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**Nonlinear Differential Equations and Applications NoDEA**

# Existence of solutions of sublinear elliptic systems with quadratic growth in the gradient

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**Abstract.** In this paper, we deal with the following sublinear elliptic system:

$$\begin{cases} -\Delta u + u + |\nabla u|^2 = a(x)|v|^p + f, & x \in \mathbb{R}^N, \\ -\Delta v + v + |\nabla v|^2 = b(x)|u|^q + g, & x \in \mathbb{R}^N, \end{cases}$$

where $0 < p < 1$ and $0 < q < 1$. Under suitable assumptions on the terms $a$, $b$, $f$ and $g$ and by using the Schauder fixed point theorem, we obtain a solution for an approximated system. The limit of the approximated solutions is a nonnegative solution.

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**Keywords.** Sublinear elliptic systems, Quadratic growth in the gradient, Nonnegative solutions.

## 1. Introduction

In this paper, we are concerned with the existence of solutions of the following system:

$$\begin{cases} -\Delta u + u + |\nabla u|^2 = a(x)|v|^p + f, & x \in \mathbb{R}^N, \\ -\Delta v + v + |\nabla v|^2 = b(x)|u|^q + g, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

Our main hypotheses are cited below:

$(H_1)$ $0 < p < 1,\ 0 < q < 1, \quad N > 2.$

$(H_2)$ $a(x)\ , b(x) \geq 0,\ a.e.$ in $\mathbb{R}^N,\ a \in L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ b \in L^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$$\text{where} \quad r = \frac{2}{1-p} \quad \text{and} \quad s = \frac{2}{1-q}.$$

$(H_3)$ We suppose also that $f,\ g \geq 0,\ f,\ g \neq 0$ and $f,\ g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The study of semilinear systems in the whole space $\mathbb{R}^N$ has received much attention recently. See, for example, [1–12]. Benrhouma [1] studied the existence of solutions of the following system:

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} a(x)|v|^{\beta}|u|^{\alpha-2}u & \text{in } \mathbb{R}^N \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} a(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N \end{cases} \tag{1.2}$$

With the help of the Nehari manifold and the linking theorem, the author proved the existence of at least two solutions of (1.2). Chen [3] studied the existence of multiple positive solutions of the system:

$$\begin{cases} \Delta u = a(|x|)g(v), & x \in \mathbb{R}^N, \\ \Delta v = b(|x|)f(u), & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

Costa [4] studied the existence of solutions of the system:

$$\begin{cases} -\Delta u + a(x)u = f(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v \ = g(x,u,v), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

By using a variant of the generalised Moutain-Pass theorem [13], the author proved the existence of a nontrivial solution of (1.4). The supersolution and subsolution method is used by Kawano and Kusano [10] to establish the existence of infinitely positive entire solutions of the semilinear system:

$$\begin{cases} \Delta u + f(x,u,v) = 0, & x \in \mathbb{R}^N, \\ \Delta v + g(x,u,v) \ = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

Similar results have been proved for quasilinear or semilinear elliptic system in bounded domains, we can for instance cite [14–20]. Other works for single equation (see [21–28] and the references therein ). A good survey on the existence results for quasilinear elliptic equations with quadratic growth in the gradient can be found in [26–28]. By contrast, it seems to us that very few results are known on the quasilinear or semilinear elliptic system with gradient term in $\mathbb{R}^N$. We can quote [29–31].

Miao and Yang [29] established the existence of infinitely positive bounded entire solutions of the following system:

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(x,u,v) = 0, & x \in \mathbb{R}^N, \\ \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + g(x,u,v) \ = 0, & x \in \mathbb{R}^N \end{cases}. \tag{1.6}$$

where the functions $f$ and $g$ are, among other assumptions, locally Lipschitz continuous in $u$ and $v$ which is not the case in system (1.1). Zhang and Liu [30] studied the existence and the nonexistence of entire positive solutions for the system:

$$\begin{cases} \Delta u + |\nabla u| = p(|x|)f(u,v), & x \in \mathbb{R}^N, \\ \Delta v + |\nabla v| \ = q(|x|)g(u,v), & x \in \mathbb{R}^N, \end{cases}. \tag{1.7}$$

with the imposed conditions on the potential functions $p$ and $q$ are not satisfied in our case. Ghergu and Ràdulescu [31] studied the existence of explosive solutions for the system:

$$\begin{cases} \Delta u + |\nabla u| = p(|x|)f(v), & x \in \mathbb{R}^N, \\ \Delta v + |\nabla v| = q(|x|)g(u), & x \in \mathbb{R}^N, \end{cases} . \tag{1.8}$$

In the present work, we are interested in finding the existence of a nonnegative solution of (1.1) [ie: $(u, v)$ is nonnegative if $u \geq 0$ and $v \geq 0$ a.e.]. Using a priori estimates technique and through an approximation process we will be able to show the existence of a nonnegative and nontrivial solution of (1.1). Applying the Schauder fixed point theorem we prove the existence of a solution $(u_n, v_n)$ of an approximated system and we show that $(u_n, v_n)$ converges to a nonnegative solution of (1.1). The main difficulty to deal with the approximated systems consists in the boundedness of $(u_n)$ and $(v_n)$ in $L^\infty(\mathbb{R}^N)$ which can not be solved by the arguments of Stampacchia [32]. In fact, these arguments are only used for bounded domains. Moreover, to prove the almost everywhere convergence of $\nabla u_n$ and of $\nabla v_n$, we give a technique which is different of the one often used [21,26,27,33,34]. Our main result is in the following

**Theorem 1.1.** *Assume $(H_1) - (H_3)$ hold. Then the system (1.1) has at least one nonnegative solution.*

We divide this paper into three sections. In Sect. 2, we establish the existence of a solution of an approximated system and we give some useful properties of this solution. In Sect. 3, we prove the Theorem 1.1. We should proceed by steps.

We introduce the Banach space $H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ equipped by the norm

$$\|(u, v)\| = \left(\|u\|^2_{H^1(\mathbb{R}^N)} + \|v\|^2_{H^1(\mathbb{R}^N)}\right)^{\frac{1}{2}}$$

We say that $(u, v)$ is a weak solution pair of the system (1.1), if $(u, v) \in H$ and

$$\int_{\mathbb{R}^N} \left(\nabla u \nabla \varphi + u\varphi + \nabla v \nabla \psi + v\psi + |\nabla u|^2 \varphi + |\nabla v|^2 \psi\right) dx$$
$$-\int_{\mathbb{R}^N} \left((a(x)v^p + f)\varphi + (b(x)u^q + g)\psi\right) dx = 0 \quad \forall (\varphi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N).$$

## 2. The approximated quasilinear systems

We devote this section to study a sequence of approximated systems to (1.1). We consider the following approximated system:

$$\begin{cases} -\Delta u + u + \frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2} = a(x)\frac{|v|^p}{1+\frac{1}{n}|v|^p} + f, & x \in B_n \\ -\Delta v + v + \frac{|\nabla v|^2}{1+\frac{1}{n}|\nabla v|^2} = b(x)\frac{|u|^q}{1+\frac{1}{n}|u|^q} + g, & x \in B_n \end{cases} \tag{2.1}$$

where $B_n = \left\{x \in \mathbb{R}^N, \ |x| < n\right\}$.

**Lemma 2.1.** *Assume $(H_1) - (H_3)$ hold. Then, there exists $(u_n, v_n) \in H_n = H_0^1(B_n) \times H_0^1(B_n)$ which is a solution pair of (2.1).*

*Proof.* To prove it, we define the function $L_n: \ H_n \to L^2(B_n) \times L^2(B_n)$ by

$$L_n(u,v) = \left(-\frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2} + a(x)\frac{|v|^p}{1+\frac{1}{n}|v|^p} + f, -\frac{|\nabla v|^2}{1+\frac{1}{n}|\nabla v|^2} + b(x)\frac{|u|^q}{1+\frac{1}{n}|u|^q} + g\right)$$

First, observe that $L_n$ is continuous on $H_n$. Indeed, let $(u_k, v_k)$ be a sequence in $H_n$ which converges to $(u,v)$ in $H_n$ as $k \to \infty$. Which implies, by a subsequence, that $u_k(x) \to u(x)$, $\nabla u_k(x) \to \nabla u(x)$, $v_k(x) \to v(x)$ and $\nabla v_k(x) \to \nabla v(x)$ a.e. in $B_n$.

By virtue of Lebesgue's dominated convergence theorem, we deduce that

$$\int_{B_n} \left| \frac{|\nabla u_k|^2}{1+\frac{1}{n}|\nabla u_k|^2} - \frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2} - a(x)\left(\frac{|v_k|^p}{1+\frac{1}{n}|v_k|^p} - \frac{|v|^p}{1+\frac{1}{n}|v|^p}\right)\right|^2 dx \to 0 \quad \text{as} \quad k \to \infty$$

and

$$\int_{B_n} \left| \frac{|\nabla v_k|^2}{1+\frac{1}{n}|\nabla v_k|^2} - \frac{|\nabla v|^2}{1+\frac{1}{n}|\nabla v|^2} - b(x)\left(\frac{|u_k|^q}{1+\frac{1}{n}|u_k|^q} - \frac{|u|^q}{1+\frac{1}{n}|u|^q}\right)\right|^2 dx \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that, $L_n$ is continuous.

Let $E = L^2(B_n) \times L^2(B_n)$. If we equipped $E$ with the norm

$$\|(u,v)\|_E = \|u\|_{L^2(B_n)} + \|v\|_{L^2(B_n)},$$

it becomes a Banach space. Define the following function

$$\begin{aligned} S: \ E &\to \quad H_n \\ (u,v) &\longmapsto \big((-\Delta + I)^{-1}(u), (\Delta + I)^{-1}(v)\big) \end{aligned}$$

where $(-\Delta + I)^{-1}$ is the inverse of the sum of the Laplacian operator and identity function. We see that the solutions of (2.1) are just the fixed points of the composition $S \circ L_n$. By classical arguments, the operator $S$ is compact and hence the composition of it with the continuous operator $L_n$ is also compact. On the other hand, we have

$$\begin{aligned} \|L_n(u,v)\|_E &= \left\| -\frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2} + a(x)\frac{|v|^p}{1+\frac{1}{n}|v|^p} + f \right\|_{L^2(B_n)} \\ &\quad + \left\| -\frac{|\nabla v|^2}{1+\frac{1}{n}|\nabla v|^2} + b(x)\frac{|u|^p}{1+\frac{1}{n}|u|^p} + g \right\|_{L^2(B_n)} \\ &\le (2n + n|a|_\infty + n|b|_\infty)\sqrt{meas(B_n)} + \|f\|_2 + \|g\|_2 . \end{aligned}$$

where $meas(B_n)$ is the Lebesgue's measure of $B_n$.

Therefore, by the continuity of $S$, there exits $R > 0$ such that

$$\|S \circ L_n(u,v)\|_n < R, \qquad \forall \ (u,v) \in H_n.$$

In particular, the compact operator $S \circ L_n$ maps the ball in $H_n$ centered at zero and with radius $R$ into itself. Which implies, by the Schauder fixed theorem, the existence of a fixed point $(u_n, v_n) \in H_n$ of $S \circ L_n$ that is a solution of (2.1).

Notice that we can extend $u_n$ and $v_n$ by zero outside of $B_n$, which provides that $(u_n, v_n) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. $\square$

**Lemma 2.2.** *The functions $u_n$ and $v_n$ are nonnegative,*

$$(i.e. \quad u_n(x) \geq 0, \quad and \quad v_n(x) \geq 0 \ a.e. \quad in \quad \mathbb{R}^N, \forall\ n \geq 1).$$

*Proof.* For $t \in \mathbb{R}$, we denote by $t^+ = \max(t, 0)$ and $t^- = \min(t, 0)$.

Taking into account that $(u_n, v_n)$ is a solution of (2.1) and putting $(u_n^- \exp(-u_n), 0)$ as test function, we get

$$\begin{aligned}
&\int_{\{x \in B_n,\ u_n \leq 0\}} |\nabla u_n|^2 \exp(-u_n) dx + \int_{B_n} |u_n^-|^2 \exp(-u_n) dx \\
&\quad + \int_{B_n} \left( \frac{1}{1 + \frac{1}{n}|\nabla u_n|^2} - 1 \right) u_n^- |\nabla u_n|^2 \exp(-u_n) dx \\
&= \int_{B_n} \left( \frac{a(x)|v_n|^p}{1 + \frac{1}{n}|v_n|^p} + f \right) u_n^- \exp(-u_n) dx \leq \int_{B_n} f u_n^- \exp(-u_n) dx.
\end{aligned}$$

Having in mind that

$$\int_{B_n} \left( \frac{1}{1 + \frac{1}{n}|\nabla u_n|^2} - 1 \right) u_n^- |\nabla u_n|^2 \exp(-u_n) dx \geq 0,$$

we deduce that

$$\int_{B_n} |u_n^-|^2 \exp(-u_n) dx \leq \int_{B_n} f u_n^- \exp(-u_n) dx \leq 0.$$

It leads to

$$u_n^- = 0 \qquad a.e. \ \text{ in } \ \mathbb{R}^N, \quad \forall\ n \geq 1.$$

By the same argument used above, we prove that

$$v_n \geq 0 \quad a.e. \ \text{ in } \ \mathbb{R}^N, \quad \forall\ n \geq 1.$$

Now, we are going to prove that $u_n$ and $v_n$ are uniformly bounded in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. $\square$

**Lemma 2.3.** *The sequences $u_n$ and $v_n$ are uniformly bounded in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

*Proof.* Taking into account that $(u_n, v_n)$ is a solution of (2.1), putting $(u_n, 0)$ as test function and having in mind that $u_n \geq 0$ $a.e.$ in $B_n$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) dx &\leq \int_{\mathbb{R}^N} a(x) v_n^p u_n dx + \int_{\mathbb{R}^N} f u_n dx \\
&\leq \|u_n\|_2 \|v_n\|_2^p \|a\|_r + \|f\|_2 \|u_n\|_2 \\
&\leq \|u_n\|_{H^1(\mathbb{R}^N)} \|v_n\|_{H^1(\mathbb{R}^N)}^p \|a\|_r + \|f\|_2 \|u_n\|_{H^1(\mathbb{R}^N)} \\
&\leq \|(u_n, v_n)\|^{p+1} \|a\|_r + \|f\|_2 \|(u_n, v_n)\| \qquad (2.2)
\end{aligned}$$

where $r$ is defined as in $(H_2)$. Also, taking $(0, v_n)$ as test function, we get

$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + |v_n|^2) dx \leq \|(u_n, v_n)\|^{q+1} \|b\|_s + \|g\|_2 \|(u_n, v_n)\|. \qquad (2.3)$$

where $s$ is defined as in $(H_2)$. Combining (2.2) and (2.3), we obtain that

$$\|(u_n, v_n)\|^2 \leq \|(u_n, v_n)\|^{p+1} \|a\|_r + \|(u_n, v_n)\|^{q+1} \|b\|_s + (\|f\|_2 + \|g\|_2) \|(u_n, v_n)\|$$

Since $p+1,\ q+1 < 2$, then $(u_n, v_n)$ is uniformly bounded in $H$, which yields that $(u_n)$ and $(v_n)$ are uniformly bounded in $H^1(\mathbb{R}^N)$. □

To prove the a priori estimate in $L^\infty(B_n)$. Choose $M > 0$, taking into account that $(u_n, v_n)$ is a solution of (2.1) and putting $((u_n + v_n - M)^+, (u_n + v_n - M)^+)$ as test function.

$$\begin{aligned}
&\int_{D_n} (|\nabla u_n|^2 + 2\nabla u_n \nabla v_n + |\nabla v_n|^2)dx + \int_{B_n} (u_n + v_n)(u_n + v_n - M)^+ dx \\
&\quad + \int_{B_n} (u_n + v_n - M)^+ \left( \frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} + \frac{|\nabla v_n|^2}{1 + \frac{1}{n}|\nabla v_n|^2} \right) dx \\
&= \int_{B_n} \left( a(x)\frac{v_n^p}{1+\frac{1}{n}v_n^p} + b(x)\frac{u_n^q}{1+\frac{1}{n}u_n^q} \right) (u_n + v_n - M)^+ dx \\
&\quad + \int_{B_n} (f+g)(u_n + v_n - M)^+ dx,
\end{aligned}$$

where $D_n = \{x \in B_n,\ u_n + v_n - M \geq 0\}$. By the previous equality we get

$$\begin{aligned}
&\int_{B_n} (u_n + v_n)(u_n + v_n - M)^+ dx \\
&\quad \leq \int_{B_n} (a(x)v_n^p + b(x)u_n^q + f + g)(u_n + v_n - M)^+ dx \\
&\quad \leq \int_{B_n} \left( \frac{1}{2}(u_n + v_n) + c_1(a(x))^{\frac{1}{1-p}} \right) (u_n + v_n - M)^+ \\
&\qquad + \int_{B_n} \left( c_2(b(x))^{\frac{1}{1-q}} + f + g \right) (u_n + v_n - M)^+,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\frac{1}{2}\int_{B_n} (u_n+v_n)(u_n+v_n - M)^+ dx \\
&\quad \leq \int_{B_n} (c_1(a(x))^{\frac{1}{1-p}} + c_2(b(x))^{\frac{1}{1-q}} + f+g)(u_n + v_n - M)^+ dx
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{B_n} |(u_n+v_n - M)^+|^2 dx \\
&\quad \leq \int_{B_n} (c_1(a(x))^{\frac{1}{1-p}} + c_2(b(x))^{\frac{1}{1-q}} + f+g - \frac{1}{2}M)(u_n + v_n - M)^+ dx.
\end{aligned}$$

Since $a,\ b,\ f$ and $g$ are in $L^\infty(\mathbb{R}^N)$, then for $M$ large enough we get

$$\int_{B_n} |(u_n + v_n - M)^+|^2 dx \leq 0.$$

So, $(u_n + v_n - M)^+ = 0 \quad a.e.$ in $B_n$. It leads, by Lemma 2.2, to

$$u_n \leq M \;\text{ and }\; v_n \leq M \; a.e. \text{ in } \mathbb{R}^N, \quad \forall\, n \geq 1.$$

## 3. Main result

In this section we study the strongly convergence of the approximated solution $(u_n, v_n)$.

Since $u_n$ and $v_n$ are uniformly bounded in $H^1(\mathbb{R}^N)$, then there exist $u$ and $v$ in $H^1(\mathbb{R}^N)$ such that up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ $a.e.$ in $\mathbb{R}^N$. By the almost convergence of $u_n$ and $v_n$, we get that $u$ and $v$ are nonnegative functions.

We need the following lemma to prove that $u_n$ and $v_n$ converge to $u$ and $v$ respectively in $H^1(\Omega)$, for every bounded domain $\Omega$ in $\mathbb{R}^N$.

**Lemma 3.1.** *Assume* $(H_1) - (H_3)$ *hold, then for every* $\psi \in C_0^\infty(\mathbb{R}^N)$ *such that* $\psi \geq 0$, *we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 \psi dx = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 \psi dx = 0$$

*Proof.* In the literature, to the author's knowledge, few results are known on the strong convergence of the gradients of solutions to elliptic equations. We can, for example, quote [21,26,27,33,34]. In these works the proofs are based on the following functions

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| \geq k \end{cases}, \quad G_k(s) = s - T_k(s), \quad \varphi_\gamma(s) = s\exp(\gamma s^2), \quad s \in \mathbb{R}$$

where $k > 0$ and $\gamma$ is a suitable constant. In the present work we should prove, in two steps, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 \psi dx = 0, \qquad \forall\, \psi \in C_0^\infty(\mathbb{R}^N)$$

□

**Step 1** $\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla(u_n - u)^-|^2 \psi dx = 0, \quad \forall\, \psi \in C_0^\infty(\mathbb{R}^N),\ \psi \geq 0.$

*Proof.* Let $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \geq 0$. There exists $n_0 \in \mathbb{N}$ such that $supp(\psi) \subset B_n$ for any $n \geq n_0$. Choosing $n \geq n_0$, taking into account that $(u_n, v_n)$ is a solution of (2.1) and putting $((u_n - u)^- \psi\, e^{-u_n}, 0)$ as test function. We get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u_n \nabla (u_n - u)^- \psi\, e^{-u_n} dx + \int_{\mathbb{R}^N} (u_n - u)^- \nabla u_n \nabla \psi\, e^{-u_n} dx \\
&\quad - \int_{\mathbb{R}^N} (u_n - u)^- \psi |\nabla u_n|^2\, e^{-u_n} dx + \int_{\mathbb{R}^N} u_n (u_n - u)^- \psi\, e^{-u_n} dx \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} (u_n - u)^- \psi\, e^{-u_n} dx \\
&= \int_{\mathbb{R}^N} \left( a(x) \frac{v_n^p}{1 + \frac{1}{n} v_n^p} + f \right) (u_n - u)^- \psi\, e^{-u_n} dx. \qquad (3.1)
\end{aligned}$$

Adding and subtracting $\int_{\mathbb{R}^N} \nabla u \nabla (u_n-u)^- \psi\ e^{-u_n} dx$ to (3.1) and having in mind that

$$\int_{\mathbb{R}^N} \left( \frac{1}{1+\frac{1}{n}|\nabla u_n|^2} - 1 \right) (u_n-u)^- \psi |\nabla u_n|^2\ e^{-u_n} dx \geq 0,$$

we derive that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla (u_n-u)^-|^2 \psi\ e^{-u_n} dx + \int_{\mathbb{R}^N} \nabla u \nabla (u_n-u)^- \psi\ e^{-u_n} dx \\ &\quad + \int_{\mathbb{R}^N} (u_n-u)^- \nabla u_n \nabla \psi\ e^{-u_n} dx + \int_{\mathbb{R}^N} u_n (u_n-u)^- \psi\ e^{-u_n} dx \\ &\leq \int_{\mathbb{R}^N} \left( a(x) \frac{v_n^p}{1+\frac{1}{n} v_n^p} + f \right) (u_n-u)^- \psi\ e^{-u_n} dx \end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u \nabla (u_n-u)^- \psi\ e^{-u_n} dx \\ &\quad = \int_{\mathbb{R}^N} \nabla u \nabla \left( (u_n-u)^-\ e^{-u_n} \right) \psi dx + \int_{\mathbb{R}^N} \nabla u \nabla u_n (u_n-u)^-\ e^{-u_n} \psi dx \end{aligned} \tag{3.3}$$

By the weak convergence of $((u_n-u)^-\ e^{-u_n})$ to 0 in $H^1(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} \nabla u \nabla \left( (u_n-u)^-\ e^{-u_n} \right) \psi dx \to 0. \tag{3.4}$$

Using Lemma 2.3, Hölder inequality and by the Lebesgue dominated convergence theorem, we get

$$\left| \int_{\mathbb{R}^N} \nabla u \nabla u_n (u_n-u)^-\ e^{-u_n} \psi dx \right| \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^2 |(u_n-u)^-|^2 \psi^2 dx \right)^{\frac{1}{2}} \to 0. \tag{3.5}$$

Combining (3.3)–(3.5), we infer that

$$\int_{\mathbb{R}^N} \nabla u \nabla (u_n-u)^- \psi\ e^{-u_n} dx \to 0. \tag{3.6}$$

Likewise, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (u_n-u)^- \nabla u_n \nabla \psi\ e^{-u_n} dx \right| &\leq C \left( \int_{\mathbb{R}^N} |(u_n-u)^-|^2 |\nabla \psi|^2 dx \right)^{\frac{1}{2}} \to 0, \\ \left| \int_{\mathbb{R}^N} u_n (u_n-u)^- \psi\ e^{-u_n} dx \right| &\leq C_1 \left( \int_{\mathbb{R}^N} |(u_n-u)^-|^2 \psi^2 dx \right)^{\frac{1}{2}} \to 0. \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \left( a(x) \frac{v_n^p}{1+\frac{1}{n} v_n^p} + f \right) (u_n-u)^- \psi\ e^{-u_n} dx \right| \\ &\quad \leq (|a|_\infty M^p + |f|_\infty) \int_{\mathbb{R}^N} |(u_n-u)^-| \psi dx \to 0, \end{aligned} \tag{3.8}$$

where $M$ is defined in the proof of Lemma 2.3. $\square$

Moreover we have

$$e^{-M}\int_{\mathbb{R}^N}|\nabla(u_n-u)^-|^2\psi dx \le \int_{\mathbb{R}^N}|\nabla(u_n-u)^-|^2\psi\, e^{-u_n}dx, \tag{3.9}$$

Combining (3.2), (3.6)–(3.9), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla(u_n-u)^-|^2\psi dx=0, \quad \forall\, \psi\in C_0^\infty(\mathbb{R}^N),\ \psi\ge 0$$

**Step 2** $\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla(u_n-u)^+|^2\psi dx=0, \quad \forall\, \psi\in C_0^\infty(\mathbb{R}^N),\ \psi\ge 0.$

*Proof.* Let $\psi\in C_0^\infty(\mathbb{R}^N)$ such that $\psi\ge 0$. There exists $n_1\in\mathbb{N}$ such that $supp(\psi)\subset B_n$ for any $n\ge n_1$. Choosing $n\ge n_1$, taking into account that $(u_n,v_n)$ is a solution of (2.1) and putting $((u_n-u)^+\psi,0)$ as test function. We get

$$\begin{aligned}&\int_{\mathbb{R}^N}\nabla u_n\nabla(u_n-u)^+\psi dx+\int_{\mathbb{R}^N}(u_n-u)^+\nabla u_n\nabla\psi dx+\int_{\mathbb{R}^N}u_n(u_n-u)^+\psi dx\\&\quad+\int_{\mathbb{R}^N}\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}(u_n-u)^+\psi dx=\int_{\mathbb{R}^N}\left(a(x)\frac{v_n^p}{1+\frac{1}{n}v_n^p}+f\right)(u_n-u)^+\psi dx.\end{aligned} \tag{3.10}$$

Adding and subtracting $\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)^+\psi dx$ to (3.10),
we derive that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n-u)^+|^2dx+\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)^+\psi dx+\int_{\mathbb{R}^N}(u_n-u)^+\nabla u_n\nabla\psi dx\\&\quad+\int_{\mathbb{R}^N}u_n(u_n-u)^+\psi dx\le\int_{\mathbb{R}^N}\left(a(x)\frac{v_n^p}{1+\frac{1}{n}v_n^p}+f\right)(u_n-u)^+\psi dx.\end{aligned} \tag{3.11}$$

By the weak convergence of $(u_n-u)^+$ to 0 in $H^1(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)^+\psi dx\to 0. \tag{3.12}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N}u_n(u_n-u)^+\psi dx\to 0\ \text{ and }\ \int_{\mathbb{R}^N}\left(a(x)\frac{v_n^p}{1+\frac{1}{n}v_n^p}+f\right)(u_n-u)^+\psi dx\to 0. \tag{3.13}$$

Also, we have

$$\left|\int_{\mathbb{R}^N}(u_n-u)^+\nabla u_n\nabla\psi dx\right|\le C_2\left(\int_{\mathbb{R}^N}|(u_n-u)^+|^2|\nabla\psi|^2dx\right)^{\frac{1}{2}}\to 0. \tag{3.14}$$

Combining (3.11)–(3.14), we infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla(u_n-u)^+|^2\psi dx=0, \quad \forall\, \psi\in C_0^\infty(\mathbb{R}^N),\ \psi\ge 0.$$

Follow the same technique used above and in two steps, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla(v_n-v)|^2\psi dx=0, \quad \forall\, \psi\in C_0^\infty(\mathbb{R}^N),\ \psi\ge 0$$

□

**Proof of Theorem 1.1 completed:** Let $(\varphi,\psi)\in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$, there exists $n_2\in\mathbb{N}$ such that

$$Supp(\varphi)\subset B_n \quad\text{and}\quad Supp(\psi)\subset B_n,\quad \forall\, n\geq n_2.$$

Choosing $n\geq n_2$, taking into account that $(u_n,v_n)$ is a solution of (2.1) and putting $(\varphi,\psi)$ as test function, we obain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\nabla u_n\nabla\varphi+u_n\varphi+\nabla v_n\nabla\psi+v_n\psi+\frac{\varphi|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}+\frac{\psi|\nabla v_n|^2}{1+\frac{1}{n}|\nabla v_n|^2|}\right)dx\\
&\quad=\int_{\mathbb{R}^N}\left(\left(a(x)\frac{v_n^p}{1+\frac{1}{n}v_n^p}+f\right)\varphi+\left(b(x)\frac{u_n^q}{1+\frac{1}{n}u_n^q}+g\right)\psi\right)dx. \qquad (3.15)
\end{aligned}$$

By the weak convergence of $u_n$ and $v_n$ to $u$ and $v$ respectively in $H^1(\mathbb{R}^N)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}(\nabla u_n\nabla\varphi+u_n\varphi+\nabla v_n\nabla\psi+v_n\psi)\,dx\\
&\quad\to\int_{\mathbb{R}^N}(\nabla u\nabla\varphi+u\varphi+\nabla v\nabla\psi+v\psi)\,dx. \qquad (3.16)
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N}\frac{a(x)v_n^p\varphi}{1+\frac{1}{n}v_n^p}dx\to\int_{\mathbb{R}^N}a(x)v^p\varphi dx. \qquad (3.17)$$

and

$$\int_{\mathbb{R}^N}\frac{b(x)u_n^q\psi}{1+\frac{1}{n}u_n^q}dx\to\int_{\mathbb{R}^N}b(x)u^q\psi dx. \qquad (3.18)$$

Using Lemma 3.1 and extracting a subsequence there exist $F\in L^2(supp(\varphi))$ and $G\in L^2(supp(\psi))$ such that

$$\begin{cases}\nabla u_n(x)\to\nabla u(x)\ \text{ and }\ \nabla v_n(x)\to\nabla v(x) & a.e.\\ |\nabla u_n|\leq|F|\ \text{ and }\ |\nabla v_n|\leq|G|\end{cases}.$$

Then, by virtue of Lebesgue's dominated convergence theorem, we get

$$\int_{\mathbb{R}^N}\frac{\varphi|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}dx\to\int_{\mathbb{R}^N}|\nabla u|^2\varphi dx \qquad (3.19)$$

and

$$\int_{\mathbb{R}^N}\frac{\psi|\nabla v_n|^2}{1+\frac{1}{n}|\nabla v_n|^2|}dx\to\int_{\mathbb{R}^N}|\nabla v|^2\psi dx. \qquad (3.20)$$

Passing to the limit in (3.15) and combining (3.16)–(3.20), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\nabla u\nabla\varphi+u\varphi+\nabla v\nabla\psi+v\psi+|\nabla u|^2\varphi+|\nabla v|^2\psi\right)dx\\
&\quad-\int_{\mathbb{R}^N}((a(x)v^p+f)\varphi+(b(x)u^q+g)\psi)\,dx=0\quad\forall(\varphi,\psi)\in C_c^\infty(\mathbb{R}^N)\times C_c^\infty(\mathbb{R}^N).
\end{aligned}$$

Besides, $u$ and $v$ are nonnegative functions and $f$ and $g$ are nontrivial functions, so $(u,v)$ is a weak nonnegative and nontrivial solution of the system (1.1).

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