimality conditions were obtained for the approximating system.

In recent works [14,15] a control system with hysteresis described by two ordinary differential equations with nonconvex constraints depending on the phase variables was investigated. In the first work the authors studied the existence of solutions and relations between solutions of the initial system and the system with convexified control constraints. In the second one the problem of minimization of an integral functional with nonconvex in control integrant and relationships between the solutions of this problem and the solutions of the problem of minimization of the convexified with respect to the control integrant over the solutions of the system with convexified constraints were considered.

Given numerous applications, it seems natural and necessary, in addition to the problems treated in [14,15], to consider for systems with hysteresis especially for those described by partial differential questions all the questions traditionally considered for control systems of other classes. The present work starts these investigations.

In this article we study the following problems:

a) existence of solutions and, in case when the control constraints do not depend on the phase variables, uniqueness and continuous data dependence of trajectories;
b) closedness of the solution sets of the system (1.1)–(1.4) with the constraints (1.6) and (1.7);
c) the property of being an absolute retract and arc-wise connectedness of the solution sets of the system (1.1)–(1.4) with the constraints (1.6) and (1.7).

The existence and uniqueness results alone extend the existing literature [2,3, and others] to the case of unbounded nonlinearities and unbounded hysteresis domains, which have not been considered before. Moreover, we prove the existence of more regular solutions than in these works.

The article is organized as follows. In Sect. 2 we recall some notions we use in the rest of the paper and give the assumptions on the data of the problem we consider. Section 3 states the main results of the paper (Theorems 3.2, 3.3) and auxiliary theorems (Theorems 3.5, 3.6) necessary to prove the main results. In Sect. 4 we construct the time discretization of our system with $\psi(v, w) \equiv 0$ and a fixed control $u$, and study some of its properties. In Sect. 5 we obtain a priori estimates for the solutions of the time discrete system independent of the discretization parameter, which allows us in Sect. 6 to show that the solutions of the discretized system converge to a solution of the initial system with $\psi(v, w) \equiv 0$ and a fixed $u$. In Sect. 7 we prove Theorems 3.5 and 3.6 giving the existence of our system with a fixed $u$ and the continuous dependence of the solutions on the initial conditions and the control. Section 8 is devoted to construction of the multivalued Nemytskii operator associated with our problem and investigation of properties of its fixed points. Finally, in Sect. 9 we prove our main results (Theorems 3.2 and 3.3).

## 2. Main notations, definitions and assumptions

Let $(X, |\cdot|_X)$ be a Banach space, $\mathcal{T} = [0, T]$ an interval of the real line with Lebesgue measure $\mu$ and $\sigma$-algebra $\Sigma$ of $\mu$-measurable subsets of $\mathcal{T}$.

In what follows we use the following notations: $\mathrm{cb}\, X$ is the family of all nonempty closed bounded subsets from $X$. We denote by $\mathrm{co}\, A$ the convex hull of a set $A \subset X$, and by $\overline{\mathrm{co}}\, A$ the closed convex hull of $A$. The collection of all extreme points of a closed convex bounded set $A \subset X$ is denoted by $\mathrm{ext}\, A$ as above. If $A \subset X$, then

$$|A|_X = \sup \{|y|_X;\ y \in A\}.$$

By $d_X(y, A)$ we mean the distance from the point $y$ to the set $A \subset X$, and by $D_X(\cdot, \cdot)$ the Hausdorff distance on the space $\mathrm{cb}\, X$.

For the space $X$ with its topological dual $X'$ the symbol $\omega$-$X$ means that the space $X$ is endowed with the weak $\sigma(X, X')$ topology, and $\omega^*$-$X'$ that the space $X'$ is endowed with the weak $\sigma(X', X)$ topology called the weak-star topology. For subsets $A \subset X$ and $B \subset X'$ the symbols $\omega$-$A$ and $\omega^*$-$B$ denote that the sets $A$ and $B$ are endowed with the topologies induced by those of the spaces $\omega$-$X$ and $\omega^*$-$X'$.

By $C(\mathcal{T}, X)$ we denote the space of all continuous functions from $\mathcal{T}$ to $X$ with the topology of uniform convergence on $\mathcal{T}$.

A subset $A$ of a Hausdorff topological space $Y$ is called a retract if there exists a continuous map $q : Y \to A$ such that $q(y) = y$, $y \in A$. Such a map $q$ is called a retraction of $Y$ onto $A$.

A subset $A$ of $Y$ is called an absolute retract if for each separable metric space $Z$, each closed subset $B$ of $Z$, and each continuous map $q : B \to A$ there exists a continuous extension of $q$ onto $Z$ taking values in $A$ [16].

A set $A$ is said to be arc-wise connected if two arbitrary points $x_0$ and $x_1$ in this set can be connected by an arc [16]. Recall that an *arc* in $A$ connecting the points $x_0$ and $x_1$ is the homeomorphic image of the interval $[0,1]$ under some mapping $\sigma : [0,1] \to A$ such that $\sigma(0) = x_0$, $\sigma(1) = x_1$.

A multivalued mapping $F : Y \to \operatorname{cb} X$ is called $D_X$-continuous, if it is continuous from $Y$ to the metric space $(\operatorname{cb} X, D_X(\cdot,\cdot))$.

Let $(\mathcal{E}, \mathcal{A})$ be a measurable space. A set $\mathcal{F}$ of measurable functions from $\mathcal{E}$ to $X$ is called decomposable if for any $v, w \in \mathcal{F}$ and any $E \in \mathcal{A}$ we have $v\chi_E + w\chi_{\mathcal{E}\setminus E} \in \mathcal{F}$, where $\chi_E$ denotes the characteristic function of the set $E$.

A multivalued mapping $F : \mathcal{E} \to X$ is called measurable if $F^{-1}(V) = \{\tau \in \mathcal{E};\ F(\tau) \cap V \neq \emptyset\} \in \mathcal{A}$ for any closed set $V \subset X$.

Let $H$ be a Hilbert space with norm $|\cdot|_H$ and scalar product $\langle\cdot,\cdot\rangle_H$. A function $\varphi : H \to \overline{\mathbb{R}} = (-\infty, +\infty]$ is called proper if its effective domain $\operatorname{dom}\varphi = \{x \in H;\ \varphi(x) < +\infty\}$ is nonempty. We denote by $\Gamma_0(H)$ the set of all functions $\varphi : H \to \overline{\mathbb{R}}$ which are proper, convex, and lower semicontinuous. For a function $\varphi \in \Gamma_0(H)$, we denote by $\partial\varphi(x)$ its subdifferential at the point $x$, which is defined by the variational inequality

$$\partial\varphi(x) = \{h \in H;\ \langle h, y - x\rangle_H \leq \varphi(y) - \varphi(x)\ \forall y \in H\}.$$

It is known [17] that $\partial\varphi$ is a maximal monotone operator, $\operatorname{dom}\partial\varphi \subset \operatorname{dom}\varphi$ and $\overline{\operatorname{dom}\partial\varphi} = \overline{\operatorname{dom}\varphi}$, where the bar stands for the closure in $H$ and $\operatorname{dom}\partial\varphi$ is the domain of $\partial\varphi$.

In the sequel $L^0(\mathcal{E})$ is the space of measurable functions from $\mathcal{E}$ to $\mathbb{R}$, $X = L^1(\Omega)$, $X^2 = X \times X$ with the norm $|(x,y)|_{X^2} = |x|_X + |y|_X$, $H = L^2(\Omega)$. As usual, we identify the space $L^0(\Omega_T)$ with the space $L^0(\mathcal{T}, L^0(\Omega))$. The same identification applies to the spaces $L^1(\Omega_T)$, $L^1(\mathcal{T}, L^1(\Omega))$ and $L^2(\Omega_T)$, $L^2(\mathcal{T}, L^2(\Omega))$. We denote the norm of a function $h \in L^p(\Omega)$, $1 \leq p \leq \infty$, by the symbol $|h|_p$.

Let $H^1(\Omega)$ and $H^2(\Omega)$ be the Sobolev spaces $W^{1,2}(\Omega)$, $W^{2,2}(\Omega)$, respectively. Denote by $V$ the space $H^1(\Omega)$. The norm on $V$ is defined as follows:

$$|v|_V = \langle v, v\rangle_V^{1/2},$$

where $\langle v, w\rangle_V = \langle v, w\rangle_H + q(v,w)$,

$$q(v,w) = \int_\Omega \langle \nabla v(x), \nabla w(x)\rangle_{\mathbb{R}^N}\, \mathrm{d}x, \quad v, w \in V. \tag{2.1}$$

Identifying $H$ with its dual space, we obtain $V \hookrightarrow H \hookrightarrow V'$ with dense, continuous and compact imbeddings.

Consider a linear continuous operator $\mathcal{L} : V \to V'$:

$$\langle \mathcal{L}v, w\rangle = q(v,w), \quad v, w \in V, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form establishing the duality between $V$ and $V'$ and $q(v,w)$ is defined by (2.1). The mapping

$$-\Delta_N :\ D(-\Delta_N) \subset H \to H \tag{2.3}$$

denotes the restriction of the operator $\mathcal{L}$ onto the subspace of the space $V$, consisting of the elements $v$ for which the inclusion $\mathcal{L}v \in H$ holds. It is known [2] that

$$D(-\Delta_N) = \{v \in H^2(\Omega);\ \partial v/\partial n = 0 \quad \text{in}\ H^{1/2}(\partial\Omega)\}$$

and

$$-\Delta_N v = -\Delta v \text{ for all } v \in D(-\Delta_N),$$

where $\partial/\partial n$ is the derivative in the direction of the outward normal to $\partial\Omega$. As usually, $|\nabla v|_H^2 = \sum_{i=1}^N |D_i v|_H^2$, $v \in V$.

Problem (1.1)–(1.4) is considered under the following hypotheses.

**Hypothesis 2.1.** The following assumptions hold throughout the paper:

(i) $f_*, f^* : \mathbb{R} \to \mathbb{R}$, $a, b, g : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz continuous in all variables with linear growth, that is, there exists a constant $G > 0$ such that

$$\begin{aligned}&|f_*(v)| + |f^*(v)| + |g(v,w)| + |a(v,w)| + |b(v,w)| \\ &\quad \le G(1 + |v| + |w|) \quad \forall (v,w) \in \mathbb{R}^2;\end{aligned}$$

Moreover, $f_*, f^*$ are nondecreasing, $f_*(v) \le f^*(v)$ for all $v \in \mathbb{R}$.

(ii) $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing and locally Lipschitz, $F(0) = 0$, $\psi : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz, $w\,\psi(v,w) \ge 0$ for all $v, w$, and there exists $k_1 > 0$ such that $\psi(v,w) = 0$ for $|v| \ge k_1$;

(iii) $v_0 \in L^\infty(\Omega) \cap V$, $w_0 \in L^\infty(\Omega)$, $w_0(x) \in K(v_0(x))$ a.e. on $\Omega$,

$$|v_0|_\infty \le r, \quad r > 0. \tag{2.4}$$

In connection with the constraint (1.5), we assume the following.

**Hypothesis 2.2.** The mapping $U : \mathcal{T} \times \Omega \times \mathbb{R} \times \mathbb{R} \to \operatorname{cb}\mathbb{R}$ has the following properties:

(i) The mapping $(t,x) \to \operatorname{co} U(t,x,v,w)$, $v, w \in \mathbb{R}$, is measurable;

(ii) There exists a function $k \in L^1(\mathcal{T}, \mathbb{R}^+)$ such that

$$D_{\mathbb{R}}(\operatorname{co} U(t,x,v_1,w_1), \operatorname{co} U(t,x,v_2,w_2)) \le k(t)(|v_1 - v_2| + |w_1 - w_2|) \tag{2.5}$$

a.e. on $\Omega_T$, $(v_i, w_i) \in \mathbb{R}$, $i = 1, 2$;

(iii) There exists $R > 0$ such that for all $(v,w) \in \mathbb{R} \times \mathbb{R}$ we have

$$|U(t,x,v,w)|_{\mathbb{R}} \le R \quad \text{a.e. on } \Omega_T,\ R > 0. \tag{2.6}$$

## 3. Main results

Let Hypothesis 2 (i), (ii) hold. Using the statement e) of [18, Theorem 3.5], it is easy to show that for any measurable functions $v, w : \Omega \to \mathbb{R}$ the multivalued mapping $(t,x) \to \operatorname{co} U(t,x,v(x),w(x))$ is measurable and has closed values. Then from [18, Theorem 5.6] it follows that there exists a sequence $f_n : \mathcal{T} \times \Omega \to \mathbb{R}$, $n \ge 1$, of measurable functions such that

$$\operatorname{co} U(t,x,v(x),w(x)) = \left\{ \overline{\bigcup_{n=1}^{\infty} f_n(t,x)} \right\},\ (t,x) \in \mathcal{T} \times \Omega,$$

where the bar stands for the closure in $\mathbb{R}$.

From Fubini's theorem we infer that for a.e. $t \in \mathcal{T}$ each function $x \to f_n(t,x)$, $n \ge 1$, is measurable. Therefore, according to Theorem 5.6 in [18] for a.e. $t \in \mathcal{T}$ the multivalued mapping $x \to \operatorname{co} U(t,x,v(x),w(x))$ is measurable.

Consequently, if Hypothesis 2 holds, we can consider a multivalued mapping $(t, v, w) \to \Gamma(t, v, w)$:

$$\Gamma(t, v, w) = \{u \in H;\ u(x) \in \operatorname{co} U(t, x, v(x), w(x)) \quad \text{a.e. on } \Omega\}, \tag{3.1}$$

whose values are convex decomposable compact subsets of $\omega$-$H$. Then, we can define the mapping $\operatorname{ext}\Gamma(t, v, w)$, which according to Corollary 5.2 in [19] satisfies the equality

$$\operatorname{ext}\Gamma(t, v, w) = \{u \in H;\ u(x) \in \operatorname{ext}\operatorname{co} U(t, x, v(x), w(x)) \quad \text{a.e. on } \Omega\}. \tag{3.2}$$

From the Krein–Milman theorem it follows that

$$\operatorname{ext}\operatorname{co} U(t, x, v(x), w(x)) \subset U(t, x, v(x), w(x)).$$

Then the multivalued mapping

$$\mathcal{U}(t, v, w) = \{u \in H;\ u(x) \in U(t, x, v(x), w(x)) \quad \text{a.e. on } \Omega\} \tag{3.3}$$

has as its values nonempty closed decomposable bounded subsets of the space $H$. Since

$$\operatorname{ext}\Gamma(t, v, w) \subset \mathcal{U}(t, v, w) \subset \Gamma(t, v, w),$$

we have $\overline{\operatorname{co}}\,\mathcal{U}(t, v, w) = \Gamma(t, v, w)$. From this equality and (3.1) we obtain

$$\overline{\operatorname{co}}\,\mathcal{U}(t, v, w) = \{u \in H;\ u(x) \in \operatorname{co} U(t, x, v(x), w(x)) \quad \text{a.e. on } \Omega\}. \tag{3.4}$$

From Hypothesis 1 it follows that, keeping our notations, we may consider the functions $f_*, f^*, F$ acting from $\mathbb{R}$ to $\mathbb{R}$ and the functions $a, b, g, \psi$ acting from $\mathbb{R}^2$ to $\mathbb{R}$ as functions from $H$ to $L^0(\Omega)$ and from $H \times H$ to $L^0(\Omega)$, respectively.

Let $I_{\mathcal{K}(v)}$ be the indicator function of the set

$$\mathcal{K}(v) = \{w \in H;\ w(x) \in K(v(x)) \quad \text{a.e. on } \Omega\}, \tag{3.5}$$

$v \in H$ and $\partial I_{\mathcal{K}(v)}(w)$ be its subdifferential at the point $w \in H$.

**Definition 3.1.** A triple $\{v, w, u\}$ is called a solution of the control system (1.1)–(1.4) with the constraint (1.5) on the control $u$, if

$$w \in W^{1,2}(\mathcal{T}, H),\ v \in W^{1,2}(\mathcal{T}, H) \cap C(\mathcal{T}, V) \cap L^2(\mathcal{T}, H^2(\Omega)),\ u \in L^2(\mathcal{T}, H), \tag{3.6}$$

and

$$w' + \partial I_{K(v)}(w) \ni g(v, w) - \psi(v, w) - F(w) \ \text{ in } H \text{ a.e. on } \mathcal{T}, \tag{3.7}$$

$$cw' + dv' - \Delta_N v = a(v, w) + b(v, w)u \text{ in } H \text{ a.e. on } \mathcal{T}, \tag{3.8}$$

$$w(0) = w_0, \quad v(0) = v_0 \quad \text{in } H, \tag{3.9}$$

$$u(t) \in \mathcal{U}(t, v(t), w(t)) \ \text{ in } H \text{ a.e. on } \mathcal{T}, \tag{3.10}$$

where $\mathcal{U}(t, v, w)$ is the multivalued mapping defined by the formula (3.3), and the prime over $w$ and $v$ denotes derivative with respect to $t$.

Solutions of the control system (1.1)–(1.4) with the control constraints (1.6) and (1.7) are defined similarly by changing the inclusion (3.10) by the inclusions

$$u(t) \in \operatorname{co} \mathcal{U}(t, v(t), w(t)) \quad \text{in}\, H \text{ a.e. on } \mathcal{T}, \tag{3.11}$$

$$u(t) \in \operatorname{ext} \operatorname{co} \mathcal{U}(t, v(t), w(t)) \quad \text{in}\, H \text{ a.e. on } \mathcal{T}, \tag{3.12}$$

respectively.

If a triple $\{v, w, u\}$ is a solution of the system (1.1)–(1.4) with the control constraints (1.5), then $\{v, w\}$ is called the trajectory corresponding to the control $u$, which we will denote as $\{v(u), w(u)\}$. Therefore, a solution of the system (1.1)–(1.4), (1.5) is the pair $\{(v(u), w(u)), u\}$ consisting of the trajectory $(v(u), w(u))$ and the control $u$. We denote by $\mathcal{R}_U(v_0, w_0)$ the set of all solutions of the control system (1.1)–(1.4), (1.5). The symbols $\mathcal{R}_{\overline{\text{co}}\, U}(v_0, w_0)$ and $\mathcal{R}_{\text{ext}\, \overline{\text{co}}\, U}(v_0, w_0)$ denote the sets of all solutions of the systems (1.1)–(1.4), (1.6) and (1.1)–(1.4), (1.7), respectively.

The main purpose of this work is to prove the following results.

**Theorem 3.2.** *Let Hypotheses* 1*,* 2 *hold. Then the set* $\mathcal{R}_{\text{ext}\, \overline{\text{co}}\, U}(v_0, w_0)$ *is nonempty, the following inclusions are valid*

$$\mathcal{R}_{\text{ext}\, \overline{\text{co}}\, U}(v_0, w_0) \subset \mathcal{R}_U(v_0, w_0) \subset \mathcal{R}_{\overline{\text{co}}\, U}(v_0, w_0) \tag{3.13}$$

*and the inequality*

$$\begin{aligned} &|v(u)|_{L^\infty(\Omega_T)} + |w(u)|_{L^\infty(\Omega_T)} + |v'(u)|_{L^2(\mathcal{T}, H)} \\ &\quad + |w'(u)|_{L^2(\mathcal{T}, H)} + |\Delta_N v(u)|_{L^2(\mathcal{T}, H)} + |\nabla v(u)|_{L^\infty(\mathcal{T}, H)} \le C(R, r) \end{aligned} \tag{3.14}$$

*is true for all* $\{v(u), w(u), u\} \in \mathcal{R}_{\overline{\text{co}}\, U}(v_0, w_0)$*, where* $C(R, r)$ *is some constant depending on the constants* $R, r$ *appearing in the inequalities* (2.4)*,* (2.6)*.*

**Theorem 3.3.** *Let Hypotheses* 1, 2 *hold. Then the sets* $\mathcal{R}_{\text{ext}\, \overline{\text{co}}\, U}(v_0, w_0)$ *and* $\mathcal{R}_{\overline{\text{co}}\, U}(v_0, w_0)$ *are closed, arc-wise connected absolute retracts in the space* $C(\mathcal{T}, H) \times C(\mathcal{T}, H) \times L^2(\mathcal{T}, H)$*.*

**Corollary 3.4.** *If Hypothesis* 2 *(i) holds also for the mapping* $U(t, x, v, w)$*, then the statement of Theorem* 3.2 *is also valid for the set* $\mathcal{R}_U(v_0, w_0)$*.*

In order to prove Theorems 3.2, 3.3 we need to consider the system of Eqs. (3.7)–(3.9) with

$$u \in S_R, \tag{3.15}$$

where

$$S_R = \{u \in L^2(\mathcal{T}, H);\ |u(t, x)| \le R \ \text{ a.e. on } \Omega_T\}. \tag{3.16}$$

This system of equations is called the *dominating system*. The set of all solutions $\{(v(u), w(u)), u\}$ of the dominating system (3.7)–(3.9), (3.15) is denoted by $\mathcal{R}(v_0, w_0)$.

The following results will serve as a basis of the proofs of Theorems 3.2, 3.3.

**Theorem 3.5.** *Let Hypothesis* 1 *hold. Then for any* $u \in S_R$*, there exists a unique solution* $\{v(u), w(u)\}$ *of the system* (3.7)*–*(3.9)*. Moreover, for each element* $\{(v(u), w(u)), u\} \in \mathcal{R}(v_0, w_0)$*, the inequality* (3.14) *is true.*

**Theorem 3.6.** *Let Hypothesis* 1 *hold. Then there exists a constant* $L^*(R,r) > 0$ *such that for any* $(v_0^i, w_0^i)$, $i = 1,2$, *satisfying Hypothesis* 1 (iii), *the following inequality holds*

$$\begin{aligned} &|(v(u_1) - v(u_2))|_X(t) + |(w(u_1) - w(u_2))|_X(t) \\ &\quad \le L^*(R,r)\left(|v_0^1 - v_0^2|_X + |w_0^1 - w_0^2|_X + \int_0^t |u_1 - u_2|_X(\tau)\,\mathrm{d}\tau\right), \ t \in \mathcal{T}, \end{aligned} \tag{3.17}$$

*for all solutions* $\{(v(u_i), w(u_i)), u_i\} \in \mathcal{R}(v_0^i, w_0^i)$, $i = 1,2$, *of the dominating system.*

## 4. Time discretization of the dominating system

In this section, we first consider the case $\psi(v,w) = 0$. In the sequel the constants which will appear in inequalities and depend on the constants $R$ and $r$ from the inequalities (2.4), (2.6) will be denoted, for simplicity, by letters with the upper index $*$.

Let $\{r_n\}_{n=1}^{\infty}$ be an increasing sequence of integers, and let $\tau_n = T/r_n$. Let $R > 0$ be fixed, and let $\{u_j; j = 1, \ldots, r_n\}$ be a given sequence such that $u_j \in L^\infty(\Omega)$, $|u_j| \le R$ a.e. on $\Omega$ for all $j = 1, \ldots, r_n$. We define the time discrete counterpart of (3.7)–(3.9) as the elliptic system in $\Omega$

$$w_j = Q_{K(v_j)}(w_{j-1} + \tau_n(g_{j-1} - F(w_j))), \tag{4.1}$$

$$\frac{c}{\tau_n}(w_j - w_{j-1}) + \frac{d}{\tau_n}(v_j - v_{j-1}) - \Delta v_j = a_{j-1} + b_{j-1}u_j, \tag{4.2}$$

$$\frac{\partial v_j}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{4.3}$$

for $j = 1, \ldots, r_n$ and for unknowns $v_j, w_j$, with initial conditions $v_0, w_0$ as in Hypothesis 1 (iii), and with the notation

$$g_{j-1} = g(v_{j-1}, w_{j-1}),\ a_{j-1} = a(v_{j-1}, w_{j-1}),\ b_{j-1} = b(v_{j-1}, w_{j-1}), \tag{4.4}$$

where for a fixed $v \in \mathbb{R}$, $Q_{K(v)} : \mathbb{R} \to K(v)$ is the projection onto $K(v)$ defined as

$$\xi = Q_{K(v)}(z) \Leftrightarrow \begin{cases} \xi = z & \text{if } z \in K(v), \\ \xi = f_*(v) & \text{if } z < f_*(v), \\ \xi = f^*(v) & \text{if } z > f^*(v). \end{cases} \tag{4.5}$$

The relation $\xi = Q_{K(v)}(z)$ admits an equivalent variational characterization

$$\xi = Q_{K(v)}(z) \Leftrightarrow \begin{cases} \xi \in K(v), \\ (z - \xi)(\xi - y) \ge 0 \quad \forall y \in K(v). \end{cases} \tag{4.6}$$

This is in turn equivalent to the inclusion

$$z \in \xi + \partial I_{K(v)}(\xi). \tag{4.7}$$

Since for any $v, w \in H$, $w \in \operatorname{dom} I_{\mathcal{K}(v)}$ the equality $\partial I_{\mathcal{K}(v)}(w) = \{h \in H;\ h(x) \in \partial I_{K(v(x))}(w(x))$ a.e. on $\Omega\}$ holds, Eq. (4.1) can also be stated as a natural discretization of (3.7) with $\psi(v, w) = 0$ in the form

$$\frac{1}{\tau_n}(w_j - w_{j-1}) + F(w_j) + \partial I_{K(v_j)}(w_j) \ni g(v_{j-1}, w_{j-1}). \tag{4.8}$$

We start with an easy lemma.

**Lemma 4.1.** *Let Hypothesis* 1 *hold, and let $\varphi, v \in \mathbb{R}$ and $\tau > 0$ be given. Then the equation*

$$w = Q_{K(v)}(\varphi - \tau F(w))$$

*has a unique solution $w \in \mathbb{R}$. If moreover*

$$w_i = Q_{K(v_i)}(\varphi_i - \tau F(w_i)), \quad i = 1, 2$$

*for some $v_1, \varphi_1, v_2, \varphi_2$, then*

$$(w_1 - w_2)(v_1 - v_2) \ge -|\varphi_1 - \varphi_2|\,|v_1 - v_2|, \tag{4.9}$$

$$|w_1 - w_2| \le |\varphi_1 - \varphi_2| + \max\{|f_*(v_1) - f_*(v_2)|, |f^*(v_1) - f^*(v_2)|\}. \tag{4.10}$$

*Proof.* For given $v, \varphi$, the function $\theta(w) = w - Q_{K(v)}(\varphi - \tau F(w))$ is increasing, $\theta(\pm\infty) = \pm\infty$, hence there exists a unique $w$ such that $\theta(w) = 0$. Let now $\varphi_1 \in \mathbb{R}$ and $v_1 > v_2$ be given, and set

$$\hat{w}_2 = Q_{K(v_2)}(\varphi_1 - \tau F(\hat{w}_2)).$$

We prove that

$$(w_1 - \hat{w}_2)(v_1 - v_2) \ge 0, \tag{4.11}$$

$$|w_1 - \hat{w}_2| \le \max\{|f_*(v_1) - f_*(v_2)|, |f^*(v_1) - f^*(v_2)|\}. \tag{4.12}$$

To this end, we distinguish the cases

(a) $\varphi_1 - \tau F(w_1) > f^*(v_1)$. Then $w_1 = f^*(v_1) \ge f^*(v_2) \ge \hat{w}_2$. Furthermore, $\varphi_1 > \tau F(w_1) + w_1 \ge \tau F(\hat{w}_2) + \hat{w}_2$, hence $w_1 - \hat{w}_2 = f^*(v_1) - f^*(v_2)$, and (4.11)–(4.12) follow.

(b) $\varphi_1 - \tau F(w_1) < f_*(v_1)$. Then $w_1 = f_*(v_1)$. If now $\hat{w}_2 > v_1$, then $\varphi_1 < \tau F(w_1) + w_1 < \tau F(\hat{w}_2) + \hat{w}_2$, hence $\hat{w}_2 = f_*(v_2) \le f_*(v_1) = w_1$, which is a contradiction. We thus have $0 \le w_1 - \hat{w}_2 \le f_*(v_1) - f_*(v_2)$, which we wanted to prove.

(c) $\varphi_1 - \tau F(w_1) \in K(v_1)$. Then $\varphi_1 = w_1 + \tau F(w_1)$. If now $\hat{w}_2 > w_1$, then $\varphi_1 < \hat{w}_2 + \tau F(\hat{w}_2)$, and we obtain $\hat{w}_2 = f_*(v_2) \le f_*(v_1) \le w_1$, which is a contradiction. Hence, $\hat{w}_2 \le w_1$. If $\hat{w}_2 < w_1$, then $\varphi_1 > \hat{w}_2 + \tau F(\hat{w}_2)$, so that $\hat{w}_2 = f^*(v_2)$, $w_1 \le f^*(v_1)$, hence $0 < w_1 - \hat{w}_2 \le f^*(v_1) - f^*(v_2)$, and the proof of (4.11)–(4.12) is complete.

It follows from (4.6) that for $\xi_1 = Q_{K(v)}(z_1)$, $\xi_2 = Q_{K(v)}(z_2)$, we have $(\xi_1 - \xi_2)(z_1 - z_2) \ge (\xi_1 - \xi_2)^2$. We now apply this inequality to the case $z_1 = \varphi_1 - \tau F(\hat{w}_2)$, $z_2 = \varphi_2 - \tau F(w_2)$, $v = v_2$, $\xi_1 = \hat{w}_2$, $\xi_2 = w_2$, and obtain

$$(\hat{w}_2 - w_2)(\varphi_1 - \varphi_2) \ge \tau(F(\hat{w}_2) - F(w_2))(\hat{w}_2 - w_2) + (\hat{w}_2 - w_2)^2 \ge (\hat{w}_2 - w_2)^2.$$

We complete the proof of Lemma 4.1 combining this inequality with (4.11)–(4.12). $\square$

The solution of (4.1)–(4.3) will be constructed by induction. Given $v_{j-1}, w_{j-1} \in L^\infty(\Omega)$, we see from (4.11)–(4.12) that Eq. (4.1) defines a function $w_j = f_j(x, v_j)$, which is nondecreasing and locally Lipschitz in $v_j$. Hence, (4.2)–(4.3) is a semilinear monotone elliptic equation of the form

$$\frac{c}{\tau_n} f_j(x, v_j) + \frac{d}{\tau_n} v_j - \Delta v_j = h_{j-1}, \tag{4.13}$$

$$\frac{\partial v_j}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \tag{4.14}$$

with a right hand $h_{j-1} \in L^\infty(\Omega)$, hence it admits a unique solution $v_j \in V$ [20]. In Proposition 5.2 below we prove that $v_j, w_j$ belong to $L^\infty(\Omega)$. In order to do so, we need the following auxiliary result which will enable us to construct supersolutions and subsolutions to (4.1)–(4.3).

**Lemma 4.2.** *Let $j \geq 1$ and numbers $0 < \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{j-1}$, $0 < v_0^{\mathrm{inc}} \leq v_1^{\mathrm{inc}} \leq \cdots \leq v_{j-1}^{\mathrm{inc}}$, $w_0^{\mathrm{inc}} \leq w_1^{\mathrm{inc}} \leq \cdots \leq w_{j-1}^{\mathrm{inc}}$ be given, $w_0^{\mathrm{inc}} \in K(v_0^{\mathrm{inc}})$, $\gamma_0 > |F(w_0^{\mathrm{inc}})|$. Let $v_j^{\mathrm{inc}}, w_j^{\mathrm{inc}}$ satisfy the system*

$$\frac{1}{\tau_n}(w_j^{\mathrm{inc}} - w_{j-1}^{\mathrm{inc}}) + F(w_j^{\mathrm{inc}}) + \partial I_{K(v_j^{\mathrm{inc}})}(w_j^{\mathrm{inc}}) \ni \gamma_{j-1}, \tag{4.15}$$

$$\frac{c}{\tau_n}(w_j^{\mathrm{inc}} - w_{j-1}^{\mathrm{inc}}) + \frac{d}{\tau_n}(v_j^{\mathrm{inc}} - v_{j-1}^{\mathrm{inc}}) = C\gamma_{j-1}, \tag{4.16}$$

*with some $C > 2c$. Then $v_j^{\mathrm{inc}} \geq v_{j-1}^{\mathrm{inc}}$, $w_j^{\mathrm{inc}} \geq w_{j-1}^{\mathrm{inc}}$.*

*Proof.* Assume first $v_j^{\mathrm{inc}} < v_{j-1}^{\mathrm{inc}}$. Then, by (4.16), $w_j^{\mathrm{inc}} > w_{j-1}^{\mathrm{inc}}$. If now $\frac{1}{\tau_n}(w_j^{\mathrm{inc}} - w_{j-1}^{\mathrm{inc}}) + F(w_j^{\mathrm{inc}}) - \gamma_{j-1} > 0$, then $w_j^{\mathrm{inc}} = f_*(v_j^{\mathrm{inc}}) \leq f_*(v_{j-1}^{\mathrm{inc}}) \leq w_{j-1}^{\mathrm{inc}}$, which is a contradiction. We thus necessarily have $\frac{1}{\tau_n}(w_j^{\mathrm{inc}} - w_{j-1}^{\mathrm{inc}}) + F(w_j^{\mathrm{inc}}) - \gamma_{j-1} \leq 0$, hence

$$\frac{1}{\tau_n}(w_j^{\mathrm{inc}} - w_{j-1}^{\mathrm{inc}}) \leq \gamma_{j-1} - F(w_j^{\mathrm{inc}}) \leq \gamma_{j-1} - F(w_0^{\mathrm{inc}}) \leq 2\gamma_{j-1}.$$

By virtue of (4.16) we have

$$\frac{d}{\tau_n}(v_j^{\mathrm{inc}} - v_{j-1}^{\mathrm{inc}}) \geq (C - 2c)\gamma_{j-1} > 0,$$

which is a contradiction. Thus, we have proved that $v_j^{\mathrm{inc}} \geq v_{j-1}^{\mathrm{inc}}$.

We now prove that $w_j^{\mathrm{inc}} \geq w_{j-1}^{\mathrm{inc}}$. To this aim, we distinguish again two cases.

(a) $w_{j-1}^{\mathrm{inc}} = f_*(v_{j-1}^{\mathrm{inc}})$. Then $w_j^{\mathrm{inc}} \geq f_*(v_j^{\mathrm{inc}}) \geq f_*(v_{j-1}^{\mathrm{inc}}) = w_{j-1}^{\mathrm{inc}}$.

(b) $w_{j-1}^{\mathrm{inc}} > f_*(v_{j-1}^{\mathrm{inc}})$.

(b1) If $j = 1$, then

$$\frac{1}{\tau_n}(w_1^{\mathrm{inc}} - w_0^{\mathrm{inc}}) + F(w_1^{\mathrm{inc}}) - F(w_0^{\mathrm{inc}}) + \partial I_{K(v_1^{\mathrm{inc}})}(w_1^{\mathrm{inc}}) \ni \gamma_0 - F(w_0^{\mathrm{inc}}) \geq 0,$$

and if $w_1^{\mathrm{inc}} < w_0^{\mathrm{inc}}$, then $w_1^{\mathrm{inc}} = f^*(v_1^{\mathrm{inc}}) \geq f^*(v_0^{\mathrm{inc}}) \geq w_0$, which is a contradiction.

(b2) If $j > 1$ and $\gamma_{j-1} - F(w_j^{\mathrm{inc}}) \geq 0$, we argue as in (b1).

(b3) If $j > 1$ and $\gamma_{j-1} - F(w_j^{\rm inc}) < 0$, we notice first that $\partial I_{K(v_{j-1}^{\rm inc})}(w_{j-1}^{\rm inc}) \cap (-\infty, 0) = \emptyset$ by hypothesis (b). By induction hypothesis, we have $w_{j-1}^{\rm inc} \ge w_{j-2}^{\rm inc}$. From (4.15) for $j$ replaced with $j-1$ we thus obtain $\gamma_{j-2} - F(w_{j-1}^{\rm inc}) \ge 0$. This yields $0 \le \gamma_{j-1} - \gamma_{j-2} < F(w_{j-1}^{\rm inc}) - F(w_j^{\rm inc})$, hence $w_j^{\rm inc} \ge w_{j-1}^{\rm inc}$ and Lemma 4.2 is proved. □

We also have the symmetric result

**Lemma 4.3.** *Let $j \ge 1$ and numbers $0 < \gamma_0 \le \gamma_1 \le \cdots \le \gamma_{j-1}$, $0 > v_0^{\rm dec} \ge v_1^{\rm dec} \ge \cdots \ge v_{j-1}^{\rm dec}$, $w_0^{\rm dec} \ge w_1^{\rm dec} \ge \cdots \ge w_{j-1}^{\rm dec}$ be given, $w_0^{\rm dec} \in K(v_0^{\rm dec})$, $\gamma_0 > |F(w_0^{\rm dec})|$. Let $v_j^{\rm dec}, w_j^{\rm dec}$ satisfy the system*

$$\frac{1}{\tau_n}(w_j^{\rm dec} - w_{j-1}^{\rm dec}) + F(w_j^{\rm dec}) + \partial I_{K(v_j^{\rm dec})}(w_j^{\rm dec}) \ni -\gamma_{j-1}, \tag{4.17}$$

$$\frac{c}{\tau_n}(w_j^{\rm dec} - w_{j-1}^{\rm dec}) + \frac{d}{\tau_n}(v_j^{\rm dec} - v_{j-1}^{\rm dec}) = -C\gamma_{j-1}, \tag{4.18}$$

*with some $C > 2c$. Then $v_j^{\rm dec} \le v_{j-1}^{\rm dec}$, $w_j^{\rm dec} \le w_{j-1}^{\rm dec}$.*

This is easily obtained from Lemma 4.2, replacing $v_j^{\rm dec}$ by $-v_j^{\rm inc}$, $w_j^{\rm dec}$ by $-w_j^{\rm inc}$, $F(w_j^{\rm dec})$ by $-F(-w_j^{\rm inc})$, $f_*(v_j^{\rm dec})$ by $-f^*(-v_j^{\rm inc})$, and $f^*(v_j^{\rm dec})$ by $-f_*(-v_j^{\rm inc})$.

Let us come back now to system (4.1)–(4.3). The linear growth condition in Hypothesis 1 (i) together with (4.4) and the inclusion $w_{j-1} \in K(v_{j-1})$ imply that

$$\begin{aligned} |g_{j-1}| + |a_{j-1}| + |b_{j-1}| &\le G(|v_{j-1}| + |w_{j-1}| + 1) \\ &\le G(|v_{j-1}| + G(|v_{j-1}| + 1) + 1). \end{aligned} \tag{4.19}$$

For $B = G(1+G)$ we thus have

$$\max\{|g_{j-1}|, |a_{j-1}|, |b_{j-1}|\} \le B(|v_{j-1}| + 1). \tag{4.20}$$

We now choose a constant $C_R > 2c$ which will be specified later, and consider auxiliary difference equations for supersolutions $v_j^\sharp, w_j^\sharp$ and subsolutions $v_j^\flat, w_j^\flat$

$$w_j^\sharp = Q_{K(v_j^\sharp)}(w_{j-1}^\sharp + \tau_n(g_{j-1}^* - F(w_j^\sharp))), \tag{4.21}$$

$$\frac{c}{\tau_n}(w_j^\sharp - w_{j-1}^\sharp) + \frac{d}{\tau_n}(v_j^\sharp - v_{j-1}^\sharp) = C_R g_{j-1}^*, \tag{4.22}$$

$$w_j^\flat = Q_{K(v_j^\flat)}(w_{j-1}^\flat - \tau_n(g_{j-1}^* + F(w_j^\flat))), \tag{4.23}$$

$$\frac{c}{\tau_n}(w_j^\flat - w_{j-1}^\flat) + \frac{d}{\tau_n}(v_j^\flat - v_{j-1}^\flat) = -C_R g_{j-1}^* \tag{4.24}$$

with the choice $g_{j-1}^* = B(\max\{|v_{j-1}^\sharp|, |v_{j-1}^\flat|\} + 1)$, and with initial conditions $v_0^\sharp \ge |v_0|_\infty$, $w_0^\sharp = f^*(v_0^\sharp)$, $v_0^\flat \le -|v_0|_\infty$, $w_0^\flat = f_*(v_0^\flat)$. The constant $B$ is possibly taken larger in order to guarantee that $g_0^* \ge \max\{|F(w_0^\sharp)|, |F(w_0^\flat)|\}$.

The solutions of (4.21)–(4.22) and (4.23)–(4.24) are constructed by induction. Assume for instance that $v_0^\sharp, \ldots, v_{j-1}^\sharp$ and $w_0^\sharp, \ldots, w_{j-1}^\sharp$ are already known. By Lemma 4.1, Eq. (4.21) defines $w_j^\sharp$ as a nondecreasing locally Lipschitz continuous function of $v_j^\sharp$. Hence, (4.22) is of the form $f_j(v_j^\sharp) = h_j$ with a locally Lipschitz continuous increasing superlinear function $f_j$, which necessarily admits a unique solution. The argument is similar for (4.23)–(4.24).

Notice that by (4.8), Eq. (4.21) is of the form (4.15), and (4.23) is of the form (4.17). Hence, by Lemmas 4.2, 4.3, the sequences $v_j^\sharp, w_j^\sharp$ are nondecreasing, and $v_j^\flat, w_j^\flat$ are nonincreasing. We therefore have

$$0 \le \frac{d}{\tau_n}(v_j^\sharp - v_{j-1}^\sharp) \le C_R g_{j-1}^*, \tag{4.25}$$

$$0 \ge \frac{d}{\tau_n}(v_j^\flat - v_{j-1}^\flat) \ge -C_R g_{j-1}^*. \tag{4.26}$$

Similar inequalities are also valid for $w_j^\sharp, w_j^\flat$. In particular,

$$\frac{d}{\tau_n}((v_j^\sharp - v_j^\flat) - (v_{j-1}^\sharp - v_{j-1}^\flat)) \le C_R B((v_{j-1}^\sharp - v_{j-1}^\flat) + 1). \tag{4.27}$$

$$\frac{c}{\tau_n}((w_j^\sharp - w_j^\flat) - (w_{j-1}^\sharp - w_{j-1}^\flat)) \le C_R B((v_{j-1}^\sharp - v_{j-1}^\flat) + 1). \tag{4.28}$$

The inequality (4.27) is a discrete Gronwall type inequality of the form $c_j - c_{j-1} \le \alpha\tau_n(c_{j-1} + 1)$ which admits the upper bound $c_j \le (c_0 + 1)\mathrm{e}^{j\alpha\tau_n} - 1 \le (c_0 + 1)\mathrm{e}^{T\alpha} - 1$ for $j = 1, \dots, r_n$. Since $v_0^\sharp \ge |v_0|_\infty$, $v_0^\flat \le -|v_0|_\infty$, according to (2.4), (4.27) there exists a constant $C^* > 0$ independent of the discretization parameter $n$ such that

$$-C^* < v_j^\flat < 0 < v_j^\sharp < C^*, \tag{4.29}$$

$$-C^* < w_j^\flat \le w_j^\sharp < C^* \tag{4.30}$$

for all $j = 0, 1, \dots, r_n$.

## 5. Estimates

We start with a discrete counterpart of the celebrated *Hilpert inequality*, the original version of which can be found in [20]. We denote by $H : \mathbb{R} \to \{0, 1\}$ the Heaviside function, that is, $H(z) = 1$ if $z > 0$, $H(z) = 0$ if $z \le 0$.

**Lemma 5.1.** *Let* (4.1) *hold for some* $v_{j-1}, v_j, w_{j-1}, w_j \in \mathbb{R}$, *and let* $v^* \in \mathbb{R}$ *and* $w^* \in K(v^*), w^* \ne w_j$ *be arbitrary. Then we have*

$$\begin{aligned}&(w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(w_j - w^*)\\ &\quad \le (w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(v_j - v^*),\end{aligned} \tag{5.1}$$

$$\begin{aligned}&-(w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(w^* - w_j)\\ &\quad \le -(w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(v^* - v_j).\end{aligned} \tag{5.2}$$

*Proof.* We distinguish the cases

(a) $w_{j-1} + \tau_n(g_{j-1} - F(w_j)) \in K(v_j)$. Then $w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)) = 0$ and the assertion holds trivially.

(b) $w_{j-1}+\tau_n(g_{j-1}-F(w_j)) > f^*(v_j)$. Then $w_j = f^*(v_j)$, hence $w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)) < 0$. If now $w_j > w^*$, then $(w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(w_j - w^*) = -|w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j))|$, and (5.2) follows. Furthermore, the left hand side of (5.3) vanishes and the right hand side is nonnegative, hence (5.3) holds as well. Consider now the case $w_j < w^*$. Then $f^*(v_j) = w_j < w^* \le f^*(v^*)$, hence $H(w_j - w^*) = H(v_j - v^*) = 0$, $H(w^* - w_j) = H(v^* - v_j) = 1$, and both (5.2), (5.3) follow.

(c) $w_{j-1} + \tau_n(g_{j-1} - F(w_j)) < f_*(v_j)$. Then $w_j = f_*(v_j)$, $w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)) > 0$, and we argue as in (b). □

**Proposition 5.2.** *Let Hypothesis* 1 *hold, and let* $(v_j, w_j)$ *be the solution of* (4.1)–(4.3)*. Then for every* $n \in \mathbb{N}$*, every* $j = 0, 1, \ldots, r_n$*, and a.e.* $x \in \Omega$*, we have, for a suitable choice of* $C_R$*, that*

$$v_j^\flat \le v_j(x) \le v_j^\sharp, \quad w_j^\flat \le w_j(x) \le w_j^\sharp.$$

*In particular, the time discrete approximations are uniformly bounded with respect to the sup-norm.*

*Proof.* We only prove the upper bound, the other case is fully symmetric. We proceed by induction, assuming that the bound is already proved for $j-1$. By Lemma 5.1, we have for $w_j(x) \neq w_j^\sharp$ that

$$\begin{aligned}
&(w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(w_j - w_j^\sharp)\\
&\quad \le (w_j - w_{j-1} - \tau_n(g_{j-1} - F(w_j)))H(v_j - v_j^\sharp), \qquad (5.3)\\
&-(w_j^\sharp - w_{j-1}^\sharp - \tau_n(g_{j-1}^* - F(w_j^\sharp)))H(w_j - w_j^\sharp)\\
&\quad \le -(w_j^\sharp - w_{j-1}^\sharp - \tau_n(g_{j-1}^* - F(w_j^\sharp)))H(v_j - v_j^\sharp). \qquad (5.4)
\end{aligned}$$

Summing up the above inequalities, we obtain

$$\begin{aligned}
&\frac{1}{\tau_n}((w_j - w_j^\sharp) - (w_{j-1} - w_{j-1}^\sharp))H(w_j - w_j^\sharp)\\
&\qquad +(g_{j-1}^* - g_{j-1})H(w_j - w_j^\sharp) + (F(w_j) - F(w_j^\sharp))H(w_j - w_j^\sharp)\\
&\quad \le \frac{1}{\tau_n}((w_j - w_j^\sharp) - (w_{j-1} - w_{j-1}^\sharp))H(v_j - v_j^\sharp)\\
&\qquad +(g_{j-1}^* - g_{j-1})H(v_j - v_j^\sharp) + (F(w_j) - F(w_j^\sharp))H(v_j - v_j^\sharp), \qquad (5.5)
\end{aligned}$$

whenever $w_j(x) \neq w_j^\sharp$. We now subtract (4.22) from (4.2) and obtain

$$\begin{aligned}
&\frac{c}{\tau_n}((w_j - w_j^\sharp) - (w_{j-1} - w_{j-1}^\sharp)) + \frac{d}{\tau_n}((v_j - v_j^\sharp) - (v_{j-1} - v_{j-1}^\sharp)) - \Delta(v_j - v_j^\sharp)\\
&\quad = a_{j-1} + b_{j-1}u_j - C_R g_{j-1}^*. \qquad (5.6)
\end{aligned}$$

We have by hypothesis $w_{j-1} \le w_{j-1}^\sharp$, $v_{j-1} \le v_{j-1}^\sharp$, $g_{j-1} \le g_{j-1}^*$ a.e. Furthermore, by monotonicity of $F$, we have

$$(F(w_j) - F(w_j^\sharp))H(w_j - w_j^\sharp) \ge (F(w_j) - F(w_j^\sharp))H(v_j - v_j^\sharp).$$

It thus follows from (5.5) for $w_j(x) \neq w_j^\sharp$ that

$$\begin{aligned}
&\frac{1}{\tau_n}(w_j - w_j^\sharp)H(w_j - w_j^\sharp)\\
&\quad \le \frac{1}{\tau_n}((w_j - w_j^\sharp) - (w_{j-1} - w_{j-1}^\sharp))H(v_j - v_j^\sharp) + (g_{j-1}^* - g_{j-1})H(v_j - v_j^\sharp),
\end{aligned} \tag{5.7}$$

and testing (5.6) by $H(v_j - v_j^\sharp)$ yields that

$$\begin{aligned}
&\frac{c}{\tau_n}((w_j - w_j^\sharp) - (w_{j-1} - w_{j-1}^\sharp))H(v_j - v_j^\sharp) + \frac{d}{\tau_n}(v_j - v_j^\sharp)H(v_j - v_j^\sharp)\\
&\quad \le \Delta(v_j - v_j^\sharp)H(v_j - v_j^\sharp) + \big(a_{j-1} + b_{j-1}u_j - C_R g_{j-1}^*\big)H(v_j - v_j^\sharp).
\end{aligned} \tag{5.8}$$

Let $\Omega_j^0 \subset \Omega$ be the set of all $x \in \Omega$ such that $w_j(x) = w_j^\sharp$. For a.e. $x \in \Omega_j^0$ we have by (5.8) that

$$\begin{aligned}
&\frac{d}{\tau_n}(v_j - v_j^\sharp)H(v_j - v_j^\sharp) - \Delta(v_j - v_j^\sharp)H(v_j - v_j^\sharp)\\
&\quad \le (a_{j-1} + b_{j-1}u_j - C_R g_{j-1}^\sharp)H(v_j - v_j^\sharp).
\end{aligned} \tag{5.9}$$

For $x \in \Omega\backslash\Omega_j^0$ we add (5.8) to (5.7) multiplied by $c$, and obtain

$$\begin{aligned}
&\frac{c}{\tau_n}(w_j - w_j^\sharp)H(w_j - w_j^\sharp) + \frac{d}{\tau_n}(v_j - v_j^\sharp)H(v_j - v_j^\sharp) - \Delta(v_j - v_j^\sharp)H(v_j - v_j^\sharp)\\
&\quad \le (a_{j-1} - c g_{j-1} + b_{j-1}u_j - (C_R - c)g_{j-1}^*)H(v_j - v_j^\sharp).
\end{aligned} \tag{5.10}$$

We have by hypothesis that $|a_{j-1}| + c|g_{j-1}| + |b_{j-1}u_j| \le (1 + c + R)g_{j-1}^*$. Hence, choosing $C_R \ge 1 + 2c + R$, we have for a.e. $x \in \Omega$ that

$$\frac{c}{\tau_n}(w_j - w_j^\sharp)^+ + \frac{d}{\tau_n}(v_j - v_j^\sharp)^+ - \Delta(v_j - v_j^\sharp)H(v_j - v_j^\sharp) \le 0, \tag{5.11}$$

where $(\cdot)^+$ denotes the positive part. We further have

$$\int_\Omega -\Delta(v_j - v_j^\sharp)H(v_j - v_j^\sharp)\,\mathrm{d}x \ge 0. \tag{5.12}$$

Indeed, this is obvious if we replace the Heaviside function $H$ by its regularization $H_\delta$ with a positive parameter $\delta$, which is defined as

$$H_\delta(z) = \begin{cases} 1 & \text{for } \ z \ge \delta,\\ z/\delta & \text{for } \ z \in (0, \delta),\\ 0 & \text{for } \ z \le 0. \end{cases}$$

Letting $\delta$ tend to 0, we obtain (5.12). We conclude that

$$\int_\Omega \left(\frac{c}{\tau_n}(w_j - w_j^\sharp)^+ + \frac{d}{\tau_n}(v_j - v_j^\sharp)^+\right)\mathrm{d}x \le 0, \tag{5.13}$$

which completes the proof of Proposition 5.2. □

The next result manifests an "almost monotone" and "almost Lipschitz" dependence of $w_j$ on $v_j$.

**Lemma 5.3.** *There exists a constant $M^* > 0$ such that for all $j = 1, \ldots, r_n$ we have*

$$(w_j - w_{j-1})(v_j - v_{j-1}) \geq -\tau_n M^* |v_j - v_{j-1}|, \tag{5.14}$$

$$|w_j - w_{j-1}| \leq M^*(\tau_n + |v_j - v_{j-1}|). \tag{5.15}$$

*Proof.* We trivially have $w_{j-1} = Q_{K(v_{j-1})}(w_{j-1})$. Hence, it suffices to apply Lemma 4.1 with $w_1 = w_{j-1}$, $v_1 = v_{j-1}$, $\varphi_1 = w_{j-1}$, $w_2 = w_j$, $v_2 = v_j$, $\varphi_2 = w_{j-1} - \tau_n(g_{j-1} - F(w_j))$, the upper bound for $|v_j|, |w_j|$ from Proposition 5.2, the inequalities (4.29), (4.30) and the (local) Lipschitz continuity of the nonlinearities. □

**Proposition 5.4.** *Let the assumptions of Proposition 5.2 hold. Then there exists a constant $C_1^*$ independent of $n$ such that*

$$\frac{1}{\tau_n} \sum_{j=1}^{r_n} |v_j - v_{j-1}|_2^2 \leq C_1^*, \tag{5.16}$$

$$|\nabla v_j|_2^2 \leq C_1^* \quad \forall j = 0, 1, \ldots, r_n, \tag{5.17}$$

$$\frac{1}{\tau_n} \sum_{j=1}^{r_n} |w_j - w_{j-1}|_2^2 \leq C_1^*, \tag{5.18}$$

$$\tau_n \sum_{j=1}^{r_n} |\Delta v_j|_2^2 \leq C_1^*. \tag{5.19}$$

*Proof.* We multiply (4.2) by $v_j - v_{j-1}$ and integrate over $\Omega$. It follows from Lemma 5.3 and from Proposition 5.2 that

$$\frac{d}{\tau_n}(v_j - v_{j-1})^2 - \Delta v_j (v_j - v_{j-1}) \leq C_2^* |v_j - v_{j-1}|,$$

with some constant $C_2^* > 0$ independent of $n$. Hence, by Cauchy–Schwarz inequality,

$$\frac{d}{\tau_n}(v_j - v_{j-1})^2 - 2\Delta v_j (v_j - v_{j-1}) \leq \tau_n C_3^*,$$

with some constant $C_3^* > 0$ independent of $n$. Integrating over $\Omega$, summing up over $j$, and using the elementary inequality

$$\int_\Omega -\Delta v_j (v_j - v_{j-1})\, \mathrm{d}x = \int_\Omega \nabla v_j \cdot (\nabla v_j - \nabla v_{j-1})\, \mathrm{d}x \geq \frac{1}{2}(|\nabla v_j|_2^2 - |\nabla v_{j-1}|_2^2),$$

we easily obtain (5.16)–(5.17). Inequality (5.18) then follows from Lemma 5.3 and Proposition 5.2, and (5.19) is obtained directly from (4.2), (5.16), and (5.18) by comparison. □

## 6. Passage to the limit

Let $u \in S_R$. For a.e. $x \in \Omega$, we can define

$$u_j(x) = \frac{1}{\tau_n} \int_{(j-1)\tau_n}^{j\tau_n} u(t, x)\, \mathrm{d}t, \quad j = 1, \ldots, r_n.$$

We now let $n$ tend to infinity and prove that the solutions to (4.1)–(4.3) converge in a suitable sense to a solution of (3.7)–(3.9) with $\psi(v,w) = 0$.

With the sequences $\{v_j\}$, $\{w_j\}$, we associate the piecewise linear and piecewise constant interpolates for $x \in \Omega$ and $t \in [(j-1)\tau_n, j\tau_n)$ by the formula

$$v^{(n)}(t,x) = v_{j-1}(x) + \frac{1}{\tau_n}(t-(j-1)\tau_n)(v_j - v_{j-1}), \tag{6.1}$$

$$\bar{v}^{(n)}(t,x) = v_j(x), \tag{6.2}$$

$$\underline{v}^{(n)}(t,x) = v_{j-1}(x), \tag{6.3}$$

for $j = 1,\ldots,r_n$, continuously extended to $t = T$, and similarly for $w^{(n)}$, $\bar{w}^{(n)}$, $\underline{w}^{(n)}$, and $\bar{u}^{(n)}$. Propositions 5.2 and 5.4 yield the estimates

$$\int_0^T \left(|v_t^{(n)}(t)|_2^2 + |w_t^{(n)}(t)|_2^2\right) \mathrm{d}t \le C_0^*, \tag{6.4}$$

$$\int_0^T |\Delta v^{(n)}(t)|_2^2 \,\mathrm{d}t \le C_0^*, \tag{6.5}$$

$$\sup_{t\in(0,T)} \operatorname{ess} |\nabla v^{(n)}(t)|_2^2 \le C_0^*, \tag{6.6}$$

$$\sup_{(t,x)\in\Omega_T} \operatorname{ess} \left(|v^{(n)}(t,x)| + |w^{(n)}(t,x)|\right) \le C_0^*, \tag{6.7}$$

for some $C_0^* > 0$.

From (4.2), (4.3) and Theorem 3.4.3 in [21] we infer that for each $t \in \mathcal{T}$ we have $\bar{v}^{(n)}(t) \in H^2(\Omega)$. Therefore, $\Delta_N \bar{v}^{(n)}(t) = \Delta \bar{v}^{(n)}(t)$, $t \in \mathcal{T}$ and the Eq. (4.2) with the boundary condition will be written in the form

$$\begin{aligned} &c(w^{(n)})' + d(v^{(n)})' - \Delta_N \bar{v}^{(n)} \\ &\quad = a(\underline{v}^{(n)}, \underline{w}^{(n)}) + b(\underline{v}^{(n)}, \underline{w}^{(n)})\bar{u}^{(n)} \quad \text{in } H \text{ a.e. on } \mathcal{T}, \end{aligned} \tag{6.8}$$

From (6.1) it directly follows that

$$\sup_{(t,x)\in\Omega_T} \operatorname{ess} |\bar{v}^{(n)}(t,x)| \le C_0^* \tag{6.9}$$

Now from Theorem 3.4.3 in [21] and (6.5), (6.9) we see that

$$|\bar{v}^{(n)}|_{L^2(\mathcal{T},H^2(\Omega))} \le C_4^*, \quad n \ge 1, \tag{6.10}$$

for some $C_4^* > 0$.

From (6.4), (6.7) it follows that $v^{(n)}, w^{(n)} \in W^{1,2}(\mathcal{T},H) \cap L^\infty(\Omega_T)$. By compact embedding $V \hookrightarrow H$ and (6.4)–(6.7), (6.10) there exists a subsequence (still indexed by $n$) and elements $v, w \in W^{1,2}(\mathcal{T},H) \cap L^\infty(\Omega_T) \cap L^2(\mathcal{T},H^2(\Omega))$ such that

$$\bar{v}^{(n)} \to v \quad \text{weakly in } L^2(\mathcal{T},H^2(\Omega)) \tag{6.11}$$

$$\Delta_N \bar{v}^{(n)} \to \Delta_N v \quad \text{weakly in } L^2(\mathcal{T},H) \tag{6.12}$$

$$v^{(n)} \to v \quad \text{strongly in } C(\mathcal{T},H) \tag{6.13}$$

$$v^{(n)} \to v \quad \text{weakly in } W^{1,2}(\mathcal{T}, H) \text{ and weakly* in } L^\infty(\mathcal{T}, V) \cap L^\infty(\Omega_T) \tag{6.14}$$

$$w^{(n)} \to w \quad \text{weakly in } W^{1,2}(\mathcal{T}, H) \text{ and weakly* in } L^\infty(\Omega_T) \tag{6.15}$$

$$\bar{u}^{(n)} \to u \quad \text{weakly* in } L^\infty(\Omega_T) \text{ and strongly in } L^2(\Omega_T). \tag{6.16}$$

By virtue of (4.6), the relation (4.1) can be equivalently rewritten as

$$\bar{w}^{(n)} \in K(\bar{v}^{(n)}) \qquad \text{a. e.}, \tag{6.17}$$

$$(g(\underline{v}^{(n)}, \underline{w}^{(n)}) - F(\bar{w}^{(n)}) - w_t^{(n)})(\bar{w}^{(n)} - y) \geq 0 \quad \text{a. e.} \, \forall y \in K(\bar{v}^{(n)}). \tag{6.18}$$

We can choose in (6.18) in particular $y = Q_{K(\bar{v}^{(n)})}(\bar{w}^{(m)})$ for some $m \neq n$. Let $L^*$ be the Lipschitz constant of all nonlinearities in the domain $|v| \leq C_0^*, |w| \leq C_0^*$ (cf. (6.7)). We then have $|\bar{w}^{(m)} - Q_{K(\bar{v}^{(n)})}(\bar{w}^{(m)})| \leq L^* |\bar{v}^{(n)} - \bar{v}^{(m)}|$ a.e., hence

$$\begin{aligned}
&(w_t^{(n)} + F(\bar{w}^{(n)}) - g(\underline{v}^{(n)}, \underline{w}^{(n)}))(\bar{w}^{(n)} - \bar{w}^{(m)}) \\
&\quad \leq L^* |w_t^{(n)} + F(\bar{w}^{(n)}) - g(\underline{v}^{(n)}, \underline{w}^{(n)})||\bar{v}^{(n)} - \bar{v}^{(m)}|.
\end{aligned}$$

Interchanging the roles of $m$ and $n$, we obtain

$$\begin{aligned}
&(w_t^{(m)} + F(\bar{w}^{(m)}) - g(\underline{v}^{(m)}, \underline{w}^{(m)}))(\bar{w}^{(m)} - \bar{w}^{(n)}) \\
&\quad \leq L^* |w_t^{(m)} + F(\bar{w}^{(m)}) - g(\underline{v}^{(m)}, \underline{w}^{(m)})||\bar{v}^{(n)} - \bar{v}^{(m)}|.
\end{aligned}$$

The sum of the two inequalities now yields the estimate

$$\begin{aligned}
&(w_t^{(n)} - w_t^{(m)})(\bar{w}^{(n)} - \bar{w}^{(m)}) + (F(\bar{w}^{(n)}) - F(\bar{w}^{(m)}))(\bar{w}^{(n)} - \bar{w}^{(m)}) \\
&\quad \leq L^*(|\underline{v}^{(n)} - \underline{v}^{(m)}| + |\underline{w}^{(n)} - \underline{w}^{(m)}|)|\bar{w}^{(n)} - \bar{w}^{(m)}| \\
&\qquad + (|w_t^{(n)}| + |w_t^{(m)}| + C_5^*)|\bar{v}^{(n)} - \bar{v}^{(m)}|
\end{aligned} \tag{6.19}$$

with some constant $C_5^* > 0$. We have for a.e. $(t, x) \in \Omega_T$ the inequality

$$\begin{aligned}
&(w_t^{(n)} - w_t^{(m)})(\bar{w}^{(n)} - \bar{w}^{(m)}) \geq \frac{1}{2} \frac{\partial}{\partial t}(w^{(n)} - w^{(m)})^2 \\
&\quad - (|w_t^{(n)}| + |w_t^{(m)}|)(|\bar{w}^{(n)} - w^{(n)}| + |\bar{w}^{(m)} - w^{(m)}|).
\end{aligned} \tag{6.20}$$

Moreover, for $t \in [(j-1)\tau_n, j\tau_n)$, we have $|\bar{w}^{(n)} - w^{(n)}|(t) \leq |w_j - w_{j-1}|$ and $|\bar{v}^{(n)} - v^{(n)}|(t) \leq |v_j - v_{j-1}|$, hence, according to (5.16)

$$\int_0^T |\bar{w}^{(n)} - w^{(n)}|_2^2(t) \, \mathrm{d}t \leq \tau_n C_6^*, \quad \int_0^T |\bar{v}^{(n)} - v^{(n)}|_2^2(t) \, \mathrm{d}t \leq \tau_n C_6^*, \tag{6.21}$$

$C_6^* > 0$, with similar estimates for $\underline{w}^{(n)}, \underline{v}^{(n)}$. Set

$$\begin{aligned}
\delta_{mn}(t) = {} & |\bar{w}^{(n)} - w^{(n)}|_2(t) + |\bar{v}^{(n)} - v^{(n)}|_2(t) + |\underline{w}^{(n)} - w^{(n)}|_2(t) \\
& + |\underline{v}^{(n)} - v^{(n)}|_2(t) + |\underline{w}^{(m)} - w^{(m)}|_2(t) + |\underline{v}^{(m)} - v^{(m)}|_2(t) \\
& + |\bar{w}^{(m)} - w^{(m)}|_2(t) + |\bar{v}^{(m)} - v^{(m)}|_2(t) + |v^{(n)} - v^{(m)}|_2(t).
\end{aligned}$$

It thus follows from (6.19), (6.20) for a.e. $t \in (0, T)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}|w^{(n)} - w^{(m)}|_2^2(t) \le 2L^*|w^{(n)} - w^{(m)}|_2^2(t) + C_7^*\delta_{mn}(t)(1 + |w_t^{(n)}|_2(t) + |w_t^{(m)}|_2(t)) \tag{6.22}$$

with some constant $C_7^* > 0$ independent of $n$ and $m$. The Gronwall argument, Hölder's inequality, and (6.4) yield

$$|w^{(n)} - w^{(m)}|_2^2(t) \le C_8^*\mathrm{e}^{2L^*T}\left(\int_0^T |\delta_{mn}(t)|^2\,\mathrm{d}t\right)^{1/2} \tag{6.23}$$

with some constant $C_8^* > 0$. By (6.21), the $L^2$-norm of $\delta_{mn}$ is small for large $n, m$, and we conclude that $w^{(n)}(t)$ is a Cauchy sequence for each $t \in \mathcal{T}$. From (6.4) it follows that the sequence $w^{(n)}$, $n \ge 1$ is equicontinuous on $\mathcal{T}$. Hence, according to the Arzela–Ascoli theorem the sequence $w^{(n)}$ converges in $C(\mathcal{T}, H)$. Using (6.15) we obtain

$$w^{(n)} \to w \quad \text{in } C(\mathcal{T}, H). \tag{6.24}$$

Denote by $S_{C_0^*}(\Omega)$ and $S_{C_0^*}(\Omega_T)$ the sets

$$S_{C_0^*}(\Omega) = \{v \in H;\ |v(x)| \le C_0^* \quad \text{a.e. on } \Omega\}, \tag{6.25}$$

and

$$S_{C_0^*}(\Omega_T) = \{v \in L^2(\mathcal{T}, H);\ |v(t,x)| \le C_0^* \quad \text{a.e. on } \Omega_T\}, \tag{6.26}$$

respectively. From Hypothesis 1 (i) it follows that for any $v \in S_{C_0^*}(\Omega)$ the set $\mathcal{K}(v)$ defined in (3.5) is a nonempty convex weakly compact subset of the space $H$ and the inequality

$$D_H(\mathcal{K}(v_1), \mathcal{K}(v_2)) \le L^*|v_1 - v_2|_H, \quad v_1, v_2 \in S_{C_0^*}(\Omega) \tag{6.27}$$

holds, where $D_H$ is the Hausdorff metric on the space $\operatorname{cb} H$. Let

$$\mathcal{K}^*(v) = \{w \in L^0(\mathcal{T}, H);\ w(t) \in \mathcal{K}(v(t)) \quad \text{a.e. on } \mathcal{T}\},\ v \in S_{C_0^*}(\Omega_T). \tag{6.28}$$

From (6.27) we see that for any $v \in S_{C_0^*}(\Omega_T)$ the set $\mathcal{K}^*(v)$ is a nonempty convex weakly compact subset of the space $L^2(\mathcal{T}, H)$ and the inequality

$$D_{L^2(\mathcal{T},H)}(\mathcal{K}^*(v_1), \mathcal{K}^*(v_2)) \le L^*|v_1 - v_2|_{L^2(\mathcal{T},H)}, \quad v_1, v_2 \in S_{C_0^*}(\Omega_T) \tag{6.29}$$

holds.

It is well known that if $v \in S_{C_0^*}(\Omega_T)$ and $w \in \mathcal{K}^*(v)$, then

$$\partial I_{\mathcal{K}^*(v)}(w) = \{h \in L^2(\mathcal{T}, H);\ h(t) \in \partial I_{\mathcal{K}(v(t))}(w(t)) \quad \text{a.e. on } \mathcal{T}\}, \tag{6.30}$$

where $I_{\mathcal{K}^*(v)}$ is the indicator function of the set $\mathcal{K}^*(v)$.

Let $v$ be a function satisfying (6.13). From (6.7) it follows that $v \in S_{C_0^*}(\Omega_T)$. Let $y \in \mathcal{K}^*(v)$ be an arbitrary point and $y^{(n)} = \operatorname{pr}_{\mathcal{K}^*(v^{(n)})}\, y$ be the projection of the point $y$ onto the set $\mathcal{K}^*(v^{(n)})$. Using (6.7), (6.8) we obtain

$$\bar{w}^{(n)} \in \mathcal{K}^*(\bar{v}^{(n)}), \tag{6.31}$$

$$\begin{aligned}&\langle g(\underline{v}^{(n)}, \underline{w}^{(n)}) - F(\bar{w}^{(n)}) - {w^{(n)}}', \bar{w}^{(n)} - y^{(n)}\rangle_{L^2(\mathcal{T},H)}\\ &\quad = \int_{\mathcal{T}} \langle g(\underline{v}^{(n)}(t), \underline{w}^{(n)}(t)) - F(\bar{w}^{(n)}(t)) - {w^{(n)}}'(t), \bar{w}^{(n)}(t) - y^{(n)}(t)\rangle_H \,\mathrm{d}t,\end{aligned} \tag{6.32}$$

where $\langle\cdot,\cdot\rangle_{L^2(\mathcal{T},H)}$ is the scalar product in $L^2(\mathcal{T},H)$.

From (6.28) and the fact that $y \in \mathcal{K}^*(v)$ we infer that

$$|y - y^{(n)}|_{L^2(\mathcal{T},H)} \le L^*|v - v^{(n)}|_{L^2(\mathcal{T},H)}. \tag{6.33}$$

Passing to the limit in (6.31), (6.32) and taking into account (6.21) and similar inequalities for $\underline{w}^{(n)}$, $\underline{v}^{(n)}$, (6.13), (6.14), (6.24), (6.29), (6.32) we obtain

$$w \in \mathcal{K}^*(v), \tag{6.34}$$

$$\langle g(v,w) - F(w) - w', w - y\rangle_{L^2(\mathcal{T},H)} \ge 0. \tag{6.35}$$

Since $y \in \mathcal{K}^*(v)$ is arbitrary, from (6.34), (6.35) it follows that

$$g(v,w) - F(w) - w' \in \partial I_{\mathcal{K}^*(v)}(w). \tag{6.36}$$

Using (6.30) we have

$$w' + \partial I_{\mathcal{K}(v)}(w) \ni g(v,w) - F(w) \quad \text{in } H \quad \text{a.e. on } \mathcal{T}. \tag{6.37}$$

Analogously, passing to the limit in (6.8) and taking into account (6.14), (6.15), (6.12), (6.16) and similar inequalities for $\underline{w}^{(n)}$, $\underline{v}^{(n)}$, we obtain (3.8) for $\psi = 0$. On the space $H$ consider the function

$$\varphi(v) = \begin{cases} \frac{1}{2}|v|_H^2 + \frac{1}{2}|\nabla v|_H^2, & \text{if } v \in V;\\ +\infty, & \text{if } v \in H \setminus V.\end{cases}$$

Then $\varphi \in \Gamma_0(H)$ and [13]

$$\operatorname{dom}\partial\varphi = D(-\Delta_N), \tag{6.38}$$

$$\partial\varphi(v) = v - \Delta_N v. \tag{6.39}$$

From (3.8), (6.38), (6.39) it follows that

$$-cw' - dv' + v + a(v,w) + b(v,w)u \in \partial\varphi(v) \quad \text{in} \quad H \quad \text{a.e on} \quad \mathcal{T}. \tag{6.40}$$

Since $v \in W^{1,2}(\mathcal{T},H)$ and

$$-cw' - dv' + v + a(v,w) + b(v,w)u \in L^2(\mathcal{T},H),$$

from (6.40) and Lemma 3.3 in [17] we infer that the function

$$t \to \varphi(v(t)) = \frac{1}{2}|v(t)|_H^2 + \frac{1}{2}|\nabla v(t)|_H^2$$

is absolutely continuous. Hence, the function

$$t \to |v(t)|_V = (|v(t)|_H^2 + |\nabla v(t)|_H^2)^{1/2} \tag{6.41}$$

is continuous. Therefore, the set $\{v(t);\ t \in \mathcal{T}\}$ is bounded in the space $V$ and, thus, relatively compact in $H$. Since the imbedding $H \hookrightarrow V'$ is dense and the function $v(t)$ is continuous from $\mathcal{T}$ to $H$, it is continuous from $\mathcal{T}$ to $V$ with weak* topology. Consequently, from (6.41) we infer that the function $t \to v(t)$

is continuous from $\mathcal{T}$ to $V$. Hence, $v \in W^{1,2}(\mathcal{T}, H) \cap C(\mathcal{T}, V) \cap L^2(\mathcal{T}, H^2(\Omega))$ and $w \in W^{1,2}(\mathcal{T}, H)$ according to (6.11), (6.14), (6.15).

Thus, the system (3.7)–(3.9), (3.15) with $\psi = 0$ has a solution $\{(v(u), w(u)), u\}$.

Obviously, according to (6.4)–(6.7) we have

$$|v'(u)|^2_{L^2(\mathcal{T},H)} + |w'(u)|^2_{L^2(\mathcal{T},H)} \le C_0^*, \tag{6.42}$$

$$|\Delta v(u)|^2_{L^2(\mathcal{T},H)} \le C_0^*, \tag{6.43}$$

$$|\nabla v(u)|_{L^\infty(\mathcal{T},H)} \le C_0^*, \tag{6.44}$$

$$(|v(u)| + |w(u)|)_{L^\infty(\Omega_T)} \le C_0^*. \tag{6.45}$$

Therefore, the existence of solution of the system (3.7)–(3.9), (3.15) and the estimate (3.14) are proved in case of $\psi = 0$.

## 7. Proofs of Theorems 3.5 and 3.6

We proceed again by comparison. First, we add a purely technical assumption

$$f_*(0) \le 0 \le f^*(0). \tag{7.1}$$

Of course, this can easily be obtained by shifting the functions $f^*$, $f_*$, and $w$ by the constant $\frac{1}{2}(f_*(0) - f^*(0))$, with corresponding modifications of the nonlinearities.

Second, by (4.21)–(4.22), (4.23)–(4.24), we have that the differences $\frac{1}{\tau_n}|v_j^\sharp - v_{j-1}^\sharp|$, $\frac{1}{\tau_n}|v_j^\flat - v_{j-1}^\flat|$, $\frac{1}{\tau_n}|w_j^\sharp - w_{j-1}^\sharp|$, $\frac{1}{\tau_n}|w_j^\flat - w_{j-1}^\flat|$, are all bounded independently of $n$ and $j$. The piecewise linear interpolates $v^{\sharp(n)}, w^{\sharp(n)}, v^{\flat(n)}, w^{\flat(n)}$ defined as in (6.1) form therefore a bounded sequence in $W^{1,\infty}(0,T)$. Selecting suitable weakly-* convergent subsequences and passing to the limit as in (6.17)–(6.23), we obtain functions $v^\sharp, w^\sharp, v^\flat, w^\flat$ as solutions to the systems

$$\dot w^\sharp + F(w^\sharp) + \partial I_{K(v^\sharp)}(w^\sharp) \ni B(\max\{|v^\sharp|, |v^\flat|\} + 1), \tag{7.2}$$

$$c\dot w^\sharp + d\dot v^\sharp = C_R B(\max\{|v^\sharp|, |v^\flat| + 1), \tag{7.3}$$

$$v^\sharp(0) = v_0^\sharp,\ w^\sharp(0) = f^*(v_0^\sharp), \tag{7.4}$$

$$\dot w^\flat + F(w^\flat) + \partial I_{K(v^\flat)}(w^\flat) \ni -B(\max\{|v^\sharp|, |v^\flat|\} + 1), \tag{7.5}$$

$$c\dot w^\flat + d\dot v^\flat = -C_R B(\max\{|v^\sharp|, |v^\flat|\} + 1), \tag{7.6}$$

$$v^\flat(0) = v_0^\flat,\ w^\flat(0) = f_*(v_0^\flat), \tag{7.7}$$

where we choose $v_0^\sharp \ge \max\{k_1, |v_0|_\infty\}$, $v_0^\flat \le -\max\{k_1, |v_0|_\infty\}$, and the constant $B$ sufficiently large, $B \ge \max\{|F(w_0^\sharp)|, |F(w_0^\flat)|\}$, such that the counterpart of (4.20)

$$\max\{|g(v,w)|, |a(v,w)|, |b(v,w)|\} \le B(|v| + 1) \tag{7.8}$$

holds for all $v \in \mathbb{R}$ and $w \in K(v)$.

The functions $v^\sharp, w^\sharp$ are nondecreasing, positive, and bounded in $[0,T]$, $v^\flat, w^\flat$ are nonincreasing, negative, and bounded in $[0,T]$. They do not depend on $\psi$, and this will play a substantial role in the proof of Theorem 3.5.

The argument will be based on another variant of the Hilpert inequality, cf. [20].

**Proposition 7.1.** *Let $g_1, v_1 \in L^\infty(\mathcal{T})$ be given, and let $w_1$ be the solution in $\mathcal{T}$ of the differential inclusion*

$$\dot{w}_1(t) + F(w_1(t)) + \partial I_{K(v_1(t))}(w_1(t)) \ni g_1(t). \tag{7.9}$$

*Let $w_2 \in K(v_2)$ for some $v_2, w_2$. Then for a.e. $t \in \mathcal{T}$ such that $w_1(t) \neq w_2$ we have*

$$(\dot{w}_1 + F(w_1) - g_1)H(w_1 - w_2) \le (\dot{w}_1 + F(w_1) - g_1)H(v_1 - v_2), \tag{7.10}$$

$$-(\dot{w}_1 + F(w_1) - g_1)H(w_2 - w_1) \le -(\dot{w}_1 + F(w_1) - g_1)H(v_2 - v_1). \tag{7.11}$$

*Proof.* The statement is obvious if $\dot{w}_1(t) + F(w_1)(t) - g_1(t) = 0$, as well as if $\dot{w}_1(t) + F(w_1)(t) - g_1(t) > 0$ and $w_1(t) < w_2$. Assume now that $\dot{w}_1(t) + F(w_1)(t) - g_1(t) > 0$ and $w_1(t) > w_2$. The only problem may occur if $v_1(t) < v_2$. Then we have $w_1(t) = f_*(v_1(t)) \le f_*(v_2) \le w_2$, which is a contradiction. Similarly, everything is clear if $\dot{w}_1(t) + F(w_1)(t) - g_1(t) < 0$ and $w_1(t) > w_2$. In the case $\dot{w}_1(t) + F(w_1)(t) - g_1(t) < 0$ and $w_1(t) < w_2$, $v_1(t) > v_2$ we obtain similarly $w_1(t) = f^*(v_1(t)) \ge f^*(v_2) \ge w_2$, that is, a contradiction again. □

**Corollary 7.2.** *Let $g_\alpha, v_\alpha \in L^\infty(\mathcal{T})$, $\alpha = 1, 2$ be given, and let $w_\alpha$ be the solution in $\mathcal{T}$ of the differential inclusion*

$$\dot{w}_\alpha(t) + \partial I_{K(v_\alpha(t))}(w_\alpha(t)) \ni g_\alpha(t). \tag{7.12}$$

*Then for a.e. $t \in \mathcal{T}$ we have*

$$(\dot{w}_1 - \dot{w}_2)(t)\,\mathrm{sign}(w_1 - w_2)(t) \le (\dot{w}_1 - \dot{w}_2)(t)\,\mathrm{sign}(v_1 - v_2)(t) + 2|g_1 - g_2|(t). \tag{7.13}$$

*Proof.* It follows from Proposition 7.1 and from the formula $\mathrm{sign}(z) = H(z) - H(-z)$ that

$$\begin{aligned}&(\dot{w}_1(t) + F(w_1)(t) - g_1(t))\,\mathrm{sign}(w_1(t) - w_2(t))\\ &\quad\le (\dot{w}_1(t) + F(w_1)(t) - g_1(t))\,\mathrm{sign}(v_1(t) - v_2(t))\end{aligned} \tag{7.14}$$

whenever $w_1(t) \neq w_2(t)$. Interchanging the indices 1 and 2 and summing the resulting inequalities, we obtain (7.13) for all $t$ such that $w_1(t) \neq w_2(t)$. For a.e. $t \in \mathcal{T}$ we have the implication

$$w_1(t) = w_2(t) \ \Rightarrow\ \dot{w}_1(t) = \dot{w}_2(t),$$

hence (7.13) holds for a.e. $t \in \mathcal{T}$. □

*Proof.* (Proof of Theorem 3.5) Consider the truncated system

$$w' + \partial I_{\mathcal{K}(v)}(w) \ni g(v, w) - \psi_M(v, w) - F(w) \quad \text{in } H \text{ a.e. on } \mathcal{T}, \tag{7.15}$$

$$cw' + dv' - \Delta_N v = a(v, w) + b(v, w)u \quad \text{in } H \quad \text{a.e. on } \mathcal{T}, \tag{7.16}$$

$$v(0) = v_0, \ \ w(0) = w_0 \quad \text{in } H, \tag{7.17}$$

with $u \in S_R$, where $M > 0$ is a fixed cutoff parameter, and

$$\psi_M(v,w) = \begin{cases} \psi(v,w) & \text{for } |w| \le M, \\ \psi(v,M) & \text{for } w > M, \\ \psi(v,-M) & \text{for } w < -M. \end{cases} \tag{7.18}$$

The function $\psi_M$ is bounded and Lipschitz continuous. Indeed, by Hypothesis 1 (ii) we have $|\psi_M(v,w)| \le \max\{|\psi(\hat{v},\hat{w})|; |\hat{v}| \le k_1, |\hat{w}| \le M\}$ for all $(v,w) \in \mathbb{R}^2$. Hence, the function $\tilde{g} := g - \psi_M$ satisfies Hypothesis 1 (i), and the existence of a solution $(v,w)$ for every fixed $M > 0$ with the regularity as in Theorem 3.5 is established in Sect. 6. It is clear that $(v,w)$ satisfy

$$w_t + \partial I_{K(v)}(w) \ni g(v,w) - \psi_M(v,w) - F(w) \qquad \text{a.e. on } \Omega_T, \tag{7.19}$$

$$cw_t + dv_t - \Delta v = a(v,w) + b(v,w)u \quad \text{a.e. on } \Omega_T, \tag{7.20}$$

$$v(x,0) = v_0(x), \;\; w(x,0) = w_0(x) \quad \text{a.e. on } \Omega. \tag{7.21}$$

We now repeat the procedure from the proof of Proposition 5.2, subtract (7.2) from (7.19), and test the difference by $H(w-w^\sharp)$ to obtain a counterpart of (5.5) in the form

$$\begin{aligned} &(w_t - w_t^\sharp)H(w-w^\sharp) + (F(w) - F(w^\sharp))H(w-w^\sharp) \\ &\quad + (g^* - g(v,w) + \psi_M(v,w))H(w-w^\sharp) \\ &\le (w_t - w_t^\sharp)H(v-v^\sharp) + (F(w) - F(w^\sharp))H(v-v^\sharp) \\ &\quad + (g^* - g(v,w) + \psi_M(v,w))H(v-v^\sharp), \end{aligned} \tag{7.22}$$

whenever $w(t,x) \ne w^\sharp(t)$, where we denote $g^* = B(\max\{|v^\sharp|, |v^\flat|\} + 1)$. By hypothesis, we have $\psi_M(v,w)H(w-w^\sharp) \ge 0$, $\psi_M(v,w)H(v-v^\sharp) = 0$, $(F(w) - F(w^\sharp))H(w-w^\sharp) - (F(w) - F(w^\sharp))H(v-v^\sharp) \ge 0$, hence

$$\begin{aligned} &(w_t - w_t^\sharp)H(w-w^\sharp) + (g^* - g(v,w))H(w-w^\sharp) \\ &\quad \le (w_t - w_t^\sharp)H(v-v^\sharp) + (g^* - g(v,w))H(v-v^\sharp). \end{aligned} \tag{7.23}$$

If $v(t,x) \le v^\sharp(t)$, then $g^*(t) - g(v,w)(t,x) \ge 0$, and (7.22) yields

$$(w_t - w_t^\sharp)H(w-w^\sharp) \le (w_t - w_t^\sharp)H(v-v^\sharp) + (g^* - g(v,w))H(v-v^\sharp). \tag{7.24}$$

If $v(t,x) > v^\sharp(t)$, then $g^*(t) - g(v,w)(t,x) \le B(v(t,x) - v^\sharp(t))$, and we have

$$\begin{aligned} (w_t - w_t^\sharp)H(w-w^\sharp) &\le (w_t - w_t^\sharp)H(v-v^\sharp) + B(v-v^\sharp) \\ &\quad + (g^* - g(v,w))H(v-v^\sharp). \end{aligned} \tag{7.25}$$

Combining (7.24) with (7.25) we obtain

$$\begin{aligned} (w_t - w_t^\sharp)H(w-w^\sharp) &\le (w_t - w_t^\sharp)H(v-v^\sharp) + B(v-v^\sharp)^+ \\ &\quad + (g^* - g(v,w))H(v-v^\sharp), \end{aligned} \tag{7.26}$$

whenever $w(t,x) \ne w^\sharp(t)$.

We further subtract (7.3) from (7.16) and test the difference by $H(v-v^\sharp)$, to obtain

$$c(w_t - w_t^\sharp)H(v - v^\sharp) + d(v_t - v_t^\sharp)H(v - v^\sharp) - \Delta(v - v^\sharp)H(v - v^\sharp)$$
$$\le (a(v,w) + b(v,w)u - C_R g^*)H(v - v^\sharp). \tag{7.27}$$

Set $\Omega_T^0 = \{(t,x) \in \Omega_T : w(t,x) = w^\sharp(t)\}$. For a.e. $(t,x) \in \Omega_T^0$ we have $w_t(t,x) = w_t^\sharp(t)$, and, as a consequence of (7.27),

$$d(v_t - v_t^\sharp)H(v - v^\sharp) - \Delta(v - v^\sharp)H(v - v^\sharp)$$
$$\le (a(v,w) + b(v,w)u - C_R g^*)H(v - v^\sharp). \tag{7.28}$$

For $(t,x) \in \Omega\backslash\Omega_T^0$, we multiply (7.26) by $c$ and add the result to (7.28) to obtain

$$c(w_t - w_t^\sharp)H(w - w^\sharp) + d(v_t - v_t^\sharp)H(v - v^\sharp) - \Delta(v - v^\sharp)H(v - v^\sharp)$$
$$\le cB(v - v^\sharp)^+ + (a(v,w) + b(v,w)u - cg(v,w) - (C_R - c)g^*)H(v - v^\sharp). \tag{7.29}$$

We now argue as in (7.26), choosing $C_R$ as in (5.10): For $v(t,x) \le v^\sharp(t)$, the right hand side of (7.29) vanishes. For $v(t,x) > v^\sharp(t)$ we have $a(v,w) + b(v,w)u - cg(v,w) - (C_R - c)g^* \le B(C_R - R - 2c - 1)(v - v^\sharp)$. We now put together (7.28) with (7.29), and find a constant $B_R$ independent of $t$ and $M$ such that for a.e. $(t,x) \in \Omega$ we have

$$c(w_t - w_t^\sharp)H(w - w^\sharp) + d(v_t - v_t^\sharp)H(v - v^\sharp)$$
$$-\Delta(v - v^\sharp)H(v - v^\sharp) \le B_R(v - v^\sharp)^+. \tag{7.30}$$

Integrating over $\Omega$ and using (5.12), we obtain

$$c\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (w - w^\sharp)^+(t,x)\,\mathrm{d}x + d\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v - v^\sharp)^+(t,x)\,\mathrm{d}x$$
$$\le B_R\int_\Omega (v - v^\sharp)^+(t,x)\,\mathrm{d}x. \tag{7.31}$$

The standard Gronwall argument yields $v(t,x) \le v^\sharp(t)$ $w(t,x) \le w^\sharp(t)$ a.e. in $\Omega_T$. We similarly prove that $v(t,x) \ge v^\flat(t)$ $w(t,x) \ge w^\flat(t)$ a.e. In particular, both $v$ and $w$ admit bounds in $L^\infty(\Omega_T)$ which are *independent of* $M$. Taking $M$ sufficiently large, we see that the constraint in (7.18) is never active, and the solution of (7.15), (7.16) with $u \in S_R$ is the desired solution of (3.7)–(3.9), (3.15) from Theorem 3.5. Uniqueness is obtained as a by-product below in the proof of Theorem 3.6. The inequality (3.14) follows from the uniqueness of a solution of the system (3.7)–(3.9), (3.15) and from the inequalities (6.42)–(6.45). □

*Proof.* (Proof of Theorem 3.6) We test the difference

$$c(w_1 - w_2)_t + d(v_1 - v_2)_t - \Delta(v_1 - v_2)$$
$$= a(v_1,w_1) + b(v_1,w_1)u_1 - a(v_2,w_2) - b(v_2,w_2)u_2$$

by $\operatorname{sign}(v_1 - v_2)$. Using Corollary 7.2 and an inequality analogous to (5.12), we obtain for a.e. $t$ that

$$c\frac{\mathrm{d}}{\mathrm{d}t}|w_1 - w_2|_X(t) + d\frac{\mathrm{d}}{\mathrm{d}t}|v_1 - v_2|_X(t)$$
$$\leq (3+R)L^*(|w_1 - w_2|_X(t) + |v_1 - v_2|_X(t)) + C_8^*|u_1 - u_2|_X(t)$$

with a constant $C_8^*$. The statement of Theorem 3.6 (and, in particular, uniqueness of solution to system (3.7)–(3.9), (3.15)) follows from the Gronwall argument. □

## 8. Nemytskii multivalued operator and properties of its fixed points

Let

$$S_R^0 = \{u \in L^0(\mathcal{T}, L^0(\Omega));\ |u(t,x)| \leq R \text{ a.e. on } \Omega_T\}.$$

**Lemma 8.1.** *The topologies of the spaces $L^1(\mathcal{T}, L^1(\Omega))$ and $L^2(\mathcal{T}, L^2(\Omega))$ coincide on the set $S_R^0$.*

*Proof.* Indeed, for $u_1, u_2 \in S_R^0$, we have by the Hölder inequality that

$$\int_{\Omega_T} |u_1 - u_2|(t,x)\,\mathrm{d}t\,\mathrm{d}x \leq \left(|\Omega_T| \int_{\Omega_T} |u_1 - u_2|^2(t,x)\,\mathrm{d}t\,\mathrm{d}x\right)^{1/2}$$
$$\leq \left(2R|\Omega_T| \int_{\Omega_T} |u_1 - u_2|(t,x)\,\mathrm{d}t\,\mathrm{d}x\right)^{1/2},$$

where $|\Omega_T|$ denotes the Lebesgue measure of $\Omega_T$, and the assertion follows. □

In what follows we denote by $S_R^1$ the set $S_R^0$ equipped with the topology of the space $L^1(\mathcal{T}, X)$.

We can consider the mappings $\overline{\mathrm{co}}\,\mathcal{U}(t,v,w)$ and $\mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}(t,v,w)$ defined by the equalities (3.1) and (3.2) as mappings defined on $\mathcal{T} \times X \times X$, $X = L^1(\Omega)$. Their values are closed decomposable bounded subsets not only of the space $H = L^2(\Omega)$, but also of the space $X = L^1(\Omega)$. In the sequel, when we consider the mappings $\overline{\mathrm{co}}\,\mathcal{U}(t,v,w)$ and $\mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}(t,v,w)$ with values in the space $X$ we denote them as $\overline{\mathrm{co}}\,\mathcal{U}^1(t,v,w)$ and $\mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}^1(t,v,w)$, respectively.

**Lemma 8.2.** *Let Hypothesis 2 hold. Then the mappings $t \to \overline{\mathrm{co}}\,\mathcal{U}^1(t,v,w)$, $t \to \mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}^1(t,v,w)$ are measurable for $v, w \in X$, and satisfy the inequalities*

$$D_X(\overline{\mathrm{co}}\,\mathcal{U}^1(t,v_1,w_1), \overline{\mathrm{co}}\,\mathcal{U}^1(t,v_2,w_2))$$
$$\leq k(t)(|v_1 - v_2|_X + |w_1 - w_2|_X), \tag{8.1}$$

$$D_X(\mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}^1(t,v_1,w_1), \mathrm{ext}\,\overline{\mathrm{co}}\,\mathcal{U}^1(t,v_2,w_2))$$
$$\leq k(t)(|v_1 - v_2|_X + |w_1 - w_2|_X), \tag{8.2}$$

*a.e. on $\mathcal{T}$ for $(v_i, w_i) \in X$, $i = 1, 2$.*

*Proof.* Since the values of the mapping $\operatorname{co} U(t,x,v,w)$, $v,w \in \mathbb{R}$ are intervals of the real line $\mathbb{R}$, the set $\operatorname{ext}\operatorname{co} U(t,x,v,w)$ consists of the boundary points of the set $\operatorname{co} U(t,x,v,w)$. Hence, from (2.5) and Hypothesis 2 it follows that the mapping $t \to \operatorname{ext}\operatorname{co} U(t,x,v,w)$ is measurable and

$$D_{\mathbb{R}}(\operatorname{ext}\operatorname{co} U(t,x,v_1,w_1), \operatorname{ext}\operatorname{co} U(t,x,v_2,w_2)) \le k(t)(|v_1-v_2| + |w_1-w_2|) \tag{8.3}$$

a.e. on $\Omega_T$, $(v_i,w_i) \in \mathbb{R}$, $i=1,2$. Fix $z \in X$. Then according to [19, Proposition 4.1],

$$d_X(z, \operatorname{ext}\overline{\operatorname{co}}\,\mathcal{U}^1(t,v,w)) = \int_\Omega d_{\mathbb{R}}(z(x), \operatorname{ext}\operatorname{co} U(t,x,v(x),w(x)))\,\mathrm{d}x.$$

Since for $z \in L^1(\Omega)$ the function $(t,x) \to d_{\mathbb{R}}(z(x), \operatorname{ext}\operatorname{co} U(t,x,v(x), w(x))$ is an element of the space $L^1(\Omega_T)$, from Fubini's theorem it follows that the function $t \to d_X(z, \operatorname{ext}\overline{\operatorname{co}}\,\mathcal{U}^1(t,v,w))$ is measurable for any $z \in X$. Then according to Theorem 3.5 from [18] the multivalued mapping $t \to \operatorname{ext}\overline{\operatorname{co}}\,\mathcal{U}^1(t,v,w))$ is measurable. Using [19, Proposition 4.2], we obtain

$$\begin{aligned} &D_X(\operatorname{ext}\overline{\operatorname{co}}\,\mathcal{U}^1(t,v_1,w_1), \operatorname{ext}\overline{\operatorname{co}}\,\mathcal{U}^1(t,v_2,w_2)) \\ &\quad \le \int_\Omega D_{\mathbb{R}}(\operatorname{ext}\operatorname{co} U(t,x,v_1(x),w_1(x)), \operatorname{ext}\operatorname{co} U(t,x,v_2(x),w_2(x)))\,\mathrm{d}x. \end{aligned} \tag{8.4}$$

Now the inequality (8.2) follows from the inequalities (8.3), (8.4). The measurability of the mapping $t \to \overline{\operatorname{co}}\,\mathcal{U}(t,v,w)$ and the inequality (8.1) are similarly proved. □

Let $\mathcal{F}$ be the operator which with each $u \in S^1_R$ associates the unique solution $(v(u), w(u))$ of the system (3.7)–(3.9), i.e.

$$(v,w) = \mathcal{F}(u), \quad u \in S^1_R. \tag{8.5}$$

According to (3.17) for $v^1_0 = v^2_0$, $w^1_0 = w^2_0$ we have

$$|\mathcal{F}(u_1) - \mathcal{F}(u_2))|_X(t) \le L^* \int_0^t |u_1-u_2|_X(\tau)\,\mathrm{d}\tau. \tag{8.6}$$

As above, we denote by $Q_{[-R,R]}$ the projection of $\mathbb{R}$ onto the interval $[-R,R]$. Then, for $u \in L^1(\mathcal{T},X)$ the function

$$\bar{u}(t,x) = Q_{[-R,R]}(u(t,x)) \tag{8.7}$$

is an element of the space $L^1(\mathcal{T},X)$. Let

$$\bar{\mathcal{F}}(u) = \mathcal{F}(\bar{u}), \quad u \in L^1(\mathcal{T},X). \tag{8.8}$$

Then, $\bar{\mathcal{F}}(u)$ is an extension of the operator $\mathcal{F}(u)$ from the set $S^1_R$ to the entire space $L^1(\mathcal{T},X)$, which with each $u \in L^1(\mathcal{T},X)$ associates the unique solution of the system (3.7)–(3.9). From (8.7), (8.6), (8.8) it follows that

$$|\bar{\mathcal{F}}(u_1) - \bar{\mathcal{F}}(u_2)|_X(t) \le L^* \int_0^t |u_1-u_2|_X(\tau)\,\mathrm{d}\tau, \tag{8.9}$$

$u_1, u_2 \in L^1(\mathcal{T},X)$.

For any $u \in L^1(\mathcal{T}, X)$ let

$$\phi_{\text{ext}}(u) = \{f(t) \in L^1(\mathcal{T}, X);\ f(t) \in \operatorname{ext} \overline{\text{co}}\,\mathcal{U}(t, \bar{\mathcal{F}}(u)(t)) \ \text{ a.e. on } \mathcal{T}\}, \quad (8.10)$$

$$\phi_{\overline{\text{co}}}(u) = \{f(t) \in L^1(\mathcal{T}, X);\ f(t) \in \overline{\text{co}}\,\mathcal{U}(t, \bar{\mathcal{F}}(u)(t)) \ \text{ a.e. on } \mathcal{T}\}. \quad (8.11)$$

From (8.9) it follows that the operator $\bar{\mathcal{F}}$ is continuous from $L^1(\mathcal{T}, X)$ to $C(\mathcal{T}, X^2)$ and thus to $L^1(\mathcal{T}, X^2)$. Consequently, according to Lemma 8.2 the mappings $t \to \operatorname{ext} \overline{\text{co}}\,\mathcal{U}(t, \bar{\mathcal{F}}(u)(t))$ and $t \to \overline{\text{co}}\,\mathcal{U}(t, \bar{\mathcal{F}}(u)(t))$ are measurable with closed values in $X$. Hence, from (8.10), (8.11) and properties of measurable multivalued mappings it follows that the sets $\phi_{\text{ext}}(u)$, $\phi_{\overline{\text{co}}}(u)$ are nonempty bounded closed decomposable subsets of the space $L^1(\mathcal{T}, X)$. Usually, the mappings $u \to \phi_{\text{ext}}(u)$, $u \to \phi_{\overline{\text{co}}}(u)$ are called the multivalued Nemytskii operators.

On the space $L^1(\mathcal{T}, X)$ consider the scalar function

$$P(u) = \int_{\mathcal{T}} \rho(t, u(t))\, \mathrm{d}t \quad (8.12)$$

with

$$\rho(t, u) = \exp\left(-2L^* \int_0^t k(\tau)\, d\tau\right) |u|, \quad (8.13)$$

where $L^*$ is the constant from the inequality (3.17) (or (8.9)) and $k(t)$ is the function from the inequality (2.5). It is clear that $P(u)$ is a norm equivalent to the norm of the space $L^1(\mathcal{T}, X)$. Denote by $D_P(\cdot,\cdot)$ the Hausdorff metric on the space $\operatorname{cb} L^1(\mathcal{T}, X)$ generated by the metric $P(u)$.

**Theorem 8.3.** *Let Hypotheses* 1 *and* 2 *hold. Then the following inequalities are valid:*

$$D_P(\phi_{\text{ext}}(u_1), \phi_{\text{ext}}(u_2)) \le \frac{1}{2} P(u_1 - u_2), \quad (8.14)$$

$$D_P(\phi_{\overline{\text{co}}}(u_1), \phi_{\overline{\text{co}}}(u_2)) \le \frac{1}{2} P(u_1 - u_2). \quad (8.15)$$

These inequalities are proved by repeating verbatim the proofs of similar inequalities in [22, Theorem 2.1] or [23, Theorem 4.1] using the integration by parts, the inequalities (8.9), (8.1), (8.2) and (8.12), (8.13).

Denote by $\operatorname{Fix} \phi_{\text{ext}}$ and $\operatorname{Fix} \phi_{\overline{\text{co}}}$ the sets of all fixed points of the operators $\phi_{\text{ext}}$ and $\phi_{\overline{\text{co}}}$, respectively.

**Theorem 8.4.** *Let Hypotheses* 1 *and* 2 *hold. Then*

(i) *the sets* $\operatorname{Fix} \phi_{\text{ext}}$ *and* $\operatorname{Fix} \phi_{\overline{\text{co}}}$ *are nonempty;*

(ii) *the sets* $\operatorname{Fix} \phi_{\text{ext}}$ *and* $\operatorname{Fix} \phi_{\overline{\text{co}}}$ *are closed arc-wise connected absolute retracts in the space* $L^2(\mathcal{T}, H)$.

*Proof.* According to Theorem 8.1 all the assumptions of Theorem 3.1 from [24] are valid for the operators $\phi_{\text{ext}}$ and $\phi_{\overline{\text{co}}}$. From this latter theorem it follows that $\operatorname{Fix} \phi_{\text{ext}} \neq \emptyset$ and $\operatorname{Fix} \phi_{\overline{\text{co}}} \neq \emptyset$.

From Theorem 4.1 and Corollary 4.1 from [24] the sets $\operatorname{Fix} \phi_{\text{ext}}$ and $\operatorname{Fix} \phi_{\overline{\text{co}}}$ are closed arc-wise connected absolute retracts in the space $L^1(\mathcal{T}, X)$. Since $\operatorname{Fix} \phi_{\text{ext}} \subset \operatorname{Fix} \phi_{\overline{\text{co}}} \subset S_R^0$, the statements of the theorem now follow from Lemma 8.1. □

## 9. Proofs of Theorems 3.2 and 3.3

According to Theorem 3.3, for any $\{(v(u), w(u)), u\} \in \mathcal{R}(v_0, w_0)$, $u \in S^0_R$ inequality (3.14) takes place. Therefore,

$$(v(u)(t), w(u)(t)) \in S^0_C \times S^0_C, \quad t \in \mathcal{T},\ u \in S^0_R, \tag{9.1}$$

where

$$S^0_C = \{z \in L^0(\Omega);\ |z(x)| \le C(R, r) \quad \text{a.e. on } \Omega\},$$

and $C(R, r)$ is the constant from inequality (3.14). From inequalities (8.5), (8.6) it follows that the operator $\mathcal{F}$ is continuous from $S^1_R$ to $C(\mathcal{T}, X^2)$. Similarly to Lemma 8.1, we can prove that the topologies of the spaces $L^1(\Omega)$ and $L^2(\Omega)$ coincide on the set $S^0_C$. Then, from (9.1) and Lemma 8.1 we infer that $\mathcal{F}$ is continuous from $S_R$ to $C(\mathcal{T}, H) \times C(\mathcal{T}, H)$, where $S_R$ is the set (3.16). From (8.8) we see that the operators $\mathcal{F}$ and $\bar{\mathcal{F}}$ coincide on the set $S_R$. Since $\operatorname{Fix}\phi_{\mathrm{ext}}, \operatorname{Fix}\phi_{\overline{\mathrm{co}}} \subset S^0_R$, from the definition of solutions of the control system (1.1)–(1.4) with the control constraints (1.6) and (1.7), (3.7)–(3.9), (3.11),(3.12) and (8.5), (8.8), (8.10), (8.11) it follows that $\{(v(u), w(u)), u\} \in \mathcal{R}_{\mathrm{ext}\,\overline{\mathrm{co}}\,U}(v_0, w_0)$ if and only if

$$(v(u), w(u)) = \mathcal{F}(u), \quad u \in \operatorname{Fix}\phi_{\mathrm{ext}}. \tag{9.2}$$

Similarly, $\{(v(u), w(u)), u\} \in \mathcal{R}_{\overline{\mathrm{co}}\,U}(v_0, w_0)$ if and only if

$$(v(u), w(u)) = \mathcal{F}(u), \quad u \in \operatorname{Fix}\phi_{\overline{\mathrm{co}}}. \tag{9.3}$$

Now, the nonemptiness of the set $\mathcal{R}_{\mathrm{ext}\,\overline{\mathrm{co}}\,U}$ follows from the statement (i) of Theorem 8.2 and equality (9.2), the nonemptiness of the sets $\mathcal{R}_U$, $\mathcal{R}_{\overline{\mathrm{co}}\,U}$ follows from the inclusions (3.13). Since

$$\mathcal{R}_{\mathrm{ext}\,\overline{\mathrm{co}}\,U}(v_0, w_0) \subset \mathcal{R}_U(v_0, w_0) \subset \mathcal{R}_{\overline{\mathrm{co}}\,U}(v_0, w_0) \subset \mathcal{R}(v_0, w_0),$$

the inequality (3.14) follows from Theorem 3.3. Theorem 3.2 is thus proved.

Theorem 3.2 follows from the statement (ii) of Theorem 8.2, (9.2), (9.3) and the continuity of the mapping $\mathcal{F}$ from $S_R$ to $C(\mathcal{T}, H)$.

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Received: 4 November 2013.
Accepted: 12 October 2014.