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Nonlinear Differential Equations and Applications NoDEA

Combined concave–convex effects in anisotropic elliptic equations with variable exponent

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Abstract. We establish the existence and multiplicity of solutions for a class of quasilinear elliptic equations involving the anisotropic $\vec{p}(\cdot)$-Laplace operator, on a bounded domain with smooth boundary. We work on the weighted anisotropic variable exponent Sobolev space and our main tools are Sobolev embeddings and the mountain pass theorem.

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1. Introduction

In this paper, we are interested in the existence and multiplicity of weak solutions for the nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = \lambda |u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, p_i, q, r are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $2 < q(x) < r(x)$ for any $x \in \overline{\Omega}$ and $i \in \{1, \ldots, N\}$, and $h : \overline{\Omega} \to \mathbb{R}$ is a continuous positive function satisfying the conditions

$$\int_\Omega \lambda^{\frac{r(x)}{r(x)-q(x)}} \frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx < \infty \tag{2}$$

and

$$\int_\Omega \left(\frac{\lambda}{h(x)^{\frac{q(x)-2}{r(x)-2}}} \right)^{\frac{r(x)}{r(x)-q(x)}} dx < \infty. \tag{3}$$

We first establish the non-existence of nontrivial weak solutions for problem (1) if λ is small enough and then we get the existence of at least two nontrivial weak solutions for problem (1) if λ is sufficiently large.

We start with preliminary basic notations and terminology on the theory of *Lebesgue–Sobolev spaces with variable exponent.*

Set

$$C_+(\overline{\Omega}) = \{g;\ g \in C(\overline{\Omega}),\ g(x) > 1 \quad \text{for all } x \in \overline{\Omega}\}.$$

For any $g \in C_+(\overline{\Omega})$ we define

$$g^+ = \sup_{x\in\Omega} g(x) \ \text{ and } \ g^- = \inf_{x\in\Omega} g(x).$$

For any $p \in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u;\ u \text{ is a measurable real-valued function and } \int_\Omega |u|^{p(x)} dx < \infty\right\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

which is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$ almost everywhere in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{4}$$

holds true.

An important role in handling the generalized Lebesgue spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx.$$

If $(u_n), u \in L^{p(\cdot)}(\Omega)$, then the following relations hold true:

$$|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1), \tag{5}$$

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{6}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}, \tag{7}$$

$$|u_n - u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{8}$$

For proofs and further related properties of variable exponent Lebesgue spaces we refer to [13].

By $L_h^{r(\cdot)}(\Omega)$, where $r:\overline{\Omega}\to(1,\infty)$ (with $r^+<+\infty$) and $h:\overline{\Omega}\to[0,\infty)$ are continuous functions, we denote the weighted Lebesgue space

$$L_h^{r(\cdot)}(\Omega)=\Big\{u;\ u \text{ is a measurable real-valued function} \\ \text{and } \int_\Omega h(x)|u|^{r(x)}dx<\infty\Big\},$$

equipped with the norm

$$|u|_{h,r(\cdot)}=\inf\left\{\mu>0;\ \int_\Omega h(x)\left|\frac{u(x)}{\mu}\right|^{r(x)}dx\le 1\right\}.$$

We notice that, if $h(x)\equiv 1$ on $\overline{\Omega}$, the resulting norm is even $|\cdot|_{r(\cdot)}$.

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the *variable exponent Sobolev space* defined by

$$W_0^{1,p(\cdot)}(\Omega)=\left\{u;\ u_{|\partial\Omega}=0,\ u\in L^{p(\cdot)}(\Omega)\text{ and }|\nabla u|\in L^{p(\cdot)}(\Omega)\right\},$$

endowed with the equivalent norms

$$\|u\|_{p(\cdot)}=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)}$$

and

$$\|u\|=\inf\left\{\mu>0;\ \int_\Omega\left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)}+\left|\frac{u(x)}{\mu}\right|^{p(x)}\right)dx\le 1\right\},$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ is the Luxemburg norm of $|\nabla u|$. We remember that $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Also, we note that if $s\in C_+(\overline{\Omega})$ and $s(x)<p^*(x)$ for all $x\in\overline{\Omega}$, then the embedding $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$.

We refer to [10–13] for more properties, details, extensions and further references.

Finally, we present the *anisotropic variable exponent Sobolev space* $W_0^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ is the vectorial function $\vec{p}(\cdot)=(p_1(\cdot),\dots,p_N(\cdot))$ and the components $p_i\in C_+(\overline{\Omega})$, $i\in\{1,\dots,N\}$, are logarithmic Hölder continuous, that is, there exists $M>0$ such that $|p_i(x)-p_i(y)|\le -M/\log(|x-y|)$ for any $x,y\in\Omega$ with $|x-y|\le 1/2$ and $i\in\{1,\dots,N\}$. $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\vec{p}(\cdot)}=\sum_{i=1}^N|\partial_{x_i}u|_{p_i(\cdot)}$$

and is a reflexive Banach space. This space is a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$.

Now, we introduce $\vec{P}_+,\vec{P}_-\in\mathbb{R}^N$ as

$$\vec{P}_+=(p_1^+,\dots,p_N^+),\qquad \vec{P}_-=(p_1^-,\dots,p_N^-),$$

and $P_+^+,P_-^+,P_-^-\in\mathbb{R}^+$ as

$$P_+^+=\max\{p_1^+,\dots,p_N^+\},\quad P_-^+=\max\{p_1^-,\dots,p_N^-\},\quad P_-^-=\min\{p_1^-,\dots,p_N^-\}.$$

We also always assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1,$$

and define $P_-^*,\ P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N} 1/p_i^- - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

2. Main results

In this paper we seek weak solutions for problem (1) in a subspace of $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Let E be the *weighted anisotropic variable exponent Sobolev space* defined by

$$E = \left\{ u \in W_0^{1,\vec{p}(\cdot)}(\Omega);\ \int_\Omega h(x)|u|^{r(x)}dx < \infty \right\},$$

equipped with the norm

$$\|u\|_E = \|u\|_{\vec{p}(\cdot)} + |u|_{h,r(\cdot)}.$$

By a *weak solution* for problem (1) we understand a function $u \in E$ with $u(x) = 0$ almost everywhere on $\partial\Omega$ so that

$$\int_\Omega \left\{ \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v - \lambda |u|^{q(x)-2} uv + h(x)|u|^{r(x)-2} uv \right\} dx = 0 \tag{9}$$

for all $u, v \in E$.

Let $\Phi : E \to \mathbb{R}$ be the energy functional defined by

$$\Phi(u) = \int_\Omega \left\{ \sum_{i=1}^{N} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} - \frac{\lambda}{q(x)} |u|^{q(x)} + \frac{h(x)}{r(x)} |u|^{r(x)} \right\} dx.$$

By standard arguments we have $\Phi \in C^1(E, \mathbb{R})$ and the derivative is given by

$$\langle \Phi'(u), v \rangle = \int_\Omega \left\{ \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v - \lambda |u|^{q(x)-2} uv + h(x)|u|^{r(x)-2} uv \right\} dx$$

for all $u, v \in E$. Thus, the weak solutions of problem (1) coincide with the critical points of Φ.

Theorem 2.1. *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$\max_{x \in \overline{\Omega}} q(x) < P_{-,\infty}.$$

Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*]$ problem (1) does not have a nontrivial weak solution.

Theorem 2.2. *Assume that $q \in C(\overline{\Omega})$ and $r \in C(\overline{\Omega})$ satisfy the inequalities*

$$\max_{x\in\overline{\Omega}} q(x) < P_{-,\infty}, \quad \max_{x\in\overline{\Omega}} r(x) < P_{-,\infty} \quad \text{and} \quad P_+^+ < q^-.$$

Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ problem(1) possesses at least two nontrivial weak solutions.

Remark 2.3. Similar results as those presented above were obtained in the linear case, for Laplace equations, in [1,8,18]. Also, for p-Laplace equations, for a general case these can be found in [20], and for a particular case we refer to [14] and [15].

The arguments in our proofs are inspired by the methods developed in the paper by Pucci and Rădulescu [14]. However, in the present paper there are several technical difficulties that are mainly due to the presence of several variable exponents. For instance, the standard energy estimates follow by more delicate computation as well as the standard L^p-estimates; in our case, due to variable exponents, it is often more efficient to use the $p(\cdot)$-modular instead of the standard L^p norm.

We conclude this section by mentioning the pioneering paper by Ambrosetti, Brezis and Cerami [2] on concave–convex nonlinearities in the semilinear case. This result was extended by Autuori and Pucci [4] to the quasilinear case and by Autuori and Pucci [5] and Brändle, Colorado, de Pablo, and Sánchez [6] to elliptic equations involving the fractional Laplacian.

3. Proof of Theorem 2.1

We assume by contradiction that $u \in E$ is a weak solution of problem (1). Taking $v = u$ in (9) we get

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx + \int_\Omega h(x)|u|^{r(x)} dx = \lambda \int_\Omega |u|^{q(x)} dx. \tag{10}$$

Let us recall now the Young's inequality

$$ab \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta}, \quad \forall a, b > 0$$

where $\alpha, \beta > 1$ fulfill $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Put $a = h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)}$, $b = \frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}}$, $\alpha = \frac{r(x)}{q(x)}$ and $\beta = \frac{r(x)}{r(x)-q(x)}$ so that

$$\lambda |u|^{q(x)} \le \frac{q(x)}{r(x)} h(x)|u|^{r(x)} + \frac{r(x)-q(x)}{r(x)} \cdot \frac{\lambda^{\frac{r(x)}{r(x)-q(x)}}}{h(x)^{\frac{q(x)}{r(x)-q(x)}}}.$$

Taking into account that $\frac{q(x)}{r(x)} < 1$ and $\frac{r(x)-q(x)}{r(x)} < 1$ and integrating over Ω we obtain

$$\lambda \int_\Omega |u|^{q(x)} dx < \int_\Omega h(x)|u|^{r(x)} dx + \int_\Omega \lambda^{\frac{r(x)}{r(x)-q(x)}} \frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx.$$

Considering the last inequality and (10) we get

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx < \int_\Omega \lambda^{\frac{r(x)}{r(x)-q(x)}} \frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx. \tag{11}$$

Clearly $\int_\Omega h(x)|u|^{r(x)} dx \geq 0$, which together with (10) yield that

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx \leq \lambda \int_\Omega |u|^{q(x)} dx. \tag{12}$$

On the other hand, the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous. Thus, there exists a constant $C > 1$ such that

$$|u|_{q(\cdot)} \leq C\|u\|_{\vec{p}(\cdot)}, \quad \forall u \in W_0^{1,\vec{p}(\cdot)}(\Omega). \tag{13}$$

Let $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ be such that $\|u\|_{\vec{p}(\cdot)} < \frac{1}{C}$. Then (13) implies

$$|u|_{q(\cdot)} < 1 \text{ for all } u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ with } \|u\|_{\vec{p}(\cdot)} < \frac{1}{C}.$$

Hence, by property (7) we have

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(\cdot)}^{q^-}. \tag{14}$$

Combining (12), (13) and (14) it follows that

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx \leq \lambda C^{q^-} \|u\|_{\vec{p}(\cdot)}^{q^-}. \tag{15}$$

Also, $\|u\|_{\vec{p}(\cdot)} < 1$ yields $|\partial_{x_i} u|_{p_i(\cdot)} < 1$ and thus, by (7), we infer that

$$\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{p_i^+} \leq \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx. \tag{16}$$

If we apply the Jensen inequality to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{P_+^+}$, $P_+^+ \geq 2$, we get

$$\frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}} = N\left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}}{N}\right)^{P_+^+} \leq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+} \leq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}^{p_i^+}. \tag{17}$$

Taking in consideration (16) and (17) we deduce that

$$\|u\|_{\vec{p}(\cdot)}^{P_+^+} \leq N^{P_+^+-1} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx. \tag{18}$$

By a simple computation, using (15) and (18), one has

$$\frac{1}{\left(\lambda C^{q^-} N^{\frac{q^-(P_+^+-1)}{P_+^+}}\right)^{\frac{P_+^+}{q^- - P_+^+}}} \leq \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx.$$

This last inequality together with (11) lead us to

$$\lambda > \left(CN^{\frac{P_+^+-1}{P_+^+}}\right)^{-q^-}\left(\int_\Omega \lambda^{\frac{r(x)}{r(x)-q(x)}}\frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}}dx\right)^{\frac{P_+^+-q^-}{P_+^+}}$$

for any $u \in E$ with $\|u\|_{\vec{p}(\cdot)} < \frac{1}{C}$. Denoting the term in the right-hand side of the previous inequality by λ^* , we conclude that Theorem 2.1 holds true.

4. Proof of Theorem 2.2

First, we will establish some auxiliary results.

Lemma 4.1. *The energy functional Φ is coercive on E.*

Proof. The proof of this Lemma relies on the inequality:

For any $k_1, k_2 > 0$ and $0 < q < r$, a straightforward computation shows that

$$k_1|t|^q - k_2|t|^r \leq Ck_1\left(\frac{k_1}{k_2}\right)^{\frac{q}{r-q}}, \quad \forall t \in \mathbb{R}, \tag{19}$$

where $C > 0$ is a constant depending on q and r.

If we choose in (19) $q = q(x)$, $r = r(x)$, $k_1 = \frac{\lambda}{q(x)}$ and $k_2 = \frac{h(x)}{2r(x)}$ we obtain

$$\begin{aligned}\frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{2r(x)}|u|^{r(x)} &\leq C\frac{\lambda}{q(x)}\left(\frac{\lambda}{q(x)}\cdot\frac{2r(x)}{h(x)}\right)^{\frac{q(x)}{r(x)-q(x)}}\\ &= C\left(\frac{1}{q(x)}\right)^{\frac{r(x)}{r(x)-q(x)}}\big(2r(x)\big)^{\frac{q(x)}{r(x)-q(x)}}\lambda^{\frac{r(x)}{r(x)-q(x)}}\frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}}.\end{aligned}$$

But the terms $\left(\frac{1}{q(x)}\right)^{\frac{r(x)}{r(x)-q(x)}}$, $(2r(x))^{\frac{q(x)}{r(x)-q(x)}}$ are bounded (since $2 < q(x) < r(x) < P_{-,\infty}$), so the above inequality becomes

$$\frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{2r(x)}|u|^{r(x)} \leq C_1\lambda^{\frac{r(x)}{r(x)-q(x)}}\frac{1}{h(x)^{\frac{q(x)}{r(x)-q(x)}}}.$$

Integrating the previous inequality over Ω and taking into account the hypothesis (2) we deduce that there exists a constant $C_2 > 0$ so that

$$\int_\Omega\left(\frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{2r(x)}|u|^{r(x)}\right)dx \leq C_2.$$

Thus

$$\begin{aligned}\Phi(u) &= \int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}dx - \lambda\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx + \int_\Omega\frac{h(x)}{r(x)}|u|^{r(x)}dx\\ &= \int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}dx - \int_\Omega\left(\frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{2r(x)}|u|^{r(x)}\right)dx\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx \\
&\geq \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}dx + \frac{1}{2}\int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx - C_2 \\
&\geq \frac{1}{P_+^+}\int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}dx + \frac{1}{2r^+}\int_\Omega h(x)|u|^{r(x)}dx - C_2. \qquad (20)
\end{aligned}$$

To proceed further, let $v \in L^{r(\cdot)}(\Omega)$, $r \in C_+(\overline{\Omega})$, provided that $|v|_{r(\cdot)} > 1$. By relation (6) we get

$$|v|_{r(\cdot)}^{r^-} \leq \int_\Omega |v|^{r(x)}dx \leq |v|_{r(\cdot)}^{r^+}.$$

Taking $v(x) = h(x)^{\frac{1}{r(x)}}u(x)$ we see that

$$|u|_{h,r(\cdot)}^{r^-} \leq \int_\Omega h(x)|u|^{r(x)}dx \leq |u|_{h,r(\cdot)}^{r^+}. \qquad (21)$$

Also, let $u \in E$ be such that $|\partial_{x_i}u|_{p_i(\cdot)} > 1$ for all $i \in \{1,\ldots,N\}$, whence $\|u\|_{\vec{p}(\cdot)} > 1$. By (6) we have

$$\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{p_i^-} \leq \int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}dx. \qquad (22)$$

Applying Jensen's inequality to the convex function $b: \mathbb{R}^+ \to \mathbb{R}^+$, $b(t) = t^{P_-^-}$, $P_-^- \geq 2$, we find that

$$\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- -1}} = N\left(\frac{\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}}{N}\right)^{P_-^-} \leq \sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_-^-} \leq \sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{p_i^-}. \qquad (23)$$

Consequently, (20), (21), (22) and (23) yield to

$$\begin{aligned}
\Phi(u) &\geq \frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{P_+^+ N^{P_-^- -1}} + \frac{1}{2r^+}|u|_{h,r(\cdot)}^{r^-} - C_2 \\
&\geq C_3\big(\|u\|_{\vec{p}(\cdot)} + |u|_{h,r(\cdot)}\big) - C_2 = C_3\|u\|_E - C_2,
\end{aligned}$$

where $C_3 = \min\left\{\frac{1}{P_+^+ N^{P_-^- -1}}, \frac{1}{2r^+}\right\}$. We infer that $\Phi(u) \to \infty$ as $\|u\|_E \to \infty$. This means that Φ is coercive on E. $\square$

Lemma 4.2. *Assume that (u_n) is a sequence in E such that $\Phi(u_n)$ is bounded. Then there exists a subsequence of (u_n), still denoted by (u_n), which converges weakly in E to some $u_0 \in E$ and*

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

Proof. Since $\Phi(u_n)$ is bounded, the inequality (20) implies that $\sum_{i=1}^N \int_\Omega |\partial_{x_i}u_n|^{p_i(x)}dx$ and $\int_\Omega h(x)|u_n|^{r(x)}dx$ are bounded.

Let (u_n) be such that $|\partial_{x_i} u_n|_{p_i(\cdot)} > 1$ for all $i \in \{1, \dots, N\}$, wherefrom $\|u_n\|_{\vec{p}(\cdot)} > 1$. By (6) we get

$$1 < |\partial_{x_i} u_n|_{p_i(\cdot)} < \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(\cdot)} < \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(\cdot)}^{p_i^-} \le \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(\cdot)} dx,$$

hence $\|u_n\|_{\vec{p}(\cdot)}$ is bounded.

Also, let $v_n \in L^{r(\cdot)}(\Omega)$, $r \in C_+(\overline{\Omega})$, provided that $|v_n|_{r(\cdot)} > 1$. Therefore, by inequality (6) and taking $v_n(x) = h(x)^{\frac{1}{r(x)}} u_n(x)$ we obtain

$$1 < |u_n|_{h,r(\cdot)} < |u_n|_{h,r(\cdot)}^{r^-} \le \int_\Omega h(x)|u_n|^{r(x)} dx,$$

whence $|u_n|_{h,r(\cdot)}$ is bounded. Hence $\|u_n\|_E$ is bounded. Consequently, there exists a subsequence of (u_n), labeled again (u_n), which converges weakly to some $u_0 \in E$. In fact, there exists $u_0 \in E$ so that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ u_n &\to u_0 \quad \text{in } L_h^{r(\cdot)}(\Omega). \end{aligned}$$

We define

$$F(x,u) = \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{r(x)}|u|^{r(x)}$$

and

$$f(x,u) = F_u(x,u) = \lambda|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u.$$

We notice that

$$f_u(x,u) = \lambda(q(x)-1)|u|^{q(x)-2} - h(x)(r(x)-1)|u|^{r(x)-2}.$$

Using again (19) for $k_1 = \lambda(q(x)-1)$, $k_2 = h(x)(r(x)-1)$, $q = q(x)-2$ and $r = r(x)-2$ it follows that

$$\begin{aligned} f_u(x,u) &\le C\lambda(q(x)-1)\left(\frac{\lambda(q(x)-1)}{h(x)(r(x)-1)}\right)^{\frac{q(x)-2}{r(x)-q(x)}} \\ &= C\left(\frac{q(x)-1}{r(x)-1}\right)^{\frac{q(x)-2}{r(x)-q(x)}} \cdot (q(x)-1) \cdot \frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}. \end{aligned}$$

Since $2 < q(x) < r(x) < P_{-,\infty}$ we can see that $\left(\frac{q(x)-1}{r(x)-1}\right)^{\frac{q(x)-2}{r(x)-q(x)}} \cdot (q(x)-1)$ is a bounded expression. It follows that

$$f_u(x,u) \le C_1 \frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}. \tag{24}$$

According to the definitions of Φ and F we obtain the following estimate

$$\Phi(u_0) - \Phi(u_n) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx + \int_\Omega [F(x,u_n) - F(x,u_0)] dx. \tag{25}$$

Clearly the following

$$\int_0^s f_u(x, u_0 + t(u_n - u_0)) dt = \frac{1}{u_n - u_0}[f(x, u_0 + s(u_n - u_0)) - f(x,u_0)] = \frac{1}{u_n - u_0}[F_u(x, u_0 + s(u_n - u_0)) - F_u(x,u_0)]$$

hold true. Integrating the previous relation over $[0,1]$ we find that

$$\begin{aligned} &\int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0)) dt\, ds \\ &= \frac{1}{u_n - u_0} \int_0^1 [F_u(x, u_0 + s(u_n - u_0)) - F_u(x,u_0)] ds \\ &= \frac{1}{(u_n - u_0)^2}[F(x,u_n) - F(x,u_0)] - \frac{f(x,u_0)}{u_n - u_0}. \end{aligned}$$

Hence

$$F(x,u_n) - F(x,u_0) = (u_n - u_0)^2 \int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0)) dt\, ds + (u_n - u_0) f(x,u_0). \tag{26}$$

Combining the relations (25) and (26), then applying (24) we infer that

$$\begin{aligned} \Phi(u_0) - \Phi(u_n) &= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx \\ &\quad + \int_\Omega (u_n - u_0)^2 \int_0^1 \int_0^s f_u(x, u_0 + t(u_n - u_0)) dt\, ds\, dx \\ &\quad + \int_\Omega (u_n - u_0) f(x,u_0) dx \\ &\le \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_0|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx \\ &\quad + C_2 \int_\Omega (u_n - u_0)^2 \frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}} dx + \int_\Omega (u_n - u_0) f(x,u_0) dx, \end{aligned} \tag{27}$$

where $C_2 > 0$ is a constant. Now, we need to demonstrate that the last two integrals converge to 0 as $n \to \infty$.

We define $J : E \to \mathbb{R}$ by

$$J(v) = \int_\Omega f(x, u_0)v\, dx.$$

Evidently, J is linear. We want to prove that J is also continuous. We have

$$\begin{aligned}|J(v)| &\le \int_\Omega |f(x, u_0)v|dx = \int_\Omega \left|\lambda|u_0|^{q(x)-2}u_0 - h(x)|u_0|^{r(x)-2}u_0\right| \cdot |v|dx \\ &\le \lambda \int_\Omega |u_0|^{q(x)-1}|v|dx + \int_\Omega h(x)|u_0|^{r(x)-1}|v|dx. \end{aligned} \tag{28}$$

Applying the Hölder type inequality (4) we find that

$$\int_\Omega |u_0|^{q(x)-1}|v|dx \le \left(\frac{1}{\left(\frac{q}{q-1}\right)^-} + \frac{1}{q^-}\right)\left||u_0|^{q(x)-1}\right|_{\frac{q(\cdot)}{q(\cdot)-1}} |v|_{q(\cdot)}.$$

On the other hand, the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and, thus, there exists a positive constant C_0 such that

$$|v|_{q(\cdot)} \le C_0\|v\|_{\vec{p}(\cdot)}, \quad \forall v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

We have also

$$\|v\|_{\vec{p}(\cdot)} \le \|v\|_E.$$

The last three inequalities imply that

$$\int_\Omega |u_0|^{q(x)-1}|v|dx \le \widetilde{C}\|v\|_E, \tag{29}$$

where $\widetilde{C} > 0$ is a constant. Using again Hölder's inequality (4) we have

$$\begin{aligned}&\int_\Omega h(x)|u_0|^{r(x)-1}|v|dx \\ &\quad = \int_\Omega \left(h(x)^{\frac{r(x)-1}{r(x)}}|u_0|^{r(x)-1}\right)\left(h(x)^{\frac{1}{r(x)}}|v|\right)dx \\ &\quad \le \left(\frac{1}{\left(\frac{r}{r-1}\right)^-} + \frac{1}{r^-}\right)\left|h(x)^{\frac{r(x)-1}{r(x)}}|u_0|^{r(x)-1}\right|_{\frac{r(\cdot)}{r(\cdot)-1}}\left|h(x)^{\frac{1}{r(x)}}|v|\right|_{r(\cdot)} \\ &\quad = \overline{C}|v|_{h,r(\cdot)} \le \overline{C}\|v\|_E, \end{aligned} \tag{30}$$

where $\overline{C}$ is a positive constant.

So, from (28), (29) and (30) we can see that there exists a constant $\widehat{C} > 0$ so that

$$|J(v)| \le \widehat{C}\|v\|_E, \quad \forall v \in E,$$

that is J is continuous. Now, considering that $u_n \rightharpoonup u_0$ in E and J is linear and continuous we conclude that

$$J(u_n) \to J(u_0).$$

This shows that

$$\lim_{n\to\infty} \int_\Omega f(x, u_0)(u_n - u_0)dx = 0. \tag{31}$$

Next, we prove that

$$\lim_{n\to\infty}\int_\Omega (u_n-u_0)^2\frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}dx=0. \tag{32}$$

We know that $u_n \rightharpoonup u_0$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Since the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)$ is compact, it follows that $u_n\to u_0$ in $L^{r(\cdot)}(\Omega)$, that is,

$$\int_\Omega |u_n-u_0|^{r(x)}dx\to 0.$$

Therefore

$$\int_\Omega \left(|u_n-u_0|^2\right)^{\frac{r(x)}{2}}dx<\infty$$

or equivalently

$$(u_n-u_0)^2\in L^{\frac{r(\cdot)}{2}}(\Omega).$$

We notice that the condition (3) may be equivalently reformulated as $\frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}\in L^{\frac{r(\cdot)}{r(\cdot)-2}}(\Omega)$. Hence, we get

$$\begin{aligned}\int_\Omega (u_n-u_0)^2\frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}dx\\ \le\left(\frac{1}{\left(\frac{r}{2}\right)^-}+\frac{1}{\left(\frac{r}{r-2}\right)^-}\right)\left|\frac{\lambda^{\frac{r(x)-2}{r(x)-q(x)}}}{h(x)^{\frac{q(x)-2}{r(x)-q(x)}}}\right|_{\frac{r(\cdot)}{r(\cdot)-2}}\left|(u_n-u_0)^2\right|_{\frac{r(\cdot)}{2}}\end{aligned}$$

via Hölder type inequality (4). We notice that also

$$\rho_{\frac{r(\cdot)}{2}}((u_n-u_0)^2)=\int_\Omega |(u_n-u_0)^2|^{\frac{r(x)}{2}}dx=\int_\Omega |u_n-u_0|^{r(x)}dx\to 0.$$

The above relation together with (8) implies

$$\left|(u_n-u_0)^2\right|_{\frac{r(\cdot)}{2}}\to 0.$$

Therefore, it turns out that (32) holds true.

Now, to conclude Lemma 4.2 we prove that the functional Λ : $W_0^{1,\vec{p}(\cdot)}(\Omega)\to\mathbb{R}$,

$$\Lambda(u)=\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}dx$$

is convex. Indeed, taking in consideration that the function

$$[0,\infty)\ni t\to t^\gamma$$

is convex for each $\gamma>1$ it follows that for any $x\in\Omega$ fixed it the inequality

$$\left|\frac{\alpha+\beta}{2}\right|^{p_i(x)}\le\left|\frac{|\alpha|+|\beta|}{2}\right|^{p_i(x)}\le\frac{1}{2}|\alpha|^{p_i(x)}+\frac{1}{2}|\beta|^{p_i(x)},\quad \forall\ \alpha,\beta\in\mathbb{R},\quad i\in\{1,\dots,N\} \tag{33}$$

holds. Using (33) we obtain

$$\left|\frac{\partial_{x_i}u+\partial_{x_i}v}{2}\right|^{p_i(x)} \le \frac{1}{2}|\partial_{x_i}u|^{p_i(x)}+\frac{1}{2}|\partial_{x_i}v|^{p_i(x)},$$
$$\forall\ u,v\in W_0^{1,\vec{p}(\cdot)}(\Omega),\ x\in\Omega,\quad i\in\{1,\dots,N\}.$$

Multiplying by $\frac{1}{p_i(x)}$, summing from 1 to N and integrating over Ω we deduce that

$$\Lambda\left(\frac{u+v}{2}\right)\le\frac{1}{2}\Lambda(u)+\frac{1}{2}\Lambda(v),\ \forall\ u,v\in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Therefore Λ is convex on $W_0^{1,\vec{p}(\cdot)}(\Omega)$. A way to prove that the functional Λ is weakly lower semicontinuous on $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is to show that this is lower semicontinuous on $W_0^{1,\vec{p}(\cdot)}(\Omega)$ via Corollary III.8 in [7]. To this aim, we fix $u\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $\varepsilon>0$. Let $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ be arbitrary. The fact that Λ is convex and inequality (4) holds lead us to

$$\begin{aligned}\Lambda(v)&\ge\Lambda(u)+\langle\Lambda'(u),v-u\rangle\\&=\Lambda(u)+\int_\Omega\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}(v-u)dx\\&\ge\Lambda(u)-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u|^{p_i(x)-1}|\partial_{x_i}(v-u)|dx\\&\ge\Lambda(u)-c_1\sum_{i=1}^N\left||\partial_{x_i}u|^{p_i(x)-1}\right|_{\frac{p_i(\cdot)}{p_i(\cdot)-1}}|\partial_{x_i}(v-u)|_{p_i(\cdot)}\\&\ge\Lambda(u)-c_2\|v-u\|_{\vec{p}(\cdot)}\\&\ge\Lambda(u)-\varepsilon\end{aligned}$$

for any $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with $\|v-u\|_{\vec{p}(\cdot)}<\varepsilon/c_2$, where c_1,c_2 are two positive constants. Thus Λ is lower semicontinuous on $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and so weakly lower semicontinuous. This means that

$$\liminf_{n\to\infty}\Lambda(u_n)\ge\Lambda(u_0),$$

that is

$$\liminf_{n\to\infty}\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u_n|^{p_i(x)}}{p_i(x)}dx\ge\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u_0|^{p_i(x)}}{p_i(x)}dx.$$

Passing to the limit in (27) and using (31), (32) and the above inequality it follows that

$$\liminf_{n\to\infty}\Phi(u_n)\ge\Phi(u_0).$$

We conclude that Lemma 4.2 holds true. $\square$

Proof of Theorem 2.2. By Lemmas 4.1, 4.2 and Theorem 1.2 in [17] we deduce that there exists a global minimizer of Φ, that is

$$\Phi(u) = \inf_{v \in E} \Phi(v).$$

It is obvious that u is weak solution of problem (1). Our goal is to show that $u \not\equiv 0$ in E. For this purpose, we will prove that $\inf_E \Phi < 0$ providing that λ is large enough.

Set

$$\overline{\lambda} = \inf \left\{ q^+ \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + q^+ \int_\Omega \frac{h(x)}{r(x)} |u|^{r(x)} dx; u \in E, \int_\Omega |u|^{q(x)} dx = 1 \right\}.$$

We want to point out that $\overline{\lambda} > 0$. For all $u \in E$ with $\int_\Omega |u|^{q(x)} dx = 1$, Hölder type inequality (4) yields

$$\lambda = \int_\Omega \frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}} h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)} dx \leq C \left| \frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}} \right|_{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}} \left| h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)} \right|_{\frac{r(\cdot)}{q(\cdot)}} \tag{34}$$

where $C = \frac{1}{(\frac{r}{r-q})^-} + \frac{1}{(\frac{r}{q})^-}$.

Next, we focus our attention on the case when $u \in E$ such that

$$\left| h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)} \right|_{\frac{r(\cdot)}{q(\cdot)}} > 1.$$

Thus, by (6), (7) and (34) it follows that

$$\lambda \leq C \left(\int_\Omega \frac{\lambda^{\frac{r(x)}{r(x)-q(x)}}}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx \right)^{1/(\frac{r}{r-q})^{\pm}} \left(\int_\Omega h(x) |u|^{r(x)} dx \right)^{1/(\frac{r}{q})^-}.$$

Hence, we get

$$\int_\Omega h(x) |u|^{r(x)} dx \geq \frac{(\frac{\lambda}{C})^{(\frac{r}{q})^-}}{\left(\int_\Omega \frac{\lambda^{\frac{r(x)}{r(x)-q(x)}}}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx \right)^{(\frac{r}{q})^- / (\frac{r}{r-q})^{\pm}}}.$$

Therefore

$$\overline{\lambda} \geq q^+ \int_\Omega \frac{h(x)}{r(x)} |u|^{r(x)} dx \geq \frac{q^+}{r^+} \left(\frac{\lambda}{C} \right)^{(\frac{r}{q})^-} \left(\int_\Omega \frac{\lambda^{\frac{r(x)}{r(x)-q(x)}}}{h(x)^{\frac{q(x)}{r(x)-q(x)}}} dx \right)^{-(\frac{r}{q})^- / (\frac{r}{r-q})^{\pm}} > 0.$$

Let $\lambda > \overline{\lambda}$. Then there exists a function $u_1 \in E$ with $\int_\Omega |u_1|^{q(x)} dx = 1$ so that

$$\lambda \int_\Omega |u_1|^{q(x)} dx = \lambda > q^+ \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_1|^{p_i(x)}}{p_i(x)} dx + q^+ \int_\Omega \frac{h(x)}{r(x)} |u_1|^{r(x)} dx.$$

Thus, we deduce that

$$\lambda \int_\Omega \frac{1}{q(x)} |u_1|^{q(x)} dx \geq \frac{\lambda}{q^+} \int_\Omega |u_1|^{q(x)} dx > \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_1|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{h(x)}{r(x)} |u_1|^{r(x)} dx.$$

Finally, we obtain

$$\Phi(u_1) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_1|^{p_i(x)}}{p_i(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u_1|^{q(x)} dx + \int_\Omega \frac{h(x)}{r(x)} |u_1|^{r(x)} dx < 0,$$

so we infer that $\inf_{u \in E} \Phi(u) < 0$. In conclusion, there exists $\lambda_0 = \overline{\lambda}$ so that the problem (1) has a nontrivial weak solution, namely $u_1 \in E$, for every $\lambda > \lambda_0$, satisfying $\Phi(u_1) < 0$. Since $\Phi(u_1) = \Phi(|u_1|)$ we may assume that $u_1 \geq 0$ almost everywhere in Ω. □

Next, we will try to find a second nontrivial weak solution for problem (1). To this aim, we fix $\lambda \geq \lambda_0$ and set

$$g(x,t) = \begin{cases} 0, & \text{for } t < 0 \\ \lambda t^{q(x)-1} - h(x) t^{r(x)-1}, & \text{for } 0 \leq t \leq u_1(x) \\ \lambda u_1(x)^{q(x)-1} - h(x) u_1(x)^{r(x)-1}, & \text{for } t > u_1(x) \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s) ds.$$

We define the functional $\Psi : E \to \mathbb{R}$ by

$$\Psi(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \int_\Omega G(x,u) dx.$$

By standard arguments, $\Psi \in C^1(E, \mathbb{R})$ and the derivative is given by

$$\langle \Psi'(u), v \rangle = \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v \, dx - \int_\Omega g(x,u) v \, dx$$

for all $u, v \in E$. It is obvious that if u is a critical point of Ψ then $u \geq 0$ almost everywhere in Ω.

Let us prove now the following lemma.

Lemma 4.3. *If u is a critical point of Ψ then $u \leq u_1$.*

Proof. Denote by $v^+(x) = \max\{v(x), 0\}$ the positive part of v. Using Theorem 7.6 in [9] it follows that if $v \in E$ then $v^+ \in E$. Next, we will apply the following inequality (see formula 2.2 in [16])

$$(|\xi_i|^{s_i-2}\xi_i - |\psi_i|^{s_i-2}\psi_i)(\xi_i - \psi_i) \geq 2^{-s_i} |\xi_i - \psi_i|^{s_i}, \ \forall \, \xi_i, \psi_i \in \mathbb{R}$$

valid for each $s_i \geq 2$. We have

$$\begin{aligned}
0 &= \langle \Psi'(u), (u-u_1)^+\rangle - \langle \Phi'(u_1), (u-u_1)^+\rangle \\
&= \int_\Omega \sum_{i=1}^N \big(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u - |\partial_{x_i} u_1|^{p_i(x)-2}\partial_{x_i} u_1\big)\partial_{x_i}(u-u_1)^+ dx \\
&\quad - \int_\Omega \big[g(x,u) - \lambda u_1^{q(x)-1} + h(x) u_1^{r(x)-1}\big](u-u_1)^+ dx \\
&= \int_{[u>u_1]} \sum_{i=1}^N \big(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u - |\partial_{x_i} u_1|^{p_i(x)-2}\partial_{x_i} u_1\big)(\partial_{x_i} u - \partial_{x_i} u_1) dx \\
&\geq \int_{[u>u_1]} \sum_{i=1}^N 2^{-p_i(x)} |\partial_{x_i} u - \partial_{x_i} u_1|^{p_i(x)} dx \geq 0.
\end{aligned}$$

This holds if and only if $(\partial_{x_i} u - \partial_{x_i} u_1)(x) = 0$ for all $x \in \omega := \{y \in \Omega;\ u(y) > u_1(y)\},\ \ i \in \{1, \ldots, N\}$. Therefore $|\partial_{x_i} u - \partial_{x_i} u_1|_{p_i(\cdot)} < 1$ in ω. Thus, by (7)

$$|\partial_{x_i} u - \partial_{x_i} u_1|_{p_i(\cdot)}^{p_i^+} \leq \int_\omega |\partial_{x_i} u - \partial_{x_i} u_1|^{p_i(x)} dx \leq |\partial_{x_i} u - \partial_{x_i} u_1|_{p_i(\cdot)}^{p_i^-}.$$

Since $\partial_{x_i} u - \partial_{x_i} u_1 = 0$ in ω we obtain that $|\partial_{x_i} u - \partial_{x_i} u_1|_{p_i(\cdot)} \leq 0$ and this is a contradiction with $|\partial_{x_i} u - \partial_{x_i} u_1|_{p_i(\cdot)} > 0$. We conclude that $u \leq u_1$ in Ω. This completes the proof. □

In what follows our goal is to determine a critical point $u_2 \in E$ of Ψ such that $\Psi(u_2) > 0$ via mountain pass theorem. By the previous lemma we deduce that $0 \leq u_2 \leq u_1$ in Ω. Hence

$$g(x, u_2) = \lambda u_2^{q(x)-1} - h(x) u_2^{r(x)-1} \quad \text{and} \quad G(x, u_2) = \frac{\lambda}{q(x)} u_2^{q(x)} - \frac{h(x)}{r(x)} u_2^{r(x)}$$

and we find that

$$\Psi(u_2) = \Phi(u_2) \quad \text{and} \quad \Psi'(u_2) = \Phi'(u_2).$$

So, we have the following

$$\Phi(u_2) = \Psi(u_2) > 0 = \Phi(0) > \Phi(u_1)$$

and

$$0 = \Psi'(u_2) = \Phi'(u_2),$$

that is u_2 is a weak solution of problem (1) such that $0 \leq u_2 \leq u_1$, $u_2 \neq 0$ and $u_2 \neq u_1$.

Now, we focus our attention to get u_2 described above.

Lemma 4.4. *There exists $\rho \in (0, \|u_1\|_{\vec{p}(\cdot)})$ and $a > 0$ so that $\Psi(u) \geq a$, for every $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.*

Proof. For every $u \in E$ we have

$$\begin{aligned}
\Psi(u) &= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \int_{[u>u_1]} G(x,u)dx - \int_{[u\le u_1]} G(x,u)dx \\
&= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \lambda \int_{[u>u_1]} u_1^{q(x)-1} u\, dx + \int_{[u>u_1]} h(x) u_1^{r(x)-1} u\, dx \\
&\quad - \lambda \int_{[u\le u_1]} \frac{1}{q(x)} u^{q(x)}\, dx + \int_{[u\le u_1]} \frac{h(x)}{r(x)} u^{r(x)} dx \\
&> \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx - \lambda \int_{[u>u_1]} u^{q(x)} dx - \frac{\lambda}{q^-} \int_{[u\le u_1]} u^{q(x)} dx \\
&> \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx - \lambda \int_\Omega |u|^{q(x)} dx. \qquad (35)
\end{aligned}$$

We know that $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$, thus there exists a constant $L > 1$ so that

$$|u|_{q(\cdot)} \le L \cdot \|u\|_{\vec{p}(\cdot)}, \quad \forall u \in W_0^{1,\vec{p}(\cdot)}(\Omega). \qquad (36)$$

We fix $\rho \in (0,1)$ such that $\rho < \frac{1}{L}$. Then the previous relation implies

$$|u|_{q(\cdot)} < 1 \text{ for all } u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ with } \|u\|_{\vec{p}(\cdot)} = \rho.$$

Evidently, $|\partial_{x_i} u|_{p_i(\cdot)} < 1$. In this case we have shown already in the proof of Theorem 2.1 that

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} dx \ge \frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{N^{P_+^+ - 1}}.$$

On the other hand, (7) and (36) imply

$$\lambda \int_\Omega |u|^{q(x)} dx \le L_1 \|u\|_{\vec{p}(\cdot)}^{q^-},$$

where $L_1 = \lambda L^{q^-}$. So, considering (35) and the last two inequalities we have

$$\Psi(u) > \frac{\|u\|_{\vec{p}(\cdot)}^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - L_1 \|u\|_{\vec{p}(\cdot)}^{q^-} = \|u\|_{\vec{p}(\cdot)}^{P_+^+} \left(\frac{1}{P_+^+ N^{P_+^+ - 1}} - L_1 \|u\|_{\vec{p}(\cdot)}^{q^- - P_+^+} \right),$$

for all $u \in E$ with $\|u\|_{\vec{p}(\cdot)} = \rho$.

We notice that the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \frac{1}{P_+^+ N^{P_+^+ - 1}} - L_1 t^{q^- - P_+^+}$$

is positive in a neighborhood of the origin, so that the choice of $\rho \in (0,1)$ is so small that $a = \rho^{P_+^+} g(\rho) > 0$. We conclude that lemma holds true. □

Lemma 4.5. *The functional Ψ is coercive in E.*

Proof. For every $u \in E$ we have

$$\begin{aligned}\Psi(u) &= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \lambda \int_{[u>u_1]} u_1^{q(x)-1} u\, dx + \int_{[u>u_1]} h(x) u_1^{r(x)-1} u\, dx \\ &\quad - \lambda \int_{[u\le u_1]} \frac{1}{q(x)} u^{q(x)}\, dx + \int_{[u\le u_1]} \frac{h(x)}{r(x)} u^{r(x)} dx \\ &> \frac{1}{P_+^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx - \lambda \int_{[u>u_1]} u_1^{q(x)} dx - \frac{\lambda}{q^-} \int_{[u\le u_1]} u_1^{q(x)} dx \\ &> \frac{1}{P_+^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx - \lambda \int_\Omega u_1^{q(x)} dx \\ &= \frac{1}{P_+^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} dx - L_2,\end{aligned}$$

where $L_2 > 0$ is a constant. Taking $u \in E$ such that $\|u\|_{\vec{p}(\cdot)} > 1$ we obtain

$$\Psi(u) > \frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - L_2.$$

We conclude that $\Psi(u) \to \infty$ as $\|u\|_{\vec{p}(\cdot)} \to \infty$. In other words, Ψ is coercive in E, completing the proof. $\square$

Proof of Theorem 2.2 completed. By Lemma 4.4 and the mountain pass theorem (see [3] with the variant given by Theorem 1.15 in [19]) we derive that there exists a sequence $(u_n) \subset E$ so that

$$\Psi(u_n) \to c > 0 \ \text{ and } \ \Psi'(u_n) \to 0 \tag{37}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

Keeping in mind (37) and relying on Lemma 4.5 we infer that (u_n) is bounded and, thus, passing eventually to a subsequence, labeled again (u_n), we may assume that there exists $u_2 \in E$ so that $u_n \rightharpoonup u_2$. Standard arguments, based on the Sobolev embeddings, help us to see that

$$\lim_{n\to\infty} \langle \Psi'(u_n), v\rangle = \langle \Psi'(u_2), v\rangle,$$

for every $v \in C_0^\infty(\Omega)$. The above piece of information together with the fact that $E \subset W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ yield to the fact that u_2 is a weak solution of problem (1). Thus, our conclusion is that problem (1) has at least two nontrivial weak solutions and the proof of Theorem 2.2 is now complete. $\square$

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