Erratum to: Structure of the solution set to impulsive functional differential inclusions on the half-line

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Abstract. The aim of the paper is twofold. At first, it is an erratum to: Gabor G., Grudzka A.: Structure of the solution set to impulsive functional differential inclusions on the half-line. NoDEA 19, 609–627 (2012). We fill a little gap in a proof of Theorem 3.5 by a change of a compactness type condition, and give a discussion on assumptions sufficient to obtain an R_δ -structure of solution sets. Simultaneously, we improve the main theorems by the use of modified proof-arguments. Namely, the assumptions on impulse maps can be weakened.

1. Introduction

In the original paper a structure of the solution set of the following impulsive Cauchy problem governed by a semilinear differential inclusion on a noncompact interval has been studied:

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.a. } t \in [0, \infty), t \neq t_k, k \in \mathbb{N}, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k \in \mathbb{N}^+, \end{cases} \quad (1)$$

where $\{A(t)\}_{t \in [0, \infty)}$ is a family of linear operators in a Banach space E generating an evolution operator; F is an upper-Carathéodory map; $\tau > 0$ is a given number, $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$; I_k are impulse functions, $k \in \mathbb{N}$,

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$y(t^+) = \lim_{s \rightarrow t^+} y(s)$, and the time sequence $(t_k)_{k \in \mathbb{N}}$ is an increasing sequence of given points in $[0, \infty)$ without accumulation points. The history of research in this topic is described in the original paper. As an intermediate step in considerations the following problem on a compact interval has been investigated

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.e. } t \in [0, t_m], \ t \neq t_k, \ k < m, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k < m. \end{cases} \quad (2)$$

One of the main results of the original paper reads as follows:

Theorem 1.1. (Theorem 3.5 in the original paper) *Let E be a Banach space and let hypothesis*

$(A)_m$ $\{A(t)\}_{t \in [0, t_m]}$ *is a family of linear (not necessarily bounded) operators*
 $(A(t) : D(A) \subset E \rightarrow E, \ t \in [0, t_m], \ D(A) \text{ a dense subset of } E \text{ not}$
depending on } t) \text{ generating an evolution operator } T : \Delta_m \rightarrow \mathcal{L}(E),

hold, where $\Delta_m = \{(t, s) \in [0, t_m] \times [0, t_m] \mid s \leq t \leq t_m\}$. *Suppose that the multivalued map* $F : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$, *where* $P_{cp, cv}(E)$ *is a family of nonempty compact convex subsets of* E , *satisfies conditions*

- $(F1)_m$ $F(\cdot, c)$ *has a strongly measurable selection for every* $c \in \mathcal{C}([-\tau, 0], E)$,
 $(F2)_m$ $F(t, \cdot)$ *is u.s.c. for a.e.* $t \in [0, t_m]$,
 $(F3)_m$ F *for almost every* $t \in [0, t_m]$ *has at most a linear growth, i.e., there exists a function* $\alpha \in L^1([0, t_m])$ *such that*

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|_{\mathcal{C}}) \quad \text{for a.e. } t \in [0, t_m],$$

$(F4)_m$ *There exists a function* $\mu \in L^1([0, t_m])$ *such that*

$$\beta(F(t, D)) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta))$$

for a.e. $t \in [0, t_m]$ *and for every bounded* $D \subset \mathcal{C}([-\tau, 0], E)$, *where* $D(\theta) = \{c(\theta) \mid c \in D\}$.

Moreover, assume that the maps $I_k : \mathcal{C}([-\tau, 0], E) \rightarrow E, k \in \mathbb{N}$, *are continuous and there exist constants* $r_k > 0$ *such that*

(I1)

$$\beta(I_k(D)) \leq r_k \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)),$$

for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$,

(I2)

$$\sum_{k=1}^m r_k < \frac{1}{B_m},$$

where $B_m = \sup_{(t, s) \in \Delta_m} \|T(t, s)\|_{\mathcal{L}(E)}$.

Then the solution set for problem (2) is an R_δ -*set in* $PC([0, t_m], E)[x]$.

Recall that $\mathcal{C}([-\tau, 0], E)$ is the space of piecewise continuous functions $c : [-\tau, 0] \rightarrow E$ with finite number of discontinuity points $\{t_*\}$ such that $t_* \neq 0$ and all values

$$c(t_*^+) = \lim_{h \rightarrow 0^+} c(t_* + h) \quad \text{and} \quad c(t_*^-) = \lim_{h \rightarrow 0^-} c(t_* + h)$$

are finite. We consider in $\mathcal{C}([-\tau, 0], E)$ the L^1 -norm $\|c\|_{\mathcal{C}} = \int_{-\tau}^0 \|c(t)\| dt$. We also denote

$$PC([0, t_m], E)[x] := \{y[x] \in PC([-\tau, t_m], E) \mid y \in PC([0, t_m], E)\},$$

where $y[x]$ is equal y on $[0, t_m]$ and $y[x] = x$ on $[-\tau, 0]$.

In Step 3 of the proof of Theorem 1.1 (see the original paper) one has applied assumption $(F4)_m$ to estimate a measure of noncompactness of some sequence of functions. Unfortunately, it has been done incorrectly, because the inequality

$$\beta\left[F\left(\{s\} \times B(\{y_{n_s}\}_{n \geq p}, 3d_p)\right)\right] \leq \mu(s) \cdot \sup_{-\tau \leq \theta \leq 0} \beta\left(B(\{y_n(s + \theta)\}_{n \geq p}, 3d_p)\right)$$

could be false in the L^1 norm in $\mathcal{C}([-\tau, 0], E)$. Below, in Sect. 2, we give an assumption which is sufficient for the proof, and discuss differences between the assumptions. In fact we really hope that assumption $(F4)_m$ is also sufficient but we have to leave this hypothesis as an open question.

In Sect. 3 we give a corrected proof, but also improve the results and show that assumption (I2) is not needed. To prove this we will use in a proof different arguments from those in the original paper.

2. Compactness type assumptions on F

Instead of $(F4)_m$ we assume

$(F4)'_m$ for every $\varepsilon > 0$ and every bounded set $D \subset \mathcal{C}([-\tau, 0], E)$ there exist $\delta > 0$ and a function $\mu \in L^1([0, t_m])$ such that

$$\beta(F(t, O_\delta(D))) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(O_\varepsilon(D(\theta))) \quad \text{for a.e. } t \in [0, t_m],$$

where $O_\delta(D)$ denotes a δ -neighborhood of D defined as $O_\delta(D) := \{z \in \mathcal{C}([-\tau, 0], E); \text{dist}(z, D) < \delta\}$.

It is easy to see that $(F4)'_m \implies (F4)_m$. Indeed, for every bounded set $D \subset \mathcal{C}([-\tau, 0], E)$ and every $\varepsilon > 0$ we take $\delta > 0$ and $\mu \in L^1([0, t_m])$ from $(F4)'_m$, and obtain, for a.e., $t \in [0, t_m]$,

$$\beta(F(t, D)) \leq \beta(F(t, O_\delta(D))) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(O_\varepsilon(D(\theta))).$$

Since ε was arbitrary, we get $(F4)_m$.

Assumption $(F4)'_m$ is satisfied by many important classes of maps. We list some of them in the following three examples.

Example 2.1. Each completely continuous map with respect to the second variable satisfies $(F4)'_m$ with arbitrary $\delta > 0$ and $\mu \in L^1([0, t_m])$.

Example 2.2. Let $F(t, y) = \alpha(t) \int_{-\tau}^0 y(\theta) d\theta$. As we can see, this perturbation F depends on the behavior of y during the whole time interval $[-\tau, 0]$. Notice that, for any bounded set D , we get

$$\begin{aligned} \beta(F(t, O_\delta(D))) &= \alpha(t) \beta \left(\left\{ \int_{-\tau}^0 z(\theta) d\theta \mid \text{dist}(z, D) \leq \delta \right\} \right) \\ &\leq \alpha(t) \left(\beta \left(\left\{ \int_{-\tau}^0 y(\theta) d\theta \mid y \in D \right\} \right) + \delta \right). \end{aligned}$$

If $\sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}) = +\infty$, then condition $(F4)'_m$ is obviously fulfilled. If $\sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}) < +\infty$, then the constant function $[-\tau, 0] \ni s \mapsto \sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\})$, is integrable. Moreover, $\{y(\theta) \mid y \in D\} \subset B(0, r)$ for some $r > 0$ and every $\theta \in [-\tau, 0]$. Therefore the family D is uniformly integrably bounded. Hence,

$$\begin{aligned} \beta \left(\left\{ \int_{-\tau}^0 y(\theta) d\theta \mid y \in D \right\} \right) &\leq 4 \int_{-\tau}^0 \sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}) d\theta \\ &= 4\tau \sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}), \end{aligned}$$

and, consequently,

$$\begin{aligned} \beta(F(t, O_\delta(D))) &\leq \alpha(t) \left[4\tau \sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}) + \delta \right] \\ &= 4\tau \alpha(t) \left[\sup_{-\tau \leq \theta \leq 0} \beta(\{y(\theta) \mid y \in D\}) + \frac{\delta}{4\tau} \right]. \end{aligned}$$

It is sufficient to take $\mu(t) := 4\tau \alpha(t)$ and $\delta := 4\tau \varepsilon$.

Example 2.3. Assume that F satisfies $(F4)_m$ and a Lipschitz condition with respect to the second variable, i.e., there exists a function $L \in L^1([0, t_m])$ such that for every $t \in [0, t_m]$ and each $y, z \in \mathcal{C}([-\tau, 0], E)$ one has

$$d_H(F(t, z), F(t, y)) \leq L(t) \|z - y\|_{\mathcal{C}}.$$

Then F satisfies $(F4)'_m$.

Indeed, by the Lipschitz condition we have

$$F(s, O_\delta(D)) \subset O_{L(s)\delta}(F(s, D))$$

for every $s \in [0, t_m]$ and $\delta > 0$. Thus

$$\begin{aligned} \beta(F(\{s\} \times O_\delta(D))) &\leq \beta(F(s, D)) + L(s)\delta \\ &\leq \mu(s) \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)) + L(s)\delta \\ &\leq \tilde{\mu}(s) \left(\sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)) + \delta \right), \end{aligned}$$

where $\tilde{\mu}(s) := \max\{\mu(s), L(s)\}$. It is sufficient to take $\delta = \varepsilon$.

3. Main results

The main goal of this paper is to prove the following result on a topological structure of the solution set for problem (2), and, as a consequence, an analogous result for the problem on a noncompact interval.

Theorem 3.1. *Let E be a Banach space, let hypotheses $(A)_m$, $(F1)_m$ – $(F3)_m$, $(F4)'_m$ hold and let the maps $I_k : \mathcal{C}([-\tau, 0], E) \rightarrow E$, $k \in \mathbb{N}$, be continuous and satisfy (I1).*

Then the solution set for problem (2) is an R_δ -set in $PC([0, t_m], E)[x]$.

Proof. We follow the steps of the proof of Theorem 3.5 in the original paper, and make suitable changes.

Step 1. We show (see the original paper) that all solutions to (2) are pointwise equibounded by some constant $\tilde{K}_m$. Let $\tilde{K}_m = \tau \bar{K}_m$. Then $\|y_t\|_C \leq \tilde{K}_m$ for every solution y to (2). We define a mapping $\tilde{F} : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp,cv}(E)$,

$$\tilde{F}(t, c) = \begin{cases} F(t, c), & \text{if } t \in [0, t_m] \text{ and } \|c\|_C \leq \tilde{K}_m, \\ F(t, \frac{\tilde{K}_m c}{\|c\|_C}), & \text{if } t \in [0, t_m] \text{ and } \|c\|_C > \tilde{K}_m, \end{cases} \quad (3)$$

which is integrably bounded and generates the same set of solutions, if we replace F by $\tilde{F}$ in (2). In conclusion, without any loss of generality we can assume

$$(F3')_m \quad \|F(t, c)\| \leq \psi_m(t) \text{ for every } t \in [0, t_m], \text{ where } \psi_m \in L^1([0, t_m])$$

instead of $(F3)_m$.

Step 2. Repeating the lines of Step 2 in the original paper we construct a sequence of multivalued maps $\{G_n\}_{n=1}^\infty$, $G_n : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cl,cv}(E)$ such that:

- (i) each multivalued map $G_n(t, \cdot) : \mathcal{C}([-\tau, 0], E) \rightarrow P_{cl,cv}(E)$, $n \geq 1$, is continuous for a.e. $t \in [0, t_m]$,
- (ii) $F(t, c) \subset \cdots \subset G_{n+1}(t, c) \subset G_n(t, c) \subset \overline{\text{conv}} F(t, B_{3d_n}(c))$, where $d_n = \frac{1}{3^n}$, $n \geq 1$,
- (iii) $F(t, c) = \bigcap_{n \geq 1} G_n(t, c)$,
- (iv) for each $n \geq 1$ there exists a selection $g_n : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow E$ of G_n , such that $g_n(\cdot, c)$ is measurable and $g_n(t, \cdot)$ is locally Lipschitz.

Step 3. We consider the differential problem:

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + G_n(t, y_t), & \text{for a.e. } t \in [0, t_m], \ t \neq t_k, \ k < m, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k < m, \end{cases} \quad (4)$$

and denote by S_m^n the solution set of problem (4). From condition (ii) it follows that there is a constant $K_m \geq 0$ such that $\|y\| \leq K_m$ for every $y \in S_m^n$.

We show that each sequence $\{y_n\}$ such that $y_n \in S_m^n$ for all $n \geq 1$ has a convergent subsequence $y_{n_k} \rightarrow y \in S_m$. Notice that $y_n(t) = x(t)$ for every $t \in [-\tau, 0]$.

Consider the sequence $\{y_n\}$ on $[0, t_1]$. Then

$$y_n(t) = T(t, 0)x(0) + \int_0^t T(t, s)f_n(s)ds$$

for $t \in [0, t_1]$, where $f_n \in L^1([0, t_1], E)$, $f_n(t) \in G_n(t, (y_n)_t)$ for a.e. $t \in [0, t_1]$.

For $d_k, k \geq 1$, from $(F4)'_m$ we get $\delta > 0$ such that

$$\beta(F(\{s\} \times O_\delta(\{(y_n)_s\}_{n \geq 1}))) \leq \mu(s) \left[\sup_{-\tau \leq \theta \leq 0} \beta(\{(y_n)_s(\theta)\}_{n \geq 1}) + d_k \right].$$

Take $d_l, l \geq k$, such that $3d_l < \delta$. Then

$$\begin{aligned} \beta(\{f_n(s)\}_{n \geq 1}) &= \beta(\{f_n(s)\}_{n \geq l}) \leq \beta \left[F \left(\{s\} \times O_{3d_l}(\{(y_n)_s\}_{n \geq l}) \right) \right] \\ &\leq \mu(s) \left[\sup_{-\tau \leq \theta \leq 0} \beta(\{y_n(s + \theta)\}_{n \geq l}) + d_k \right] \\ &\leq \mu(s) \left[\max \left(\sup_{-\tau \leq \sigma \leq 0} \beta(\{x(\sigma)\}), \sup_{0 \leq \sigma \leq s} \beta(\{y_n(\sigma)\}_{n \geq l}) \right) + d_k \right] \\ &= \mu(s)(\bar{\rho}(s) + d_k), \end{aligned}$$

where $\bar{\rho}(s) := \sup_{0 \leq \sigma \leq s} \beta(\{y_n(\sigma)\}_{n \geq 1})$. This implies that

$$\begin{aligned} \beta(\{y_n(t)\}_{n \geq 1}) &= \beta \left(\left\{ T(t, 0)x(0) + \int_0^t T(t, s)f_n(s)ds \right\}_{n \geq 1} \right) \\ &\leq 2B_m \int_0^t \mu(s)(\bar{\rho}(s) + d_k)ds \end{aligned}$$

for every $k \geq 1$. But $d_k \searrow 0$ when $k \rightarrow \infty$, so

$$\bar{\rho}(t) \leq 2B_m \int_0^t \mu(s)\bar{\rho}(s)ds.$$

From the Gronwall inequality we obtain $\bar{\rho}(t) = 0$, which means that, in the interval $[0, t_1]$,

$$\beta(\{y_n(t)\}_{n \geq 1}) = 0.$$

Hence, $\beta(\{f_n(s)\}_{n \geq 1}) = 0$ for $s \leq t_1$.

For $t = t_1$ one has

$$\begin{aligned} \beta(\{T(t, t_1)I_1((y_n)_{t_1})\}_{n \geq 1}) &= \beta(\{T(t, t_1)I_1((y_n)_{t_1})\}_{n \geq l}) \\ &\leq \|T(t, t_1)\| \cdot \beta(\{I_1((y_n)_{t_1})\}_{n \geq l}) \\ &\leq B_m p_1 \sup_{-\tau \leq \theta \leq 0} \beta(\{(y_n)_{t_1}(\theta)\}_{n \geq l}) \\ &\leq B_m p_1 \cdot \max \left(\sup_{-\tau \leq \sigma \leq 0} \beta(\{x(\sigma)\}), \sup_{0 \leq \sigma \leq t_1} \beta(\{y_n(\sigma)\}_{n \geq 1}) \right) \\ &= B_m p_1 \bar{\rho}(t_1) = 0. \end{aligned}$$

Every solution y_n has the form

$$y_n(t) = T(t, t_1)y_n(t_1) + T(t, t_1)I_1((y_n)_{t_1}) + \int_{t_1}^t T(t, s)f_n(s)ds,$$

for $t \in [t_1, t_2]$, where $f_n \in L^1([t_1, t_2], E)$, $f_n(t) \in G_n(t, (y_n)_t)$ for a.e. $t \in [t_1, t_2]$. Analogously as before one shows that $\beta(\{f_n(s)\}_{n \geq 1}) = 0$. Next, we consider $t = t_2$ and so on. Finally, $\beta(\{y_n(t)\}_{n \geq 1}) = 0$ for $t \in [0, t_m]$, and hence $\beta(\{f_n(t)\}_{n \geq 1}) = 0$.

For $t \leq t'$ in $[0, t_1]$ we have

$$\|y_n(t') - y_n(t)\| = \left\| \int_t^{t'} \dot{y}_n(s) ds \right\| \leq B_m \int_t^{t'} \psi_m(s) ds,$$

since $\|G_n(t, c)\| \leq \psi_m(t)$ for a.e. $t \in [0, t_m]$ from (ii) and $(F3)'_m$. Therefore, the family $\{y_n\}$ is equicontinuous. This implies the existence of a convergent subsequence on $[0, t_1]$ denoted by $(y_{n_{q_1}}^1)_{q_1 \in Q_1}$. Define

$$y_{n_{q_1},1}(t) := \begin{cases} y_{n_{q_1}}^1(t), & \text{for } t \in [0, t_1], \\ y_{n_{q_1}}^1(t_1) + I_1((y_{n_{q_1}}^1)_{t_1}) \\ \quad + \sum_{t_1 < t_j < t} T(t, t_j) I_j((y_{n_{q_1}})_{t_j}) \\ \quad + \int_{t_1}^t T(t, s) f_{n_{q_1}}(s) ds, & \text{for } t > t_1. \end{cases}$$

Let $y_{n_{q_1}}^2 := y_{n_{q_1},1}|_{[t_1, t_2]}$. Notice that $y_{n_{q_1}}^2(t_1^+) = y_{n_{q_1}}^1(t_1^+)$ for every $k \geq 1$. For $t \leq t'$ w $[t_1, t_2]$ we have

$$\|y_{n_{q_1}}^2(t') - y_{n_{q_1}}^2(t)\| \leq B_m \int_t^{t'} \psi_m(s) ds,$$

so the family $\{y_{n_{q_1}}^2\}$ is equicontinuous. Thus it contains a convergent subsequence $(y_{n_{q_2}}^2)_{q_2 \in Q_2}$ on $[t_1, t_2]$, where $Q_2 \subset Q_1$. We take a concatenation

$$y_{n_{q_2},2}(t) := \begin{cases} y_{n_{q_2}}^1(t), & \text{dla } t \in [0, t_1], \\ y_{n_{q_2}}^2(t), & \text{dla } t \in (t_1, t_2], \\ y_{n_{q_2}}^2(t_2) + I_2((y_{n_{q_2}}^2)_{t_2}) \\ \quad + \sum_{t_2 < t_j < t} T(t, t_j) I_j((y_{n_{q_2}})_{t_j}) \\ \quad + \int_{t_2}^t T(t, s) f_{n_{q_2}}(s) ds, & \text{dla } t > t_2. \end{cases}$$

We proceed up to m and find a convergent subsequence $(y_{n_{q_m},m})_{q_m \in Q_m}$ of (y_n) , where $Q_m \subset Q_{m-1}$. Thus we have

$$y_{n_{q_m},m}(t) := \begin{cases} y_{n_{q_m}}^1(t), & \text{dla } t \in [0, t_1], \\ y_{n_{q_m}}^2(t), & \text{dla } t \in (t_1, t_2], \\ \vdots \\ y_{n_{q_m}}^m(t), & \text{dla } t \in (t_{m-1}, t_m]. \end{cases}$$

Let y be a limit of the sequence $(y_{n_{q_m},m})_{q_m \in Q_m}$. Denote the sequence $(f_{n_{q_m}})_{q_m \in Q_m}$ by $(f_{n_{q_m}(l)})_{l \geq 1}$.

From $(F3)'_m$, the equality $\beta(\{f_{n_{q_m}(l)}(s)\}_{l \geq 1}) = 0$ and the Dunford–Pettis theorem it follows that, up to subsequence, $f_{n_{q_m}(l)} \rightharpoonup f_0 \in L^1([0, t_m], E)$. By the continuity of impulse functions I_j

$$y(t) = T(t, 0)x(0) + \sum_{0 < t_j < t} T(t, t_j) I_j(y_{t_j}) + \int_0^t T(t, s) f_0(s) ds.$$

In the original paper we have showed the way how to prove that $f_0(s) \in F(s, (y[x])_s)$ for a.e. $s \in [0, t_m]$. We omit details.

Step 4. As in the original paper we show that $S_m = \bigcap_{n=1}^{\infty} \text{cl } S_m^n$ and $\tilde{\beta}(\text{cl } S_m^n) \xrightarrow{n \rightarrow \infty} 0$, where $\tilde{\beta}$ is the Hausdorff measure of noncompactness in $PC([0, t_m], E)[x]$.

Step 5. We show (see the original paper) that the sets $\text{cl } S_m^n$ are contractible, and finish the proof of the theorem by application of the Hyman and Bothe theorems (see Lemma 5 in [1]). $\square$

Theorem 1.1 was applied in the original paper to prove the R_δ structure of solution sets for problems on noncompact intervals. Now Theorem 3.1 can be used in an analogous way to obtain

Theorem 3.2. (Comp. Theorem 3.7 in the original paper) *Let E be a Banach space and let hypothesis*

$(A)^\infty$ *$\{A(t)\}_{t \in [0, \infty)}$ is a family of linear (not necessarily bounded) operators $(A(t) : D(A) \subset E \rightarrow E, t \in [0, \infty), D(A)$ a dense subset of E not depending on t) generating an evolution operator $T : \Delta_\infty \rightarrow \mathcal{L}(E)$,*

hold, where $\Delta_\infty = \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid 0 \leq s \leq t\}$. Suppose that the multivalued map $F : [0, \infty) \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ satisfies conditions

$(F1)^\infty$ *$F(\cdot, c)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau, 0], E)$,*

$(F2)^\infty$ *$F(t, \cdot)$ is u.s.c. for a.e. $t \geq 0$,*

$(F3)^\infty$ *F for almost every $t \geq 0$ has at most a linear growth, i.e., there exists a function $\alpha \in L^1_{loc}([0, \infty))$ such that*

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|_C) \quad \text{for a.e. } t \in [0, \infty),$$

$(F4)^\infty$ *for every $\varepsilon > 0$ and every bounded set $D \subset \mathcal{C}([-\tau, 0], E)$ there exist $\delta > 0$ and a function $\mu \in L^1_{loc}([0, \infty))$ such that*

$$\beta(F(t, O_\delta(D))) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(O_\varepsilon(D(\theta))) \quad \text{for a.e. } t \geq 0.$$

Moreover, assume that the maps $I_k : \mathcal{C}([-\tau, 0], E) \rightarrow E, k \in \mathbb{N}$, are continuous and there exist constants $r_k > 0$ such that

(II)

$$\beta(I_k(D)) \leq r_k \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)),$$

for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$.

Then the solution set for problem (2) is an R_δ -set in $PC([0, \infty), E)[x]$, where

$$PC([0, \infty), E)[x] := \{y[x] \in PC([-\tau, \infty), E) \mid y \in PC([0, \infty), E)\}.$$

Proof. As in the original paper we observe that $\{S_m, p_m^k, \mathbb{N}\}$, where $p_m^k : S_k \rightarrow S_m, p_m^k(y) = y|_{[-\tau, t_m]}$ for $k \geq m$, forms an inverse system with the limit $S = \varprojlim S_m$ equal to the solution set for problem (1). Now, we apply Proposition 3.2 in [2] to conclude that S is also an R_δ -set. $\square$

References

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