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Existence results for parametric boundary value problems involving the mean curvature operator

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Abstract. In this note we propose a variational approach to a parametric differential problem where a prescribed mean curvature equation is considered. In particular, without asymptotic assumptions at zero and at infinity on the potential, we obtain an explicit positive interval of parameters for which the problem under examination has at least one nontrivial and nonnegative solution.

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1. Introduction

In this paper we look for solutions of the following Dirichlet problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda\gamma(t)f(u) \quad \text{in }]0,1[, \quad u(0)=u(1)=0, \tag{1.1}$$

where $\gamma : [0,1] \to \mathbb{R}$ is a continuous positive function, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and λ is a positive parameter. Without loss of generality, we can assume that

$$\int_0^1 \gamma(t)dt = 1. \tag{1.2}$$

Clearly, problem (1.1) represents the one-dimensional version of the elliptic differential problem

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda\gamma(x)f(u) \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega, \tag{1.3}$$

and, in these latest years, both (1.1) and (1.3), or even more general problems, have been widely investigated in order to assure the existence as well as the multiplicity of solutions (for example see [1,7,8,12,16–21]). In particular, in [8], problem (1.1) has been treated in a very deep way obtaining different results when the potential function is required to satisfy some asymptotic assumptions at zero or at infinity, as well as both at zero and at infinity. Subsequently, in [19], this analysis has been extended to the case of higher dimensions, namely problem (1.3). For the existence and the multiplicity of solutions when the prescribed curvature is sublinear we refer to [10,11,17,20].

Here, inspired by a technical device adopted in [19] and exploited also in [15], we propose a variational approach to guarantee the existence of an explicit interval Λ of positive parameters such that, for every $\lambda \in \Lambda$, problem (1.1) admits at least one non trivial and nonnegative solution u_λ. More precisely, we first consider a modified problem on the basis of a suitable truncation of the functional related to the mean curvature operator, and then we obtain particular solutions of this new problem that, indeed, are solutions of problem (1.1) too. We emphasize that the truncation mentioned above is different from that considered in [19], in view of the fact that, thanks to the technique that we use, we can exploit some a priori information on the values of u_λ as well as of u'_λ.

It worth noticing that our solution u_λ is found by requiring only a suitable algebraic inequality (see condition (3.18) of Corollary 3.1) and avoiding any kind of asymptotic condition at zero and at infinity.

Our main result is Theorem 3.1 and we explicitly observe that it contains the case when $\gamma \equiv 1$ and $F(s) = \int_0^s f(t)dt$, $s \in \mathbb{R}$, is sub-quadratic at zero (see [8, Thorem 3.7] as well as [8, Thorem 3.2]). Indeed, as a consequence of Theorem 3.1 we can state the following special case for the autonomous problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda f(u) \quad \text{in }]0,1[, \quad u(0) = u(1) = 0. \tag{1.4}$$

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, nonnegative function and $F(s) = \int_0^s f(t)dt$ for every $s \in \mathbb{R}$. Assume that*

$$\limsup_{s \to 0^+} \frac{F(s)}{s^2} = +\infty. \tag{1.5}$$

Then, for each $\lambda \in]0, \lambda^*[$, where

$$\lambda^* = 2\left\{[F(1)]^2 + [2\max_{[0,1]} f]^2\right\}^{-1/2},$$

problem (1.4) admits at least one positive solution $u_\lambda \in C^2([0,1])$, such that

$$\|u_\lambda\|_{C^0} < 1, \quad \|u'_\lambda\|_{C^0} \leq \frac{\max_{[0,1]} f}{F(1)}.$$

Notice that, in comparison with the results contained in [8], where λ^* is only stated to exist, we have here an explicit expression of it (see Remark 3.4).

Theorem 3.1 is also accompanied by a concrete example (see Example 3.1) with a simple case of possible nonlinearity, not covered by the results contained in [8,19], and for which our result applies (see Remark 3.2).

The main tool in studying problem (1.1) is a local minimum theorem established in [2] (see [2, Theorem 3.1]) that we recall here in a convenient form given in [6] (see Theorem 2.1). Such a local minimum result has been successfully exploited also in treating several other types of differential problems (see [3–5, 13, 14]).

2. Preliminaries and variational settings

Here and in the sequel X denotes the usual Sobolev space $W_0^{1,2}([0,1])$ endowed with the norm

$$\|u\| := \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2}, \tag{2.1}$$

for every $u \in X$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We will assume that

$$f(0) \geq 0. \tag{2.2}$$

Put

$$g(t) = \begin{cases} f(t) & \text{if } t \geq 0 \\ f(0) & \text{if } t < 0, \end{cases}$$

$$F(s) = \int_0^s f(t)\,dt, \qquad G(s) = \int_0^s g(t)\,dt \quad \forall s \in \mathbb{R},$$

$$\Psi(u) = \int_0^1 \gamma(t)G(u(t))\,dt \quad \forall u \in X. \tag{2.3}$$

Since $X \hookrightarrow\hookrightarrow C^0([0,1])$ compactly, with

$$\|u\|_{C^0} \leq \frac{1}{2}\|u\| \quad \forall u \in X, \tag{2.4}$$

it is easy to verify that Ψ is continuously Gâteaux differentiable, Ψ' is a compact operator and, in particular,

$$\Psi'(u)(v) = \int_0^1 \gamma(t)g(u(t))v(t)\,dt \quad \forall u, v \in X. \tag{2.5}$$

Moreover Ψ is sequentially weakly semicontinuous.

If $\bar{z} > 0$ and $\varepsilon > 0$ are fixed, we define the function $\alpha : [0,+\infty[\to]0,+\infty[$ by

$$\alpha(z) = \alpha_{\bar{z},\varepsilon}(z) = \begin{cases} \frac{1}{\sqrt{1+z}} & \text{if } 0 \leq z < \bar{z} \\ k_1\left(z - (1+\varepsilon)\bar{z}\right)^2 + k_2 & \text{if } \bar{z} \leq z < (1+\varepsilon)\bar{z} \\ k_2 & \text{if } z \geq (1+\varepsilon)\bar{z}, \end{cases}$$

where

$$k_1 = \frac{1}{4\varepsilon\bar{z}(1+\bar{z})\sqrt{1+\bar{z}}}, \quad k_2 = \frac{4 + (4-\varepsilon)\bar{z}}{4(1+\bar{z})\sqrt{1+\bar{z}}}.$$

It is simple to show that $\alpha \in C^{1,1}([0,+\infty[)$ is a non increasing function such that

$$k_2 \leq \alpha(z) \leq 1 \quad \forall z \in [0,+\infty[. \tag{2.6}$$

Let $a : \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to [0, +\infty[$ be two functions defined by

$$a(z) = \alpha(z^2)z, \qquad A(z) = \int_0^{|z|} \alpha(s)\, ds \quad \forall z \in \mathbb{R}.$$

Clearly, in view of (2.6),

$$k_2 z^2 \le A(z^2) \le z^2 \quad \forall z \in \mathbb{R}, \tag{2.7}$$

and a simple computation shows that the function $z \mapsto A(z^2)$ is convex.

Let $\Phi : X \to \mathbb{R}$ be the functional defined by

$$\Phi(u) = \frac{1}{2} \int_0^1 A(|u'(t)|^2)\, dt \tag{2.8}$$

for every $u \in X$. It results that Φ is well-defined, convex and continuously Gâteaux differentiable, with

$$\Phi'(u)(v) = \int_0^1 a(|u'(t)|)v'(t)\, dt = \int_0^1 \alpha(|u'(t)|^2)u'(t)v'(t)\, dt \tag{2.9}$$

for each $u, v \in X$. Moreover, in view of (2.7) one has

$$\frac{k_2}{2}\|u\|^2 \le \Phi(u) \le \frac{1}{2}\|u\|^2 \tag{2.10}$$

for each $u \in X$. This means that, taking in mind (2.4) and that $\frac{4+(4-\varepsilon)\bar{z}}{4(1+\bar{z})} > \frac{4-\varepsilon}{4}$ for each $\varepsilon > 0$, we obtain, for every $r > 0$ and for every $u \in X \setminus \{0\}$ such that $\Phi(u) \le r$

$$\begin{aligned} \|u\|_{C^0}^2 &\le \frac{1}{4}\|u\|^2 = \frac{1}{4}\frac{4}{4-\varepsilon}\frac{4-\varepsilon}{4}\frac{1}{\sqrt{1+\bar{z}}}\|u\|^2\sqrt{1+\bar{z}} \\ &< \frac{1}{4-\varepsilon}k_2\|u\|^2\sqrt{1+\bar{z}} \le \frac{2}{4-\varepsilon}\Phi(u)\sqrt{1+\bar{z}} \\ &\le \frac{2}{4-\varepsilon}r\sqrt{1+\bar{z}}, \end{aligned}$$

for every $\varepsilon \in]0, 4[$. In other words,

$$\begin{aligned} \Phi^{-1}(]-\infty, r]) &= \{u \in X : \ \Phi(u) \le r\} \\ &\subseteq \left\{ u \in X : \ \|u\|_{C^0} < \sqrt{\frac{2}{4-\varepsilon}r\sqrt{1+\bar{z}}} \right\}, \end{aligned} \tag{2.11}$$

for every $\varepsilon \in]0, 4[$. Fix $\lambda > 0$ and recall that a solution to problem (1.1) is any function $u \in X$ such that

$$\int_0^1 \frac{u'(t)}{\sqrt{1+|u'(t)|^2}}v'(t)\, dt = \lambda \int_0^1 \gamma(t)f(u(t))v(t)\, dt$$

for every $v \in X$. We find the solutions to problem (1.1) considering the following modified problem

$$-\left(\alpha(|u'|^2)u'\right)' = \lambda\gamma(t)g(u) \quad \text{in }]0,1[, \quad u(0) = u(1) = 0. \tag{2.12}$$

In particular, we first point out the following fact.

Lemma 2.1. *Each solution to (2.12) is nonnegative.*

Proof. Indeed, if u solves (2.12) one has

$$\int_0^1 \alpha(|u'(t)|^2)u'(t)v'(t)\,dt \qquad = \lambda \int_0^1 \gamma(t)g(u(t))v(t)\,dt \tag{2.13}$$

for every $v \in X$. Testing (2.13) with $v = u^- = \max\{0, -u\}$ one has

$$\begin{aligned}\int_0^1 \alpha(|u'(t)|^2)u'(t)(u^-(t))'\,dt &= \lambda \int_0^1 \gamma(t)g(u(t))u^-(t)\,dt \\ = \lambda \int_{\{u<0\}} \gamma(t)g(u(t))u(t)\,dt &= -\lambda f(0) \int_{\{u<0\}} \gamma(t)u(t)\,dt. \end{aligned}\tag{2.14}$$

On the other hand

$$\begin{aligned}\int_0^1 \alpha(|u'(t)|^2)u'(t)(u^-(t))'\,dt &= \int_{\{u\geq 0\}} \alpha(|u'(t)|^2)u'(t)(u^-(t))'\,dt \\ &\quad + \int_{\{u<0\}} \alpha(|u'(t)|^2)u'(t)(u^-(t))'\,dt \\ &= -\int_{\{u<0\}} \alpha(|(u^-(t))'|^2)(u^-(t))'(u^-(t))'\,dt \\ &= -\int_0^1 \alpha(|(u^-(t))'|^2)(u^-(t))'(u^-(t))'\,dt \end{aligned}\tag{2.15}$$

Putting together (2.6), (2.15), (2.14) and (2.2) we conclude that

$$k_2\|u^-\|^2 \leq \int_0^1 \alpha(|(u^-(t))'|^2)(u^-(t))'(u^-(t))'\,dt = \lambda f(0) \int_{\{u<0\}} \gamma(t)u(t)\,dt \leq 0,$$

that is u^- is a.e. zero in $[0, 1]$ and (2.1) holds. □

At this point, according to the definitions of α, g, Φ and Ψ it is clear that any solution u to problem (2.12), namely every critical point of the functional $I_\lambda = \Phi - \lambda\Psi$, that, in addition, satisfies the condition

$$\|u'\|_{C^0} \leq \sqrt{\bar{z}} \tag{2.16}$$

is a solution to problem (1.1).

Remark **2.1.** We explicitly observe that the described technique furnishes regular solutions. Indeed, if u is a solution of (2.12) satisfying (2.16), because the function $s \mapsto \alpha(s^2)\cdot s$ is invertible and the right-hand side of (2.12) is assumed to be continuous, the function $v(t) = \varphi(u'(t))$ belongs to $C^1([0, 1])$ (see [9, Theorem VIII.2]) as well as $u' = \varphi^{-1}(v)$, namely $u \in C^2([0, 1])$. Moreover, whenever the right-hand side is nonnegative, it is simple to verify that the solution is concave and therefore, if nontrivial it must be positive.

We conclude this section recalling our main tool (see [6] and [2]) for proving the existence of a solution to problem (1.1).

Theorem 2.1. [Theorem 2.1 of [6]] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function, $\Psi : X \to \mathbb{R}$ be a sequentially weakly* *upper semicontinuous and continuously Gâteaux differentiable functional. Put* $I_\lambda = \Phi - \lambda\Psi$ *and assume that there are* $r_1, r_2 \in \mathbb{R}$, *with* $r_1 < r_2$, *such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2), \tag{2.17}$$

where

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1}.$$

Then, for each $\lambda \in \left] \frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right[$ *there is* $u_\lambda \in \Phi^{-1}(]r_1, r_2[)$ *such that* $I_\lambda(u_\lambda) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r_1, r_2[)$ *and* $I'_\lambda(u_\lambda) = 0$.

Remark **2.2.** As we pointed out previously, the introduction of a truncation of the mean curvature operator function through our α has been already done in [19]. Here, the function α is considered in a different way with respect to that of [19], in view of the fact that we apply Theorem 2.1 and obtain particular solutions u of the modified problem (2.12), in the sense that a priori information on the values of u and u' are available.

3. Main results

Here is our main result.

Theorem 3.1. *Assume that* (2.2) *hold and put*

$$\widetilde{\gamma} := \int_{1/4}^{3/4} \gamma(t)\, dt. \tag{3.1}$$

Assume furthermore that there exist two positive constants c, d *with* $d < c$, *such that*

$$\int_0^d F(s)\gamma\left(\frac{s}{4d}\right) ds \geq 0, \quad \int_0^d F(s)\gamma\left(1 - \frac{s}{4d}\right) ds \geq 0 \tag{3.2}$$

and

$$\left[\left(\frac{\max_{[0,c]} F}{c^2}\right)^2 + \left(2 \max_{[0,c]} |f|\right)^2\right]^{1/2} < \frac{\widetilde{\gamma}}{2} \frac{F(d)}{d^2}. \tag{3.3}$$

Then, for every

$$\lambda \in \left] \frac{4d^2}{\widetilde{\gamma} F(d)}, 2\left[\left(\frac{\max_{[0,c]} F}{c^2}\right)^2 + \left(2 \max_{[0,c]} |f|\right)^2\right]^{-1/2} \right[,$$

problem (1.1) *admits at least one nontrivial, nonnegative solution* $u_\lambda \in C^2([0,1])$, *such that*

$$\|u_\lambda\|_{C^0} < c, \quad \|u'_\lambda\|_{C^0} < \frac{2 \max_{[0,c]} |f|}{\max_{[0,c]} F} c^2.$$

Proof. Notice first that $\max_{[0,c]} F \geq F(0) = 0$. Put

$$L = L(c) = \frac{\max_{[0,c]} F}{c^2}, \quad M = M(c) = \max_{[0,c]} |f|, \tag{3.4}$$

$$\bar{z} = \bar{z}(c) = 4\frac{M^2}{L^2}. \tag{3.5}$$

Clearly, one has

$$\sqrt{1+\bar{z}} = \left[\left(\frac{\max_{[0,c]} F}{c^2}\right)^2 + \left(2\max_{[0,c]} |f|\right)^2\right]^{1/2} \frac{c^2}{\max_{[0,c]} F},$$

hence

$$2\left[\left(\frac{\max_{[0,c]} F}{c^2}\right)^2 + \left(2\max_{[0,c]} |f|\right)^2\right]^{-1/2} = \frac{1}{\sqrt{1+\bar{z}}}\frac{2c^2}{\max_{[0,c]} F},$$

and Assumption (3.3) implies the following inequality

$$\frac{F(d)}{2} > \frac{1}{\tilde{\gamma}}\sqrt{1+\bar{z}}\max_{[0,c]} F\frac{d^2}{c^2}. \tag{3.6}$$

Preliminarily we can also observe that

$$d < \frac{1}{\sqrt{2}\sqrt[4]{1+\bar{z}}}c. \tag{3.7}$$

Indeed, if (3.7) was false then, exploiting (3.3) and recalling that $d < c$, one should obtain

$$\frac{1}{2\sqrt{1+\bar{z}}}\frac{F(d)}{d^2} \leq \frac{\max_{[0,c]} F}{c^2} < \frac{\tilde{\gamma}}{2\sqrt{1+\bar{z}}}\frac{F(d)}{d^2} \leq \frac{1}{2\sqrt{1+\bar{z}}}\frac{F(d)}{d^2},$$

a contradiction. From (3.7) and (3.6), there exists $\bar{\varepsilon} \in]0,4[$ such that, for every $\varepsilon \in]0,\bar{\varepsilon}[$

$$4d^2 < \frac{c^2(4-\varepsilon)}{2\sqrt{1+\bar{z}}} \tag{3.8}$$

and

$$\frac{1}{\tilde{\gamma}}\sqrt{1+\bar{z}}\max_{[0,c]} F\frac{d^2}{c^2}\frac{4}{4-\varepsilon} < \frac{F(d)}{2}. \tag{3.9}$$

Now, fix $\lambda \in \left]\frac{4d^2}{\tilde{\gamma}F(d)}, \frac{2c^2}{\sqrt{1+\bar{z}}\max_{[0,c]} F}\right[$ and pick $\varepsilon \in]0,\bar{\varepsilon}[$ such that

$$\lambda\frac{4}{4-\varepsilon} < \frac{2c^2}{\sqrt{1+\bar{z}}\max_{[0,c]} F}. \tag{3.10}$$

Our aim is to apply Theorem 2.1 with $X = W_0^{1,2}([0,1])$ and Φ, Ψ as in (2.8) and (2.3) respectively.

Let $u_0 \in X$ be defined by

$$u_0(t) = \begin{cases} 4dt & \text{if } 0 \leq t \leq \frac{1}{4} \\ d & \text{if } \frac{1}{4} < t \leq \frac{3}{4} \\ 4d(1-t) & \text{if } \frac{3}{4} < t \leq 1, \end{cases}$$

and observe that

$$\|u_0\|^2 = 8d^2.$$

Put

$$r_1 = 0, \qquad r_2 = \frac{c^2(4-\varepsilon)}{2\sqrt{1+\bar{z}}}.$$

Clearly $r_1 < r_2$ and, in view of (2.7) and (3.8),

$$r_1 = 0 < \Phi(u_0) \le \frac{\|u_0\|^2}{2} = 4d^2 < \frac{c^2(4-\varepsilon)}{2\sqrt{1+\bar{z}}} = r_2. \tag{3.11}$$

Moreover $\Phi^{-1}(]-\infty, 0]) = \{0\}$ and $\Psi(0) = 0$. Since

$$\int_0^{1/4} \gamma(t)G(u_0(t))\,dt = \int_0^{1/4} \gamma(t)F(u_0(t))\,dt = \frac{1}{4d}\int_0^d F(s)\gamma\left(\frac{s}{4d}\right)\,ds,$$

$$\int_{3/4}^{1} \gamma(t)G(u_0(t))\,dt = \int_{3/4}^{1} \gamma(t)F(u_0(t))\,dt = \frac{1}{4d}\int_0^d F(s)\gamma\left(1-\frac{s}{4d}\right)\,ds,$$

taking in mind Assumption (3.2) one has that

$$\Psi(u_0) = \int_0^1 \gamma(t)G(u_0(t))dt \ge \int_{1/4}^{3/4} \gamma(t)G(u_0(t))dt = \widetilde{\gamma}F(d). \tag{3.12}$$

Exploiting (3.12) and (3.11) we can observe that

$$\rho(r_1, r_2) \ge \frac{\Psi(u_0)}{\Phi(u_0)} \ge \frac{\widetilde{\gamma}F(d)}{\Phi(u_0)} \ge \frac{\widetilde{\gamma}F(d)}{4d^2}. \tag{3.13}$$

It is simple to verify that

$$\max_{[-c,c]} G = \max_{[0,c]} F. \tag{3.14}$$

Indeed,

$$\max_{[-c,c]} G = \max\left\{\max_{[0,c]} G, \max_{[-c,0]} G\right\},$$

and

$$\max_{[0,c]} G = \max_{[0,c]} F \ge F(0) = 0,$$

as well as, in view of (2.2),

$$\max_{[-c,0]} G = \max_{s\in[-c,0]} \int_0^s g(t)\,dt = \max_{s\in[-c,0]} f(0)s = 0,$$

and (3.14) holds.

Hence, thanks to (3.11), (2.11), (3.9), (3.14), (3.6), (3.10) and the choice of λ, one has

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)-\Psi(u_0)}{r_2-\Phi(u_0)} \le \frac{\max_{|s|\le c}G(s)-\widetilde{\gamma}F(d)}{r_2-\Phi(u_0)}\\ &= \frac{\max_{[0,c]}F-\widetilde{\gamma}F(d)}{r_2-\Phi(u_0)} \le \frac{\max_{[0,c]}F-\widetilde{\gamma}F(d)}{\frac{\widetilde{\gamma}c^2(4-\varepsilon)}{\sqrt{1+\overline{z}}}-4d^2}\\ &< \frac{\max_{[0,c]}F-\frac{8}{4-\varepsilon}\sqrt{1+\overline{z}}\frac{d^2}{c^2}\max_{[0,c]}F}{\frac{c^2(4-\varepsilon)}{2\sqrt{1+\overline{z}}}\left(1-\frac{8}{4-\varepsilon}\sqrt{1+\overline{z}}\frac{d^2}{c^2}\right)} \qquad (3.15)\\ &= \frac{\max_{[0,c]}F\left(1-\frac{8}{4-\varepsilon}\sqrt{1+\overline{z}}\frac{d^2}{c^2}\right)}{\frac{c^2(4-\varepsilon)}{2\sqrt{1+\overline{z}}}\left(1-\frac{8}{4-\varepsilon}\sqrt{1+\overline{z}}\frac{d^2}{c^2}\right)}\\ &= \frac{2}{4-\varepsilon}\sqrt{1+\overline{z}}\frac{\max_{[0,c]}F}{c^2}\\ &< \frac{1}{\lambda} < \frac{\widetilde{\gamma}F(d)}{4d^2}.\end{aligned}$$

Putting together (3.13) and (3.15) one has

$$\beta(r_1,r_2) < \frac{1}{\lambda} < \rho(r_1,r_2)$$

and, from Theorem 2.1 there exists $u_\lambda \in X$ such that $\Phi'(u_\lambda)-\lambda\Psi'(u_\lambda)=0$, with, in particular,

$$\begin{aligned}\Phi(u_\lambda)-\lambda\Psi(u_\lambda) &= \inf_{v\in\Phi^{-1}(]r_1,r_2[)}(\Phi(v)-\lambda\Psi(v)) \le \Phi(u_0)-\lambda\Psi(u_0) \qquad (3.16)\\ &\le 4d^2-\lambda\widetilde{\gamma}F(d)<0,\end{aligned}$$

and

$$0<\Phi(u_\lambda)<\frac{c^2(4-\varepsilon)}{2\sqrt{1+\overline{z}}},$$

namely $u_\lambda \neq 0$ and, in view of (2.11),

$$\|u_\lambda\|^2_{C^0} < \frac{2}{4-\varepsilon}r_2\sqrt{1+\overline{z}} = \frac{2}{4-\varepsilon}\frac{c^2(4-\varepsilon)}{2\sqrt{1+\overline{z}}}\sqrt{1+\overline{z}} = c^2.$$

Taking in mind the considerations made in Sect. 2, see in particular Lemma 2.1 and Remark 2.1, we can claim that $u_\lambda \in C^2([0,1])$ and it is a (nontrivial) nonnegative solution to problem (2.12) provided that

$$|u'_\lambda(t)| \le \sqrt{\overline{z}} \qquad (3.17)$$

for every $t\in[0,1]$. To this end, we already pointed out in Remark 2.1 that the function $v(\cdot)=\varphi(u'(\cdot))=\alpha(|u'(\cdot)|^2)\cdot u'(\cdot)$ is continuously differentiable. Hence, $u'=\varphi^{-1}(v)\in C^0([0,1])$, namely $u_\lambda\in C^1([0,1])$ and $u_\lambda(0)=u_\lambda(1)=0$. Thus, there exists $t_0 \in]0,1[$ such that $u'_\lambda(t_0)=0$, and, in view of (3.10), for every

$t \in [0,1]$ one has

$$\begin{aligned}\frac{1}{\sqrt{1+\bar{z}}}|u'_\lambda(t)| \leq & \frac{4}{4-\varepsilon}k_2|u'_\lambda(t)| \leq \frac{4}{4-\varepsilon}\alpha(|u'_\lambda(t)|^2)|u'_\lambda(t)| \\ = & \frac{4}{4-\varepsilon}\lambda\left|\int_{t_0}^t \gamma(t)f(u_\lambda(\sigma))\,d\sigma\right| \leq \frac{4}{4-\varepsilon}\lambda\max_{[0,c]}|f| \\ < & \frac{2c^2}{\sqrt{1+\bar{z}}\max_{[0,c]}F}\max_{[0,c]}|f|.\end{aligned}$$

That is, for every $t \in [0,1]$

$$|u'_\lambda(t)| \leq 2\frac{M}{L} = \sqrt{\bar{z}},$$

and the proof is complete. □

Remark **3.1.** We explicitly observe that in order to study problem (1.1) phase plane or time-maps methods can not be used.

We point out the following immediate consequence of Theorem 3.1. Observe that, when $\gamma \equiv 1$ then $\widetilde{\gamma} = \frac{1}{2}$.

Corollary 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nonnegative function and assume that there exist two positive constants c, d, with $d < c$, such that*

$$\left[\left(\frac{F(c)}{c^2}\right)^2 + (2\max_{[0,c]} f)^2\right]^{1/2} < \frac{1}{4}\frac{F(d)}{d^2}. \tag{3.18}$$

Then, for every

$$\lambda \in \left]\frac{8d^2}{F(d)}, 2\left[\left(\frac{F(c)}{c^2}\right)^2 + (2\max_{[0,c]} f)^2\right]^{-1/2}\right[$$

problem (1.4) *admits at least one positive solution $u_\lambda \in C^2([0,1])$, such that*

$$\|u_\lambda\|_{C^0} < c, \quad \|u'_\lambda\|_{C^0} \leq \frac{2\max_{[0,c]} f}{F(c)}c^2.$$

Here we show an example of possible nonlinearity satisfying all the assumptions of Theorem 3.1.

Example **3.1.** Let $m > 0$ and put

$$f_m(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ mt & \text{if } 0 < t \leq \frac{1}{2^5} \\ m\left(\frac{1}{2^4} - t\right) & \text{if } \frac{1}{2^5} < t \leq \frac{3}{2^5} \\ -\frac{m}{2^5} & \text{if } \frac{3}{2^5} < t \leq 1 \\ g(t) & \text{if } t > 1 \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is any function. Then, for every $\lambda \in \left]\frac{2^4}{m}, \frac{2^6}{\sqrt{5}m}\right[$ the problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda f(u) \quad \text{in }]0,1[, \quad u(0) = u(1) = 0, \tag{P_λ}$$

admits at least one (nontrivial) nonnegative solution $u_\lambda \in C^{1,\tau}([0,1])$ for some $\tau \in]0,1]$, such that $\|u_\lambda\|_{C^0} < 1$ and $\|u'_\lambda\|_{C^1} < 2$. In fact, if we consider the following function

$$\bar{f}_m(t) = \begin{cases} f_m(t) & \text{if } t \le 1 \\ -\frac{m}{2^5} & \text{if } t > 1 \end{cases}$$

we can apply Theorem 3.1 to $\bar{f}_m$ in the case $c = 1$, $d = \frac{1}{2^5}$. Hence,

$$\frac{\max_{[0,c]} \bar{F}_m}{c^2} = \frac{m}{2^{10}}, \quad \max_{[0,c]} 2|\bar{f}_m| = \frac{m}{2^4}, \quad \bar{F}_m(d) = \frac{m}{2^{11}}$$

and it is easy to verify that all the assumptions of Theorem 3.1 hold. In particular,

$$\left[\left(\frac{\max_{[0,c]} \bar{F}_m}{c^2}\right)^2 + \left(\max_{[0,c]} 2|\bar{f}_m|\right)^2\right]^{1/2} = \left[\frac{m^2}{2^{20}} + \frac{m^2}{2^8}\right]^{1/2} = \frac{m}{2^4}\left(\frac{1}{2^{12}} + 1\right)^{1/2}$$
$$< \frac{m}{8} = \frac{1}{4}\frac{m}{2^{11}}2^{10} = \frac{1}{4}\frac{F(d)}{d^2},$$

and (3.3) is satisfied. At this point we conclude that for every $\lambda \in \left]\frac{2^4}{m}, \frac{2^6}{\sqrt{5}m}\right[$ the following problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda \bar{f}_m(u) \quad \text{in }]0,1[, \quad u(0) = u(1) = 0,$$

admits at least one nontrivial and nonnegative solution u_λ such that $\|u_\lambda\|_{C^0} < 1$ and $\|u'_\lambda\|_{C^0} < 2$. The conclusion is achieved once observed that $\bar{f}_m \equiv f_m$ in the interval $[0,1]$.
In particular, if $\tilde{f} = f_{20}$, the problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \tilde{f}(u) \quad \text{in }]0,1[, \quad u(0) = u(1) = 0, \qquad (\tilde{P})$$

has at least one nonnegative and nontrivial solution.

Remark **3.2.** The previous example shows that our result is independent from those contained in [8,19]. Indeed, we can observe that Theorem 3.1 of [19] cannot be applied to problem (P_λ) since the potential of the nonlinearity has a quadratic behaviour at zero. Moreover, in Theorem 3.6 of [19], among other ones, the case of quadratic potential at zero is investigated too, but, in order to assure the existence of at least one positive solution, a suitable condition at infinity on the potential is required; hence the freedom of choice of the function g considered in defining f prevent to treat problem $(\tilde{P})$. Furthermore, all the results contained in [8] consider various cases where the potential of the nonlinearity is either sub-quadratic or super-quadratic at zero, hence it is excluded a case as proposed in Example 3.1.

Finally, we wish to stress that in both [8,19] the behaviour of the function $F(u)/u^p$, for some $p \ge 1$, at zero or at infinity is at the basis of the whole analysis, while the assumptions of Theorem 3.1 do not require a particular asymptotic behaviour of $F(u)/u^p$.

As a consequence of Theorem 3.1 we can obtain the following result where the case when F is subquadratic at zero is considered.

Theorem 3.2. *Assume that there exists $\delta > 0$ such that*

$$F(s) \geq 0 \text{ for every } s \in [0, \delta], \tag{3.19}$$

and that condition (1.5) holds. Put

$$\bar{\lambda} = 2 \sup_{c>0} \left[\left(\frac{\max_{[0,c]} F}{c^2} \right)^2 + \left(2 \max_{[0,c]} |f| \right)^2 \right]^{-1/2}.$$

Then, for every $\lambda \in]0, \bar{\lambda}[$, problem (1.4) admits at least one nontrivial, nonnegative solution $u_\lambda \in C^2([0,1])$. Moreover, $\|u_\lambda\| \to 0$ as $\lambda \to 0$.

Proof. First of all observe that from (3.19) and the continuity of f one has that (2.2) holds. Fix $\lambda \in]0, \bar{\lambda}[$ and pick $c > 0$ such that $0 < \lambda < \lambda^*(c)$, with

$$\lambda^*(c) = 2 \left[\left(\frac{\max_{[0,c]} F}{c^2} \right)^2 + \left(2 \max_{[0,c]} |f| \right)^2 \right]^{-1/2}.$$

From (3.19) and (1.5), corresponding to the positive number $8/\lambda$ there exists $d \in]0, c[$ such that

$$4 \left[\left(\frac{\max_{[0,c]} F}{c^2} \right)^2 + \left(2 \max_{[0,c]} |f| \right)^2 \right]^{1/2} < \frac{8}{\lambda} < \frac{F(d)}{d^2}, \tag{3.20}$$

and

$$F(s) \geq 0, \tag{3.21}$$

for every $s \in [0, d]$. Hence, in particular, Assumptions (3.2) and (3.3) are verified as well as

$$\lambda \in \left] \frac{8d^2}{F(d)}, 2 \left[\left(\frac{\max_{[0,c]} F}{c^2} \right)^2 + \left(2 \max_{[0,c]} |f| \right)^2 \right]^{-1/2} \right[.$$

By applying Theorem 3.1 we obtain the existence of at least a nontrivial, nonnegative positive solution $u_\lambda \in C^{1,\tau}([0,1])$ for some $\tau \in]0, 1]$, such that

$$\|u_\lambda\|_{C^0} < c \quad \text{and} \quad \|u'_\lambda\|_{C^0} < \frac{2 \max_{[0,c]} |f|}{\max_{[0,c]} F} c^2,$$

namely the first part of the theorem is proved.

Fix now $\mu \in]0, \bar{\lambda}[$. Reasoning as above, there exists $\bar{c} = \bar{c}(\mu) > 0$ such that $0 < \mu < \lambda^*(\bar{c})$ and, for every $\lambda \in]0, \mu] \subset]0, \lambda^*(\bar{c})[$ the solution u_λ, whose existence has been already shown, satisfies, in particular, the following additional conditions

$$\|u_\lambda\|_{C^0} < \bar{c}, \quad \|u'_\lambda\|_{C^0} \leq \frac{2 \max_{[0,\bar{c}]} |f|}{\max_{[0,\bar{c}]} F} \bar{c}^2 = \bar{d}.$$

Hence, for every $\lambda \in]0, \mu[$ one has

$$\frac{1}{\sqrt{1+\bar{d}^2}} \int_0^1 |u_\lambda'(t)|^2\, dt \le \int_0^1 \frac{|u_\lambda'(t)|^2}{\sqrt{1+|u'(t)|^2}}\, dt$$
$$= \lambda \int_0^1 f(u_\lambda(t))u_\lambda(t)\, dt \le \lambda \bar{c} \max_{[0,\bar{c}]} f,$$

from which one immediately obtains that $\|u_\lambda\| \to 0$ as $\lambda \to 0$. $\square$

Example **3.2.** The problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \sqrt{u} + u^2 \quad \text{in }]0,1[, \quad u(0)=u(1)=0,$$

admits at least one positive solution. Indeed, we can apply Theorem 3.2 (recall also Remark 2.1) to the function

$$f(t) = \sqrt{|t|} + t^2$$

for every $t \in \mathbb{R}$. In this case, it is very simple to verify that $F(s) = \frac{2}{3}s\sqrt{|s|} + \frac{s^3}{3}$ for every $s \in \mathbb{R}$. Hence, F is an increasing function satisfying (3.19) and (1.5). Moreover, a simple computation shows that $\bar{\lambda} > 1$ and the conclusion is achieved.

Remark **3.3.** We explicitly wish to point out that the interval $]0, \bar{\lambda}[$ obtained in Theorem 3.2 may be not the best possible. Indeed, if we consider the function $f(t) = 1$ for every $t \in \mathbb{R}$, it is easy to verify that all the assumptions of Theorem 3.2 are satisfied and, in addition, $\bar{\lambda} = 1$. Hence, according to Theorem 3.2 and Remark 2.1, for every $\lambda \in]0, 1[$ the problem

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda \quad \text{in }]0,1[, \quad u(0)=u(1)=0 \tag{P}$$

admits at least one positive solution. On the other hand, Example 4 of [1] assures that (P) admits a classical solution if and only if $|\lambda| < 2$, with

$$u(t) = \frac{1}{\lambda}\left[\sqrt{1-\lambda^2\left(t-\frac{1}{2}\right)^2} - \sqrt{1-\frac{\lambda^2}{4}}\right]$$

the unique solution.

Looking at the proof of the previous theorem it is possible to point out the following

Theorem 3.3. *Assume that* (3.19) and (1.5) *hold. Then, for every* $c > 0$, *and for every*

$$\lambda \in \left]0, 2\left[\left(\frac{\max_{[0,c]} F}{c^2}\right)^2 + \left(2\max_{[0,c]}|f|\right)^2\right]^{-1/2}\right[$$

problem (1.4) *admits at least one nontrivial, nonnegative solution* $u_\lambda \in C^2([0,1])$, *such that*

$$\|u_\lambda\|_{C^0} < c, \quad \|u'_\lambda\|_{C^0} \leq \frac{2\max_{[0,c]}|f|}{\max_{[0,c]}F}c^2.$$

At this point it is very simple to produce the proof of Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1. Apply Theorem 3.3 with $c = 1$. □

Remark **3.4.** In Theorem 3.7 of [8] (see also [19, Theorem 3.1]), where the existence of a λ^* such that, for every $\lambda \in]0, \lambda^*[$, the differential problem has at least one positive solution is only stated. We emphasize that the previous Theorem 3.3, under slightly more restrictive assumptions, furnishes a numerical information about a possible choice of such a λ^*.

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