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Nonlinear Differential Equations and Applications NoDEA

Error estimates for approximations of nonlinear uniformly parabolic equations

Olga Turanova

Abstract. We introduce the notion of δ-viscosity solutions for fully nonlinear uniformly parabolic PDE on bounded domains. We prove that δ-viscosity solutions are uniformly close to the actual viscosity solution, with an explicit error of order δ^{α}. As a consequence we obtain an error estimate for implicit monotone finite difference approximations of uniformly parabolic PDE.

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1. Introduction

We introduce δ-viscosity solutions for the nonlinear parabolic problem

$$u_t - F(D^2u) = 0 \quad \text{in } \Omega \times (0,T). \tag{1}$$

We prove an estimate between viscosity solutions and δ-viscosity solutions of (1) under the assumption that the nonlinearity F is uniformly elliptic (see (F1) below). As a consequence, we find a rate of convergence for monotone and consistent implicit finite difference approximations to (1). Both results generalize the work of Caffarelli and Souganidis in [8,9], who consider the time-independent case.

The nonlinearity F is a continuous function on $\mathcal{S}_{n\times n}$, where $\mathcal{S}_{n\times n}$ is the set of $n \times n$ real symmetric matrices endowed with the usual order and norm (for $X \in \mathcal{S}_{n\times n}$, $||X|| = \sup_{|v|=1} |Xv|$). We make the following assumptions:

(F1) F is uniformly elliptic with constants $0 < \lambda \leq \Lambda$, which means that for any $X \in \mathcal{S}_{n\times n}$ and for all $Y \geq 0$,

$$\lambda||Y|| \leq F(X+Y) - F(X) \leq \Lambda||Y||;$$

and,

(F2) $F(0) = 0$.

We also assume:

(U1) Ω is a bounded subset of $\mathbb{R}^n$.

The main result is an error estimate between viscosity solutions and δ-viscosity solutions of (1). The definition of δ-viscosity solutions is in Sect. 2.2. Next we present a statement of our main result that has been simplified for the introduction; the full statement is in Sect. 6.

Theorem 1.1. *Assume (F1), (F2) and (U1). Assume u is a viscosity solution of* (1) *that is Lipschitz continuous in x and Hölder continuous in t. Assume $\{u_\delta\}_{\delta>0}$ is a family of Hölder continuous δ-viscosity solutions of* (1) *that satisfy*

$$u_\delta = u \text{ on the parabolic boundary of } \Omega \times (0, T)$$

for all δ. There exist positive constants C and α that do not depend on δ such that for all δ small enough,

$$||u - u_\delta||_{L^\infty(\Omega\times(0,T))} \leq C\delta^\alpha.$$

We remark that if u is the solution of a boundary value problem with sufficiently regular boundary data, then the regularity conditions on u of Theorem 1.1 are satisfied (see Remark 6.2 for more details). We make this an assumption in order to avoid discussing boundary value problems and to keep the setting as simple as possible.

The notion of δ-viscosity solutions was introduced by Caffarelli and Souganidis in [8] as a tool for obtaining uniform estimates for viscosity solutions. In [8], δ-viscosity solutions were used to establish an error estimate for finite difference approximations of nonlinear uniformly elliptic equations. An error estimate between viscosity and δ-viscosity solutions of uniformly elliptic equations was obtained by Caffarelli and Souganidis in [9]. This was an important step in establishing a rate for homogenization in random media [9].

The main challenge in obtaining an error estimate between viscosity and δ-viscosity solutions, in both the elliptic and parabolic setting, is overcoming the lack of regularity of the viscosity solution u. Indeed, the proof of the error estimate in [8] is based on a regularity result [8, Theorem A], which says that outside of sets of small measure, solutions of uniformly elliptic equations have second-order expansions with controlled error. We prove a similar result for solutions of equation (1). This is Theorem 3.2 of this paper, and it is an essential part of our proof of Theorem 1.1.

But even once it is known that the solution u of (1) has second-order expansions with controlled error, it is necessary to further regularize u and the δ-viscosity solution u_δ in order establish an estimate for their difference. For this we use the classical inf- and sup- convolutions, along with another regularization of inf-sup type, which we call x-sup- and x-inf- convolutions. Both of these regularizations preserve the notions of viscosity and δ-viscosity solution. Moreover, we show that the regularity we establish in Theorem 3.2 for solutions u of (1) is also enjoyed by the x-inf and x-sup convolutions of u (Proposition 4.6).

In this paper we also study implicit finite difference approximations to (1) and prove an error estimate between the solution u of (1) and approximate solutions to (1). We have simplified the notation in the introduction in order to state this result here; see Sect. 7 for all the details about approximation schemes and for the full statement of the error estimate.

We write the finite difference approximations as

$$\mathcal{S}_h[u](x,t) = \delta_\tau^- u(x,t) - \mathcal{F}_h(\delta^2 u(x,t)) = 0 \quad \text{in } (\Omega\times(0,T))\cap E. \tag{2}$$

Here $E = h\mathbb{Z}^n \times h^2\mathbb{Z}$ is the mesh of discretization, $\mathcal{S}_h[u]$ is the implicit finite difference operator, and $\delta_\tau^- u$ and $\delta^2 u$ are finite difference quotients associated to a function u. We assume:

(S1) if u_h^1 and u_h^2 are solutions of (2) with $u_h^1 \le u_h^2$ on the discrete boundary of $\Omega\times(0,T)$, then $u_h^1 \le u_h^2$ in $(\Omega\times(0,T))\cap E$; and

(S2) there exists a positive constant K such that for all $\phi \in C^3(\Omega\times(0,T))$ and all $h > 0$,

$$\sup|\phi_t - F(D^2\phi) - \mathcal{S}_h[\phi]| \le K(h + h||D_x^3\phi||_{L^\infty(\Omega\times(0,T))} + h^2||\phi_{tt}||_{L^\infty(\Omega\times(0,T))}).$$

Schemes that satisfy (S1) and (S2) are said to be, respectively, *monotone* and *consistent with an error estimate for* F. We prove:

Theorem 1.2. *Assume (F1), (F2) and (U1). Assume that* $\mathcal{S}_h$ *is a monotone implicit approximation scheme for* (1) *that is consistent with an error estimate. Assume* u *is a viscosity solution of* (1) *that is Lipschitz continuous in* x *and Hölder continuous in* t *and that* u_h *is a Hölder continuous solution of* (2)*. Assume that for each* $h > 0$,

$$u_h = u \text{ on the discrete boundary of } \Omega\times(0,T).$$

There exist positive constants C *and* α *that do not depend on* h *such that for every* h *small enough,*

$$||u - u_h||_{L^\infty((\Omega\times(0,T))\cap E)} \le Ch^\alpha.$$

The convergence of monotone and consistent approximations of fully nonlinear second order PDE was first established by Barles and Souganidis [5]. Kuo and Trudinger later studied the existence of monotone and consistent approximations for nonlinear elliptic and parabolic equations and the regularity of the approximate solutions u_h (see [21–25]). They showed, both in the elliptic and in the parabolic cases, that if F is uniformly elliptic, then there exists a monotone finite difference scheme $\mathcal{S}_h$ that is consistent with F, and that the approximate solutions u_h are uniformly equicontinuous. However, obtaining an error estimate remained an open problem.

The first error estimate for approximation schemes was established by Krylov in [18,19] for nonlinearities F that are either convex or concave, but possibly degenerate. Krylov used stochastic control methods that apply in the convex or concave case, but not in the general setting. Barles and Jakobsen in [2–4] improved Krylov's error estimates for convex or concave equations. In [20] Krylov improved the error estimate to be of order $h^{1/2}$, but still in the convex/concave case. In addition, Jakobsen [15,16] and Bonnans et al. [6] established error estimates for special equations or for special dimensions. The first error estimate without a convexity or concavity assumption was obtained

Caffarelli and Souganidis in [8] for nonlinear elliptic equations. To our knowledge, Theorem 1.2 is the first error estimate for nonlinear parabolic equations that are neither convex nor concave.

While this paper was in preparation, we learned of the preprint of Daniel [12]. It contains a result [12, Theorem 1.2] that is similar to our Theorem 3.2. His proof also uses methods similar to those of Caffarelli and Souganidis in [8, Theorem A]. Theorem 3.2 is the only overlap between this paper and [12].

Our paper is structured as follows. In Sect. 2 we establish notation, give the definition of δ-viscosity solutions, and state several known results. In Sect. 3 we prove Theorem 3.2, the regularity result that is essential for the rest of our arguments. In Sect. 4 we discuss regularizations of inf-sup type and establish Proposition 4.6, a version of Theorem 3.2 for x-sup and x-inf convolutions. In Sect. 5 we prove a key estimate between viscosity solutions of (1) and sufficiently regular δ-viscosity solutions of (1). Theorems 1.1 and 1.2 are straightforward consequences of this estimate. In Sect. 6 we give the precise statement and proof of Theorem 1.1. Section 7 is devoted to introducing the necessary notation and stating known results about approximation schemes. The precise statement and proof of the error estimate for approximation schemes is in Sect. 8. We also include an appendix with known results about inf- and sup-convolutions.

2. Preliminaries

In this section we establish notation, recall the definition of viscosity solutions for parabolic equations, give the definition of δ-viscosity solutions, and recall some known regularity results for solutions of (1).

2.1. Notation

Points in $\mathbb{R}^{n+1}$ are denoted (x,t) with $x \in \mathbb{R}^n$. The parabolic distance between two points is

$$d((x,t),(y,s)) = (|x-y|^2 + |s-t|)^{1/2}.$$

We denote the usual Euclidean distance in $\mathbb{R}^{n+1}$ by $d_e((x,t),(y,s)) = (|x-y|^2 + |s-t|^2)^{1/2}$. Throughout the argument we consider parabolic cubes, denoted by

$$Q_r(x,t) = [x-r, x+r]^n \times (t-r^2, t],$$

and forward and backward cylinders,

$$Y_r(x,t) = B_r(x) \times (t, t+r^2] \text{ and } Y_r^-(x,t) = B_r(x) \times (t-r^2, t],$$

where $B_r(x) = \{y : |x-y| < r\}$ is the open ball in $\mathbb{R}^n$. We write B_r, Q_r, Y_r and Y_r^- to mean $B_r(0)$, $Q_r(0)$, $Y_r(0)$ and $Y_r^-(0)$, respectively.

For $\Omega \subset \mathbb{R}^n$, the parabolic boundary of $\Omega \times (a,b)$ is defined as

$$\partial_p(\Omega \times (a,b)) = (\Omega \times \{a\}) \cup (\partial\Omega \times (a,b)).$$

The time derivative of a function u is denoted by u_t or $\partial_t u$. The gradient of u with respect to the space variable $x = (x_1, \ldots, x_n)$ is denoted by $Du =$

$(u_{x_1}, \ldots, u_{x_n})$, and the Hessian of u with respect to x is denoted by D^2u. In addition, we use u^+ and u^- to denote, respectively, the positive and negative parts of a function u.

We say that a constant is *universal* if it is positive and depends only on Λ, λ, and n.

We introduce notation for several families of paraboloids.

Definition 2.1. Let $M > 0$. We define:

(1) the class of *convex paraboloids of opening* M:

$$\mathcal{P}_M^+ = \left\{ P(x,t) = c + l \cdot x + mt + \frac{M}{2}|x|^2 \text{ where } l \in \mathbb{R}^n, c, m \in \mathbb{R}, |m| \le M \right\};$$

(2) the class of *concave paraboloids of opening* M:

$$\mathcal{P}_M^- = \{-P(x,t) | P \in \mathcal{P}_M^+\};$$

(3) the class of paraboloids of arbitrary opening:

$$\mathcal{P}_\infty = \left\{ P(x,t) = c + l \cdot x + mt + x \cdot Qx^T \text{ where } l \in \mathbb{R}^n, c, m \in \mathbb{R}, Q \in \mathcal{S}_{n\times n} \right\};$$

(4) the set of polynomials that are quadratic in x and linear in t with the only mixed term of the form $a \cdot xt$:

$$\mathcal{P} = \left\{ P(x,t) = c + l \cdot x + mt + a \cdot xt + x \cdot Qx^T \text{ where } l, a \in \mathbb{R}^n, c, m \in \mathbb{R}, Q \in \mathcal{S}_{n\times n} \right\}.$$

Remark **2.2.** We remark that if $P(x,t)$ is a paraboloid in $\mathcal{P}_\infty$, then P_t and D^2P are constants.

We will make use of the following seminorms, norms, and function spaces:

Definition 2.3. The class of continuous functions on $U \subset \mathbb{R}^{n+1}$ is denoted $C(U)$. The class $C^2(U)$ is the set of functions ϕ that are differentiable in t and twice differentiable in x, with $\phi_t \in C(U)$ and $D^2\phi \in C(U)$.

Definition 2.4. For $\eta \in (0,1]$ and $u \in C(\Omega \times (a,b))$, we define:

$$[u]_{C^{0,\eta}(\Omega\times(a,b))} = \sup_{(x,t),(y,s)\in\Omega\times(a,b)} \frac{|u(x,t) - u(y,s)|}{d((x,t),(y,s))^\eta},$$

$$[u]_{C_x^{0,\eta}(\Omega\times(a,b))} = \sup_{x,y\in\Omega; t\in(a,b)} \frac{|u(x,t) - u(y,t)|}{|x-y|^\eta},$$

$$[u]_{C_t^{0,\eta}(\Omega\times(a,b))} = \sup_{t,s\in(a,b); x\in\Omega} \frac{|u(x,t) - u(x,s)|}{|t-s|^\eta}, \text{ and,}$$

$$||u||_{C^{0,\eta}(\Omega\times(a,b))} = ||u||_{L^\infty(\Omega\times(a,b))} + [u]_{C^{0,\eta}(\Omega\times(a,b))}.$$

Definition 2.5. For $\eta \in (0,1]$, we define:

$$C^{0,\eta}(\Omega \times (a,b)) = \{u \in C(\Omega \times (a,b)) : ||u||_{C^{0,\eta}(\Omega\times(a,b))} < \infty\},$$

$$C_x^{0,\eta}(\Omega \times (a,b)) = \left\{ u \in C(\Omega \times (a,b)) : [u]_{C_x^{0,\eta}(\Omega\times(a,b))} + ||u||_{L^\infty(\Omega\times(a,b))} < \infty \right\}.$$

We will also need the following notion of differentiability:

Definition 2.6. For a constant K, we say that u satisfies $D^2u(x,t) \geq KI$ (resp. $D^2u(x,t) \leq KI$) *in the sense of distributions* if there exists a quadratic polynomial

$$P(y) = c + p \cdot y + \frac{K|x-y|^2}{2}$$

such that $u(x,t) = P(x)$ and $u(y,t) \geq P(y)$ (resp. $u(y,t) \leq P(y)$) for all $y \in B_r(x)$, for some $r > 0$.

2.2. Viscosity and δ-viscosity solutions

We recall the definition of viscosity solutions for parabolic equations.

Definition 2.7. (1) A function $u \in C(\Omega \times (0,T))$ is a *viscosity supersolution* of (1) if for all $(x,t) \in \Omega \times (0,T)$, any $\phi \in C^2(\Omega \times (0,T))$ with $\phi \leq u$ on $Y_\rho^-(x,t)$ for some $\rho > 0$ and $\phi(x,t) = u(x,t)$ satisfies

$$\phi_t(x,t) - F(D^2\phi(x,t)) \geq 0.$$

(2) A function $u \in C(\Omega \times (0,T))$ is a *viscosity subsolution* of (1) if for all $(x,t) \in \Omega \times (0,T)$, any $\phi \in C^2(\Omega \times (0,T))$ with $\phi \geq u$ on $Y_\rho^-(x,t)$ for some $\rho > 0$ and $\phi(x,t) = u(x,t)$ satisfies

$$\phi_t(x,t) - F(D^2\phi(x,t)) \leq 0.$$

(3) We say that $u \in C(\Omega \times (0,T))$ is a *viscosity solution* of (1) if u is both a sub- and a super- solution of (1).

Remark **2.8.** In the above definitions, we require the test function ϕ to stay above (or below) u on a backward cylinder $Y_\rho^-(x,t)$. However, this definition is equivalent to the usual one, which requires the test function to stay above (or below) u on a Euclidean open set around (x,t). This equivalence is proven by Crandall et al. [10, Lemma 1.4] for L^p viscosity solutions. Their proof carries over into our setting with no modifications.

See Section 8 of Crandall et al. [11] for further discussion of viscosity solutions of parabolic equations.

We now introduce δ-viscosity solutions for (1), following the definition of [8,9].

Definition 2.9. Fix $\delta > 0$.

(1) A function $v \in C(\Omega \times (0,T))$ is a *δ-viscosity supersolution* of (1) if for all $(x,t) \in \Omega \times (0,T)$ such that $Y_\delta^-(x,t) \subset \Omega \times (0,T)$, any $P \in \mathcal{P}_\infty$ with $P \leq v$ on $Y_\delta^-(x,t)$ and $P(x,t) = v(x,t)$ satisfies

$$P_t - F(D^2P) \geq 0.$$

(2) A function $v \in C(\Omega \times (0,T))$ is a *δ-viscosity subsolution* of (1) if for all $(x,t) \in \Omega \times (0,T)$ such that $Y_\delta^-(x,t) \subset \Omega \times (0,T)$, any $P \in \mathcal{P}_\infty$ with $P \geq v$ on $Y_\delta^-(x,t)$ and $P(x,t) = v(x,t)$ satisfies

$$P_t - F(D^2P) \leq 0.$$

(3) We say v is a *δ-viscosity solution* of (1) if it is both a δ-viscosity sub- and super- solution.

From now on we will say "solution" to mean viscosity solution and "δ-solution" to mean δ-viscosity solution.

From the definitions, it is clear that a viscosity solution of (1) is a δ-solution of (1). The main difference between the definitions of viscosity and δ-viscosity solution is that for v to be a δ-supersolution (resp. subsolution), any test polynomial must stay below (resp. above) v on a set of fixed size.

2.3. Known results

We introduce the Pucci extremal operators, which are defined for constants $0 < \lambda \leq \Lambda$ and $X \in \mathcal{S}_{n\times n}$ by

$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$
$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where the e_i are the eigenvalues of X (see Caffarelli and Cabre [7] and Wang [26]). We also introduce the so-called upper and lower monotone envelopes of a function, which play the role of the convex and concave envelopes in the regularity theory of elliptic equations. We follow the notes of Imbert and Silvestre in [14, Section 4] for our presentation of Definition 2.10, Lemma 2.11 and Proposition 2.12 below.

Definition 2.10. Let Ω be a convex subset of $\mathbb{R}^n$ and let $u : \Omega \times (a, b) \to \mathbb{R}$ be continuous.

(1) The *lower monotone envelope* of u is the largest function $v : \Omega\times(a, b) \to \mathbb{R}$ that lies below u and is non-increasing with respect to t and convex with respect to x. It is often denoted by $\underline{\Gamma}(u)$.
(2) The *upper monotone envelope* of u is the smallest function $v : \Omega\times(a, b) \to \mathbb{R}$ that lies above u and is non-decreasing with respect to t and concave with respect to x. It is often denoted by $\bar{\Gamma}(u)$.

We have the following representation formulas for the upper and lower envelopes of u ([14, Section 4]):

Lemma 2.11. *Assume $u \in C(Y^-_\rho)$. Then*

$$\underline{\Gamma}(u)(x, t) = \sup\{\zeta \cdot x + h : \ \zeta \cdot y + h \leq u(y, s) \ \textit{for all} \ (y, s) \in Y^-_\rho \cap \{s \leq t\}\}$$

and

$$\bar{\Gamma}(u)(x, t) = \inf\{\zeta \cdot x + h : \ \zeta \cdot y + h \geq u(y, s) \ \textit{for all} \ (y, s) \in Y^-_\rho \cap \{s \leq t\}\}.$$

The following fact will play a central role in our arguments. In Subsect. A.1 of the Appendix, we explain how Proposition 2.12 follows from parts of the proof of the parabolic version of the Alexandroff–Bakelman–Pucci estimate as presented in [14, Section 4.1.2]. We refer the reader to [14, Section 4] for a history of the parabolic version of the Alexandroff–Bakelman–Pucci estimate and the relevant references.

Proposition 2.12. *Assume $u \in C(Y_\rho^-)$ is such that $u \geq 0$ on $\partial_p Y_\rho^-$. Assume that there exists a constant K so that $[u]_{C_t^{0,1}(Y_\rho^-)} \leq K$ and $D^2u(x,t) \leq KI$ in the sense of distributions for every $(x,t) \in Y_\rho^-$. There exists a universal constant C such that*

$$\sup_{Y_\rho^-} u^- \leq C\rho^{\frac{n}{n+1}} |\{u = \Gamma\}|^{\frac{1}{n+1}} K, \tag{3}$$

where Γ is the lower monotone envelope of $\min(u,0)$ *extended by 0 to $Y_{2\rho}^-$.*

Next we state several regularity results. We will use the following interior Hölder gradient estimate [27, Theorems 4.8 and 4.9].

Theorem 2.13. *Assume (F1) and (F2). There exist universal constants $\alpha \in (0,1)$ and C such that if $u \in C(Q_1)$ is a solution of $u_t - F(D^2u) = 0$ in Q_1, then $u_t \in C^{0,\alpha}(Q_1)$ and $Du \in C^{0,\alpha}(Q_1)$ with*

$$||Du||_{C^{0,\alpha}(Q_{1/2})} + ||u_t||_{C^{0,\alpha}(Q_{1/2})} \leq C(||u||_{L^\infty(Q_1)} + 1).$$

The following proposition follows from Theorem 2.13 by a standard rescaling and covering argument.

Proposition 2.14. *Assume (F1) and (F2). Assume that V is a subset of $\Omega \times (0,T)$ with $d(V, \partial_p\Omega \times (0,T)) > r$. There exist universal constants $\alpha \in (0,1)$ and C such that if $u \in C(\Omega \times (0,T))$ is a solution of $u_t - F(D^2u) = 0$ in $\Omega \times (0,T)$, then*

$$r^{1+\alpha}[Du]_{C^{0,\alpha}(V)} + r||Du||_{L^\infty(V)} \leq C(||u||_{L^\infty(\Omega\times(0,T))} + 1)$$
$$r^{2+\alpha}[u_t]_{C^{0,\alpha}(V)} + r^2||u_t||_{L^\infty(V)} \leq C(||u||_{L^\infty(\Omega\times(0,T))} + 1).$$

3. The regularity result

In this section, we establish the parabolic version of the regularity result [8, Theorem A], which says that outside of sets of small measure, solutions of uniformly elliptic equations have second-order expansions with controlled error.

Definition 3.1. Given $u \in C(Y_1)$, we define the set $\Psi_M(u, Y_1) \subset Y_1$ by

$$\begin{aligned}\Psi_M(u, Y_1) = \{(x,t) \in Y_1 :\ & \text{there exists } P \in \mathcal{P} \text{ such that for all} \\ & (y,s) \in Y_1 \cap \{s \leq t\}, |u(y,s) - u(x,t) - P(y,s)| \leq \\ & \leq nM(|x-y|^3 + |x-y|^2|t-s| + |x-y||t-s| + |t-s|^2)\}.\end{aligned}$$

In order to state our result, we introduce another family of subsets of $\mathbb{R}^{n+1}$. We define

$$K_r(x,t) = \left[x - \frac{r}{9\sqrt{n}}, x + \frac{r}{9\sqrt{n}}\right]^n \times \left(t, t + \frac{r^2}{81n}\right].$$

We denote $K_r(0,0)$ by K_r.

We prove the following growth estimate for $|\Psi_M(u, Y_1)|$:

Theorem 3.2. *Assume F satisfies (F1) and (F2). Assume u is a solution of $u_t - F(D^2u) = 0$ in Y_1 and Du and u_t exist and are continuous in Y_1. There exist universal constants C, M_0 and σ such that for every $M \geq M_0$,*

$$|K_1 \setminus \Psi_M(u, Y_1)| \leq CM^{-\sigma}\left(||Du||^{\sigma}_{L^\infty(Y_1)} + ||u_t||^{\sigma}_{L^\infty(Y_1)}\right). \tag{4}$$

Remark **3.3.** If u is a solution of $u_t - F(D^2u) = 0$ in $\Omega \times (0, T)$ and Y_1 is compactly contained in $\Omega \times (0, T)$, then u satisfies the hypotheses of Theorem 3.2 because, according to Proposition 2.14, Du and u_t exist and are continuous in Y_1.

Remark **3.4.** Suppose $(x, t) \in \Psi_M(u, Y_1)$ and P is the paraboloid given by the definition of $\Psi_M(u, Y_1)$. If u is a viscosity solution of $u_t - F(D^2u) = 0$ in Y_1, then of course $P_t(x, t) - F(D^2P(x, t)) = 0$. On the other hand, suppose u is only a δ-solution of $u_t - F(D^2u) = 0$ in Y_1, and $(x, t) \in \Psi_M(u, Y_1)$ with $Y_\delta^-(x, t) \subset Y_1$. In this case we *cannot* conclude that $P_t(x, t) - F(D^2P(x, t)) = 0$. This is because in the definition of $\Psi_M(u, Y_1)$ we consider paraboloids P in the class $\mathcal{P}$, while in the definition of δ-solution we allow only paraboloids in the smaller class $\mathcal{P}_\infty$. In Corollary 3.7 we establish a version of Theorem 3.2 with $P \in \mathcal{P}_\infty$ instead of $P \in \mathcal{P}$.

An essential element of our proof is [26, Theorem 4.11], which says that solutions of uniformly parabolic equations have *first* order expansions with controlled error on large sets (see also [12, Section 3]). To state this result, we recall the following definitions:

$$\begin{aligned}\overline{G}_M(u, Y_1) = \{(x, t) \in Y_1 :\ &\text{there exists } P \in \mathcal{P}^+_M \\ &\text{with } P(x, t) = u(x, t) \text{ and } P \geq u \text{ on } Y_1 \cap \{s \leq t\}\},\\ \underline{G}_M(u, Y_1) = \overline{G}_M(-u, Y_1), &\text{ and}\\ G_M(u, Y_1) = \overline{G}_M(u, Y_1) &\cap \underline{G}_M(u, Y_1).\end{aligned}$$

We observe that if $(x, t) \in G_M(u, Y_1)$, then there exists $p \in \mathbb{R}^n$ such that for all $(y, s) \in Y_1 \cap \{s \leq t\}$,

$$|u(y, s) - u(x, t) - p \cdot (y - x)| \leq \frac{1}{2}M|x - y|^2 + M|t - s|.$$

We now present [26, Theorem 4.11], with notation adapted for our setting:

Theorem 3.5. *Assume (F1) and (F2). Assume that v is a solution of*

$$\begin{cases} v_t - \mathcal{M}^-(D^2v) \geq 0,\\ v_t - \mathcal{M}^+(D^2v) \leq 0\end{cases} \tag{5}$$

in Y_1. There exist universal constants C, σ, and M_0 such that for any $M \geq M_0$,

$$|K_1 \setminus G_M(v, Y_1)| \leq CM^{-\sigma}(||v||_{L^\infty(Y_1)})^{\sigma}.$$

The main idea of the proof of Theorem 3.2 is to apply the growth estimate of Theorem 3.5 to the derivatives of u. Formally, upon differentiating $u_t - F(D^2u) = 0$, we obtain that the derivatives u_{x_i} and u_t solve the linear parabolic equation $\partial_t u_{x_i} - \mathrm{tr}(DF \cdot D^2u_{x_i}) = 0$. Therefore, the estimates of Theorem 3.5 apply to u_{x_i} and u_t, and from this the estimates on u are deduced.

To carry out this plan, we will first use the following proposition [27, Theorem 4.6].

Proposition 3.6. *Assume that u is a solution of $u_t - F(D^2u) = 0$ in Y_1. Then u_t and u_{x_i}, for $i = 1, \ldots, n$, satisfy (5) in Y_1.*

Next we proceed with the proof of Theorem 3.2. We remark that a result similar to [8, Theorem A] was established later, via similar methods, by Armstrong, Silvestre and Smart [1, Section 5]. Our proof is based on the arguments in [8, Theorem A] and our presentation follows that of [1, Section 5].

Proof of Theorem 3.2. By Proposition 3.6, the derivatives u_{x_i} for $i = 1, \ldots, n$ and u_t satisfy (5) in Y_1. Therefore, by Theorem 3.5, for any $M \geq M_0$ the size of the "bad sets" of the u_{x_i} and of u_t are controlled: we have

$$\begin{aligned} &|K_1 \setminus G_M(u_{x_i}, Y_1)| \leq CM^{-\sigma}||Du||^{\sigma}_{L^\infty(Y_1)} \quad \text{for } i = 1, .., n, \\ &|K_1 \setminus G_M(u_t, Y_1)| \leq CM^{-\sigma}||u_t||^{\sigma}_{L^\infty(Y_1)}, \end{aligned} \tag{6}$$

where C is a universal constant. We define the set G_M to be the intersection of the "good sets" of the derivatives of u:

$$G_M = \cap_{i=1}^n G_M(u_{x_i}, Y_1) \cap G_M(u_t, Y_1).$$

The estimates of (6) imply the following bound for the size of complement of G_M:

$$|K_1 \setminus G_M| \leq CM^{-\sigma}\left(||Du||^{\sigma}_{L^\infty(Y_1)} + ||u_t||^{\sigma}_{L^\infty(Y_1)}\right), \tag{7}$$

where C is a universal constant. To prove that the desired upper bound (4) on the size of the complement of $\Psi_M(u, Y_1)$ holds, it will be enough to establish the following relationship between G_M and $\Psi_M(u, Y_1)$:

$$(G_M \cap K_1) \subset (\Psi_M(u, Y_1) \cap K_1). \tag{8}$$

Indeed, we may use (8) and then the estimate (7) to obtain:

$$|K_1 \setminus \Psi_M(u, Y_1)| \leq |K_1 \setminus G_M| \leq CM^{-\sigma}\left(||Du||^{\sigma}_{L^\infty(Y_1)} + ||u_t||^{\sigma}_{L^\infty(Y_1)}\right).$$

Let us now prove that (8) holds. To this end, fix $(x, t) \in G_M \cap K_1$. The goal is to show $(x, t) \in \Psi(u, Y_1) \cap K_1$; in other words, to produce a polynomial P that is the second order expansion of u at (x, t).

Because $(x, t) \in G_M$, there exist $p^1, \ldots, p^{n+1} \in \mathbb{R}^n$ such that for $i = 1, \ldots, n$ and for all $(y, s) \in Y_1 \cap \{s \leq t\}$,

$$\begin{aligned} &|u_{x_i}(y, s) - u_{x_i}(x, t) - p^i \cdot (y - x)| \leq \frac{1}{2}M|x - y|^2 + M|t - s|, \text{ and} \\ &|u_t(y, s) - u_t(x, t) - p^{n+1} \cdot (y - x)| \leq \frac{1}{2}M|x - y|^2 + M|t - s|. \end{aligned} \tag{9}$$

We point out that $D^2u(x,t) = (p^1, \ldots, p^n)$ and $Du_t(x,t) = p^{n+1}$. We define the $n \times n$ matrix Q by $Q_{ij} = p_i^j$ and define the paraboloid $P \in \mathcal{P}$ by

$$P(y,s) = Du(x,t)\cdot(y-x) + \frac{1}{2}(y-x)\cdot Q(y-x)^T + (s-t)u_t(x,t) + \frac{1}{2}p^{n+1}\cdot(y-x)(s-t). \tag{10}$$

We will now show that this polynomial P is the second order expansion of u at (x,t); more precisely, we will prove

$$|u(y,s)-u(x,t)-P(y,s)| \le nM(|x-y|^3+|x-y|^2|t-s|+|x-y||s-t|+|t-s|^2) \quad \text{for all } (y,s) \in Y_1 \cap \{s \le t\}. \tag{11}$$

Once (11) is established, we may conclude $(x,t) \in \Psi(u, Y_1) \cap K_1$, and the proof of the theorem will be complete.

To prove (11), let us fix any $(y,s) \in Y_1 \cap \{s \le t\}$. First, we express the difference between $u(y,s)$ and $u(x,t)$ in the following way:

$$u(y,s) - u(x,t) = \int_0^1 \frac{d}{d\tau} u(x+\tau(y-x), t+\tau(s-t))\, d\tau.$$

Expanding the right-hand side of the previous line in terms of Du and u_t we find

$$u(y,s) - u(x,t) = \int_0^1 (y-x)\cdot Du(x+\tau(y-x), t+\tau(s-t))\, d\tau + \int_0^1 (s-t)u_t(x+\tau(y-x), t+\tau(s-t))d\tau.$$

Now we subtract $P(y,s)$ from both sides of the previous line. Using the definition of P (equation (10)) and rearranging yields,

$$\begin{aligned} &u(y,s) - u(x,t) - P(y,s) \\ &\quad = (y-x)\cdot \int_0^1 \left(Du(x+\tau(y-x), t+\tau(s-t)) - Du(x,t) - \tau Q(y-x)^T\right)\, d\tau + \\ &\quad + (s-t)\int_0^1 u_t(x+\tau(y-x), t+\tau(s-t)) - u_t(x,t) - \tau p^{n+1}\cdot(y-x)\, d\tau. \end{aligned}$$

We take the absolute value of both sides of the previous line and find

$$\begin{aligned} &|u(y,s) - u(x,t) - P(y,s)| \\ &\quad \le |y-x| \int_0^1 \left(\sum_{i=1}^n |u_{x_i}(x+\tau(y-x), t+\tau(s-t)) - u_{x_i}(x,t) - \tau p^i\cdot(y-x)|^2\, d\tau\right)^{1/2} + \\ &\quad + |s-t| \int_0^1 |u_t(x+\tau(y-x), t+\tau(s-t)) - u_t(x,t) - \tau p^{n+1}\cdot(y-x)|\, d\tau. \end{aligned}$$

We use (9) to bound each term on the right-hand side of the previous line, and obtain

$$|u(y,s) - u(x,t) - P(y,s)| \leq n|y-x| \int_0^1 \frac{1}{2} M\tau^2 |x-y|^2 + M\tau|t-s|\, d\tau$$
$$+ |s-t| \int_0^1 \frac{1}{2} M\tau^2|x-y|^2 + M\tau|t-s|\, d\tau.$$

Integrating in τ and combining terms yields

$$|u(y,s) - u(x,t) - P(y,s)| \leq nM(|x-y|^3 + |x-y||s-t| + |x-y|^2|t-s| + |t-s|^2).$$

Since $(y,s) \in Y_1 \cap \{s \leq t\}$ was arbitrary, we have established (11) and so the proof of the theorem is complete. □

Corollary 3.7. *Assume F satisfies (F1) and (F2). Assume u is a solution of $u_t - F(D^2u) = 0$ in $Y_r(\bar{x},\bar{t})$ and Du and u_t exist and are continuous in $Y_r(\bar{x},\bar{t})$. There exists a subset $\Psi_M(u, Y_r(\bar{x},\bar{t}))$ of $Y_r(\bar{x},\bar{t})$ and a universal constant C such that for any $(x,t) \in \Psi_M(u, Y_r(\bar{x},\bar{t}))$, there exists $Q \in \mathcal{P}_\infty$ with*

$$Q_t - F(D^2Q) = 0$$

and such that on $Y_r(\bar{x},\bar{t}) \cap \{s \leq t\}$,

$$|u(y,s) - u(x,t) - Q(y,s)| \leq \frac{nM}{r^2}(r|x-y|^3 + |x-y|^2|t-s| + |t-s|^2) + $$
$$+ \frac{C}{r}(M + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))})|x-y|(t-s). \quad (12)$$

Moreover,

$$|K_r(\bar{x},\bar{t}) \setminus \Psi_M(u, Y_r(\bar{x},\bar{t}))| \leq \frac{Cr^{n+2}}{M^\sigma}(r^{-\sigma}||Du||^\sigma_{L^\infty(Y_r(\bar{x},\bar{t}))} + ||u_t||^\sigma_{L^\infty(Y_r(\bar{x},\bar{t}))}). \quad (13)$$

Proof. We prove the statement for $r = 1$ and $(\bar{x},\bar{t}) = (0,0)$. The general case follows by translation and rescaling: the equation that u satisfies is translation invariant, and given u that solves $u_t - F(D^2u) = 0$ in Y_r, the rescaled function $\hat{u}(\hat{x},\hat{t}) = \frac{1}{r^2}u(r\hat{x}, r^2\hat{t})$ solves the same equation in Y_1. We omit the details of the rescaling argument and proceed with the proof in the case $r = 1$ and $(\bar{x},\bar{t}) = (0,0)$.

Fix $(x,t) \in \Psi_M(u, Y_1)$. We define the paraboloid $Q \in \mathcal{P}_\infty$ by $Q(y,s) = P(y,s) - \frac{1}{2}p^{n+1}\cdot(y-x)(s-t)$, where P and p^{n+1} are as in the proof of Theorem 3.2. By the definition of Q, we have,

$$|u(y,s) - u(x,t) - Q(y,s)| \leq |u(y,s) - u(x,t) - P(y,s)| + \frac{1}{2}|p^{n+1}||y-x||s-t|.$$

We use (11) to bound from above the first term on the right-hand side of the previous line, and find, for all $(y,s) \in Y_1 \cap \{s \leq t\}$,

$$|u(y,s) - u(x,t) - Q(y,s)| \leq nM(|x-y|^3 + |x-y||s-t| + |x-y|^2|t-s| + |t-s|^2) +$$
$$+ \frac{1}{2}|p^{n+1}||y-x||s-t|. \quad (14)$$

Let us now estimate $|p^{n+1}|$. To this end, we use the second inequality in (9) to obtain, for any $(y,s) \in Y_1 \cap \{s \le t\}$,

$$|p^{n+1} \cdot (y-x)| \le \frac{1}{2}M|x-y|^2 + M|t-s| + 2||u_t||_{L^\infty(Y_1)}.$$

Let us evaluate the previous inequality at $s=t$ and at a point y such that $|x-y| = \frac{1}{2}$. This yields,

$$|p^{n+1}| \le \frac{1}{4}M + 4||u_t||_{L^\infty(Y_1)}.$$

We use this to bound the last term on the right-hand side of (14). We also rearrange so that both terms involving $|y-x||s-t|$ are together. This yields the bound (12) with $r=1$ and $(\bar{x},\bar{t}) = (0,0)$, and so the proof is complete. □

4. Two regularizations of inf- sup- type

This section is devoted to introducing and establishing the properties of two regularizations of inf-sup type. In Subsect. 4.1 we recall the definitions of inf- and sup- convolutions in both the space and time variables, which are quite similar to those used in the regularity theory of elliptic equations, and to the regularizations of [27, Section 4]. In the proof of our main result, Theorem 1.1, we will use inf- and sup- convolutions to regularize the δ-solution u_δ.

In Subsect. 4.2 we define regularizations of inf-sup type that we call x-inf and x-sup convolutions. We will use them to regularize the viscosity solution u in the proofs of Theorems 1.1 and 1.2.

In Subsect. 4.3, we prove that if u is a solution of $u_t - F(D^2u) = 0$ then the x-inf and x-sup convolutions of u enjoy regularity properties similar to those established in Theorem 3.2 for u. This result is a very important ingredient in our proofs of Theorems 1.1 and 1.2.

4.1. Regularization in both the space and time variables

We recall the definitions of inf- and sup- convolutions. We use the notation $v^-_{\theta,\theta}$, instead of the expected v^-_θ, to distinguish these from the x-inf- and x-sup-convolutions that we introduce in the next subsection.

Definition 4.1. For $v \in C(\Omega \times (0,T))$ and $\theta > 0$, we define the *inf-convolution* $v^-_{\theta,\theta}$ and the *sup-convolution* $v^+_{\theta,\theta}$ by

$$v^-_{\theta,\theta}(x,t) = \inf_{\Omega\times(0,T)} \left\{ v(y,s) + \frac{|x-y|^2}{2\theta} + \frac{|t-s|^2}{2\theta} \right\}$$

and

$$v^+_{\theta,\theta}(x,t) = \sup_{\Omega\times(0,T)} \left\{ v(y,s) - \frac{|x-y|^2}{2\theta} - \frac{|t-s|^2}{2\theta} \right\}.$$

Definition 4.2. Given $\theta > 0$, $\delta > 0$ and $v \in C(\Omega \times (0,T))$, we define

$$U^{\theta,\delta} = \left\{ (x,t) \in \Omega \times (0,T) : \quad d_e((x,t), \partial_p(\Omega \times (0,T))) \ge 2\theta^{1/2}||v||^{1/2}_{L^\infty(\Omega\times(0,T))} + \delta \right\}.$$

We summarize the basic properties of inf- and sup- convolutions in Proposition A.4 of the Appendix. One property that is particularly important and useful to us is that taking inf- and sup- convolution preserves the notion of δ-super and δ-sub solutions. We state this precisely in item (5) of Proposition A.4.

4.2. Regularization in only the space variable

We now define the x-inf and x-sup convolutions.

Definition 4.3. For $u \in C(\Omega \times (0,T))$ and $\theta > 0$, we define the *x-sup convolution* u_θ^+ and *x-inf-convolution* u_θ^- of u by

$$u_\theta^+(x,t) = \sup_{y\in\Omega}\left\{u(y,t) - \frac{|x-y|^2}{2\theta}\right\} \text{ and } u_\theta^-(x,t) = \inf_{y\in\Omega}\left\{u(y,t) + \frac{|x-y|^2}{2\theta}\right\}.$$

Definition 4.4. For $u \in C_x^{0,1}(\Omega \times (0,T))$ and $\tilde{\Omega} \subseteq \Omega$ we define the subset $\tilde{\Omega}^\theta$ of $\tilde{\Omega}$ by

$$\tilde{\Omega}^\theta = \left\{x \in \tilde{\Omega} : \inf_{y\in\partial\tilde{\Omega}} |x-y| \geq 2\theta ||Du||_{L^\infty(\Omega\times(0,T))}\right\}.$$

In addition, for $U = \tilde{\Omega} \times I$ where $I \subseteq (0,T)$ is an interval, we define the subset U^θ of U by

$$U^\theta = \tilde{\Omega}^\theta \times I.$$

Note that for the definition of U^θ to be meaningful, the size of U cannot be too small compared to θ. Whenever we use this notation we make sure this is not the case.

In addition, we denote $Y_r^\theta(\bar{x},\bar{t}) = (Y_r(\bar{x},\bar{t}))^\theta$ and $K_r^\theta(\bar{x},\bar{t}) = (K_r(\bar{x},\bar{t}))^\theta$. We will be using these families of sets quite frequently, so we write down their definitions explicitly for the convenience of the reader:

$$\begin{aligned} Y_r^\theta(\bar{x},\bar{t}) &= B(\bar{x}, r - 2\theta||Du||_{L^\infty(\Omega\times(0,T))}) \times (\bar{t}, \bar{t}+r^2] \\ K_r^\theta(\bar{x},\bar{t}) &= \left[\bar{x} - \left(\frac{r}{9\sqrt{n}} - 2\theta||Du||_{L^\infty(\Omega\times(0,T))}\right),\right. \\ &\qquad \left.\bar{x} + \left(\frac{r}{9\sqrt{n}} - 2\theta||Du||_{L^\infty(\Omega\times(0,T))}\right)\right]^n \times \left(\bar{t}, \bar{t} + \frac{r^2}{81n}\right]. \end{aligned}$$

In the following proposition we state the facts about x-inf- and x-sup-convolutions that we will use in this paper. Their proofs are very similar to those in the elliptic case (see [9, Chapter 4] or [7, Proposition 5.3]) and we omit them.

Proposition 4.5. *Assume $u \in C_x^{0,1}(\Omega \times (0,T))$. Then:*

(1) *If (x^*,t) is any point at which the infimum (resp. supremum) is achieved in the definition of $u_\theta^-(x,t)$ (resp. $u_\theta^+(x,t)$), then*

$$|x - x^*| \leq 2\theta ||Du||_{L^\infty(\Omega\times(0,T))}.$$

Moreover, if $(x,t) \in U^\theta$ then $(x^*,t) \in U$.

(2) *We have $u_\theta^-(x,t) \leq u(x,t) \leq u_\theta^+(x,t)$ for all $(x,t) \in \Omega \times (0,T)$.*

(3) *For all* $(x,t) \in \Omega^\theta \times (0,T)$, *we have*
$u_\theta^+(x,t) - 2\theta||Du||^2_{L^\infty(\Omega\times(0,T))} \le u(x,t) \le u_\theta^-(x,t) + 2\theta||Du||^2_{L^\infty(\Omega\times(0,T))}$.

(4) *In the sense of distributions,* $D^2u_\theta^+(x,t) \ge -\theta^{-1}I$ *and* $D^2u_\theta^-(x,t) \le \theta^{-1}I$ *for all* $(x,t) \in \Omega \times (0,T)$.

(5) *If* u *is a subsolution of*

$$u_t - F(D^2u) = c \tag{15}$$

in $\Omega \times (0,T)$, *then* u_θ^+ *is a subsolution of* (15) *in* $\Omega^\theta \times (0,T)$. *If* u *is a supersolution of* (15) *in* $\Omega \times (0,T)$, *then* u_θ^- *is a supersolution of* (15) *in* $\Omega^\theta \times (0,T)$.

4.3. Regularity of x-inf and x-sup convolutions

This subsection is devoted to the proof of Proposition 4.6, which states that the extra regularity established in Theorem 3.2 for a solution u of $u_t - F(D^2u) = 0$ carries over to u_θ^+ and u_θ^-.

Proposition 4.6. *Assume (F1) and (F2). Suppose* $u \in C(\Omega \times (0,T))$ *is a solution of* $u_t - F(D^2u) = 0$ *in* $\Omega \times (0,T)$. *Assume* $Y_r(\bar{x},\bar{t}) \subset \Omega \times (0,T)$ *and*

$$d(Y_r(\bar{x},\bar{t}), \partial_p(\Omega \times (0,T))) \ge \theta. \tag{16}$$

There exist universal constants M_0, σ, *and* C *such that for every* $M \ge M_0$, *there exists a set* $\Psi_M^{+,\theta} \subset K_r^\theta(\bar{x},\bar{t})$ *(resp.* $\Psi_M^{-,\theta} \subset K_r^\theta(\bar{x},\bar{t})$*) such that for any* $(x,t) \in \Psi_M^{+,\theta}$ *(resp.* $(x,t) \in \Psi_M^{-,\theta}$*), there exists a polynomial* $P \in \mathcal{P}_\infty$ *such that* $P(x,t) = 0$,

$$P_t - F(D^2P) = 0, \tag{17}$$

and if $Y_\rho^-(x,t) \subset Y_r^\theta(\bar{x},\bar{t})$, *then for all* $(y,s) \in Y_\rho^-(x,t)$, *we have*

$$u_\theta^+(y,s) - u_\theta^+(x,t) \ge P(y,s) - nM\frac{\rho}{r}|x-y|^2 - \frac{\rho}{r}\left(CM + \frac{1 + ||u||_{L^\infty(\Omega\times(0,T))}}{\theta^2}\right)(t-s) \tag{18}$$

$$\left(\textit{resp. } u_\theta^-(y,s) - u_\theta^-(x,t) \le P(y,s) + nM\frac{\rho}{r}|x-y|^2 + \frac{\rho}{r}\left(CM + \frac{1 + ||u||_{L^\infty(\Omega\times(0,T))}}{\theta^2}\right)(t-s).\right)$$

Moreover,

$$|K_r^\theta(\bar{x},\bar{t}) \setminus \Psi_M^{\pm,\theta}| \le \frac{Cr^{n+2}}{M^\sigma \theta^{n+\sigma}}(r^{-\sigma} + \theta^{-\sigma})\left(||u||_{L^\infty(\Omega\times(0,T))} + 1\right)^{n+\sigma}. \tag{19}$$

Outline of the proof of Proposition 4.6.

We prove that if the supremum in the definition of $u_\theta^+(x,t)$ is achieved at a point (x^*,t) that is contained in the "good set" $\Psi_M(u)$ of u, then an estimate similar to (18) holds at (x,t). We state this as Lemma 4.7 below. This follows from Corollary 3.7 and the observation that if u is touched from below at (x^*, t) by some function ϕ, then u_θ^+ is touched by below at (x, t) by a translate of ϕ. We make this observation precise in Lemma 4.11, which we use in the proof of Lemma 4.7.

The previous discussion indicates that $\Psi_M^{+,\theta}$, the "good set" for u_θ^+, should be defined as the points (x, t) for which (x^*, t) is in the good set $\Psi_M(u)$ of u. To obtain a bound on the size of $\Psi_M^{+,\theta}$ we thus consider a map that takes x^* to x and investigate what happens to the size of $\Psi_M(u)$ under this map. For this we need Lemma 4.8, which says that u is twice differentiable in x at any point (x^*, t) where the supremum or infimum is achieved in the regularizations u_θ^+ or u_θ^-, and gives a two-sided estimate for D^2u at such points.

Statement of lemmas that we use in the proof of Proposition 4.6

Lemma 4.7. *Let $Y_r(\bar{x}, \bar{t}) \subset \Omega \times (0, T)$ and let $\Psi(u, Y_r(\bar{x}, \bar{t}))$ be the set given by Corollary 3.7. Let us suppose that the supremum in the definition of $u_\theta^+(x, t)$ is achieved at $(x^*, t) \in \Psi_M(u, Y_r(\bar{x}, \bar{t}))$. Then there exists a paraboloid $P \in \mathcal{P}_\infty$ that satisfies (17) and such that, for all $(y, s) \in Y_r^\theta(\bar{x}, \bar{t}) \cap \{s \le t\}$,*

$$u_\theta^+(y, s) - u_\theta^+(x, t) \ge P(y, s) - \frac{nM}{r^2}(r|x-y|^3 + |x-y|^2|t-s| + |t-s|^2) - C\frac{M + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}}{r}|x-y|(t-s). \tag{20}$$

Lemma 4.8. *Under the assumptions of Proposition 4.6, let $(x, t) \in Y_r^\theta(\bar{x}, \bar{t})$. There exists a universal constant C such that if x^* is any point where the supremum (resp. infimum) is achieved in the definition of $u_\theta^+(x, t)$ (resp. $u_\theta^-(x, t)$), then $u(y, t)$ is twice differentiable in y at x^* with*

$$-C(||u||_{L^\infty(\Omega\times(0,T))} + 1)\theta^{-2}I \le D^2u(x^*, t) \le \theta^{-1}I$$

$$\big(\textit{resp. } -\theta^{-1}I \le D^2u(x^*, t) \le C(||u||_{L^\infty(\Omega\times(0,T))} + 1)\theta^{-2}I.\big)$$

We postpone the proof of Lemmas 4.7 and 4.8 and proceed with:

Proof of Proposition 4.6. We will give the proof for u_θ^+; the arguments for u_θ^- are similar. Fix M and, to simplify the notation, denote the set $\Psi_M(u, Y_r(\bar{x}, \bar{t}))$ given by Corollary 3.7 by just Ψ_M. We define the set $\mathcal{C}$ by

$$\mathcal{C} = \big\{(x^*, t) : (x^*, t) \text{is a point at which the supremum in the definition of } u_\theta^+(x, t) \text{ is achieved, for some } (x, t) \in Y_r^\theta(\bar{x}, \bar{t})\big\}.$$

Item (1) of Proposition 4.5 implies $\mathcal{C} \subset Y_r(\bar{x}, \bar{t})$. For any $(x^*, t) \in \mathcal{C}$, the map

$$y \mapsto \left(u(y, t) - \frac{|x-y|^2}{2\theta}\right)$$

has a local maximum at x^*, because the supremum in the definition of $u_\theta^+(x, t)$ is achieved at (x^*, t). Hence its derivative has a zero at x^*. Computing the derivative and setting it equal to zero gives,

$$Du(x^*, t) + \frac{(x - x^*)}{\theta} = 0.$$

Rearranging this equality yields $x = x^* - \theta Du(x^*, t)$. We define the map $T : \Omega \times (0, T) \to \mathbb{R}^n \times (0, T)$ by

$$T(y, t) = (y - \theta Du(y, t), t),$$

so that $(x, t) = T(x^*, t)$. By Lemma 4.8, Du is Lipschitz on $\mathcal{C}$; hence, the map T is too. Let us denote by $D_{y,t}T$ the matrix of derivatives of T in both the space and time variables, and by D_yT the matrix of derivatives of T in only the space variable. We will now estimate the determinant of $D_{y,t}T$ on $\mathcal{C}$. We observe that $\det D_{y,t}T = \det D_yT$. Let us fix $(y, t) \in \mathcal{C}$. We use the definition of the map T to express D_yT in terms of D^2u:

$$D_yT(y, t) = I - \theta D^2u(y, t).$$

We use Lemma 4.8 to bound $-\theta D^2u(y, t)$ from above and below and obtain,

$$0 \le D_yT(y, t) \le I \cdot C\theta^{-1}(||u||_{L^\infty(\Omega\times(0,T))} + 1).$$

The left-most inequality in the previous line says $D_yT(y, t)$ is a non-negative definite matrix. Hence, taking determinants preserves inequalities, and so we find

$$0 \le \det D_{y,t}T(y, t) \le C\theta^{-n}(||u||_{L^\infty(\Omega\times(0,T))} + 1)^n, \tag{21}$$

where we have also used $\det D_yT = \det D_{y,t}T$.

We define the set A by

$$A = T(\mathcal{C} \cap (K_r(\bar{x}, \bar{t}) \setminus \Psi_M)).$$

By the Area Formula ([13, Chapter 3]), we have

$$|A| \le \int_{\mathcal{C}\cap(K_r(\bar{x},\bar{t})\setminus\Psi_M)} |\det D_{y,t}T(y, t)| \, dy \, dt.$$

We use the bound (21) on $\det D_{y,t}T(y, t)$ and the bound (13) on $|K_r(\bar{x}, \bar{t})\setminus\Psi_M|$ of Corollary 3.7 to estimate the right-hand side of the previous line from above and obtain,

$$|A| \le \frac{Cr^{n+2}}{M^\sigma}(r^{-\sigma}||Du||^\sigma_{L^\infty(Y_r(\bar{x},\bar{t}))} + ||u_t||^\sigma_{L^\infty(Y_r(\bar{x},\bar{t}))})\frac{(||u||_{L^\infty(\Omega\times(0,T))} + 1)^n}{\theta^n}. \tag{22}$$

We define the set $\Psi^{+,\theta}_M$ by

$$\Psi^{+,\theta}_M := K^\theta_r(\bar{x}, \bar{t}) \setminus A.$$

We will now use Lemma 4.7 to prove that for each point (x, t) of $\Psi^{+,\theta}_M$ there exists $P \in \mathcal{P}_\infty$ satisfying (17) and (18). To this end, let us fix $(x, t) \in \Psi^{+,\theta}_M$. Let (x^*, t) be any point at which the supremum in the definition of $u^+_\theta(x, t)$ is achieved. We have $(x^*, t) \in \mathcal{C}$. Since $(x, t) \in K^\theta_r(\bar{x}, \bar{t})$, by item (1) of Proposition 4.5 we also have $(x^*, t) \in K_r(\bar{x}, \bar{t})$. In addition, since $(x, t) = T(x^*, t)$, and $(x, t) \notin A$ by the definition of $\Psi^{+,\theta}_M$, we find

$$T(x^*, t) = (x, t) \notin A = T(\mathcal{C} \cap (K_r(\bar{x}, \bar{t}) \setminus \Psi_M)),$$

where the second equality follows simply by recalling the definition of A. Therefore, $(x^*,t) \notin \mathcal{C} \cap (K_r(\bar{x},\bar{t}) \setminus \Psi_M)$. Since we've already shown that (x^*,t) is contained in $\mathcal{C} \cap K_r(\bar{x},\bar{t})$, this implies $(x^*,t) \in \Psi_M$. Thus, we have

$$(x^*,t) \in K_r(\bar{x},\bar{t}) \cap \Psi_M.$$

Since (x^*,t) is a point at which the supremum in the definition of $u^+_\theta(x,t)$ is achieved, this means we are in exactly the situation of Lemma 4.7. Therefore, we apply the lemma and conclude that there exists a paraboloid $P \in \mathcal{P}_\infty$ satisfying (17) and such that the inequality (20) holds for all $(y,s) \in Y^\theta_r(\bar{x},\bar{t}) \cap \{s \le t\}$. Finally, if the backward cylinder $Y^-_\rho(x,t)$ is contained in $Y^\theta_r(\bar{x},\bar{t})$, then for any $(y,s) \in Y^-_\rho(x,t)$ we have $(y,s) \in Y^\theta_r(\bar{x},\bar{t}) \cap \{s \le t\}$ and so the inequality (20) holds at (y,s). Moreover, we have $|x-y| \le \rho$ and $|t-s| \le \rho^2$. Using this to bound the right-hand side of (20) from below, we obtain, for all $(y,s) \in Y^-_\rho(x,t)$,

$$\begin{aligned} u^+_\theta(y,s) - u^+_\theta(x,t) \ge P(y,s) &- \frac{nM}{r^2}(r\rho|x-y|^2 + \rho^2|t-s|) \\ &- C\rho\frac{M + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}}{r}(t-s). \end{aligned}$$

By assumption we have $\rho \le r$; therefore, we have $\frac{\rho^2}{r^2} \le \frac{\rho}{r}$. We use this to estimate the right-hand side of the previous line from below, and rearrange so that all the terms with $(t-s)$ are together, to obtain, for all $(y,s) \in Y^-_\rho(x,t)$,

$$u^+_\theta(y,s) - u^+_\theta(x,t) \ge P(y,s) - nM\tfrac{\rho}{r}|x-y|^2 - \tfrac{\rho}{r}\left(CM + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}\right)(t-s). \tag{23}$$

Since, by assumption, $d(Y_r(\bar{x},\bar{t}), \partial_p\Omega \times (0,T)) \ge \theta$, Proposition 2.13 implies

$$||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))} \le \theta^{-2}(||u||_{L^\infty(\Omega\times(0,T))} + 1)$$

and

$$||Du||_{L^\infty(Y_r(\bar{x},\bar{t}))} \le \theta^{-1}(||u||_{L^\infty(\Omega\times(0,T))} + 1).$$

We use these bounds in (22) and (23) to complete the proof. □

We now give the proofs of the two lemmas, starting with Lemma 4.8. First, we need to establish the following basic fact about how parabolic viscosity solutions relate to viscosity solutions in only x:

Lemma 4.9. *If u is a Lipschitz in t viscosity solution of $u_t - F(D^2u) = 0$ in Y_r, then for any $t \in (0,r^2)$ the function $x \mapsto u(x,t)$ satisfies, in the viscosity sense,*

$$\begin{aligned} \mathcal{M}^-(D^2u(x,t)) &\le ||u_t||_{L^\infty(Y_r)} \quad \text{in } B_r, \\ \mathcal{M}^+(D^2u(x,t)) &\ge -||u_t||_{L^\infty(Y_r)} \quad \text{in } B_r. \end{aligned}$$

Proof. We will check that $u(\cdot,t)$ is a subsolution; checking that $u(\cdot,t)$ is a supersolution is analogous. Let us suppose that $\phi(y) \in C^2(B_r)$ touches $u(y,t)$ from above at $x \in B_r$, so that

$$u(y,t) \le \phi(y) \quad \text{for all } y \in B_r, \text{ with equality at } y = x. \tag{24}$$

Since u is Lipschitz in t, we have

$$u(y,s) \le u(y,t) + ||u_t||_{L^\infty(Y_r)}(t-s) \quad \text{for all } y \in B_r \text{ and for all } s \le t, \quad \text{with equality at } s = t.$$

Using the inequality (24) to bound the right-hand side of the previous line from above yields that $\phi(y) + ||u_t||_{L^\infty(Y_r)}(t-s)$ touches $u(y,s)$ from above at (x,t) in $B_r \times (0,t]$. Because u is a viscosity solution of $u_t - F(D^2u) = 0$, we obtain

$$-||u_t||_{L^\infty(Y_r)} - F(D^2\phi(x)) \le 0.$$

The uniform ellipticity of F implies

$$0 \ge -||u_t||_{L^\infty(Y_r)} - \mathcal{M}^+(D^2\phi(x)),$$

as desired. □

We also recall the Harnack inequality for *elliptic* equations (see [7, Theorem 4.3]).

Theorem 4.10. *Assume that $u : B_s(0) \subset \mathbb{R}^n \to \mathbb{R}$ satisfies $u \ge 0$ in $B_s(0)$ and, for some positive constant c,*

$$\mathcal{M}^-(D^2u) \le c \quad \text{in } B_s(0)$$
$$\mathcal{M}^+(D^2u) \ge -c \quad \text{in } B_s(0).$$

There exits a universal constant C so that

$$\sup_{B_{s/2}(0)} u \le C\left(\inf_{B_{s/2}(0)} u + s^2c\right).$$

We are now ready to proceed with:

Proof of Lemma 4.8. We will give the proof for $u_\theta^+(x,t)$; the argument for $u_\theta^-(x,t)$ is very similar.

Let x^* be a point at which the supremum in the definition of $u_\theta^+(x,t)$ is achieved, so that

$$u(x^*,t) - \frac{|x-x^*|^2}{2\theta} \ge u(y,t) - \frac{|x-y|^2}{2\theta} \quad \text{for all } y \in \Omega. \tag{25}$$

Therefore, as a function of y, $u(y,t)$ is touched from above at x^* by the convex paraboloid $Q(y)$, where

$$Q(y) = u(x^*,t) - \frac{|x-x^*|^2}{2\theta} + \frac{|x-y|^2}{2\theta}.$$

Thus, we have a bound from above on $D^2u(x^*,t)$:

$$D^2u(x^*,t) \le \theta^{-1}I.$$

Next we will show, using the Harnack inequality, that $u(\cdot,t)$ is also touched from below at x^* by a paraboloid with bounded opening, thus obtaining a bound from below on $D^2u(x^*,t)$.

There exists an affine function $l(y)$ with $l(x^*) = u(x^*, t)$ and such that

$$Q(y) = l(y) + \frac{|y - x^*|^2}{2\theta}.$$

There exists $\rho > 0$ such that $B_\rho(x^*) \subset B_r(\bar{x})$. We fix any $0 < s < \rho$ and define the function $w(y)$ by

$$w(y) = l(y) + \frac{s^2}{2\theta} - u(y, t).$$

We will apply the Harnack inequality, Theorem 4.10, to w in $B_s(x^*)$ once we verify its hypotheses. First, we verify that w is non-negative on $B_s(x^*)$. Let us fix $y \in B_s(x^*)$. We use the definition of w, the fact that $u \leq Q$ on $B_s(x^*)$ and the definition of Q, to obtain:

$$w(y) = l(y) + \frac{s^2}{2\theta} - u(y,t) \geq l(y) + \frac{s^2}{2\theta} - Q(y) = \frac{s^2}{2\theta} - \frac{|y - x^*|^2}{2\theta} \geq 0.$$

Next, by the properties of the extremal operators and Lemma 4.9, we see that w satisfies

$$\mathcal{M}^-(D^2 w) = \mathcal{M}^-(-D^2 u) = -\mathcal{M}^+(D^2 u) \leq ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))} \quad \text{in } B_s(x^*),$$
$$\mathcal{M}^+(D^2 w) = \mathcal{M}^+(-D^2 u) = -\mathcal{M}^-(D^2 u) \geq -||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))} \quad \text{in } B_s(x^*).$$

Therefore, the Harnack inequality implies that there exists a universal constant C such that

$$\sup_{B_{s/2}(x^*)} w \leq C \left(\inf_{B_{s/2}(x^*)} w + s^2 ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))} \right). \tag{26}$$

We evaluate w at x^* and use that $l(x^*) = u(x^*, t)$ to obtain a upper bound on the infimum of w:

$$\inf_{B_{s/2}(x^*)} w \leq w(x^*) = l(x^*) + \frac{s^2}{2\theta} - u(x^*, t) = \frac{s^2}{2\theta}.$$

We use this to bound the first term on the left-hand side of (26) from above and thus obtain an upper bound on the supremum of w:

$$\sup_{B_{s/2}(x^*)} w \leq C s^2 (\theta^{-1} + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}).$$

Next we use this upper bound to estimate the difference between Q and u from above. We use the definitions of Q and of w, and the estimate on the supremum of w in the previous line, to find, for all $y \in B_{s/2}(x^*)$,

$$Q(y) - u(y,t) = l(y) + \frac{|y - x^*|^2}{2\theta} - u(y,t) \leq w(y) \leq C s^2 (\theta^{-1} + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}).$$

Since this holds for *all* $0 < s < \rho$, we have that $(Q(\cdot) - u(\cdot, t))$ is touched from above at x^* by a convex paraboloid of opening $C(\theta^{-1} + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))})$. Therefore,

$$D^2 Q(x^*) - D^2 u(x^*, t) \leq C(\theta^{-1} + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}) I.$$

Rearranging the previous inequality and using $D^2 Q(x^*) = \theta^{-1} I$ yields

$$D^2 u(x^*, t) \geq -C(\theta^{-1} + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}) I. \tag{27}$$

Since assumption (16) of this proposition says that $Y_r(\bar{x},\bar{t})$ is distance θ away from the boundary of $\Omega \times (0,T)$, Proposition 2.14 implies

$$||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))} \le C\theta^{-2}(||u||_{L^\infty(\Omega\times(0,T))} + 1).$$

We use this to bound the right-hand side of (27) from below and obtain the desired estimate. □

Next, we give the proof of Lemma 4.7. We need the following lemma, which we mentioned in the outline of the proof of Proposition 4.6.

Lemma 4.11. *Let us suppose that the supremum in the definition of $u_\theta^+(x,t)$ is achieved at $(x^*,t) \in Y_r(\bar{x},\bar{t}) \subset \Omega \times (0,T)$. In addition, suppose that for some $\phi \in C(\Omega \times (0,T))$ we have*

$$u(z,s) - u(x^*,t) \ge \phi(z,s) \tag{28}$$

for all $(z,s) \in Y_r(\bar{x},\bar{t}) \cap \{s \le t\}$ and $\phi(x^*,t) = 0$. Then, for all $(y,s) \in Y_r^\theta(\bar{x},\bar{t}) \cap \{s \le t\}$,

$$u_\theta^+(y,s) - u_\theta^+(x,t) \ge \phi(y + x^* - x, s).$$

Proof of Lemma 4.11. Since (x^*,t) is a point at which the supremum is achieved in the definition of $u_\theta^+(x,t)$, we have

$$u_\theta^+(x,t) = u(x^*,t) - \frac{|x-x^*|^2}{2\theta}. \tag{29}$$

Let (y,s) be any point in $Y_r^\theta(\bar{x},\bar{t}) \cap \{s \le t\}$. By item (1) of Proposition 4.5, we have $(y + x - x^*, s) \in Y_r(\bar{x},\bar{t})$. In particular, we have $(y + x - x^*) \in \Omega$, so we may use $(y + x^* - x)$ as a "test point" in the definition of $u_\theta^+(y,s)$. We obtain

$$u_\theta^+(y,s) \ge u(y + x^* - x, s) - \frac{|x-x^*|^2}{2\theta}.$$

Subtracting the equality (29) from the above gives, for all $y \in Y_r^\theta(\bar{x},\bar{t})$,

$$u_\theta^+(y,s) - u_\theta^+(x,t) \ge u(y + x^* - x, s) - u(x^*,t). \tag{30}$$

Since $(y + x^* - x, s) \in Y_r(\bar{x},\bar{t}) \cap \{s \le t\}$, we may evaluate (28) at $z = y + x^* - x$ to find,

$$u(y + x^* - x, s) - u(x^*,t) \ge \phi(y + x^* - x, s).$$

We use this to bound the right-hand side of (30) and obtain,

$$u_\theta^+(y,s) - u_\theta^+(x,t) \ge \phi(y + x^* - x, s).$$

This holds for any (y,s) in $Y_r^\theta(\bar{x},\bar{t}) \cap \{s \le t\}$; hence the proof of the lemma is complete. □

We now proceed with:

Proof of Lemma 4.7. Since $(x^*,t) \in \Psi_M(u, Y_r(\bar{x},\bar{t}))$, Corollary 3.7 implies that there exists a polynomial $Q \in \mathcal{P}_\infty$ with $Q_t - F(D^2Q) = 0$ such that for $(z,s) \in Y_r(\bar{x},\bar{t}) \cap \{s \le t\}$,

$$|u(z,s) - u(x^*,t) - Q(z,s)| \le \frac{nM}{r^2}(r|x^*-z|^3 + |x^*-z|^2|t-s| + |t-s|^2) + \\ + C\frac{M + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}}{r}|x^*-z|(t-s).$$

In particular, we have $u(z,s) - u(x^*,t) \ge \phi(z,s)$ for all $(z,s) \in Y_r(\bar{x},\bar{t}) \cap \{s \le t\}$, where we take ϕ to be

$$\phi(z,s) = Q(z,s) - \frac{nM}{r^2}(r|x^*-z|^3 + |x^*-z|^2|t-s| + |t-s|^2) - \\ - C\frac{M + ||u_t||_{L^\infty(Y_r(\bar{x},\bar{t}))}}{r}|x^*-z|(t-s).$$

We note $\phi(x^*,t) = Q(x^*,t) = 0$. Therefore, according to Lemma 4.11, we have,

$$u^+_\theta(y,s) - u^+_\theta(x,t) \ge \phi(y + x^* - x, s) \text{ for all } (y,s) \in Y^\theta_r(\bar{x},\bar{t}) \cap \{s \le t\}. \quad (31)$$

Let us now use the definition of ϕ to obtain an alternate expression for the right-hand side of (31):

$$\phi(y+x^*-x,s) = Q(y+x^*-x,s) - \frac{nM}{r^2}(r|x-y|^3 + |x-y|^2|t-s| + |t-s|^2) - \\ - C\frac{M + ||u||_{L^\infty(Y_r(\bar{x},\bar{t}))}}{r}|x-y|(t-s).$$

Let us define the paraboloid $P(y,s) \in \mathcal{P}_\infty$ by $P(y,s) = Q(y-x+x^*,s)$. Using this definition and the above expression for $\phi(y+x^*-x,s)$ in the inequality (31) gives that (20) holds for all $(y,s) \in Y^\theta_r(\bar{x},\bar{t}) \cap \{s \le t\}$. Finally, we observe that P satisfies the equation (17): indeed, P is only a translate of Q, and since $Q \in \mathcal{P}_\infty$, we have that $Q_t(y,s)$ and $D^2Q(y,s)$ do not depend on (y,s). Therefore, we have $P_t - F(D^2P) = Q_t - F(D^2Q) = 0$. □

5. A key estimate between solutions and sufficiently regular δ-solutions

In this section we state and prove Proposition 5.1, a key part of the proofs of the two main results, Theorems 1.1 and 1.2. Roughly, Proposition 5.1 says that if w, u, and v are, respectively, a δ-subsolution, a solution and a δ-supersolution of (33) that satisfy

$$w \le u \le v$$

on the parabolic boundary of a region, and v and w are sufficiently regular, then we have

$$w - \tilde{c}\delta^{\tilde{\alpha}} \le u \le v + \tilde{c}\delta^{\tilde{\alpha}}$$

on the interior of the region. Once we have proven Proposition 5.1, we will only need to use the basic properties of inf- and sup- convolutions to verify its hypotheses and establish the two main results. We define the constant ζ by

$$\zeta = \frac{\sigma}{2(4n+3+3\sigma)}, \tag{32}$$

where σ is the constant from Proposition 4.6. We remark that since σ is universal, so is ζ.

Proposition 5.1. *Assume (F1), (F2). Assume that Ω' satisfies (U1) and let (a,b) be a subset of $(0,T)$ for some $T>0$. We denote*

$$U = \Omega' \times (a,b).$$

Assume $u \in C_x^{0,1}(U)$ is a solution of

$$u_t - F(D^2u) = 0 \tag{33}$$

in U. For $\delta \in (0,1)$, define the quantity r_δ by

$$r_\delta = 9\sqrt{n}(1 + ||u||_{L^\infty(U)} + 2||Du||_{L^\infty(U)})\delta^\zeta$$

and the set $\tilde{U}$ by

$$\tilde{U} = \left\{(x,t) \in U : \quad t \le b - r_\delta^2 \text{ and } d((x,t), \partial_p U) > 2r_\delta\right\}.$$

There exist universal constants $\tilde{\alpha}$ and $\tilde{\delta}$ and a positive constant $\tilde{c}$ that depends on n, λ, Λ, $\operatorname{diam}\Omega'$, T, $||u||_{L^\infty(U)}$ and $||Du||_{L^\infty(U)}$ such that for all $\delta \le \tilde{\delta}$ and:

(1) for $v \in C^{0,1}(U)$ a δ-supersolution of (33) in U that satisfies
   - *(a) $D^2v(x,t) \le \delta^{-\zeta}I$ in the sense of distributions for every $(x,t) \in U$,*
   - *(b) $[v]_{C_t^{0,1}(U)} \le 3T\delta^{-\zeta}$, and*
   - *(c) $\sup_{\partial_p\tilde{U}}(u-v) \le 0$,*

   we have

$$\sup_{\tilde{U}}(u-v) \le \tilde{c}\delta^{\tilde{\alpha}};$$

(2) for $w \in C^{0,1}(U)$ a δ-subsolution of (33) in U with $D^2w(x,t) \ge -\delta^{-\zeta}I$ for every $(x,t) \in U$ in the sense of distributions, $[w]_{C_t^{0,1}(U)} \le 3T\delta^{-\zeta}$ and $\sup_{\partial_p\tilde{U}}(w-u) \le 0$, we have

$$\sup_{\tilde{U}}(w-u) \le \tilde{c}\delta^{\tilde{\alpha}}.$$

5.1. Outline

We outline the proof of part (1) of the Proposition; the proof of part (2) is very similar. The main idea of the proof is to control the supremum of $(u-v)$ by the size of the contact set of $(u-v)$ with the upper monotone envelope of $(u-v)$.

We first regularize u by x-sup convolution and obtain $u^+_{\delta\zeta}$ (see Definition 4.3). We then perturb $u^+_{\delta\zeta}$ by subtracting $\delta^{1/4}t$ to obtain a strict sub-solution.

We will bound $\sup(u^+_{\delta\varsigma} - \delta^{1/4}t - v)$. Since $u^+_{\delta\varsigma}$ and v are sufficiently regular, we are able to apply Proposition 2.12 and find

$$\sup_{\tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v) \leq C|\{u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v = \bar{\Gamma}\}|^{\frac{1}{n+1}}\delta^{-C\zeta},$$

where $\bar{\Gamma}$ is the upper monotone envelope of $u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v$ and C and c_1 are positive constants. This is Lemma 5.2 below. We will proceed by contradiction and assume that $\sup_{\tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v)$ is "large". By the estimate above, this implies that the size of the contact set $\{u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v = \bar{\Gamma}\}$ is large as well.

The key part of our argument is Proposition 4.6, which states that there exists a large subset Ψ of U on which $u^+_{\delta\varsigma}$ is very close to being a polynomial. We use this proposition, together with the fact that the size of the contact set is large, to find a point in the intersection of Ψ and the contact set. This is Lemma 5.3 below. It follows from Proposition 4.6 by a covering argument.

Our last ingredient is the fact that if (x, t) is a point in the contact set $\{u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v = \bar{\Gamma}\}$ at which $u^+_{\delta\varsigma}$ is touched from below by some ϕ, then v is touched from below by ϕ at that point as well. This is Lemma 5.4 below. We use this fact, applied at the point (x, t) in the intersection of Ψ and the contact set, to obtain the desired contradiction.

5.2. Proof of Proposition 5.1

Throughout the rest of this section, we will use C and C_i with $i = 1, 2, \ldots$, to denote positive constants that depend only n, λ, Λ, $\operatorname{diam}\Omega'$, T, $||u||_{L^\infty(U)}$ and $||Du||_{L^\infty(U)}$. For the proof of Proposition 5.1 we need three lemmas.

Lemma 5.2. *Under the assumptions of Proposition 5.1, let $\bar{x} \in \mathbb{R}^n$ and $\rho > 0$ be such that $\tilde{U} \subset Y^-_\rho(\bar{x}, b - r^2_\delta)$. There exists a constant $c_1 > 0$ that depends only on $||Du||_{L^\infty(U)}$ and a constant C_1 so that*

$$\sup_{\partial_p\tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - v) \leq c_1\delta^{\zeta}, \tag{34}$$

and

$$|\{u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v = \bar{\Gamma}\} \cap \tilde{U}| \geq C_1\delta^{2\zeta(n+1)}\left(\sup_{\tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v)\right)^{(n+1)}, \tag{35}$$

where $\bar{\Gamma}$ is the upper monotone envelope of $(u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v)^+$ extended by 0 to $Y^-_{2\rho}(\bar{x}, b - r^2_\delta)$.

Lemma 5.3. *Under the assumptions of Proposition 5.1, there exists a positive constant δ_1 that depends only on n, a universal constant $\bar{C}$, and a positive constant $\bar{c}$ that depends on n, λ, Λ, $\operatorname{diam}\Omega'$, $||u||_{L^\infty(U)}$ and $||Du||_{L^\infty(U)}$, such that, if $\delta \leq \delta_1$ and $\mathcal{C} \subset \tilde{U}$ with*

$$|\mathcal{C}| \geq \bar{c}\delta^{3\zeta(n+1)},$$

there then there exists $(x_0, t_0) \in \mathcal{C}$ and a paraboloid $P \in \mathcal{P}_\infty$ with

(1) $Y^-_\delta(x_0,t_0)\subset U$,
(2) $P(x_0,t_0)=0$,
(3) $P_t - F(D^2P)=0$, *and*
(4) *for all* $(x,t)\in Y^-_\delta(x_0,t_0)$,

$$u^+_{\delta\varsigma}(x,t)-u^+_{\delta\varsigma}(x_0,t_0)\geq P(x,t)-\bar{C}\delta^{1/2}(|x-x_0|^2+(t_0-t)). \quad (36)$$

Lemma 5.3 follows from Proposition 4.6 and a covering argument. The covering argument is slightly more involved than in the elliptic case because of the need to avoid times t that are close to the terminal time T.

Lemma 5.4. *Under the assumptions of Proposition* 5.1*, suppose there exists a point* (x_0,t_0) *and a function* $\phi(x,t)$ *such that*

$$(x_0,t_0)\in\{u^+_{\delta\varsigma}-\delta^{1/4}t-c_1\delta^\zeta-v=\bar{\Gamma}\}\cap\tilde{U},$$

where $\bar{\Gamma}$ *is as in Lemma* 5.2*, and*
(1) $Y^-_\delta(x_0,t_0)\subset U$,
(2) $\phi(x_0,t_0)=0$,
(3) *for all* $(x,t)\in Y^-_\delta(x_0,t_0)$,

$$u^+_{\delta\varsigma}(x,t)-u^+_{\delta\varsigma}(x_0,t_0)\geq \phi(x,t). \quad (37)$$

Then

$$\phi_t(x_0,t_0)-F(D^2\phi(x_0,t_0))\geq\delta^{1/4}. \quad (38)$$

In the situation of Lemma 5.4, we have that $u^+_{\delta\varsigma}$ is a viscosity *subsolution*, so we cannot directly deduce (38) from (37). The proof of the lemma uses that (x_0,t_0) is in the contact set and the fact that v is a δ-supersolution.

We postpone the proofs of the lemmas and proceed with the proof of Proposition 5.1.

Proof of Proposition 5.1. We will give the proof for part (1) of the proposition; the proof of part (2) is very similar. Let c_1 and C_1 be the constants from Lemma 5.2 and let δ_1, $\bar{c}$ and $\bar{C}$ be the constants from Lemma 5.3. Let us take

$$\tilde{\delta}=\min\left\{\delta_1,\frac{1}{2}\left(\bar{C}\left(1+\lambda/2\right)\right)^{-2}\right\}. \quad (39)$$

We point out that $\tilde{\delta}$ is universal, because δ_1 depends only on n and $\bar{C}$ is universal. Let us fix $\delta\leq\tilde{\delta}$. We claim

$$\sup_{\tilde{U}}(u^+_{\delta\varsigma}-\delta^{1/4}t-c_1\delta^\zeta-v)\leq\left(\frac{\bar{c}}{C_1}\right)^{\frac{1}{n+1}}\delta^\zeta. \quad (40)$$

If (40) holds, then we obtain the desired bound on $\sup(u-v)$: indeed, since $u\leq u^+_{\delta\varsigma}$ and $t\leq b\leq T$, we have,

$$\begin{aligned}\sup_{\tilde{U}}(u-v)&\leq\sup_{\tilde{U}}(u^+_{\delta\varsigma}-\delta^{1/4}t-c_1\delta^\zeta-v)+\delta^{1/4}T+c_1\delta^\zeta\\&\leq\left(\frac{\bar{c}}{C_1}\right)^{\frac{1}{n+1}}\delta^\zeta+\delta^{1/4}T+c_1\delta^\zeta\leq\tilde{c}\delta^{\tilde{\alpha}},\end{aligned}$$

where $\tilde{c} = c_1 + T + \left(\frac{\bar{c}}{C_1}\right)^{\frac{1}{n+1}}$ and $\tilde{\alpha} = \min\{1/4, \zeta\}$. Therefore, to complete the proof of this proposition it will suffice to show that (40) holds. To this end, we proceed by contradiction and assume

$$\sup_{\tilde{U}}(u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v) > \left(\frac{\bar{c}}{C_1}\right)^{\frac{1}{n+1}} \delta^\zeta. \tag{41}$$

By Lemma 5.2, we have the following lower bound on the size of the contact set:

$$|\{u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v = \bar{\Gamma}\} \cap \tilde{U}| \geq C_1\delta^{2\zeta(n+1)} \left(\sup_{\tilde{U}}(u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v)\right)^{(n+1)}.$$

We use (41) to bound the right-hand side of the previous line from below and obtain,

$$|\{u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v = \bar{\Gamma}\} \cap \tilde{U}| \geq \bar{c}\delta^{3\zeta(n+1)}.$$

We now apply Lemma 5.3 with $\mathcal{C} = \{u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v = \bar{\Gamma}\} \cap \tilde{U}$ and obtain that there exists a point $(x_0, t_0) \in \{u^+_{\delta^\zeta} - \delta^{1/4}t - c_1\delta^\zeta - v = \bar{\Gamma}\}$ with

$$Y^-_\delta(x_0, t_0) \subset \tilde{U}, \tag{42}$$

and a paraboloid $P \in \mathcal{P}_\infty$ with $P(x_0, t_0) = 0$,

$$P_t - F(D^2P) = 0, \tag{43}$$

and

$$u^+_{\delta^\zeta}(x,t) - u^+_{\delta^\zeta}(x_0,t_0) \geq P(y,s) - \bar{C}\delta^{1/2} \cdot (|x_0 - x|^2 + (t_0 - t)), \tag{44}$$

where $\bar{C}$ is a universal constant. We see that the hypotheses of Lemma 5.4 are satisfied, with $\phi(y,s) = P(y,s) - \bar{C}\delta^{1/2} \cdot (|x_0 - x|^2 + (t_0 - t))$. Therefore, applying Lemma 5.4 yields that inequality (38) holds. With our choice of ϕ, (38) reads,

$$P_t + \bar{C}\delta^{1/2} - F\left(D^2P - \frac{\bar{C}\delta^{1/2}}{2}I\right) \geq \delta^{1/4}$$

(we remark that since $P \in \mathcal{P}_\infty$, we have that P_t and D^2P are constant.) We use the uniform ellipticity of F to obtain an upper bound for the left-hand side of the previous line, and find

$$P_t + \bar{C}\delta^{1/2} - F(D^2P) + \lambda\frac{\bar{C}\delta^{1/2}}{2} \geq \delta^{1/4}.$$

But, according to (43), we have $P_t - F(D^2P) = 0$, so we use this to simplify the left-hand side of the previous line and obtain,

$$\bar{C}\delta^{1/2}(1 + \lambda/2) \geq \delta^{1/4}.$$

Multiplying both sides by $\delta^{-1/2}$ we find,

$$\bar{C}(1 + \lambda/2) \geq \delta^{-1/2}.$$

Since we took $\delta \leq \tilde{\delta}$, this inequality contradicts our choice of $\tilde{\delta}$ in (39). Therefore, we have obtained the desired contradiction to (41), so (40) must hold. The proof of the proposition is thus complete. □

We now proceed with the proofs of Lemmas 5.2, 5.3 and 5.4.

Proof of Lemma 5.2. We first establish the estimate (34). To this end, by item (3) of Proposition 4.5 and assumption (c) of Proposition 5.1 we have

$$\sup_{\partial_p \tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - v) \leq \sup_{\partial_p \tilde{U}}(u^+_{\delta\varsigma} - v) \leq \sup_{\partial_p \tilde{U}}(u - v) + 2\delta^{\zeta}||Du||^2_{L^\infty(U)} \leq c_1\delta^{\zeta},$$

where $c_1 = 2||Du||^2_{L^\infty(U)}$. Let us define the function w on $Y^-_\rho(\bar{x}, b - r^2_\delta)$ by

$$w(x,t) = \begin{cases} \max\{u^+_{\delta\varsigma}(x,t) - \delta^{1/4}t - v(x,t) - c_1\delta^{\zeta}, 0\} & \text{for } (x,t) \in \tilde{U} \\ 0 & \text{on } Y^-_\rho(\bar{x}, b - r^2_\delta) \setminus \tilde{U}. \end{cases}$$

The previous estimate implies that w is continuous on $Y^-_\rho(\bar{x}, b - r^2_\delta)$.

To establish the second assertion of the lemma, estimate (35), we seek to apply Proposition 2.12 to $-w$. Since $d(\tilde{U}, \partial_p U) = 2r_\delta$, Proposition 2.14 implies that u is differentiable in t in $\tilde{U}$, with

$$||u_t||_{L^\infty(\tilde{U})} \leq Cr_\delta^{-2}(||u||_{L^\infty(U)} + 1) \leq C\delta^{-2\zeta}.$$

Therefore, $[u^+_{\delta\varsigma}]_{C^{0,1}_t(\tilde{U})} \leq C\delta^{-2\zeta}$ as well. Together with assumption (c) of Proposition 5.1, this implies $[w]_{C^{0,1}_t(\tilde{U})} \leq C\delta^{-2\zeta}$. In addition, assumption (a) of Proposition 5.1 and the properties of x-sup convolutions (item (4) of Proposition 4.5)) imply $-D^2w = D^2(-u^+_{\delta\varsigma} + v) \leq 2\delta^{-\zeta}I$ in the sense of distributions on all of $\tilde{U}$. Since $w \equiv 0$ outside of $\tilde{U}$, the hypotheses of Proposition 2.12 hold with $K = C\delta^{-2\zeta}$. Applying Proposition 2.12 gives a bound on the supremum of $(-w)^-$ in terms of the size of the contact set of $-w$ with Γ, the lower monotone envelope of $\min(-w, 0)$:

$$\sup_{Y^-_{2\rho}(\bar{x}, b - r^2_\delta)} (-w)^- \leq C|\{-w = \Gamma\}|^{\frac{1}{n+1}}\delta^{-2\zeta}.$$

It is easy to see $\bar{\Gamma} \equiv -\Gamma$, where $\bar{\Gamma}$ is defined in the statement of this lemma, and $\sup_{Y^-_{2\rho}(\bar{x}, b - r^2_\delta)}(-w)^- = \sup_{\tilde{U}} w$. Using this, together with the previous estimate and the definition of w yields,

$$\sup_{\tilde{U}}(u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v) \leq C|\{u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^{\zeta} - v = \bar{\Gamma}\} \cap \tilde{U}|^{\frac{1}{n+1}}\delta^{-2\zeta}.$$

Rearranging, we find that (35) holds. □

Proof of Lemma 5.3. Let us fix a subset $\mathcal{C}$ of $\tilde{U}$ that satisfies

$$|\mathcal{C}| \geq \bar{c}\delta^{3\zeta(n+1)}, \tag{45}$$

where the constant $\bar{c}$ is specified in line (51) below, and depends only on n, λ, Λ, $\operatorname{diam}\Omega'$, T, $||u||_{L^\infty(U)}$ and $||Du||_{L^\infty(U)}$. To simplify notation, for the

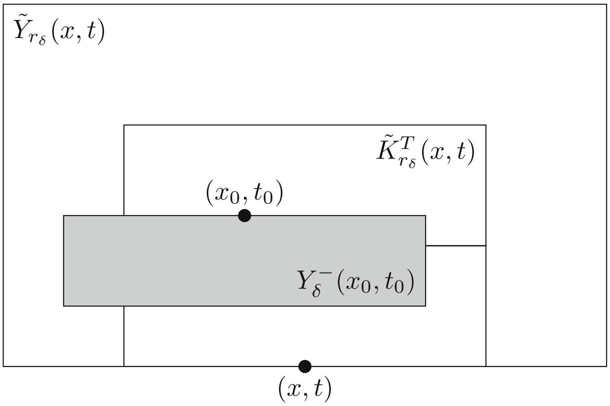

Figure 1. The sets involved in the proof of Lemma 5.3

remainder of this proof we denote the x-sup convolution $u^+_{\delta^\zeta}$ by simply u^+. In addition, we need the following families of sets:

$$\tilde{Y}_{r_\delta}(x,t) := Y^{\delta^\zeta}_{r_\delta}(x,t) = B\left(x, r_\delta - 2\delta^\zeta ||Du||_{L^\infty(U)}\right) \times \left(t, t + r_\delta^2\right],$$

$$\tilde{K}_{r_\delta}(x,t) := K^{\delta^\zeta}_{r_\delta}(x,t)$$

$$= \left[x - \delta^\zeta(1 + ||u||_{L^\infty(U)}), x + \delta^\zeta(1 + ||u||_{L^\infty(U)})\right]^n \times \left(t, t + \frac{r_\delta^2}{81n}\right],$$

$$\tilde{K}^T_{r_\delta}(x,t) := \left[x - \delta^\zeta(1 + ||u||_{L^\infty(U)}), x + \delta^\zeta(1 + ||u||_{L^\infty(U)})\right]^n \times \left(t + \frac{r_\delta^2}{162n}, t + \frac{r_\delta^2}{81n}\right].$$

We point out that $\tilde{K}^T_{r_\delta}(x,t)$ is the top half (in terms of t) of $\tilde{K}_{r_\delta}(x,t)$. See Fig. 1. We remark that there exists a constant δ_1 that depends only on n such that if $\delta \le \delta_1$, then

$$\text{if } (x,t) \in \tilde{U} \text{ and } (x_0,t_0) \in \tilde{K}^T_{r_\delta}(x,t), \text{ then } Y^-_\delta(x_0,t_0) \subset \tilde{Y}_{r_\delta}(x,t) \cap U. \quad (46)$$

(We point out that in order for (46) to hold, it is important that (x_0,t_0) is contained in $\tilde{K}^T_{r_\delta}(x,t)$, and not only in $\tilde{K}_{r_\delta}(x,t)$.) We take $\delta \le \delta_1$.

We will be applying Proposition 4.6. We first remark that although it is stated in $\Omega \times (0,T)$, it holds in $U = \Omega' \times (a,b)$ as well, with no other modifications.

We recall that Proposition 4.6 gives a lower bound on the size of $\Psi^{+,\delta^{2\zeta}}_M(u^+)$, the set where u^+ has second-order expansions from below. For the rest of the argument we will denote $\Psi^{+,\delta^{2\zeta}}_M(u^+)$ simply by Ψ^+_M. We will show that for M large enough there exists a point in the intersection of Ψ^+_M and $\mathcal{C}$ by using the lower bounds on the sizes of $\mathcal{C}$ and Ψ^+_M (the latter is proved in Proposition 4.6). Since Proposition 4.6 bounds the size of the complement of Ψ^+_M inside of sets of the form $\tilde{K}_{r_\delta}(x,t)$, we first cover $\tilde{U}$ by $\{\tilde{K}^T_{r_\delta}(x,t) : (x,t) \in \tilde{U}\}$. It is clear that there exits a finite collection $\{\tilde{K}^T_{r_\delta}(x_j,t_j) : (x_j,t_j) \in \tilde{U}\}_{j=1}^s$

that covers $\tilde{U}$, where $s \leq C_0\delta^{-\zeta(n+2)}$, for C_0 a constant that depends on $\operatorname{diam}\Omega'$, T, and n. Since $\mathcal{C} \subset \tilde{U}$ and the $\tilde{K}^T_{r_\delta}(x_j, t_j)$ cover $\tilde{U}$, there exists an i such that

$$|\mathcal{C} \cap \tilde{K}^T_{r_\delta}(x_i, t_i)| \geq \frac{|\mathcal{C}|}{s}.$$

We use (45) and the upper bound on s to estimate the right-hand side of the previous line from below. We find,

$$|\mathcal{C} \cap \tilde{K}^T_{r_\delta}(x_i, t_i)| \geq \frac{\bar{c}\delta^{3\zeta(n+1)}}{C_0\delta^{-\zeta(n+2)}} = \frac{\bar{c}}{C_0}\delta^{\zeta(4n+5)}.$$

In view of the definition of $\tilde{U}$, we have $d((x_i, t_i), \partial_p U) \geq 2r_\delta$ and $t_i \leq b - r_\delta^2$; therefore, $Y_{r_\delta}(x_i, t_i) \subset U$ and

$$d(Y_{r_\delta}(x_i, t_i), \partial_p U) \geq r_\delta > \delta^\zeta.$$

Therefore, u satisfies the hypotheses of Proposition 4.6 in $Y_{r_\delta}(x_i, t_i)$ with $\theta = \delta^\zeta$, $\rho = \delta$ and $r = r_\delta$. Let M_0 be the universal constant from Proposition 4.6. Thus, for any $M > M_0$, there exists a set Ψ^+_M (the "good set" of u^+) with

$$|\tilde{K}_{r_\delta}(x_i, t_i) \setminus \Psi^+_M| \leq \frac{C_1 r_\delta^{n+2}}{M^\sigma \delta^{\zeta(n+\sigma)}}(r_\delta^{-\sigma} + \delta^{-\zeta\sigma})\left(||u||_{L^\infty(U)} + 1\right)^{n+\sigma}, \quad (47)$$

where C_1 is a universal constant. We take

$$M = \left(\frac{2C_1C_0 r_\delta^{n+2}\delta^{-\zeta(n+\sigma)}(r_\delta^{-\sigma} + \delta^{-\zeta\sigma})(||u||_{L^\infty(U)} + 1)^{n+\sigma}}{\bar{c}\delta^{\zeta(4n+5)}}\right)^{1/\sigma} + M_0.$$

We use our choice of M to bound the right-hand side of (47) from above, and obtain

$$|\tilde{K}_{r_\delta}(x_i, t_i) \setminus \Psi^+_M| < \frac{\bar{c}}{C_0}\delta^{\zeta(n+\sigma)} = |\mathcal{C} \cap \tilde{K}^T_{r_\delta}(x_i, t_i)|.$$

Since we have $\tilde{K}^T_{r_\delta}(x_i, t_i) \subset \tilde{K}_{r_\delta}(x_i, t_i)$, the previous inequality implies,

$$|\tilde{K}^T_{r_\delta}(x_i, t_i) \setminus \Psi^+_M| < |\tilde{K}_{r_\delta}(x_i, t_i) \setminus \Psi^+_M| < |\mathcal{C} \cap \tilde{K}^T_{r_\delta}(x_i, t_i)|.$$

Therefore, there exists a point $(x_0, t_0) \in \mathcal{C} \cap \Psi^+_M \cap \tilde{K}^T_{r_\delta}(x_i, t_i)$. Because of (46), we have $Y^-_\delta(x_0, t_0) \subset \tilde{Y}_{r_\delta}(x, t) \cap U$. Thus, by Proposition 4.6, there exists a polynomial $P \in \mathcal{P}_\infty$ such that

$$P_t - F(D^2P) = 0$$

and, on $Y^-_\delta(x_0, t_0)$,

$$u^+(x, t) - u^+(x_0, t_0) \geq P(y, s) - C_2 M\frac{\delta}{r_\delta}|x - x_0|^2 - \left(\frac{\delta}{r_\delta}C_2 M + \frac{\delta}{r_\delta}\frac{1 + ||u||_{L^\infty(U)}}{\delta^{2\zeta}}\right)(t_0 - t), \quad (48)$$

where C_2 is a universal constant. We will now estimate the sizes of the coefficients of the error terms on the right-hand side of the previous line. We first observe that, by the definition of r_δ, there exists a constant C_3 that depends on n, $||u||_{L^\infty(U)}$ and $||Du||_{L^\infty(U)}$ so that

$$\delta^\zeta < r_\delta \leq C_3\delta^\zeta. \tag{49}$$

Therefore,

$$M \leq C_4\bar{c}^{-1/\sigma}\delta^{-\frac{\zeta}{\sigma}(4n+3+2\sigma)} + M_0, \tag{50}$$

where C_4 depends on n, λ, Λ, $\operatorname{diam}\Omega'$, T, $||u||_{L^\infty(U)}$, and $||Du||_{L^\infty(U)}$. Let us choose

$$\bar{c} = (C_2C_4)^\sigma. \tag{51}$$

We claim that, with this choice of $\bar{c}$, we have the following two bounds on the coefficients of the error terms in (48):

$$C_2M\frac{\delta}{r_\delta} \leq (1 + C_2M_0)\delta^{1/2}, \tag{52}$$

and,

$$\frac{\delta}{r_\delta}\frac{1 + ||u||_{L^\infty(U)}}{\delta^{2\zeta}} \leq \delta^{1/2}. \tag{53}$$

Once these bounds are established, we may use them to bound the right-hand side of (48) from below, and then take $\bar{C} = 2 + C_2M_0$ to complete the proof of the lemma.

Let us first prove that (52) holds. Since $r_\delta > \delta^\zeta$, we have

$$\frac{\delta}{r_\delta} \leq \delta^{1-\zeta}.$$

We use this, together with (50), to obtain,

$$M\frac{\delta}{r_\delta} \leq C_4\bar{c}^{-1/\sigma}\delta^{1-\zeta-\frac{\zeta}{\sigma}(4n+3+2\sigma)} + M_0\delta^{1-\zeta} = C_4\bar{c}^{-1/\sigma}\delta^{1-\frac{\zeta}{\sigma}(4n+3+3\sigma)} + M_0\delta^{1-\zeta}.$$

Using our choice of ζ to simplify the exponent in the first term of the previous line yields,

$$M\frac{\delta}{r_\delta} \leq C_4\bar{c}^{-1/\sigma}\delta^{1/2} + M_0\delta^{1-\zeta}.$$

In addition, since $\zeta \leq 1/6$ (which is clear from the definition of ζ), we have $1 - \zeta \geq 5/6 \geq 1/2$, so that $\delta^{1-\zeta} \leq \delta^{1/2}$. Therefore, we find

$$M\frac{\delta}{r_\delta} \leq (C_4\bar{c}^{-1/\sigma} + M_0)\delta^{1/2}.$$

Multiplying by C_2 and using our choice of $\bar{c}$ yields the estimate (52). We will now establish (53). We have $r_\delta > (1 + ||u||_{L^\infty(U)})\delta^\zeta$. Thus, we obtain

$$\frac{\delta}{r_\delta}\frac{1 + ||u||_{L^\infty(U)}}{\delta^{2\zeta}} < \frac{\delta}{(1 + ||u||_{L^\infty(U)})\delta^\zeta}\frac{1 + ||u||_{L^\infty(U)}}{\delta^{2\zeta}} = \delta^{1-3\zeta}.$$

Next, we use $\zeta \leq 1/6$ to bound $\delta^{1-3\zeta}$ from above by $\delta^{1/2}$ and obtain (53). The proof of the lemma is therefore complete. □

Proof of Lemma 5.4. We recall the representation formula for the upper monotone envelope of a function (Lemma 2.11):

$$\bar{\Gamma}(w)(x_0,t_0) = \inf\{\zeta\cdot x_0 + h:\ \zeta\cdot x + h \geq w(x,t) \text{ for all } (x,t) \in Y^-_{2\rho}(x_0, b - r_\delta^2)\cap\{t\leq t_0\}\}.$$

Since (x_0,t_0) is in the contact set of $u^+_{\delta\varsigma} - \delta^{1/4}t - c_1\delta^\zeta - v$ with its upper monotone envelope, the representation formula implies that there exist $\zeta\in\mathbb{R}^n$ and $h\in\mathbb{R}$ such that

$$\zeta\cdot x_0 + h = \bar{\Gamma}(w)(x_0,t_0) = u^+_{\delta\varsigma}(x_0,t_0) - \delta^{1/4}t_0 - c_1\delta^\zeta - v(x_0,t_0)$$

and

$$\zeta\cdot x + h \geq u^+_{\delta\varsigma}(x,t) - \delta^{1/4}t - c_1\delta^\zeta - v(x,t)$$

for all x and for all $t\leq t_0$ (so in particular, for all points in $Y^-_\delta(x_0,t_0)$). Rearranging the previous inequality we find, for all $(x,t)\in Y^-_\delta(x_0,t_0)$,

$$v(x,t) \geq u^+_{\delta\varsigma}(x,t) - \delta^{1/4}t - c_1\delta^\zeta - \zeta\cdot x - h,$$

with equality holding at (x_0,t_0). Next, we use that $u^+_{\delta\varsigma}$ is touched from below at (x_0,t_0) by $\phi(x,t)$ (this is assumption (37) of this lemma) to bound the first term on the right-hand side of the previous inequality from below. Thus we find, for all $(x,t)\in Y^-_\delta(x_0,t_0)$,

$$v(x,t) \geq u^+_{\delta\varsigma}(x_0,t_0) + \phi(x,t) - \delta^{1/4}t - c_1\delta^\zeta - \zeta\cdot x - h, \tag{54}$$

with equality at (x_0,t_0). By assumption, we have $Y^-_\delta(x_0,t_0)\subset U$. Since v is a δ-supersolution of (33) on U and (54) holds on $Y^-_\delta(x_0,t_0)$ with equality at (x_0,t_0), we find

$$\phi_t(x_0,t_0) - \delta^{1/4} - F\left(D^2\phi(x_0,t_0)\right) \geq 0.$$

Thus the proof of the lemma is complete. □

6. Error estimate

In this section we give the precise statement and the proof of our main result.

Theorem 6.1. *Assume (F1), (F2) and (U1). Assume* $u\in C^{0,\eta}(\Omega\times(0,T))\cap C^{0,1}_x(\Omega\times(0,T))$ *is a solution of*

$$u_t - F(D^2u) = 0 \quad \text{in } \Omega\times(0,T) \tag{55}$$

and assume $\{v_\delta\}_{\delta>0}$ *is a family of* δ*-supersolutions (resp.* δ*-subsolutions) of* (55) *such that, for some* $M<\infty$ *and for all* $\delta>0$,

$$||v_\delta||_{C^{0,\eta}(\Omega\times(0,T))} \leq M \tag{56}$$

and

$$u - v_\delta \leq 0 \text{ (resp. } v_\delta - u \leq 0) \quad \text{on } \partial_p(\Omega\times(0,T)).$$

There exist a universal constant $\bar{\delta}$, a constant $\bar{\alpha}$ that depends on n, λ, Λ and η, and a constant $\bar{c}$ that depends on n, λ, Λ, $\operatorname{diam}\Omega$, T, M, $||u||_{C^{0,\eta}(\Omega\times(0,T))}$, and $||Du||_{L^\infty(\Omega\times(0,T))}$, such that for all $\delta \le \bar{\delta}$,

$$\sup_{\Omega\times(0,T)} u - v_\delta \le \bar{c}\delta^{\bar{\alpha}} \quad \left(\textit{resp.} \sup_{\Omega\times(0,T)} v_\delta - u \le \bar{c}\delta^{\bar{\alpha}} \right). \tag{57}$$

Remark **6.2.** The assumption $u \in C^{0,\eta}(\Omega\times(0,T))\cap C^{0,1}_x(\Omega\times(0,T))$ is satisfied in the following situation. Assume (F1), (F2), (U1), and that $\partial\Omega$ is sufficiently regular. Take $g \in C^{1,\alpha}(\partial_p(\Omega\times(0,T)))$. Then, according to the interior regularity estimates of Theorem 2.13 and the boundary estimates of [27, Section 2], the solution u of the boundary value problem

$$\begin{cases} u_t - F(D^2u) = 0 & \text{in } \Omega\times(0,T), \\ u = g & \text{on } \partial_p(\Omega\times(0,T)), \end{cases}$$

is indeed Lipschitz continuous in x and Hölder continuous in t on all of $\Omega\times(0,T)$, with

$$||u||_{C^{0,\eta}(\Omega\times(0,T))}, ||Du||_{L^\infty(\Omega\times(0,T))} \le C,$$

where C depends on n, λ, Λ, $||g||_{C^{1,\alpha}(\partial_p(\Omega\times(0,T)))}$, $\operatorname{diam}\Omega$ and the regularity of $\partial\Omega$.

Proof of Theorem 6.1. We give the proof of the bound on $\sup(u - v_\delta)$ in the case that v_δ is a δ-supersolution; the proof of the other case is very similar. Let us take $\delta \le \tilde{\delta}$, where $\tilde{\delta}$ is the universal constant given by Proposition 5.1. To simplify notation, throughout the remainder of this proof we will use c and c_i with $i = 0, 1, 2, \ldots$, to denote positive constants that depend only on n, λ, Λ, $\operatorname{diam}\Omega$, T, M, $||u||_{C^{0,\eta}(\Omega\times(0,T))}$, and $||Du||_{L^\infty(\Omega\times(0,T))}$.

We regularize v_δ by taking inf-convolution: let v^- denote $(v_\delta)^-_{\delta^\zeta,\delta^\zeta}$ and let U denote $U^{\delta^\zeta,\delta}$. For the convenience of the reader, we write down the definition of U explicitly:

$$U = \left\{ (x,t) \in \Omega\times(0,T) : \ d_e((x,t), \partial_p(\Omega\times(0,T))) \ge 2\delta^{\zeta/2}||v_\delta||^{1/2}_{L^\infty(\Omega\times(0,T))} + \delta \right\}.$$

We will apply part (1) of Proposition 5.1 to v^- in U once we verify its hypotheses. First, we see that U is of the form $\Omega'\times(a,b)$, where Ω' satisfies (U1) and $(a,b)\subset(0,T)$. Next, we use items (3) and (4) of Proposition A.4 with $\theta = \delta^\zeta$ and find that we have $D^2v^- \le \delta^{-\zeta}I$ in U in the sense of distributions and $[v^-]_{C^{0,1}_t(U)} \le 3T\delta^{-\zeta}$. Therefore, v^- satisfies assumptions (a) and (c) of Proposition 5.1. Finally, item (5) of Proposition A.4 says that v^- is a δ-supersolution of (33) in U.

Now let r_δ and $\tilde{U}$ be as in the statement of Proposition 5.1:

$$r_\delta = 9\sqrt{n}(1 + ||u||_{L^\infty(U)} + 2||Du||_{L^\infty(U)})\delta^\zeta, \text{ and}$$
$$\tilde{U} = \left\{ (x,t) \in U : \ d((x,t), \partial_p U) \ge 2r_\delta, \text{ and } t \le T - r_\delta^2 \right\}.$$

Let us apply Proposition 5.1 with $u-\left(\sup_{\partial_p\tilde{U}}(u-v^-)\right)$ instead of just u itself (this ensures that the hypothesis (c) is satisfied.) We conclude

$$\sup_{\tilde{U}}(u-v^-) \leq \tilde{c}\delta^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v^-). \tag{58}$$

We will now use this estimate, the regularity properties of u and v_δ, and the fact that the distance between $\partial_p\tilde{U}$ and $\partial_p(\Omega\times(0,T))$ is small to deduce the bound (57).

First, we use the definitions of U, $\tilde{U}$ and r_δ to estimate the distance between $\partial_p\tilde{U}$ and $\partial_p(\Omega\times(0,T))$:

$$\begin{aligned} d(\tilde{U},\partial_p(\Omega\times(0,T))) &= \inf_{(x,t)\in\tilde{U}}\{d((x,t),\partial_p(\Omega\times(0,T))\} \\ &\leq 2r_\delta + 2\delta^{\frac{\zeta}{2}}||v_\delta||^{1/2}_{L^\infty(\Omega\times(0,T))} + \delta \leq c_0\delta^{\zeta/2}, \end{aligned} \tag{59}$$

where the last inequality follows from the definition of r_δ and since $\frac{\zeta}{2}<\zeta<1$.

Next, since $v^-\leq v_\delta$ holds on $\Omega\times(0,T)$, and we have $u\in C^{0,\eta}(\Omega\times(0,T))$, $v_\delta\in C^{0,\eta}(\Omega\times(0,T))$, and $u\leq v_\delta$ on $\partial_p(\Omega\times(0,T))$, the estimate (59) implies that there exists a constant c_1 such that

$$\sup_{\partial_p\tilde{U}}(u-v_\delta) \leq c_1\delta^{\frac{\zeta\eta}{2}} \tag{60}$$

and

$$\sup_{\Omega\times(0,T)}(u-v_\delta) \leq \sup_{\tilde{U}}(u-v^-) + c_1\delta^{\frac{\zeta\eta}{2}}. \tag{61}$$

We use (58) to bound the first term on the right-hand side of (61) from above and obtain,

$$\sup_{\Omega\times(0,T)}(u-v_\delta) \leq \tilde{c}\delta^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v^-) + c_1\delta^{\frac{\zeta\eta}{2}}. \tag{62}$$

Thus, to establish (57) it is left to bound the right-hand side of (62) from above. To this end, item (2) of Proposition A.4 applied with $\theta=\delta^\zeta$, and the assumption (56) of this theorem, imply, for all $(x,t)\in U$,

$$v^-(x,t) \geq v_\delta(x,t) - [v_\delta]^{\frac{2}{2-\eta}}_{C^{0,\eta}(\Omega\times(0,T))}\delta^{\frac{\zeta\eta}{2-\eta}} \geq v_\delta(x,t) - M^{\frac{2}{2-\eta}}\delta^{\frac{\zeta\eta}{2-\eta}}.$$

We apply this estimate with $(x,t)\in\partial_p\tilde{U}\subset U$ to obtain a bound from above for the second term on the right-hand side of (62):

$$\sup_{\partial_p\tilde{U}}(u-v^-) \leq \sup_{\partial_p\tilde{U}}(u-v_\delta) + M^{\frac{2}{2-\eta}}\delta^{\frac{\zeta\eta}{2-\eta}}.$$

We now use (60) to bound the first term on the right-hand side of the previous line and obtain

$$\sup_{\partial_p\tilde{U}}(u-v^-) \leq c_1\delta^{\frac{\zeta\eta}{2}} + M^{\frac{2}{2-\eta}}\delta^{\frac{\zeta\eta}{2-\eta}} \leq c_2\delta^{\frac{\zeta\eta}{2}}.$$

Next we use the previous line to bound the second term on the right-hand side of (62) from above, and obtain

$$\sup_{\Omega\times(0,T)} (u - v_\delta) \le \tilde{c}\delta^{\tilde{\alpha}} + c_2\delta^{\frac{\zeta\eta}{2}} + c_1\delta^{\frac{\zeta\eta}{2}}.$$

We thus take $\bar{c} = \tilde{c} + c_2 + c_1$ and $\bar{\alpha} = \min\{\tilde{\alpha}, \frac{\zeta\eta}{2}\}$ to complete the proof of the theorem. □

7. Discrete approximation schemes

We now present the error estimate for finite difference approximation schemes. First, we introduce the necessary notation and assumptions, closely following [8,25]. In the next section we give the full statement of the error estimate and its proof. The space-time mesh is denoted by E:

$$E = h\mathbb{Z}^n \times h^2\mathbb{Z} = \{(x,t):\ x = (m_1,\ldots,m_n)h, t = mh^2, \text{where}\, m, m_1,\ldots,m_n \in \mathbb{Z}\}.$$

We fix some $N > 1$ and define the subset Y of E by

$$Y = \{y \in h\mathbb{Z}^n :\quad 0 < |y| < hN\}.$$

Next we introduce finite difference operators for a function u:

$$\begin{aligned}
\delta_\tau^- u(x,t) &= \frac{1}{h}(u(x,t) - u(x,t-h^2)),\\
\delta_y^2 u(x,t) &= \frac{1}{|y|^2}(u(x+y,t) + u(x-y,t) - 2u(x,t)), \text{ and}\\
\delta^2 u(x,t) &= \{\delta_y^2 u(x,t) :\quad y \in Y\}.
\end{aligned}$$

An *implicit finite difference operator* is an operator of the form

$$\mathcal{S}_h[u](x,t) = \delta_\tau^- u(x,t) - \mathcal{F}_h(\delta^2 u(x,t)),$$

where $\mathcal{F}_h : \mathbb{R}^Y \to \mathbb{R}$ is locally Lipschitz. We denote points in $\mathbb{R}^Y$ by $r = (r_1,\ldots,r_{|Y|})$. We say an operator is *monotone* if it satisfies:

(S1) there exists a constant λ_0 and Λ_0 such that for all $i = 1,\ldots,|Y|$,

$$\lambda_0 \le \frac{\partial\mathcal{F}_h}{\partial r_i} \le \Lambda_0.$$

This is equivalent to the definition given in the introduction.

A scheme $\mathcal{S}_h$ is said to be *consistent with* F if for all $\phi \in C^3(\Omega\times(0,T))$,

$$\sup|\phi_t - F(D^2\phi) - \mathcal{S}_h[\phi]| \to 0 \quad \text{as } h \to 0.$$

In [25, Section 4], it was shown that if a nonlinearity F satisfies (F1), then there exists a monotone implicit scheme $\mathcal{S}_h$ that is consistent with $u_t - F(D^2u) = 0$, and the constants λ_0 and Λ_0 depend only on n, λ and Λ. In [23], a monotone and consistent approximation scheme for elliptic equations is explicitly constructed, and the construction in the parabolic case is analogous.

In order to obtain an error estimate, we need to make an assumption that quantifies the above rate of convergence. As in [8], we assume:

(S2) there exists a positive constant K such that for all $\phi \in C^3(\Omega \times (0,T))$,

$\sup |\phi_t - F(D^2\phi) - \mathcal{S}_h[\phi]| \leq K(h + h||D_x^3\phi||_{L^\infty(\Omega\times(0,T))} + h^2||\phi_{tt}||_{L^\infty(\Omega\times(0,T))})$.

Schemes that satisfy (S2) are said to be *consistent with an error estimate for F in $\Omega \times (0,T)$ with constant K.*

Given $\Omega \times (0,T) \subset \mathbb{R}^{n+1}$, we denote the mesh points $(\Omega \times (0,T)) \cap E$ by $\mathcal{U}_h$. We divide $\mathcal{U}_h$ into interior and boundary points relative to the operator $\mathcal{S}_h$. We define

$$\mathcal{U}_h^i = \{p \in \mathcal{U}_h : \quad d(p, \partial_p\Omega \times (0,T)) \geq Nh\}$$

to be the interior points and

$$\mathcal{U}_h^b = \mathcal{U}_h \setminus \mathcal{U}_h^i$$

to be the boundary points. Notice that $\mathcal{S}_h[u](x,t)$ depends only on $u(x+y,s)$ for $0 \leq |y| < hN$ and for $t - h^2 \leq s \leq t$.

The discrete Hölder seminorm of a function u on $\mathcal{U}_h$ is defined to be

$$[u]_{C^{0,\eta}(\mathcal{U}_h)} = \sup_{p,q\in\mathcal{U}_h} \frac{|u(p) - u(q)|}{d(p,q)^\eta}$$

and the discrete Hölder norm is

$$||u||_{C^{0,\eta}(\mathcal{U}_h)} = ||u||_{L^\infty(\mathcal{U}_h)} + [u]_{C^{0,\eta}(\mathcal{U}_h)}.$$

In [25, Section 4] it is shown that solutions of the discrete equation $\mathcal{S}_h[v_h] = 0$ are uniformly equicontinuous. We summarize this result:

Theorem 7.1. *Assume that $\mathcal{S}_h$ is an implicit monotone finite difference scheme and that v_h is a solution of $\mathcal{S}_h[v_h] = 0$. There exists a constant C that depends only on n, λ_0 and Λ_0 such that for all $h \in (0,1)$,*

$$||v_h||_{C^{0,\eta}(\mathcal{U}_h^i)} \leq C.$$

7.1. Inf and sup convolutions for approximation schemes

Definition 7.2. Given $\theta > 0$ and a mesh function $v : \mathcal{U}_h \to \mathbb{R}$, we define the *inf- convolution* $v_{\theta,\theta}^-$ and the *sup- convolution* $v_{\theta,\theta}^+$ of v at $(x,t) \in \Omega \times (0,T)$ by

$$v_{\theta,\theta}^-(x,t) = \inf_{(y,s)\in\mathcal{U}_h} \left\{ v(y,s) + \frac{|x-y|^2}{2\theta} + \frac{|t-s|^2}{2\theta} \right\}$$

and

$$v_{\theta,\theta}^+(x,t) = \sup_{(y,s)\in\mathcal{U}_h} \left\{ v(y,s) - \frac{|x-y|^2}{2\theta} - \frac{|t-s|^2}{2\theta} \right\}.$$

Definition 7.3. Given $\theta > 0$ and $v \in C^{0,\eta}(\mathcal{U}_h)$, we introduce the quantity

$$\omega(h,\theta) = nh + 2\theta^{1/2}||v||_{L^\infty(\mathcal{U}_h)}$$

and the set

$$U_\theta^h = \{p \in \Omega \times (0,T) : \quad d_e(p, \partial_p(\Omega \times (0,T))) \geq \omega(h,\theta) + Nh\}.$$

In the appendix we summarize the basic properties of inf- and sup- convolutions of mesh functions (see Proposition A.5).

It is a classical fact of the theory of viscosity solutions that if u is the viscosity solution of $u_t - F(D^2u) = 0$ in $\Omega \times (0,T)$, then the sup-convolution of u is a subsolution of the same equation. In the following proposition, we establish a similar relationship between solutions v_h of discrete equations and δ-solutions of $u_t - F(D^2u) = 0$. This is a key step in establishing an error estimate for discrete approximation schemes.

Proposition 7.4. *Assume that $\mathcal{S}_h$ is a monotone and implicit scheme consistent with an error estimate for F with constant K. Assume $v \in C^{0,\eta}(\mathcal{U}_h)$.*

(1) If v satisfies $\mathcal{S}_h[v] \geq 0$ in $\mathcal{U}_h^i$ then $v^-_{\theta,\theta}$ is a δ-supersolution of

$$u_t - F(D^2u) = -Kh$$

in U_θ^h, with $\delta = Nh$.

(2) If v satisfies $\mathcal{S}_h[v] \leq 0$ in $\mathcal{U}_h^i$, then $v^+_{\theta,\theta}$ is δ-subsolution of

$$u_t - F(D^2u) = Kh$$

in U_θ^h, with $\delta = Nh$.

Proof. We give the proof of item (1); the proof of item (2) is analogous.

Let us suppose $(x,t) \in U_\theta^h$ is such that $Y^-_{Nh}(x,t) \subset U_\theta^h$ and that $P \in \mathcal{P}_\infty$ is such that

$$P \leq v^-_{\theta,\theta} \text{ on } Y^-_{Nh}(x,t), \text{ and } P(x,t) = v^-_{\theta,\theta}(x,t). \tag{63}$$

We use (63) and the definition of the operators δ_y^2 and δ_τ^- to obtain, for each y with $0 < |y| \leq hN$,

$$\delta_y^2 P(x,t) \leq \delta_y^2 v^-_{\theta,\theta}(x,t), \text{ and } \delta_\tau^- P(x,t) \geq \delta_\tau^- v^-_{\theta,\theta}(x,t). \tag{64}$$

(We remark that for this to hold, it is essential that $P \leq v$ on *all* of $Y^-_{Nh}(x,t)$.) Now let (x^*,t^*) be a point at which the infimum is achieved in the definition of $v^-_{\theta,\theta}(x,t)$. According to item (6) of Proposition A.5, we have

$$\delta_y^2 v^-_{\theta,\theta}(x,t) \leq \delta_y^2 v(x^*,t^*) \text{ and } \delta_\tau^- v^-_{\theta,\theta}(x,t) \geq \delta_\tau^- v(x^*,t^*). \tag{65}$$

We now use the first inequality in (65) to estimate the right-hand side of the first inequality in (64) from above, and we use the second inequality in (65) to estimate the right-hand side of the second inequality in (64) from below. We find,

$$\delta_y^2 P(x,t) \leq \delta_y^2 v(x^*,t^*), \text{ and } \delta_\tau^- P(x,t) \geq \delta_\tau^- v(x^*,t^*). \tag{66}$$

Item (1) of Proposition A.5 yields $(x^*,t^*) \in \mathcal{U}_h^i$. Since v satisfies $\mathcal{S}_h[v] \geq 0$ in $\mathcal{U}_h^i$, we obtain,

$$0 \leq \mathcal{S}_h[v](x^*,t^*) = \delta_\tau^- v(x^*,t^*) - \mathcal{F}_h(\delta^2 v(x^*,t^*)).$$

Using (66) and the monotonicity of $\mathcal{F}_h$ to bound the right-hand side of the previous line yields

$$0 \leq \delta_\tau^- P(x,t) - \mathcal{F}_h(\delta^2 P(x,t)). \tag{67}$$

We use that $\mathcal{S}_h$ is consistent with an error estimate and find,

$$|P_t(x,t) - F(D^2P(x,t)) - (\delta_\tau^- P(x,t) - \mathcal{F}_h(\delta^2 P(x,t)))| \leq Kh.$$

Together with (67), this implies

$$P_t - F(D^2P) \geq -Kh,$$

as desired. □

8. Error estimate for approximation schemes

In this section we give the precise statement and proof of Theorem 1.2.

Theorem 8.1. *Assume (F1), (F2) and (U1). Assume $u \in C^{0,\eta}(\Omega \times (0,T)) \cap C_x^{0,1}(\Omega \times (0,T))$ is a solution of*

$$u_t - F(D^2u) = 0 \tag{68}$$

in $\Omega \times (0,T)$. Assume that $\mathcal{S}_h$ is an implicit monotone scheme consistent with an error estimate for (68) and that v_h satisfies $\mathcal{S}_h[v_h] \geq 0$ (resp. $\mathcal{S}_h[v_h] \leq 0$) in $\mathcal{U}_h$ for all $h > 0$. Assume that for some constant $M < \infty$ and for all $h > 0$,

$$||v_h||_{C^{0,\eta}(\mathcal{U}_h^i)} \leq M$$

and

$$u - v_h \leq 0 \ (\text{resp. } u - v_h \geq 0) \text{ on } \mathcal{U}_h^b. \tag{69}$$

There exists a constant $\bar{h}$ that depends only on λ, Λ, n and N, a constant $\bar{\alpha}$ that depends on n, λ, Λ and η, and a constant $\bar{c}$ that depends on n, N, λ, Λ, $\operatorname{diam}\Omega$, T, K, M, $||u||_{C^{0,\eta}(\Omega\times(0,T))}$, and $||Du||_{L^\infty(\Omega\times(0,T))}$, such that for all $h \leq \bar{h}$,

$$\sup_{\mathcal{U}_h} u - v_h \leq \bar{c}h^{\bar{\alpha}} \quad \left(\text{resp. } \sup_{\mathcal{U}_h} v_h - u \leq \bar{c}h^{\bar{\alpha}}\right).$$

Remark **8.2.** Suppose that u is the solution of the boundary value problem

$$\begin{cases} u_t - F(D^2u) = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial_p(\Omega \times (0,T)), \end{cases}$$

and v_h satisfies

$$\begin{cases} \mathcal{S}_h[v_h] = 0 & \text{in } \mathcal{U}_h^i, \\ v_h = g & \text{on } \mathcal{U}_h^b, \end{cases}$$

where $\partial\Omega$ is sufficiently regular and $g \in C^{1,\alpha}(\Omega \times (0,T))$. Then, as explained in Remark 6.2, u satisfies the hypotheses of Theorem 8.1, and v_h satisfies the hypotheses of Theorem 8.1 because of Theorem 7.1.

Proof of Theorem 8.1. We give the proof of the bound on $\sup(u - v_h)$; the proof of the other case is very similar. We will apply Proposition 5.1 with $\delta = Nh$. We define $\bar{h}$ to be $\bar{h} = \tilde{\delta}N^{-1}$, where $\tilde{\delta}$ is the universal constant given by Proposition 5.1. To simplify notation, throughout the remainder of this proof we will use c and c_i with $i = 0, 1, 2, \ldots,$ to denote positive constants that depend only on n, N, λ, Λ, $\operatorname{diam}\Omega$, T, K, M, $||u||_{C^{0,\eta}(\Omega\times(0,T))}$, and $||Du||_{L^\infty(\Omega\times(0,T))}$. In addition, c may change from line to line.

First we regularize v_h by taking inf-convolution: we denote $v^- = (v_h)^-_{h^\zeta,h^\zeta}$ and $U^h_{h^\zeta}$ by U. For the convenience of the reader, we write down the definition of U explicitly:

$$U = \left\{(x,t) \in \Omega \times (0,T) : \quad d((x,t), \partial_p(\Omega \times (0,T))) \geq \omega(h, h^\zeta) + Nh\right\},$$

and

$$\omega(h, h^\zeta) = nh + 2h^{\zeta/2}||v_h||_{L^\infty(\mathcal{U}_h)}.$$

Then, according to item (5) of Proposition (7.4), v^- is a δ-supersolution of

$$v^-_t - F(D^2 v^-) \geq -Kh$$

in U. We point out that U is of the form $\Omega' \times (a,b)$, where Ω' satisfies (U1). Therefore, Proposition 5.1 holds in U. Since Proposition 5.1 applies to δ-solutions with right-hand side 0, we have to perturb v^-. We introduce

$$v(x,t) = v^-(x,t) + Kht,$$

so that v is a δ-supersolution of (68) in U. Moreover, by items (4) and (5) of Proposition A.5, v satisfies the regularity assumptions (a) and (c) of Proposition 5.1. Let r_δ and $\tilde{U}$ be as in Proposition 5.1 (recall we are taking $\delta = Nh$):

$$r_\delta = 9\sqrt{n}(1 + ||u||_{L^\infty(U)} + 2||Du||_{L^\infty(U)})(Nh)^\zeta, \text{ and}$$
$$\tilde{U} = \left\{(x,t) \in U : \quad d((x,t), \partial_p U) \geq 2r_\delta, \text{ and } t \leq T - r_\delta^2\right\}.$$

We apply Proposition 5.1 to with $u - \left(\sup_{\partial_p\tilde{U}}(u-v)\right)$ instead of u, so that the hypothesis (c) of Proposition (5.1) is satisfied. We obtain the bound

$$\sup_{\tilde{U}}(u-v) \leq c(Nh)^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v).$$

By the definition of v, we have $v^- \leq v \leq v^- + KhT$ on $\tilde{U}$. We use this to bound the left-hand side of the previous line from below and the right-hand side of the previous line from above and find,

$$\sup_{\tilde{U}}(u-v^-) \leq KhT + c(Nh)^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v^-) \leq c_1 h^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v^-), \qquad (70)$$

where the second inequality follows since $\tilde{\alpha} \leq 1$. We will now deduce the conclusion of the theorem from (70). This step is a bit more complicated than the corresponding step in the proof of Theorem 6.1 because v_h is a function on the mesh $\mathcal{U}_h$ and not on all of $\Omega \times (0,T)$. By the definitions of U and $\tilde{U}$, we have the following estimate, which we'll be applying frequently in the remainder of the argument:

$$d(\tilde{U}, \mathcal{U}^b_h) = \inf_{(x,t)\in\tilde{U}}\{d((x,t), \mathcal{U}^b_h)\} \leq c(h + h^{\zeta/2} + h^\zeta) \leq c_0 h^{\zeta/2}.$$

We first bound $\sup_{\mathcal{U}_h}(u-v_h)$ from above. Since u and v_h are Hölder-continuous, and since $v^- \leq v^h$ holds on $\mathcal{U}_h$, we have:

$$\sup_{\mathcal{U}_h}(u-v_h) \leq \sup_{\tilde{U}\cap E}(u-v_h) + c(d(\tilde{U},\mathcal{U}_h^b))^\eta \leq \sup_{\tilde{U}\cap E}(u-v_h) + ch^{\frac{\zeta\eta}{2}}$$
$$\leq \sup_{\tilde{U}\cap E}(u-v^-) + c_2 h^{\frac{\zeta\eta}{2}}.$$

We next use (70) to estimate the first term on right-hand side of the previous line from above. We find,

$$\sup_{\mathcal{U}_h}(u-v_h) \leq c_1 h^{\tilde{\alpha}} + \sup_{\partial_p\tilde{U}}(u-v^-) + c_2 h^{\frac{\zeta\eta}{2}}. \tag{71}$$

We will now establish the following upper bound for the middle term of right-hand side of (71):

$$\sup_{\partial_p\tilde{U}}(u-v^-) \leq \sup_{\mathcal{U}_h^b}(u-v_h) + c_3 h^{\frac{\zeta\eta}{2}}. \tag{72}$$

To this end, let us fix some $(x,t) \in \partial_p\tilde{U}$. Let (y,s) be a nearest mesh point to (x,t) (recall that v_h is only defined on the mesh E, and (x,t) need not be a mesh point). By item (3) of Proposition A.5, we have a bound from below on the difference between $v^-(x,t)$ and $v_h(y,s)$:

$$v^-(x,t) \geq v_h(y,s) - M\omega(h,h^\zeta)^\eta \geq v_h(y,s) - ch^{\frac{\zeta\eta}{2}}, \tag{73}$$

where the second inequality follows from the definition of ω. In addition, since $(x,t) \in \partial_p\tilde{U}$ is "near" $\mathcal{U}_h^b$ and (y,s) is a nearest mesh point to h, we have that (y,s) is "near" $\mathcal{U}_h^b$ too. Precisely, there exists a point $(z,r) \in \mathcal{U}_h^b$ with

$$d((y,s),(z,r)) \leq nh + d(\tilde{U},\mathcal{U}_h^b) \leq ch^{\zeta/2}.$$

We use this estimate and the fact that v_h is Hölder-continuous to bound the difference between $v_h(y,s)$ and $v_h(z,t)$ and obtain,

$$v_h(y,s) \geq v_h(z,t) - ch^{\frac{\zeta\eta}{2}}.$$

Using this to bound the right-hand side of (73), we find,

$$v^-(x,t) \geq v_h(z,r) - ch^{\frac{\zeta\eta}{2}}.$$

Using the previous estimate and the fact that u is also Hölder continuous and $d(\tilde{U},\mathcal{U}_h^b) \leq c_0 h^{\zeta/2}$, we obtain

$$u(x,t) - v(x,t) \leq u(z,r) - v_h(z,r) + ch^{\frac{\zeta\eta}{2}} \leq \sup_{\mathcal{U}_h^b}(u-v_h) + c_3 h^{\frac{\zeta\eta}{2}},$$

where the second inequality follows since $(z,r) \in \mathcal{U}_h^b$. Since this holds for any $(x,t) \in \partial_p\tilde{U}$, we find that (72) holds as well. The assumption (69) says that the first term on the right-hand side of (72) is non-positive. The estimate (72) therefore becomes,

$$\sup_{\partial_p\tilde{U}}(u-v^-) \leq c_3 h^{\frac{\zeta\eta}{2}}. \tag{74}$$

Using (74) to bound the right-hand side of (71) and taking $\bar{c} = c_1 + c_2 + c_3$ and $\bar{\alpha} = \min\{\tilde{\alpha}, \frac{\zeta\eta}{2}\}$ completes the proof the theorem. □

Appendix A.

A.1. Proof of Proposition 2.12

We outline the proof Proposition 2.12, which is a slight modification of part of the proof of the parabolic version of the ABP estimate in [14, Section 4.1.2]. We assume $\rho = 1$; the general case follows by rescaling.

As in [14, Section 4.1.2], we define the function $G\Gamma : Y_2^- \to \mathbb{R}^{n+1}$ by

$$G\Gamma(x,t) = (\Gamma(x,t) - x \cdot D\Gamma(x,t), D\Gamma(x,t)).$$

Proposition 2.12 follows from the following three lemmas.

Lemma A.1. *Suppose u is as in Proposition 2.12. Let M denote $\sup_{Y_1^-} u^-$. Then,*

$$\{(h,\xi) \in \mathbb{R}^{n+1} : |\xi| \leq \frac{M}{2} \leq -h \leq M\} \subset G\Gamma(Y_1^- \cap \{u = \Gamma\}).$$

Lemma A.2. *If Γ is $C^{1,1}$ with respect to x and Lipschitz with respect to t, then $G\Gamma$ is Lipschitz in (x,t) and, for almost every $(x,t) \in Y_1^-$,*

$$\det D_{x,t} G\Gamma = \partial_t \Gamma \det D^2\Gamma.$$

Lemma A.3. *Suppose u is as in Proposition 2.12. Then Γ is $C^{1,1}$ with respect to x and Lipschitz in t.*

Lemma A.1 is very similar to [14, Lemma 4.13]—the latter is stated for u a supersolution of $u_t - \mathcal{M}^-(D^2u) \geq f(x)$ in Y_1^-. However, it is easy to verify that the proof of [14, Lemma 4.13] holds for a general continuous function u, not only for supersolutions. Lemma A.2 is exactly [14, Lemma 4.4]. The proof of Lemma A.3 is very similar to that of Lemma [14, Lemma 4.11], which states that if u is a supersolution of $u_t - \mathcal{M}^-(D^2u) \geq f(x)$, then Γ is $C^{1,1}$ in x and Lipschitz in t. We explain the modifications needed to obtain Lemma A.3:

Proof of Lemma A.3. As in the proof of Lemma [14, Lemma 4.11], it will suffice to show that there exists a constant K such that for all $(x,t) \in Y_1^-$, the following holds:

$$\begin{cases} \text{if } P(y,s) = c + \alpha t + p \cdot x + x \cdot X x^T \text{ satisfies } P(x,t) = \Gamma(x,t) \text{ and} \\ P(y,s) \leq \Gamma(y,s) \text{ for all } (y,s) \text{ in a neighborhood of } (x,t), \\ \text{then } \alpha \geq -K \text{ and } X \leq KI. \end{cases} \tag{75}$$

In the proof of [14, Lemma 4.11], the assumption that u is a supersolution of $u_t - \mathcal{M}^-(D^2u) \geq f(x)$ is used only to show that (75) holds for $(x,t) \in \{u = \Gamma\}$. Thus, we need to verify (75) for such points. To this end, assume $(x,t) \in \{u = \Gamma\}$ and $P(y,s) = c + \alpha t + p \cdot x + x \cdot X x^T$ satisfies $P(x,t) = \Gamma(x,t)$ and $P(y,s) \leq \Gamma(y,s)$ for all (y,s) in a neighborhood of (x,t). Since Γ is the lower monotone envelope of $\min(u,0)$ and $(x,t) \in \{u = \Gamma\}$, we have

$$P(x,t) = u(x,t) \text{ and } P(y,s) \le u(y,s) \text{ for all } (y,s) \text{ in a neighborhood of } (x,t). \tag{76}$$

Since we have $D^2u(x,t) \le KI$ in the sense of distributions, there exists a paraboloid $Q(x)$ of opening K that touches $u(\cdot,t)$ from above at x. Together with (76), this implies that Q touches $P(\cdot,t)$ from above at x. Therefore, $X \le KI$. In addition, (76) implies

$$|P_t(x,t)| \le [u]_{C_t^{0,1}(Y_1^-)}.$$

Since $[u]_{C_t^{0,1}(Y_1^-)} \le K$, we obtain $\alpha = P_t(x,t) \ge -K$. Thus we've verified (75) and the proof of the lemma is complete. □

Proof of Proposition 2.12. Lemma A.1, Lemma A.2 and the Area Formula [13, Chapter 3], which applies to $G\Gamma$ because of Lemma A.3, imply

$$\begin{aligned} CM^{n+1} &= |\{(h,\xi) \in \mathbb{R}^{n+1} : |\xi| \le \frac{M}{2} \le -h \le M\}| \\ &\le |G\Gamma(Y_1^- \cap \{u = \Gamma\})| \\ &\le \int_{Y_1^- \cap \{u=\Gamma\}} |\det D_{x,t}G\Gamma| \\ &\le \int_{Y_1^- \cap \{u=\Gamma\}} -\partial_t\Gamma \det D^2\Gamma. \end{aligned} \tag{77}$$

Because Γ is non-increasing in t and convex in x, the Alexandroff theorem for functions depending on x and t (see Krylov [17, Appendix 2]) implies that there exists $A \subset Y_1^-$ with $|Y_1^- \setminus A| = 0$ such that for every $(x,t) \in A$,

$$\begin{aligned} |\Gamma(y,s) - \Gamma(x,t) - D\Gamma(x,t)\cdot(y-x) - (y-x)\cdot D^2\Gamma(x,t)(y-x)^T \\ - \partial_t\Gamma(x,t)(s-t)| \le o(|x-y|^2 + |t-s|) \end{aligned}$$

(in other words, Γ is twice differentiable almost everywhere on Y_1^-). We claim

$$\text{if } (x,t) \in A \cap \{u = \Gamma\}, \text{ then } -\partial_t\Gamma(x,t)\det D^2\Gamma(x,t) \le K^{n+1}. \tag{78}$$

Let us fix $(x,t) \in A \cap \{u = \Gamma\}$. We have

$$u(x,t) = \Gamma(x,t) \text{ and } u(y,s) \ge \Gamma(y,s) \text{ for all } (y,s) \text{ in a neighborhood of } (x,t). \tag{79}$$

Since we have $D^2u(x,t) \le KI$ in the sense of distributions, we find that there exists a paraboloid $Q(x)$ of opening K that touches $u(\cdot,t)$ from above at x. Therefore, Q touches $\Gamma(\cdot,t)$ from above at x. Thus, $D^2\Gamma(x,t) \le KI$. Since Γ is convex in x, we obtain $D^2\Gamma(x,t) \ge 0$. Therefore,

$$0 \le \det D^2\Gamma \le K^n.$$

In addition, (79) implies

$$|\partial_t\Gamma(x,t)| \le [u]_{C_t^{0,1}(Y_1^-)}.$$

Since $[u]_{C_t^{0,1}(Y_1^-)} \le K$, we obtain $\partial_t\Gamma(x,t) \ge -K$. Since Γ is non-increasing in t, we have $\partial_t\Gamma(x,t) \le 0$. Together with the estimate on $\det D^2\Gamma$, this implies that (78) holds. The bound (3) follows by using (78) to bound the right-hand side of (77), recalling the definition of $M = \sup_{Y_1^-} u^-$, and rearranging. □

A.2. Inf and sup convolutions

In the following proposition we state the facts about inf- and sup- convolutions that are used in this paper. Their proofs are very similar to those in the elliptic case (see [9, Propositions 5.3 and 5.5] and [7, Lemma 5.2]) and we omit them.

Proposition A.4. *Assume $v \in C(\Omega \times (0,T))$. Then:*

(1) We have $v^-_{\theta,\theta}(x,t) \leq v(x,t) \leq v^+_{\theta,\theta}(x,t)$ for all $(x,t) \in \Omega \times (0,T)$.

(2) If $v \in C^{0,\eta}(\Omega \times (0,T))$, then for any $(x,t) \in U^{\theta,\delta}$,

$$v^+_{\theta,\theta}(x,t) - [v]^{\frac{2}{2-\eta}}_{C^{0,\eta}(\Omega\times(0,T))}(2\theta)^{\frac{\eta}{2-\eta}} \leq v(x,t) \leq v^-_{\theta,\theta}(x,t) + [v]^{\frac{2}{2-\eta}}_{C^{0,\eta}(\Omega\times(0,T))}(2\theta)^{\frac{\eta}{2-\eta}}.$$

(3) In the sense of distributions, $D^2 v^-_{\theta,\theta}(x,t) \leq \theta^{-1} I$ and $D^2 v^+_{\theta,\theta}(x,t) \geq -\theta^{-1} I$ for every $(x,t) \in U^{\theta,\delta}$.

(4) We have $[v^{\pm}_{\theta,\theta}]_{C^{0,1}_t(\Omega\times(0,T))} \leq 3T\theta^{-1}$.

(5) If v is a δ-supersolution of $v_t - F(D^2 v) = 0$ in $\Omega \times (0,T)$, then $v^-_{\theta,\theta}$ is a δ-supersolution of $v_t - F(D^2 v) = 0$ in $U^{\theta,\delta}$. If v is a δ-subsolution of $v_t - F(D^2 v) = 0$ in $\Omega \times (0,T)$, then $v^+_{\theta,\theta}$ is a δ-subsolution of $v_t - F(D^2 v) = 0$ in $U^{\theta,\delta}$.

A.3. Inf and sup convolutions of mesh functions

In the following proposition we summarize the facts that we need about inf and sup convolutions of mesh functions.

Proposition A.5. *Let us take $\theta > 0$ and $v \in C^{0,\eta}(\mathcal{U}_h)$. Then:*

(1) If (x^*,t^*) is a point at which the infimum (resp. supremum) is achieved in the definition of $v^-_{\theta,\theta}(x,t)$ (resp. $v^+_{\theta,\theta}(x,t)$), then

$$d_e((x^*,t^*),(x,t)) \leq \omega(h,\theta).$$

In particular, if $(x,t) \in U^h_\theta$ then $(x^*,t^*) \in \mathcal{U}^i_h$.

(2) We have $v^-_{\theta,\theta}(x,t) \leq v(x,t) \leq v^+_{\theta,\theta}(x,t)$ for all $(x,t) \in \mathcal{U}_h$.

(3) Assume $(x,t) \in \Omega \times (0,T)$ and $(y,s) \in \mathcal{U}_h$ is a closest mesh point to (x,t). Then

$$v^+_{\theta,\theta}(x,t) - ||v||_{C^{0,\eta}(\mathcal{U}_h)}\omega(h,\theta)^\eta \leq v(x,t) \leq v^-_{\theta,\theta}(x,t) + ||v||_{C^{0,\eta}(\mathcal{U}_h)}\omega(h,\theta)^\eta.$$

(4) In the sense of distributions, $D^2 v^-_{\theta,\theta}(x,t) \leq \theta^{-1} I$ and $D^2 v^+_{\theta,\theta}(x,t) \geq -\theta^{-1} I$ for every $(x,t) \in U^h_\theta$.

(5) We have $[v^{\pm}_{\theta,\theta}]_{C^{0,1}_t(\Omega\times(0,T))} \leq 3T\theta^{-1}$.

(6) If $(x,t) \in U^h_\theta$ is such that $Y^-_{Nh}(x,t) \subset U^h_\theta$ and (x^*,t^*) is a point at which the infimum (resp. supremum) is achieved in the definition of $v^-_{\theta,\theta}(x,t)$ (resp. $v^+_{\theta,\theta}(x,t)$), then

$$\delta^-_\tau v^-_{\theta,\theta}(x,t) \geq \delta^-_\tau v(x^*,t^*) \text{ and } \delta^2_y v^-_{\theta,\theta}(x,t) \leq \delta^2_y v(x^*,t^*)$$
$$(\text{resp. } \delta^-_\tau v^+_{\theta,\theta}(x,t) \leq \delta^-_\tau v(x^*,t^*) \text{ and } \delta^2_y v^+_{\theta,\theta}(x,t) \geq \delta^2_y v(x^*,t^*)).$$

The proofs of items (1)-(5) are very similar to the elliptic case (see Proposition 2.3 of [8] and [7, Lemma 5.2]) so we omit them. We provide the proof of item (6).

Proof of item (6) of Proposition A.5. We will give the proof of the first inequality; the proofs of the others are analogous. By the definition of the discrete operator δ_τ^- and the definition of (x^*, t^*), we have

$$\begin{aligned}\delta_\tau^- v_{\theta,\theta}^-(x,t) &= \frac{1}{h}\left(v_{\theta,\theta}^-(x,t) - v_{\theta,\theta}^-(x,t-h^2)\right) \\ &= \frac{1}{h}\left(v(x^*,t^*) + \frac{|x-x^*|^2}{2\theta} + \frac{|t-t^*|^2}{2\theta} - v_{\theta,\theta}^-(x,t-h^2)\right).\end{aligned}$$

We now estimate $v_{\theta,\theta}^-(x,t-h^2)$ from above. Since $(x,t) \in U_\theta^h$, item (1) of this proposition implies $(x^*,t^*) \in \mathcal{U}_h^i$. In addition, since we have $Y_{Nh}^-(x,t) \subset U_\theta^h$, we find $(x^*, t^*-h^2) \in \mathcal{U}_h$. Therefore, we may use (x^*, t^*-h^2) as a "test point" in the definition of $v_{\theta,\theta}^-(x,t-h^2)$ and obtain,

$$\begin{aligned}\delta_\tau^- v_{\theta,\theta}^-(x,t) &\geq \frac{1}{h}\left(v(x^*,t^*) + \frac{|x-x^*|^2}{2\theta} + \frac{|t-t^*|^2}{2\theta}\right. \\ &\qquad \left. - v(x^*,t^*-h^2) - \frac{|x-x^*|^2}{2\theta} - \frac{|t-t^*|^2}{2\theta}\right) \\ &= \delta_\tau^- v(x^*,t^*),\end{aligned}$$

where the equality follows from the definition of the operator δ_τ^-. □

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Olga Turanova
University of Chicago
Chicago, IL
USA
e-mail: turanova@math.uchicago.edu

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