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Positive solutions for a modified Leslie–Gower prey–predator model with Crowley–Martin functional responses

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Abstract. In this paper, we study a modified Leslie–Gower prey–predator model with Crowley–Martin functional response. The stability and instability of the trivial and semi-trivial solutions was studied by analyzing the eigenvalues of the linearized system. The existence, multiplicity and uniqueness of positive steady state solutions were shown by using bifurcation theory, degree theory, energy estimate and asymptotic behavior analysis. Furthermore, all results were characterized in parameter plane.

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1. Introduction

A prey–predator system that incorporates conservation of mass and division of population rates of change into birth and death processes has the following canonical form (see for instance [1,4])

$$\begin{cases} \frac{du}{dt} = ru(1-u/K) - vp(u,v), & t>0, \\ \frac{dv}{dt} = \varepsilon vp(u,v) - \mu v, & t>0, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent the population density of prey and predator at time t, respectively; r and K are the intrinsic growth rate and the carrying capacity of the prey respectively; μ is the mortality rate of predator; ε is the conversion efficiency ($0 < \varepsilon < 1$); $p(u,v)$ is the consumption rate of prey by a predator or the functional response of the predator, which includes Holling types, ratio-dependent type, Beddington–DeAngelis type, Hassell–Varley type,

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Ivlev type, Watt type and so on (see [27, Appendix A] for a collection of prey–predator models).

Another type of prey–predator model was first introduced by Leslie and Gower in [32]. In which the authors replace the right term of the equation for v in (1.1) by $r_2 v(1 - v/u)$. By [32], the biological interpretation of this term is that in the case of severe scarcity, v can switch over to other populations. So this type of models do not follows the "mass conservation" principle. The term $-r_2 v^2/u$ is called the Leslie–Gower term, which measures the loss in the predator population due to rarity of its favorite food. Recently, there has been a growing interest in the study of mathematical models incorporates a modified version of Leslie–Gower functional response with Holling-type II functional response [25]. This model can be written as follows (see [5])

$$\begin{cases} \frac{du}{dt} = r_1 u\left(1 - \frac{u}{K}\right) - \frac{auv}{b+u}, & t > 0, \\ \frac{dv}{dt} = r_2 v\left(1 - \frac{v}{u+c}\right), & t > 0, \end{cases} \tag{1.2}$$

where r_1, r_2, a, b, c and K are all positive constant and $u(t)$, $v(t)$ represent the population densities at time t. The Holling type II functional response $p(u) = a/(b+u)$ in (1.2) is classified as one of prey-dependent functional response, it assumes that predators do not interfere with one another's activities; thus competition among predators for food occurs only via the depletion of prey. However, Crowley and Martin proposed a functional that can accommodate interference among predators (see [14]). This type of functional response is classified as one of predator-dependent functional response, i.e. they are functions of the abundance of both prey and predator due to predator interference. It is assumed that predator-feeding rate decreases by higher predator density even when prey density is high, and therefore the effects of predator interference on feeding rate remain important all the time whether an individual predator is handing or searching for a prey at a given instant of time. The per capita feeding rate in this formulation is given by

$$p(u, v) = \frac{mu}{1 + Au + Bv + ABuv},$$

where m, A and B are positive parameters that describe the effects of capture rate handling time and the magnitude of interference among predators, respectively, on the feeding rate (see [17,37,38,50–52] for related works).

By take above considerations, Ali and Jazar [2] consider a prey–predator model which incorporates a modified version of the Leslie–Gower with Crowley–Martin functional response:

$$\begin{cases} \frac{du}{dt} = a_1 u\left(1 - \frac{u}{K}\right) - \frac{buv}{(1+cu)(1+dv)}, & t > 0, \\ \frac{dv}{dt} = v\left(a_2 - \frac{ev}{u+f}\right), & t > 0, \end{cases} \tag{1.3}$$

where u and v represent the population densities at time t; a_1, a_2, b, c, d, e, f and K are positive constants, and their biological meaning are as follows: a_1 and K are the intrinsic growth rate and the carrying capacity of prey population u respectively. The constants b, c and d are the saturating Crowley–Martin type functional response parameters, in which c measures the magnitude of interference among prey. Further, a_2 describes the growth rate of predator v; e is the maximum value which per capita reduction rate of v can attain, f measure the extent to which environment provides protection to predator v.

On the other hand, the spatial component of ecological interactions has been identified as an important factor in how ecological communities are shaped, and understanding the role of space in challenging both theoretically and empirically [42]. Empirical evidence suggests that the spatial scale and structure of environment can influence population interactions [9]. By considering the above reasons, we consider the following reaction–diffusion equations:

$$\begin{cases} \tilde{u}_t - d_1\Delta\tilde{u} = \tilde{a}\tilde{u}\left(1 - \frac{\tilde{u}}{K}\right) - \frac{\tilde{\lambda}\tilde{u}\tilde{v}}{(1+\tilde{c}\tilde{u})(1+\tilde{d}\tilde{v})}, & x \in \Omega,\ t > 0, \\ \tilde{v}_t - d_2\Delta\tilde{v} = \tilde{v}\left(\tilde{b} - e_2\tilde{v} - \frac{\tilde{\mu}\tilde{v}}{\tilde{u}+\tilde{r}}\right), & x \in \Omega,\ t > 0, \\ \tilde{u} = \tilde{v} = 0, & x \in \partial\Omega,\ t > 0, \\ \tilde{u}(x,0) = \tilde{u}_0(x),\ \tilde{v}(x,0) = \tilde{v}_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$; $d_1, d_2, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, K, e_2,\ \tilde{\lambda}, \tilde{\mu}$, and $\tilde{r}$ are positive constants. The problem (1.4) models the interactions between a predator, with population density $\tilde{v}(x,t)$, and a prey, with population density $\tilde{u}(x,t)$, inhabiting the region Ω. The biological meanings of the parameters are

- d_1 and d_2 are the diffusion coefficients of prey u and predator v, respectively;
- $\tilde{a}$ denotes birth rate of prey, and $\tilde{b}$ denotes the birth rate of predator; K is the carrying capacity of prey population $\tilde{u}, e_2$ represents intra-specific pressures of predator;
- the term $\tilde{\lambda}\tilde{v}/[(1+\tilde{c}\tilde{u})(1+\tilde{d}\tilde{v})]$ is known as Crowley–Martin interactions, in which $\tilde{\lambda}$ and $\tilde{d}$ describe the effects of capture rate handling time and $\tilde{c}$ describes the magnitude of interference among predators (see [17,37,38,50–52]);
- the term $\tilde{\mu}\tilde{v}/(\tilde{u}+\tilde{r})$ is a modified version of Leslie–Gower functional response, in which $\tilde{\mu}$ is the maximum value which per capita reduction rate of $\tilde{v}$ can attain, $\tilde{r}$ measure the extent to which environment provides protection to predator $\tilde{v}$ (see [5]).

The homogeneous Dirichlet boundary condition "$\tilde{u} = \tilde{v} = 0$ on $\partial\Omega$" is sometimes said to correspond to a lethal boundary, which can be considered as such a condition under which neither of the two species can exist on the boundary (see [9, page 31] or [54]). The initial values $\tilde{u}_0(x)$ and $\tilde{v}_0(x)$ are continuous functions, which are assumed nonnegative and nontrivial. Comparing the reaction terms between (1.3) and (1.4), we add a term $-e_2\tilde{v}^2$, which is referred as a "closure term", describes either a self-limitation of the consumer, v, or the influence of predation (see for example [46,57]).

In this paper, we mainly consider the associated stationary problem:

$$\begin{cases} -d_1\Delta\tilde{u} = \tilde{a}\tilde{u}\left(1-\frac{\tilde{u}}{K}\right) - \frac{\tilde{\lambda}\tilde{u}\tilde{v}}{(1+\tilde{c}\tilde{u})(1+\tilde{d}\tilde{v})}, & x\in\Omega,\\ -d_2\Delta\tilde{v} = \tilde{v}\left(\tilde{b} - e_2\tilde{v} - \frac{\tilde{\mu}\tilde{v}}{\tilde{u}+\tilde{r}}\right), & x\in\Omega,\\ \tilde{u}=\tilde{v}=0, & x\in\partial\Omega. \end{cases} \tag{1.5}$$

It is known that elliptic systems with homogeneous Dirichlet boundary condition are usually more difficult to analyze, as shown in [8,10,11,16,18–20,29,30,33,40,41] for several other diffusive ecological interaction models, since the only possible constant steady state is $(0,0)$ and the ODE dynamics is not embedded in the PDE models. There are more references for the corresponding Neumann boundary value problems, which we do not list here but refer to [9,21,43]. Problem (1.5) with $e_2=\tilde{d}=0$, i.e. the elliptic equation corresponding to (1.2), was studied in [55,56,58]. The existence of one positive solution under some conditions on the parameters was shown in [55], while the nonexistence of positive solutions in some other parameter regions was also proved. Furthermore, the multiplicity and uniqueness of positive solutions were studied in [58], and the existence of multiple steady state solutions indicate the system can have possible bistable dynamics.

By rescaling as follows

$$u=\frac{\tilde{a}}{d_1K}\tilde{u},\ a=\frac{\tilde{a}}{d_1},\ \lambda=\frac{d_2\tilde{\lambda}}{d_1e_2},\ c=\frac{d_1K\tilde{c}}{\tilde{a}},\ d=\frac{d_2\tilde{d}}{e_2},$$
$$v=\frac{e_2}{d_2}\tilde{v},\ b=\frac{\tilde{b}}{d_2},\ r=\frac{\tilde{a}\tilde{r}}{d_1K},\ \mu=\frac{\tilde{a}\tilde{\mu}}{d_1Ke_2r},$$

we have the following system:

$$\begin{cases} -\Delta u = u\left(a-u-\frac{\lambda v}{(1+cu)(1+dv)}\right), & x\in\Omega,\\ -\Delta v = v\left(b-v-\frac{\mu r v}{u+r}\right), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{1.6}$$

The organization of the remaining part of the paper is as follows. The stability of the trivial and semi-trivial solutions to (1.6) is studied in Sect. 2. In Sect. 3, we consider the non-existence conditions of positive solutions to (1.6). Section 4 is devoted to study the existence of positive solutions to (1.6). In Sect. 5, we study the multiplicity of positive solutions of problem (1.6), and the uniqueness of the positive solution under certain condition is studied in Sect. 6. The conclusions are stated in Sect. 7. Finally, in Sect. 8, we list some basic results which are used in this paper.

2. Stability analysis of the trivial and semi-trivial solutions

In this section we study the trivial and semi-trivial solutions of (1.6). Before which, we will introduce some notations used in this paper, we use $||\cdot||_{\mathbf{X}}$ as the norm of Banach space $\mathbf{X}$, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space $\mathbf{X}$ and its dual space $\mathbf{X}^*$. For a linear operator $\mathfrak{L}$, we use $\mathbf{N}(\mathfrak{L})$ as the null space of $\mathfrak{L}$ and

$\mathbf{R}(\mathfrak{L})$ as the range space of $\mathfrak{L}$, and we use $\mathfrak{L}[w]$ to denote the image of w under the linear mapping $\mathfrak{L}$. For a multilinear operator $\mathfrak{L}$, we use $\mathfrak{L}[w_1, w_2, \ldots, w_k]$ to denote the image of $(w_1, w_2, \ldots, w_k)$ under $\mathfrak{L}$, and when $w_1 = w_2 = \cdots = w_k$, we use $\mathfrak{L}[w_1]^k$ instead of $\mathfrak{L}[w_1, \ldots, w_1]$. For a nonlinear operator $\mathfrak{F}$, we use $\mathfrak{F}_u$ as the partial derivative of $\mathfrak{F}$ with respect to u.

For each $q \in C(\overline{\Omega})$, let $\rho_1(q)$ be the principle eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = au, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

As is well known, the principal eigenvalue $\rho_1(q)$ is given by the following variational characterization:

$$\rho_1(q) = \inf_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{L^2(\Omega)} = 1}} \int_\Omega (|\nabla\phi|^2 + q(x)\phi^2)dx. \tag{2.2}$$

We denote $\rho_1(0)$ by ρ_1 and let $\phi_1(x)$ be the positive eigenfunction corresponding to ρ_1 with $\|\phi_1\|_{L^2(\Omega)} = 1$.

It is obvious that (1.6) has, in addition to the trivial steady state $(0, 0)$, two semi-trivial steady states

$$(\theta_a, 0) \quad \text{if } a > \rho_1 \ \text{ and } \ (0, \theta_b^*) \quad \text{if } b > \rho_1,$$

where θ_a is defined in Theorem 8.6 and $\theta_b^* := \theta_b/(\mu + 1)$, in which θ_b is the unique positive solution of (8.3) with a replacing by b when $b > \rho_1$.

In order to study the stability of the solutions to (1.6), we let $(u(x, t), v(x, t))$ be the unique solution of the following problem

$$\begin{cases} u_t - \Delta u = u\left(a - u - \frac{\lambda v}{(1+cu)(1+dv)}\right), & x \in \Omega,\ t > 0 \\ v_t - \Delta v = v\left(b - v - \frac{\mu r v}{u+r}\right), & x \in \Omega,\ t > 0 \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \not\equiv 0,\ v(x, 0) = v_0(x) \geq 0, \not\equiv 0, & x \in \Omega, \end{cases} \tag{2.3}$$

which is the evolutionary equation corresponding to (1.6). Clearly, $(u(x, t), v(x, t))$ exists globally and $(u(x, t), v(x, t)) > (0, 0)$ in $\Omega \times (0, \infty)$. The following statements hold.

Theorem 2.1. *Assume $(u(x, t), v(x, t))$ is the unique solution of (2.3). Then the following conclusions hold.*

(i) *If $a \leq \rho_1$ and $b \leq \rho_1$, then $(u(x, t), v(x, t))$ converges to the trivial steady state $(0, 0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, while $(0, 0)$ is unstable if $a > \rho_1$ or $b > \rho_1$.*

(ii) *Assume $b > \rho_1$. Then the semi-trivial steady state $(0, \theta_b^*)$ is asymptotically stable if $a < \rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$, while it is unstable if $a > \rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$. Moreover, $(u(x, t), v(x, t))$ converges $(0, \theta_b^*)$ as $t \to \infty$ when $a \leq \rho_1$.*

(iii) *Assume $a > \rho_1$. Then, if $b \leq \rho_1$, $(u(x, t), v(x, t))$ converges to the semi-trivial steady state $(\theta_a, 0)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, while $(\theta_a, 0)$ is unstable if $b > \rho_1$.*

Proof. Throughout the proof, we will use the notations in Theorem 8.7.

(i) By the first equation of (2.3),

$$u_t - \Delta u \le u(a-u),\ (x,t) \in \Omega \times (0,\infty),\ \ u = 0,\ (x,t) \in \partial\Omega \times (0,\infty).$$

Then $0 \le u(x,t) \le u_a^1(x,t)$ for $(x,t) \in \overline{\Omega} \times [0,\infty)$ by using comparison principle for parabolic equations. By Theorem 8.7, it follows from $a \le \rho_1$ that $u_a^1(x,t)$ converges to 0 uniformly on $\overline{\Omega}$ as $t \to \infty$. So $u(x,t)$ converges to 0 uniformly on $\overline{\Omega}$ as $t \to \infty$. Then for any $\epsilon > 0$, there exists a constant $T_1 = T_1(\epsilon) \gg 1$ such that $u(x,t) \le \epsilon$ in $\overline{\Omega} \times [T_1,\infty)$. By the second equation of (2.3),

$$\begin{aligned} &v_t - \Delta v \le v\left[b - \left(1 + \frac{\mu r}{r+\epsilon}\right)v\right],\ (x,t) \in \Omega \times (T_1,\infty),\\ &v = 0,\ (x,t) \in \partial\Omega \times (T_1,\infty). \end{aligned}$$

In view of $b \le \rho_1$, we conclude that $v(x,t)$ converges to 0 uniformly on $\overline{\Omega}$ as $t \to \infty$.

To study the instability of $(0,0)$, we use the linearization principle. The stability of $(0,0)$ is determined by studying the following spectral problem:

$$\begin{cases} -\Delta\phi - a\phi = \rho\phi, & x \in \Omega,\\ -\Delta\psi - b\psi = \rho\psi, & x \in \Omega,\\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

It is easy to see that both $\rho_1 - a$ and $\rho_1 - b$ are eigenvalues of (8.4), and at least one of them is negative by the fact that $a > \rho_1$ or $b > \rho_1$, which implies the instability of $(0,0)$.

(ii) By linearization principle, the stability of $(0,\theta_b^*)$ is determined by studying the following spectral problem:

$$\begin{cases} -\Delta\phi + \left(\frac{\lambda\theta_b}{\mu+1+d\theta_b} - a\right)\phi = \rho\phi, & x \in \Omega,\\ -\Delta\psi - \frac{\mu(\theta_b)^2}{r(\mu+1)^2}\phi + (2\theta_b - b)\psi = \rho\psi, & x \in \Omega,\\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Let ρ be an eigenvalue of (2.5) and let (ϕ,ψ) be the corresponding eigenfunction.

1. Assume $\phi = 0$, then ρ must be an eigenvalue of $-\Delta + (2\theta_b - b)I$ with zero Dirichlet boundary condition. Then Theorem 8.3 assures $\rho > \rho_1(\theta_b - b) = 0$.
2. If $\phi \ne 0$, then ρ is an eigenvalue for the first equation of (2.5). The least eigenvalue ρ^* among such eigenvalues is given by

$$\rho^* = \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b} - a\right) = \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right) - a.$$

Combining the above results one can prove that, if $a < \rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$, then all eigenvalues of (2.5) are positive, which implies the asymptotical stability of $(0,\theta_b^*)$. On the other hand, if $a > \rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$, then (2.5) has a negative eigenvalue, which implies the instability of $(0,\theta_b^*)$.

Next we consider the globally asymptotical stability of $(0,\theta_b^*)$ by using a similar method as in (i). Since $a \le \rho_1$, it follows from the proof of (i) that $u(x,t)$ converges to 0 uniformly on $\overline{\Omega}$ as $t \to \infty$. Then for any $\epsilon > 0$, there exists a constant $T_2 = T_2(\epsilon) \gg 1$ such that $u(x,t) \le \epsilon$ in $\overline{\Omega} \times [T_1, \infty)$. So, it follows from the second equation of (2.3) that

$$\begin{cases} v[b-(1+\mu)v] \le v_t - \Delta v \le v\left[b - \left(1+\frac{\mu r}{r+\epsilon}\right)v\right], & x \in \Omega,\ t \ge T_2 \\ u = v = 0, & x \in \partial\Omega,\ t \ge T_2. \end{cases}$$

Then it follows from comparison principle of parabolic equations that $u_b^{\delta_1}(x,t) \le v(x,t) \le u_b^{\delta_2}(x,t)$ in $\Omega \times [T_2, \infty)$, where

$$\delta_1 := 1+\mu, \quad \delta_2 := 1 + \frac{\mu r}{r+\epsilon} = \frac{(1+\mu)r+\epsilon}{r+\epsilon}.$$

By Theorem 8.7, $u_b^{\delta_1}(x,t)$ and $u_b^{\delta_2}(x,t)$ converges to $\theta_b^*(x)$ and $(r+\epsilon)\theta_b/[(\mu+1)r+\epsilon](x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$, respectively. By letting $\epsilon \to 0$ and using the fact that $u_b^{\delta_1}(x,t) \le v(x,t) \le u_b^{\delta_2}(x,t)$, we obtain $v(x,t)$ converges to $\theta_b^*(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$.

The proof of (iii) is similar to the proof in the second part of (ii). So we omit the details. $\square$

In order to understand the meaning of Theorem 2.1, we define a function:

$$a = f(b) = \rho_1 \left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right), \quad b \ge \rho_1. \tag{2.6}$$

Then f has the following properties:

Lemma 2.2. *The function defined in* (2.6) *is a strictly increasing function of class C^1, which satisfies:*

(i) $f(\rho_1) = \rho_1$, $\lim_{b\to\infty} f(b) = \rho_1 + \frac{\lambda}{d}$, $f'(\rho_1) = \frac{\lambda}{\mu+1}$.

(ii) *Assume $\lambda \ge \mu+1$, then*

$$f(b) \le \begin{cases} \varphi_1(b), & \text{if } \rho_1 \le b < b^{\#}; \\ \varphi_2(b), & \text{if } b \ge b^{\#}, \end{cases}$$

where $b^{\#} := \frac{d\rho_1 + \sqrt{(d\rho_1)^2 + 4d\rho_1(\mu+1)}}{2d}$.

(iii) *Assume $\lambda < \mu+1$, then there exists a constant $b^* \in (\rho_1, \infty)$ such that*

$$f(b) \le \begin{cases} \varphi_3(b), & \text{if } \rho_1 \le b < b^*; \\ \varphi_2(b), & \text{if } b \ge b^*. \end{cases}$$

Here,

$$\varphi_1(b) := \rho_1 + \frac{\lambda}{\mu+1}(b-\rho_1), \quad \varphi_2(b) := \rho_1 + \frac{\lambda b}{\mu+1+bd},$$
$$\varphi_3(b) := b\left[1 - \left(1 - \frac{\lambda}{\mu+1}\right)\left(1-\frac{\rho_1}{b}\right)^3 \kappa\right],$$

where κ stands for the constant

$$\kappa = |\Omega|^{-1} \int_\Omega \omega^3 dx < 1,$$

$|\Omega|$ *denotes the measure of* Ω *and* $\omega(x)$ *is the positive eigenfunction corresponding to* ρ_1 *such that*

$$\max_{x\in\overline{\Omega}} \omega(x) = 1.$$

Proof. By Theorem 8.6, θ_b is a continuous and strictly increasing function with respect to b such that $\lim_{b\to\rho_1^+} \theta_b = 0$ uniformly in Ω and $\lim_{b\to\infty} \theta_b = \infty$ uniformly in any compact subsets of Ω. Then it follows from Theorem 8.3 and the fact that $z \mapsto \frac{\lambda z}{\mu+1+dz}$ is strictly increasing that $f(b)$ is a strictly increasing function of class C^1.

Proof of (i). Since

$$\lim_{b\to\rho_1^+} \frac{\lambda\theta_b}{\mu+1+d\theta_b} = 0 \ \text{ uniformly in } \ \Omega,$$

it follows from Theorem 8.3 that

$$f(\rho_1) = \lim_{b\to\rho_1^+} f(b) = \rho_1(0) = \rho_1.$$

By variational characterization of the principal eigenvalue and the fact that $\frac{\lambda\theta_b}{\mu+1+d\theta_b} \le \frac{\lambda}{d}$, we obtain

$$\begin{aligned} f(b) = & \inf_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{L^2(\Omega)}=1}} \int_\Omega \left(|\nabla\phi|^2 + \frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi^2\right) dx \\ \le & \inf_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{L^2(\Omega)}=1}} \int_\Omega \left(|\nabla\phi|^2 + \frac{\lambda}{d}\phi^2\right) dx \\ = & \, \rho_1 + \frac{\lambda}{d}, \end{aligned} \tag{2.7}$$

which means

$$\limsup_{b\to\infty} f(b) \le \rho_1 + \frac{\lambda}{d}. \tag{2.8}$$

Next we will prove $\liminf_{b\to\infty} f(b) \ge \rho_1 + \frac{\lambda}{d}$. Let $\phi_b \in H_0^1(\Omega)$ with $\|\phi_b\|_{L^2(\Omega)} = 1$ be a positive function satisfying

$$f(b) = \int_\Omega \left(|\nabla\phi_b|^2 + \frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi_b^2\right) dx. \tag{2.9}$$

Then it follows from (2.7) and (2.9) that

$$\|\nabla\phi_b\|^2_{L^2(\Omega)} = f(b) - \int_\Omega \frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi_b^2 dx \le \rho_1 + \frac{\lambda}{d}.$$

By the reflexive property of $H_0^1(\Omega)$, there exists a function $\phi_\infty \in H_0^1(\Omega)$ with $\|\phi_\infty\|_{L^2(\Omega)} = 1$ and a subsequence of $\{\phi_b\}_b$, denoted by $\{\phi_b\}_b$ again, such that $\nabla\phi_b \rightharpoonup \nabla\phi_\infty$ weakly and $\phi_b \to \phi_\infty$ strongly in $L^2(\Omega)$.

Then, by (2.9), weakly lower continuous of $\|\cdot\|_{L^2(\Omega)}$ and Lebesgue's dominated convergence theorem, we obtain

$$\liminf_{b\to\infty} f(b) \geq \int_\Omega \left(|\nabla\phi_\infty|^2 + \frac{\lambda}{d}\phi_\infty^2\right) dx \geq \rho_1 + \frac{\lambda}{d}. \tag{2.10}$$

Thus, $\lim_{b\to\infty} f(b) = \rho_1 + \lambda/d$ follows from (2.8) and (2.10).

By Theorem 8.3, we have

$$f'(b) = \int_\Omega \frac{\lambda(\mu+1)}{(\mu+1+d\theta_b)^2}\frac{\partial\theta_b}{\partial b}\phi_b^2 dx. \tag{2.11}$$

It follows from [53, page 432] that

1. $\lim_{b\to\rho_1^+}\theta_b = 0$ uniformly in Ω;
2. $\lim_{b\to\rho_1^+}\phi_b = \phi_1$ strongly in $H_0^1(\Omega)$;
3. $\lim_{b\to\rho_1^+}\frac{\partial\theta_b}{\partial b} = \left(\int_\Omega \phi_1^3 dx\right)^{-1}\phi_1$ uniformly in Ω.

Then combining above facts and (2.11), $f'(\rho_1) = \lambda/(\mu+1)$ follows.

Proofs of (ii) and (iii). By variational characterization of the principal eigenvalue and the fact that $\theta_b \leq b$, we obtain

$$\begin{aligned} f(b) &= \inf_{\phi\in H_0^1(\Omega),\,\phi\not\equiv 0} \|\phi\|_{L^2(\Omega)}^{-2}\left(\|\nabla\phi\|_{L^2(\Omega)}^2 + \int_\Omega \frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi^2 dx\right)\\ &\leq \inf_{\phi\in H_0^1(\Omega),\,\phi\not\equiv 0} \|\phi\|_{L^2(\Omega)}^{-2}\left(\|\nabla\phi\|_{L^2(\Omega)}^2 + \frac{\lambda b}{\mu+1+bd}\int_\Omega \phi^2 dx\right)\\ &\leq \rho_1 + \frac{\lambda b}{\mu+1+bd} = \varphi_2(b). \end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned} f(b) &= \inf_{\phi\in H_0^1(\Omega),\,\phi\not\equiv 0} \|\phi\|_{L^2(\Omega)}^{-2}\left(\|\nabla\phi\|_{L^2(\Omega)}^2 + \int_\Omega \frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi^2 dx\right)\\ &\leq \|\theta_b\|_{L^2(\Omega)}^{-2}\left(\|\nabla\theta_b\|_{L^2(\Omega)}^2 + \frac{\lambda}{\mu+1}\int_\Omega (\theta_b)^3 dx\right). \end{aligned} \tag{2.13}$$

It follows from $-\Delta\theta_b = \theta_b(b-\theta_b)$ and $\theta_b|_{\partial\Omega} = 0$ that

$$\int_\Omega (\theta_b)^3 dx \leq b\|\theta_b\|_{L^2(\Omega)}^2 - \|\nabla\theta_b\|_{L^2(\Omega)}^2. \tag{2.14}$$

In view above inequality and (2.13), we get

$$f(b) \leq \left(1 - \frac{\lambda}{\mu+1}\right)\|\theta_b\|_{L^2(\Omega)}^{-2}\|\nabla\theta_b\|_{L^2(\Omega)}^2 + \frac{\lambda b}{\mu+1}. \tag{2.15}$$

If $\lambda \geq \mu+1$, then it follows from Poincaré's inequality and (2.15) that

$$f(b) \leq \left(1 - \frac{\lambda}{\mu+1}\right)\rho_1 + \frac{\lambda b}{\mu+1} = \varphi_1(b). \tag{2.16}$$

If $\lambda < \mu+1$, it follows from (2.13) and (2.14) that

$$f(b) \leq b - \left(1 - \frac{\lambda}{\mu+1}\right)\frac{\int_\Omega (\theta_b)^3 dx}{\int_\Omega (\theta_b)^2 dx}.$$

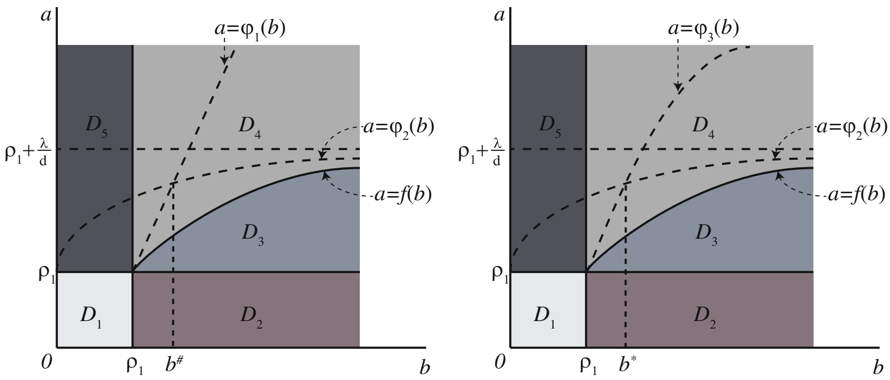

Figure 1. Illusions of the sets in Corollary 2.3. *Left* is the case $\lambda \geq \mu + 1$; *right* is the case $\lambda < \mu + 1$

By comparison principle $(b - \rho_1)\omega(x) \leq \theta_b \leq b$ for all $x \in \overline{\Omega}$. Then,

$$f(b) \leq b\left[1 - \left(1 - \frac{\lambda}{\mu+1}\right)\left(1 - \frac{\rho_1}{b}\right)^3 \kappa\right] = \varphi_3(b). \tag{2.17}$$

By (2.12), (2.16) and (2.17), we get

$$f(b) \leq \begin{cases} \min\{\varphi_1(b), \varphi_2(b)\}, & \text{if } \lambda \geq \mu + 1; \\ \min\{\varphi_2(b), \varphi_3(b)\}, & \text{if } \lambda < \mu + 1. \end{cases}$$

Since

$$\min\{\varphi_1(b), \varphi_2(b)\} = \begin{cases} \varphi_1(b), & \rho_1 \leq b < b^{\#}; \\ \varphi_2(b), & b \geq b^{\#}, \end{cases}$$

the conclusion (ii) follows.

By the methods developed by López-Gómez and Pardo San Gil [40], functions $a = \varphi_2(b),\ b > \rho_1$ and $a = \varphi_3(b),\ b > \rho_1$ intersect only at one point $(a^*, b^*) \in (\rho_1, \rho_1 + \lambda/d) \times (\rho_1, \infty)$ such that $a^* = \varphi_2(b^*) = \varphi_3(b^*)$ and

$$\min\{\varphi_2(b), \varphi_3(b)\} = \begin{cases} \varphi_3(b), & \text{if } \rho_1 \leq b < b^*; \\ \varphi_2(b), & \text{if } b \geq b^*. \end{cases}$$

Then the conclusion (iii) follows. □

Based on Lemma 2.2, we define five sets as follows (see Fig.1):

$$\begin{aligned} D_1 &:= \{(a, b) : 0 < a \leq \rho_1,\ 0 < b \leq \rho_1\}, \\ D_2 &:= \{(a, b) : 0 < a \leq \rho_1,\ b > \rho_1\}, \\ D_3 &:= \{(a, b) : \rho_1 < a < f(b),\ b > \rho_1\}, \\ D_4 &:= \{(a, b) : a > f(b),\ b > \rho_1\}, \\ D_5 &:= \{(a, b) : a > \rho_1,\ 0 < b \leq \rho_1\}. \end{aligned} \tag{2.18}$$

Then the following results follows from Theorem 2.1.

Corollary 2.3. *Let $D_i, i = 1, 2, \ldots, 5$ be the sets defined in* (2.18)*. Then the following results hold.*

1. $(0, 0)$ *is globally asymptotically stable if* $(a, b) \in D_1$*, while it is unstable if* $(a, b) \in (0, \infty) \times (0, \infty) \backslash \overline{D_1}$*.*
2. $(0, \theta_b^*)$ *is asymptotically stable if* $(a, b) \in D_2 \cup D_3$*;* $(0, \theta_b^*)$ *is globally asymptotically stable if* $(a, b) \in D_2$*, while* $(0, \theta_b^*)$ *is unstable if* $(a, b) \in D_4$*.*
3. $(\theta_a, 0)$ *is globally asymptotically stable if* $(a, b) \in D_5$*, while it is unstable if* $(a, b) \in D_3 \cup D_4 \cup \{(a, b) : a = f(b), b > \rho_1\}$*.*

3. Non-existence of positive solutions

In this section we will study the results about the non-existence of positive solutions to (1.6), and the main result of this section is as follows.

Theorem 3.1. *Problem* (1.6) *has no positive solution if one of the following conditions holds.*

(i) $a \leq \rho_1$ *or* $b \leq \rho_1$*;*

(ii) $b > \rho_1$ *and* $a \leq \rho_1 (2\sqrt{\frac{\lambda \theta_b}{c(\mu + 1 + d\theta_b)}}) - \frac{1}{c}$*.*

In order to prove above theorem, we first give some a priori estimates of positive solutions of (1.6).

Lemma 3.2. *Assume* $a > \rho_1$ *and* $b > \rho_1$*. Let* (u, v) *be a positive solution of* (1.6)*. Then*

$$u(x) \leq \theta_a < a \ \textit{and} \ v(x) \leq \theta_b < b, \quad x \in \overline{\Omega}.$$

Proof. It follows from (1.6) that

$$-\Delta u \leq u(a - u), \ -\Delta v \leq v(b - v), \ x \in \Omega, \quad u = v = 0, \ x \in \partial\Omega.$$

Then the conclusion follows by comparison principle and Theorem 8.6. □

Proof of Theorem 3.1. Let (u, v) be a positive solution of problem (1.6).

(i) Taking $L^2(\Omega)$-inner product to the first equation of (1.6) leads to

$$\|\nabla u\|_{L^2(\Omega)}^2 = \int_\Omega u^2 \left(a - u - \frac{\lambda v}{(1 + cu)(1 + dv)} \right) dx < a\|u\|_{L^2(\Omega)}^2.$$

Note $\|\nabla u\|_{L^2(\Omega)}^2 \geq \rho_1 \|u\|_{L^2(\Omega)}^2$. Hence, if (1.6) has a positive solution, then a must satisfy $a > \rho_1$. This fact implies that (1.6) admits no positive solution for $a \leq \rho_1$. Similarly, one can prove (1.6) admits no positive solution for $b \leq \rho_1$.

(ii) From the second equation of (1.6), we can derive

$$-\Delta v > v(b - (\mu + 1)v), \ x \in \Omega, \ \ v = 0, \ x \in \partial\Omega.$$

Then

$$v(x) > \frac{1}{\mu + 1} \theta_b \tag{3.1}$$

for $x \in \Omega$ by comparison principle.

By the first equation of (1.6), we have

$$\begin{aligned} -\Delta u &< u\left(a-u-\frac{\lambda\theta_b}{c(1/c+u)(\mu+1+d\theta_b)}\right) \\ &= u\left(a+\frac{1}{c}-\left(u+\frac{1}{c}+\frac{\lambda\theta_b}{c(1/c+u)(\mu+1+d\theta_b)}\right)\right) \\ &\le u\left(a+\frac{1}{c}-2\sqrt{\frac{\lambda\theta_b}{c(\mu+1+d\theta_b)}}\right), \end{aligned}$$

which implies

$$\rho_1\left(2\sqrt{\frac{\lambda\theta_b}{c(\mu+1+d\theta_b)}}-a-\frac{1}{c}\right)<0.$$

his fact implies that (1.6) admits no positive solution if (ii) holds. □

In order to understand the meaning of Theorem 3.1, we define a function:

$$a=g(b)=\rho_1\left(2\sqrt{\frac{\lambda\theta_b}{c(\mu+1+d\theta_b)}}\right)-\frac{1}{c},\quad b\ge\rho_1. \tag{3.2}$$

Then $g(b)$ satisfies the following properties.

Lemma 3.3. *The function defined in* (3.2) *is a strictly increasing function of class C^1, which satisfies:*

$$g(\rho_1)=\rho_1-\frac{1}{c},\ \lim_{b\to\infty}g(b)=\rho_1+2\sqrt{\frac{\lambda}{cd}}-\frac{1}{c},\ g'(\rho_1)=\infty. \tag{3.3}$$

Furthermore,

(1) *$g(b)\le f(b)$, where $f(b)$ is the function defined in* (2.6)*;*
(2) *If $d\ge 4\lambda c$, then $g(b)\le\rho_1$ for all $b\ge\rho_1$; and if $d<4\lambda c$, then there exists a unique constant $b_\#\in(\rho_1,\infty)$ such that*

$$g(b)\begin{cases} <\rho_1, & when\, b\in[\rho_1,b_\#);\\ =\rho_1, & when\, b=b_\#;\\ >\rho_1, & when\, b>b_\#.\end{cases}$$

Proof. Similar to the proof of (i) in Lemma 2.2, we can get (3.3). Since

$$\frac{\lambda\theta_b}{\mu+1+d\theta_b}\ge 2\sqrt{\frac{\lambda\theta_b}{c(\mu+1+d\theta_b)}}-\frac{1}{c}\Leftrightarrow\left(\sqrt{\frac{\lambda\theta_b}{\mu+1+d\theta_b}}-\frac{1}{\sqrt{c}}\right)^2\ge 0.$$

Then (1) follows from Theorem 8.3. Finally, (2) follows by the fact that

$$\rho_1+2\sqrt{\frac{\lambda}{cd}}-\frac{1}{c}\le\rho_1\Leftrightarrow d\ge 4\lambda c$$

and the monotone property of $g(b)$. □

The following result follows from Theorem 3.1 and Lemma 3.3.

Corollary 3.4. *Let $D_1 D_2$ and D_5 be the sets defined in* (2.18)*. Define a set D_6 by*

$$D_6 := \{(a,b) : b > b_{\#},\ \rho_1 < a \leq g(b)\}$$

for the case $d < 4\lambda c$, where $g(b)$ and $b_{\#}$ are defined in Lemma 3.3*. Then* (1.6) *has no positive solution if $(a,b) \in D_1 \cup D_2 \cup D_5$ for any $\lambda, c, d > 0$; and $(a,b) \in D_1 \cup D_2 \cup D_5 \cup D_6$ for the case $d < 4\lambda c$.*

4. Existence of positive solutions

In this section, we will study the existence of positive solutions to problem (1.6) by degree theory which has been prepared in Sect. 8. Throughout this section and next two sections, we will use the notations in Theorem 8.8 and

$E := C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, where $C_0(\overline{\Omega})$ is defined in Sect. 4. It is obvious that E is a Banach space with the norm $\|(u,v)\|_E = \max_{x\in\overline{\Omega}} |u(x)| + \max_{x\in\overline{\Omega}} |v(x)|$.
$\mathcal{W} = K \times K$, where $K := \{u \in C_0(\overline{\Omega}) : u(x) \geq 0 \text{ for } x \in \overline{\Omega}\}$.
$D := \{(u,v) \in \mathcal{W} : u(x) \leq a+1,\ v(x) \leq b+1 \text{ for } x \in \overline{\Omega}\}$.

We can see from Lemma 3.2 that all positive solutions of (1.6) lie in the interior of D ($= \mathrm{int}D$). In order to apply the degree theory, we choose a sufficiently large number p such that

$$p + a - u - \frac{\lambda v}{(1+cu)(1+dv)} > 0 \quad \text{and} \quad p + b - v - \frac{\mu r v}{u+r} > 0$$

for $(u,v) \in D$. Define a mapping A in E by

$$\begin{aligned} A(u,v) \ &= (-\Delta + pI)^{-1}\left(\left(p + a - u - \frac{\lambda v}{(1+cu)(1+dv)}\right)u,\right.\\ &\qquad \left.\times\left(p + b - v - \frac{\mu r v}{u+r}\right)v\right) = ((-\Delta + pI)^{-1}(pu + F(u,v)),\\ &\qquad (-\Delta + pI)^{-1}(pv + G(u,v))), \end{aligned} \tag{4.1}$$

where

$$F(u,v) = u\left(a - u - \frac{\lambda v}{(1+cu)(1+dv)}\right),\ G(u,v) = v\left(b - v - \frac{\mu r v}{u+r}\right).$$

By (4.1) and the maximum principle for elliptic equations, A maps D into $\mathcal{W}$, and the strong maximum principle implies that A maps $D \setminus (0,0)$ into the demi-interior of $\mathcal{W}$ (see [15,16]). Moreover, the regularity theory for elliptic equations tells us that A is completely continuous in E. Therefore, one can define the degree, $\deg_{\mathcal{W}}(I - A, \mathrm{int}D)$, for A with respect to $\mathcal{W}$ because A has no fixed point on the boundary of D with respect to $\mathcal{W}$ by Lemma 3.2

Lemma 4.1. $\deg_{\mathcal{W}}(I-A, \mathrm{int}D) = 1$.

Proof. For $t \in [0,1]$, we define a completely continuous mapping in E by

$$A_t(u,v) = (-\Delta + pI)^{-1}\left(\left(p + at - u - \frac{\lambda v}{(1+cu)(1+dv)}\right)u,\right.$$
$$\left.\times\left(p + bt - v - \frac{\mu r v}{u+r}\right)v\right).$$

As in the study of A, one can show that A_t maps D into $\mathcal{W}$ and A_t has no fixed point on the boundary of D with respect to $\mathcal{W}$. Hence it follows from the homotopy invariance that $\deg_{\mathcal{W}}(I - A_t, \mathrm{int}D)$ is independent of $t \in [0,1]$ (see [3, Theorem 11.1]); so it follows from $(0,0)$ is the unique fixed point of A_0 in D that

$$\begin{aligned}\deg_{\mathcal{W}}(I - A, \mathrm{int}D) &= \deg_{\mathcal{W}}(I - A_1, \mathrm{int}D)\\ &= \deg_{\mathcal{W}}(I - A_0, \mathrm{int}D) = \mathrm{index}_{\mathcal{W}}(A_0, (0,0)).\end{aligned} \tag{4.2}$$

Let $A_0'(0,0)$ be the Fréchet derivative of A_0, which is given by $A_0'(0,0)(u,v) = (-\Delta + pI)^{-1}(pu, pv)$. Observe that $r(A_0'(0,0)) = \frac{p}{\rho_1 + p} < 1$. Then $\mathrm{index}_{\mathcal{W}}(A_0, (0,0)) = 1$ is obtained by Theorem 8.8. Thus the conclusion follows from (4.2). $\square$

By Theorem 3.1, we assume $a > \rho_1$ and $b > \rho_1$ to study the existence of positive solutions of (1.6), which means that (1.6) has one trivial solution $(0,0)$ and two semi-trivial solutions $(\theta_a, 0)$ and $(0, \theta_b^*)$. Next we will study the index of the trial and semi-trivial solutions listed above.

Lemma 4.2. *Let $a > \rho_1$ and $b > \rho_1$.*

(i) $\mathrm{index}_{\mathcal{W}}(A, (0,0)) = 0$ *and* $\mathrm{index}_{\mathcal{W}}(A, (\theta_a, 0)) = 0$.

(ii) *It holds that* $\begin{cases} \mathrm{index}_{\mathcal{W}}(A, (0, \theta_b^*)) = 0, & \text{if } a > \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right);\\ \mathrm{index}_{\mathcal{W}}(A, (0, \theta_b^*)) = 1, & \text{if } a < \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right).\end{cases}$

Proof. (i) We begin with the study of $\mathrm{index}_{\mathcal{W}}(A, (0,0))$. A direct calculation shows that

$$A'(0,0)(u,v) = (-\Delta + pI)^{-1}((p+a)u, (p+b)v).$$

Clearly, $\overline{\mathcal{W}_{(0,0)}} = \mathcal{W}$ and $S_{(0,0)} = \{(0,0)\}$; so that $\widetilde{A'(0,0)}$ is identical with $A'(0,0)$. Observe that

$$r(A'(0,0)) = \max\left\{\frac{a+p}{\rho_1 + p}, \frac{b+p}{\rho_1+p}\right\}.$$

For $a > \rho_1$ and $b > \rho_1$, it is easy to see that $A'(0,0)y \neq y$ on $\mathcal{W}\backslash\{(0,0)\}$ and that $r(A'(0,0)) = r(\widetilde{A'(0,0)}) > 1$. Hence Theorem 8.8 yields $\mathrm{index}_{\mathcal{W}}(A, (0,0)) = 0$.

The proof of $\mathrm{index}_{\mathcal{W}}(A, (\theta_a, 0)) = 0$ is similar to the proof of $\mathrm{index}_{\mathcal{W}}(A, (0, \theta_b^*)) = 0$; so we omit it and give the details of the computation of the index for $(0, \theta_b^*)$.

(ii) After a series of calculations, one can show

$$A'(0,\theta_b^*)(u,v) = (-\Delta + pI)^{-1}\left(\left(p + a - \frac{\lambda\theta_b}{\mu+1+d\theta_b}\right)u,\right.$$
$$\left.\frac{\mu(\theta_b)^2}{r(\mu+1)^2}u + (p+b-2\theta_b)v\right).$$

Define T_1 and T_2 by

$$T_1 := (-\Delta + pI)^{-1}\left(p + a - \frac{\lambda\theta_b}{\mu+1+d\theta_b}\right),\ T_2 := (-\Delta + pI)^{-1}(p+b-2\theta_b).$$

Since $\overline{\mathcal{W}_{(0,\theta_b^*)}} = K \times C_0(\overline{\Omega})$ and $S_{(0,\theta_b^*)} = \{0\} \times C_0(\overline{\Omega})$; so that we can identify $\widetilde{A'(0,\theta_b^*)}$ with T_1.

We will show by contradiction that $I - A'(0,\theta_b^*) \neq 0$ on $\overline{\mathcal{W}_{(0,\theta_b^*)}}\backslash\{(0,0)\}$ if

$$\rho_1(\kappa) \neq 0 \ \text{ with } \ \kappa = \frac{\lambda\theta_b}{\mu+1+d\theta_b}.$$

Assume that there exists $(\xi_1,\xi_2) \in \overline{\mathcal{W}_{(0,\theta_b^*)}}\backslash\{(0,0)\}$ such that $A'(0,\theta_b^*)(\xi_1,\xi_2) = (\xi_1,\xi_2)$. If $\xi_1 = 0$, then $T_2\xi_2 = \xi_2$, which implies

$$-\Delta\xi_2 + (2\theta_b - b)\xi_2 = 0,\ x \in \Omega,\ \ \xi = 0,\ x \in \partial\Omega.$$

Therefore $\xi_2 = 0$ by the fact that $\rho_1(2\theta_b - b) > 0$, which is a contradiction. So $\xi_1 \in K$ must satisfy $\xi_1 \neq 0$ and $T_1\xi_1 = \xi_1$. Hence it follows from the Krein–Rutman theorem $r(T_1) = 1$, which, together with Theorem 8.4, implies $\rho_1(\kappa) = 0$. This is also a contradiction. Thus we have shown $I - A'(0,\theta_b^*) \neq 0$ on $\overline{\mathcal{W}_{(0,\theta_b^*)}}\backslash\{(0,0)\}$.

We apply Theorem 8.8 to calculate the index of $(0,\theta_b^*)$. In the case $a > \rho_1(\kappa)$, Theorem 8.4 gives $r(T_1) > 1$. Since $\widetilde{A'(0,\theta_b^*)}$ is identical with T_1, we know that $r(\widetilde{A'(0,\theta_b^*)}) > 1$. Then it follows from (8.8) that $\text{index}_{\mathcal{W}}(A,(0,\theta_b^*)) = 0$. Next, we consider the case $a < \rho_1(\kappa)$, which is equivalent to $r(T_1) < 1$. Furthermore, one can get $r(T_1) < 1$ by the fact that $\rho_1(2\theta_b - b) > 0$. Then it follows from [35, page 76] that

$$r(A'(0,\theta_b^*)) = \max\{r(T_1),\ r(T_2)\} < 1.$$

Hence it is sufficient to employ Theorem 8.8 to get $\text{index}_{\mathcal{W}}(A,(0,\theta_b^*)) = 1$. □

We are now ready to derive an existence result of positive steady states for (1.6).

Theorem 4.3. *If $b > \rho_1$ and $a > \rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$, then (1.6) admits at least one positive solution.*

Proof. We will prove this theorem by the degree theory. Assume on the contrary that (1.6) has no positive solution. It follows from the definition of degree that

$$\deg_{\mathcal{W}}(I - A, \text{int}D) = \text{index}_{\mathcal{W}}(A,(0,0)) + \text{index}_{\mathcal{W}}(A,(\theta_a,0)) + \text{index}_{\mathcal{W}}(A,(0,\theta_b^*)) \tag{4.3}$$

By Lemma 4.1, the left-hand side of (4.3) is equal to 1. On the other hand, Lemma 4.2 implies the right-hand side of (4.3) is equal to zero. This contradicts to (4.3); so that (1.6) must possess at least one positive solution. □

Remark 4.4. *In this remark, we give some biological interpretations of Theorem 4.3. Firstly, we consider the second equation of (1.6) for any $u \geq 0$, i.e., the following problem*

$$\begin{cases} -\Delta v = v\left(b - v - \frac{\mu r v}{u+r}\right), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

It follows from the Krein–Rutman theorem (see [22]) that $v > 0$ in Ω if and only if $b > \rho_1$. Furthermore, we get $v \geq \theta_b^* := \theta_b/(\mu+1)$ by comparison principle. So, the growth rate b of predator v is larger than ρ_1 in Theorem 4.3 ensures they can survive no matter how little the prey u is. Moreover, we conjecture there must be other food sources for predator v.

Secondly, we consider the first equation of (1.6) for any $v \geq \theta_b^*$ since $b > \rho_1$, i.e., the following problem

$$\begin{cases} -\Delta u + \frac{\lambda v}{(1+cu)(1+dv)} u = u\,(a-u)\,, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

Then $u > 0$ in Ω if and only if

$$a > \rho_1\left(\frac{\lambda v}{(1+cu)(1+dv)}\right). \tag{4.6}$$

Theorem 4.3 tells us if the growth rate a of prey u is greater than $\rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b})$, condition (4.6) is ensured. Furthermore, we can infer

$$\rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right) \geq \rho_1\left(\frac{\lambda v}{(1+cu)(1+dv)}\right)$$

i.e.,

$$\theta_b u < \frac{\mu+1}{cd} \quad \text{and} \quad v \leq \frac{\theta_b(1+cu)}{\mu+1-cd\theta_b u},$$

which give the upper bounds of the positive solutions of problem (1.6).

By Lemma 2.2, Theorem 4.3 can be re-stated as follows.

Corollary 4.5. *(1.6) admits at least one positive solution if $(a,b) \in D_4$, where D_4 is defined in (2.18).*

Remark 4.6. *Corollaries 2.3 and 4.5 imply that (1.6) admits a positive solution when both the trivial and semi-trivial solutions of (1.6) are unstable.*

The next corollary is some specific inequalities to ensure the existence of positive solution of problem (1.6), which follows from Lemma 2.2 and Theorem 4.3.

Corollary 4.7. *Problem (1.6) admits at least one positive solution if $b > \rho_1$ and any one of the following conditions holds.*

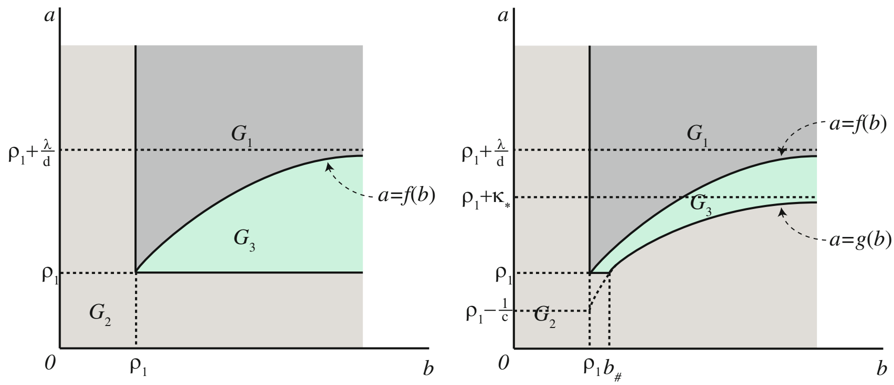

FIGURE 2. Illusions of the sets in Corollary 4.8. *Left* is the case $d \geq 4\lambda c$; *right* is the case $d < 4\lambda c$. Here, $b_\#$ is defined in Lemma 3.3 and $\kappa_* = 2\sqrt{\frac{\lambda}{cd}} - \frac{1}{c}$

(i) $\lambda \geq \mu + 1$ *and* $a > \begin{cases} \varphi_1(b), & \textit{if } \rho_1 \leq b < b^\#; \\ \varphi_2(b), & \textit{if } b \geq b^\#. \end{cases}$

(ii) $\lambda < \mu + 1$ *and* $a > \begin{cases} \varphi_3(b), & \textit{if } \rho_1 \leq b < b^*; \\ \varphi_2(b), & \textit{if } b \geq b^*. \end{cases}$

Here $\varphi_i(b)$, $i = 1, 2, 3$, $b^\#$ *and* b^* *were defined in Lemma* 2.2.

At the end of this section, we make some comments on Sects. 4 and 5. Firstly, we define some sets by using the notations in Lemmas 2.2 and 3.3 (see Fig. 2).

$$G_1 := \{(a, b) : a > f(b), b > \rho_1\} = D_4, \text{ where } D_4 \text{ is the set defined in (2.18).}$$

$$G_2 := \begin{cases} \{(a, b) : a \leq \rho_1 \text{ or } b \leq \rho_1\}, & \text{if } d \geq 4\lambda c; \\ \{(a, b) : a \leq \rho_1 \text{ or } b \leq \rho_1 \text{ or } a \leq g(b)\}, & \text{if } d < 4\lambda c. \end{cases}$$

$$G_3 := \begin{cases} \{(a, b) : \rho_1 < a \leq f(b),\ b > \rho_1\}, & \text{if } d \geq 4\lambda c; \\ \{(a, b) : \max\{\rho_1,\ g(b)\} < a \leq f(b)\}, & \text{if } d < 4\lambda c. \end{cases} \quad (4.7)$$

Then the following results follows from Theorems 3.1 and 4.3.

Corollary 4.8. *Let* G_i, $i = 1, 2, 3$, *be the sets defined in* (4.7). *Then* (1.6) *admits at least one positive solution if* $(a, b) \in G_1$, *while it has no positive solution if* $(a, b) \in G_2$.

5. Multiplicity of positive solutions

In Corollary 4.8, we have shown that (1.6) has a positive solution if $(a, b) \in G_1$, while it has no positive solution if $(a, b) \in G_2$. However, the above result gives no information on the existence or non-existence of positive solutions for $(a, b) \in G_3$. Our main interest is to know what happens if $(a, b) \in G_3$, and we will show that (1.6) may has at least two positive solution when (a, b) lie in some subset of G_3. Throughout this section, we will fixed $b > \rho_1$, and study bifurcating solutions from $(0, \theta_b^*)$ by regarding a as a bifurcation parameter.

By linearizing (1.6) at $(0, \theta_b^*) = (0, \theta_b/(\mu+1))$, we obtain the following eigenvalue problem:

$$\begin{cases} \Delta\phi + \left(a - \frac{\lambda\theta_b}{\mu+1+d\theta_b}\right)\phi = \rho\phi, & x \in \Omega, \\ \Delta\psi + \frac{\mu}{r}\left(\frac{\theta_b}{\mu+1}\right)^2 \phi + (b - 2\theta_b)\psi = \rho\psi, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

A necessary condition for bifurcation is that the principal eigenvalue of (5.1) is zero, which occurs if

$$a = \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right) = f(b),$$

where $f(b)$ is defined in Lemma 2.2. Let Φ be a unique positive solution of

$$\begin{cases} \Delta\Phi + \left(f(b) - \frac{\lambda\theta_b}{\mu+1+d\theta_b}\right)\Phi = 0, & x \in \Omega, \\ \Phi = 0, & x \in \partial\Omega, \\ \Phi > 0, & x \in \Omega, \\ \|\Phi\|_{L^2(\Omega)} = 1. \end{cases} \tag{5.2}$$

We also define Ψ as the unique positive solution of

$$\begin{cases} -\Delta\Psi + (2\theta_b - b)\Psi = \frac{\mu}{r}\left(\frac{\theta_b}{\mu+1}\right)^2 \Phi, & x \in \Omega, \\ \Psi = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

Here we have used the invertibility of $-\Delta + (2\theta_b - b)I$ with zero Dirichlet boundary condition since $\rho_1(2\theta_b - b) > \rho_1(\theta_b - b) = 0$. Furthermore, for $p > N$, set

$$\begin{cases} \mathbf{X} := \left[W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right] \times \left[W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right], \\ \mathbf{Y} := L^p(\Omega) \times L^p(\Omega). \end{cases} \tag{5.4}$$

By Sobolev's theorem, $\mathbf{X}$ is continuously embedded into $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$, where $C_0^1(\overline{\Omega}) = C^1(\Omega) \cap C_0(\overline{\Omega})$.

With functions defined above, we have the following result regarding the bifurcation solutions of (1.6) from $(0, \theta_b^*)$ at $a = f(b)$.

Lemma 5.1. *Let $b > \rho_1$ be fixed. Then positive solutions of (1.6) bifurcate from a semi-trivial solution curve $\{(0, \theta_b^*, a) : a > \rho_1\}$ if and only if $a = f(b)$. More precisely, there exists a positive number δ such that all positive solutions of (1.6) near $(0, \theta_b^*, f(b)) \in \mathbf{X} \times \mathbb{R}$ lie on a smooth curve*

$$\Gamma_1 = \{(u, v, a) = (u(s), v(s), a(s)) : 0 < s < \delta\}$$

with

$$\begin{aligned} u(s) &= s\Phi + s\hat{u}(s) = s\Phi + O(s^2), \\ v(s) &= \theta_b^* + s\Psi + s\hat{v}(s) = \theta_b^* + s\Psi + O(s^2), \\ a(s) &= f(b) + sa_1 + O(s^2). \end{aligned}$$

Here $(\hat{u}(s), \hat{v}(s), a(s))$ *for* $0 < s \le \delta$ *is a family of smooth functions with respect to* s *satisfying* $(\hat{u}(0), \hat{v}(0), a(0)) = (0, 0, f(b))$, $\int_\Omega \hat{u}(s)\Phi dx = 0$, $\int_\Omega \hat{v}(s)\Psi dx = 0$, *and*

$$a_1 = \int_\Omega \left[\left(1 - \frac{\lambda c\theta_b}{\mu + 1 + d\theta_b}\right)\Phi^2 + \frac{\lambda(\mu+1)^2}{(\mu+1+d\theta_b)^2}\Phi^2\Psi\right] dx. \tag{5.5}$$

Proof. Define a nonlinear mapping $\mathfrak{F} : \mathbf{X} \times \mathbb{R} \mapsto \mathbf{Y}$ by

$$\mathfrak{F}(u, v, a) = \begin{pmatrix} \Delta u + F(u, v, a) \\ \Delta v + G(u, v) \end{pmatrix},$$

where

$$F(u, v) = u\left(a - u - \frac{\lambda v}{(1 + cu)(1 + dv)}\right), \; G(u, v) = v\left(b - v - \frac{\mu r v}{u + r}\right).$$

From straightforward calculation, we obtain

$$\mathfrak{F}_{(u,v)}(u, v, a)[\xi, \eta] = \begin{pmatrix} \Delta\xi + \left(a - 2u - \frac{\lambda v}{(1+cu)^2(1+dv)}\right)\xi - \frac{\lambda u}{(1+cu)(1+dv)^2}\eta \\ \Delta\eta + \frac{\mu r v^2}{(u+r)^2}\xi + \left(b - 2v - \frac{2\mu r v}{u+r}\right)\eta \end{pmatrix},$$

$$\mathfrak{F}_a(u, v, a) = \begin{pmatrix} u \\ 0 \end{pmatrix}, \; F_{a(u,v)}(u, v, a)[\xi, \eta] = \begin{pmatrix} \xi \\ 0 \end{pmatrix},$$

$$\begin{aligned} &\mathfrak{F}_{(u,v)(u,v)}(a, u, v)[\xi, \eta]^2 \\ &= 2\begin{pmatrix} \left(\frac{\lambda c v}{(1+cu)^3(1+dv)} - 1\right)\xi^2 - \frac{\lambda}{(1+cu)^2(1+dv)^2}\xi\eta + \frac{\lambda d u}{(1+cu)(1+dv)^3}\eta^2 \\ -\frac{\mu r v^2}{(u+r)^3}\xi^2 + \frac{2\mu r v}{(u+r)^2}\xi\eta - \left(1 + \frac{\mu r}{u+r}\right)\eta^2 \end{pmatrix}. \end{aligned}$$

At $(u, v, a) = (0, \theta_b^*, f(b))$, it is easy to see that the kernel

$$\mathbf{N}(\mathfrak{F}_{(u,v)}(0, \theta_b^*, f(b))) = \text{span}\{(\Phi, \Psi)\},$$

and the range

$$\mathbf{R}(\mathfrak{F}_{(u,v)}(0, \theta_b^*, f(b))) = \left\{(\phi, \psi) \in \mathbf{Y} : \int_\Omega \phi(x)\Phi(x) dx = 0\right\}.$$

Furthermore,

$$\mathfrak{F}_{a(u,v)}(0, \theta_b^*, f(b))[\Phi, \Psi] = (\Phi, 0) \notin \mathbf{R}(\mathfrak{F}_{(u,v)}(0, \theta_b^*, f(b)))$$

since $\int_\Omega \Phi^2 dx = 1 \neq 0$. Then we can apply Theorem 8.1 to conclude the set of positive solutions to (1.6) near $(0, \theta_b^*, f(b))$. Moreover, by Theorem 8.1,

$$a_1 = a'(0) = -\frac{\langle \ell, \mathfrak{F}_{(u,v)(u,v)}(0, \theta_b^*, f(b))[\Phi, \Psi]^2\rangle}{2\langle \ell, \mathfrak{F}_{a(u,v)}(0, \theta_b^*, f(b))[\Phi, \Psi]\rangle},$$

where ℓ is a linear functional on $\mathbf{Y}$ defined as $\langle \ell, (\phi, \psi)\rangle = \int_\Omega \phi(x)\Phi(x)dx$. Thus, (5.5) follows. $\square$

Lemma 5.2. *For any $a > \rho_1$, there is no positive solution of (1.6) bifurcating from $(\theta_a, 0)$. Similarly, positive solutions of (1.6) cannot bifurcate from $(0, 0)$.*

Proof. By linearizing (1.6) at $(\theta_a, 0)$, we obtain the following eigenvalue problem:

$$\begin{cases} \Delta\phi + (a - 2\theta_a)\,\phi - \dfrac{\lambda\theta_a}{1 + c\theta_a}\psi = \rho\phi, & x \in \Omega, \\ \Delta\psi + b\psi = \rho\psi, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases}$$

A necessary condition for bifurcation is that the principal eigenvalue of (5.1) is zero, which occurs if $b = \rho_1$. However, it follows from Theorem 2.1 that $(\theta_a, 0)$ is globally asymptotically stable if $a > \rho_1$ and $b = \rho_1$, which means bifurcation cannot happen at $(\theta_a, 0)$ when $b = \rho_1$. The second conclusion can be obtained by the same way. $\square$

By Theorem 8.1, the curve Γ_1 of bifurcating positive solutions is contained in a connected component Σ of the set of positive solutions of (1.6). Our next result is about Σ. To this end, we denote

$$\mathbf{P} = \left\{ w \in C_0^1(\Omega),\ w(x) > 0 \text{ for } x \in \Omega \text{ and } \frac{\partial w}{\partial \nu}(x) < 0 \text{ for } x \in \partial\Omega \right\}, \quad (5.6)$$

where $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$.

Theorem 5.3. *Let $b > \rho_1$ be fixed. Then the global bifurcation curve Σ is unbounded in $\mathbf{X}\times\mathbb{R}$, where $\mathbf{X}$ is defined in (5.4). Moreover, $\Sigma\backslash\{(0, \theta_b^*, f(b))\} =: \widetilde{\Sigma} \subset \mathbf{P}\times\mathbf{P}\times[\rho_1, \infty)$ and $[f(b), \infty) \subset \mathrm{Prj}_a\Sigma$, where $\mathrm{Prj}_a\Sigma$ means the projection of Σ on a-axis.*

Proof. To conclude the conclusion, we only need to prove $\widetilde{\Sigma} \subset \mathbf{P}\times\mathbf{P}\times[\rho_1, \infty)$ and $[f(b), \infty) \subset \mathrm{Prj}_a\Sigma$. Let $(u(s), v(s), a(s))$ be the bifurcation solutions got in Theorem 5.1. It should be noted that $a(s) \in [\rho_1, \infty)$ by Theorem 3.1 and $(u(s), v(s)) \in \mathbf{P}\times\mathbf{P}$ for sufficiently small $s > 0$ because Φ and θ_b^* belong to $\mathbf{P}$ by the strong maximum principle.

On the contrary, we suppose $\widetilde{\Sigma}$ is not contained in $\mathbf{P}\times\mathbf{P}\times[\rho_1, \infty)$. Then the above arguments show that there exists a point $(\hat{u}, \hat{v}, \hat{a}) \in \partial(\mathbf{P}\times\mathbf{P})\times[\rho_1, \infty)$, which is the limit of a sequence of points $\{(u_n, v_n, a_n)\}_{n=1}^\infty \subset \widetilde{\Sigma}\cap(\mathbf{P}\times\mathbf{P}\times[\rho_1, \infty))$. Then $\hat{u} \in \partial\mathbf{P}$ or $\hat{v} \in \partial\mathbf{P}$.

If $\hat{u} \in \partial\mathbf{P}$, then $\hat{u} \geq 0$ for $x \in \overline{\Omega}$ and either $\hat{u}(x_0) = 0$ for some $x_0 \in \Omega$ or $\partial\hat{u}/\partial\nu(x_0) = 0$ for some $x_0 \in \partial\Omega$. Since $\hat{u}$ satisfies

$$-\Delta\hat{u} = \hat{u}\left(a - \hat{u} - \frac{\lambda\hat{v}}{(1 + c\hat{u})(1 + d\hat{v})}\right),\ x \in \Omega,\ \ \hat{u} = 0,\ x \in \partial\Omega.$$

The strong maximum principle that $\hat{u} \equiv 0$. Similarly, we can show $\hat{v} \equiv 0$ if $\hat{v} \in \partial\mathbf{P}$. Thus we the following three cases:

$$(i)\ (\hat{u}, \hat{v}) = (\theta_a, 0)\ \ (ii)\ (\hat{u}, \hat{v}) = (0, \theta_b^*)\ \ (iii)\ (\hat{u}, \hat{v}) = (0, 0).$$

By Lemma 5.2, (i) and (iii) cannot happen. Since $(0, \theta_b^*, f(b)) \notin \widetilde{\Sigma}$, (ii) cannot happen. Thus, $\widetilde{\Sigma} \subset \mathbf{P} \times \mathbf{P} \times [\rho_1, \infty)$.

Next we will prove $[f(b), \infty) \subset \mathrm{Prj}_a \Sigma$. For any positive solution (u, v) of (1.5), Lemma 3.2 gives $0 \le u \le a$ and $0 \le v \le b$. Then the regularity theory of elliptic equations ensure that there exists a positive constant C depending only on a such that any $(u, v, a) \in \widetilde{\Sigma}$ satisfies

$$\|u\|_{W^{2,p}(\Omega)} \le C \text{ and } \|v\|_{W^{2,p}(\Omega)} \le C.$$

This estimates together with the unboundedness of Σ implies $[f(b), \infty) \subset \mathrm{Prj}_a \Sigma$. □

Let $\{(u(s), v(s), a(s))\}_{0<s<\delta}$ be the family of positive solutions of (1.6) got by Lemma 5.1. Next we will study the stability properties of this family.

Theorem 5.4. *Let a_1 and δ be defined in Lemma 5.1 and $\{(u(s), v(s), a(s))\}_{0<s<\delta}$ be the family of positive solutions of (1.6) got by Lemma 5.1. Then there exists a constant $\tilde{\delta} \in (0, \delta]$ such that $(u(s), v(s))$ with $0 < s < \tilde{\delta}$ is asymptotically stable is the bifurcation is supercritical (i.e., $a_1 > 0$), while it is unstable if the bifurcation is subcritical (i.e., $a_1 < 0$).*

Proof. We will apply Theorem 8.2 to prove this theorem. Throughout the proof, we will use the notations in the proof of Lemma 5.1. In order to study the stability, we consider the following eigenvalue problem

$$\begin{cases} \mathfrak{L}(s) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \varrho(s) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}, & x \in \Omega, \\ u(s) = v(s) = 0, & x \in \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \mathfrak{L}(s) &= -\mathfrak{F}_{(u,v)}(u(s), v(s), a(s)) \\ &= \begin{pmatrix} -\Delta - \left(a(s) - 2u(s) - \frac{\lambda v(s)}{(1+cu(s))^2(1+dv(s))}\right) & \frac{\lambda u(s)}{(1+cu(s))(1+dv(s))^2} \\ -\frac{\mu r v^2(s)}{(u(s)+r)^2} & -\Delta - \left(b - 2v(s) - \frac{2\mu r v(s)}{u(s)+r}\right) \end{pmatrix}. \end{aligned}$$

Furthermore,

$$\lim_{s \to 0} \mathfrak{L}(s) = \begin{pmatrix} -\Delta - f(b) + \frac{\lambda\theta_b}{\mu+1+d\theta_b} & 0 \\ -\frac{\mu}{r}\left(\frac{\theta_b}{\mu+1}\right)^2 & -\Delta + 2\theta_b - b \end{pmatrix} =: \mathfrak{L}_0.$$

Since

$$\rho_1 \left(-f(b) + \frac{\lambda\theta_b}{\mu + 1 + d\theta_b}\right) = 0 \text{ and } \rho_1(2\theta_b - b) > 0,$$

0 is the first eigenvalue of $\mathfrak{L}_0$ with corresponding eigenfunction (Φ, Ψ). Moreover the real part of all other eigenvalues of $\mathfrak{L}_0$ are positive and apart from 0. By perturbation theory of linear operators [28], we know that there exists a small constant $\delta_1 \in (0, \delta]$ such that for $s \in (0, \delta_1), \mathfrak{L}(s)$ has a unique eigenvalue $\varrho(s)$ such that $\lim_{s\to 0} \varrho(s) = 0$ and all other eigenvalues of $\mathfrak{L}(s)$ have positive real part and apart from 0.

Now we determine the sign of $\varrho(s)$ by using Theorem 8.2. Consider the following eigenvalue problem

$$\begin{cases} -\mathfrak{F}_{(u,v)}(0,\theta_b^*,a)\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix} = m(a)\begin{pmatrix}\phi(a)\\ \psi(a)\end{pmatrix}, & x\in\Omega,\\ \phi(a)=\psi(a)=0, & x\in\partial\Omega.\end{cases}$$

Similar to the analysis for $\mathfrak{L}(s)$, there exists a small constant $\delta_2\in(0,\delta]$ such that for $|a-f(b)|<\delta_2$, $-\mathfrak{F}_{(u,v)}(0,\theta_b^*,a)$ has a unique eigenvalue $m(a)$ such that $\lim_{a\to f(b)} m(a)=0$ and all other eigenvalues of $-\mathfrak{F}_{(u,v)}(0,\theta_b^*,a)$ have positive real part and apart from 0. Furthermore, $m(a)$ is determined by the following eigenvalue problem

$$\begin{cases} -\Delta\phi(a)+\frac{\lambda\theta_b}{\mu+1+d\theta_b}\phi(a)-a\phi(a)=m(a)\phi(a), & x\in\Omega,\\ \phi(a)=0, & x\in\partial\Omega.\end{cases} \tag{5.7}$$

Since $m(f(b))=0$ and $\phi(f(b))=\Phi$, then by differentiating (5.7) with respect to a at $a=f(b)$ we obtain that

$$\begin{cases} -\Delta\chi+\frac{\lambda\theta_b}{\mu+1+d\theta_b}\chi-f(b)\chi-\Phi=m'(f(b))\Phi, & x\in\Omega,\\ \chi=0, & x\in\partial\Omega,\end{cases} \tag{5.8}$$

where $\chi=\phi'(f(b))$. Multiplying both sides of (5.8) and integrating the result over Ω yields

$$\begin{aligned} m'(f(b))\int_\Omega \Phi^2dx &= -\int_\Omega\Phi^2dx+\int_\Omega\left(-\Delta\chi\Phi+\frac{\lambda\theta_b}{\mu+1+d\theta_b}\chi\Phi-f(b)\chi\Phi\right)dx\\ &= -\int_\Omega\Phi^2dx+\int_\Omega\chi\left(-\Delta\Phi+\frac{\lambda\theta_b}{\mu+1+d\theta_b}\Phi-f(b)\Phi\right)dx\\ &= -\int_\Omega\Phi^2dx,\end{aligned}$$

where the last equality follows from (5.2). So $m'(f(b))=-1$. Since $a_1\neq 0$, then if follows from Theorem (8.2) that there exists a small positive constant $\delta_3\le\min\{\delta_1,\delta_2\}$ such that $\mu(s)\neq 0$ for $s\in(0,\delta_3)$ and

$$\lim_{s\to 0}\frac{\varrho(s)}{s}=-m'(f(a))a'(0)=a_1.$$

Then there exists a positive constant $\tilde{\delta}\le\delta_3$ such that $\mu(s)$ has same sign with a_1 for $s\in(0,\tilde{\delta})$, and the conclusion follows. □

Based on above preparations, we can give our multiplicity result as follows.

Theorem 5.5. *Assume $b>\rho_1$ be fixed. Let $f(b)$ and a_1 be defined in Lemma 2.2 and (5.5), respectively. If $a_1<0$, then there exists a positive constant $\varepsilon=\varepsilon(b)<f(b)-\rho_1$ such that (1.6) has at least two positive solution if $f(b)-\varepsilon<a<f(b)$, and it has at least one positive solution if $a\ge f(b)-\varepsilon$.*

Proof. From Lemma 5.1, (1.6) has a curve $\Gamma_1=\{(u(s),v(s),a(s)):0<s<\delta\}$ of positive solutions near $(0,\theta_b^*,f(b))$. Since $a_1<0$, then $a(s)<f(b)$ for $s>0$ small. Assume on the contrary that (1.6) has a unique positive solution $(\hat{u},\hat{v})$

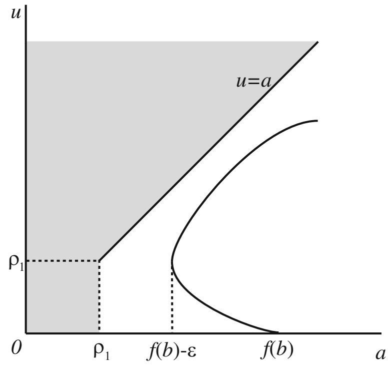

FIGURE 3. Bifurcation curve of u when $a_1 < 0$

when $a < f(b)$ but near $f(b)$, then it is obvious that $(\hat{u}, \hat{v}) = (u(s), v(s))$, and it is not degenerate by Theorem 5.4. Let A be the operator defined in (4.1) and D be the region defined at the beginning Sect. 5, then $I - A_{(u,v)}(\hat{u}, \hat{v}) : \overline{\mathcal{W}_{(\hat{u},\hat{v})}} \mapsto \overline{\mathcal{W}_{(\hat{u},\hat{v})}}$ is invertible. Since $(\hat{u}, \hat{v})$ is an interior point of D, it follows from Theorem 8.8 that $\text{index}_{\mathcal{W}}(A, (\hat{u}, \hat{v})) = \pm 1$. Notice $\rho_1 < a < f(b)$ for $s > 0$ small and $b > \rho_1$. It follows from Lemmas 4.1 and 4.2 that

$$\begin{aligned}1 &= \deg_{\mathcal{W}}(I - A, \text{int}D) \\ &= \text{index}_{\mathcal{W}}(A, (0,0)) + \text{index}_{\mathcal{W}}(A, (\theta_a, 0)) + \text{index}_{\mathcal{W}}(A, (0, \theta_b^*)) \\ &\quad + \text{index}_{\mathcal{W}}(A, (\hat{u}, \hat{v})), = 0 + 0 + 1 \pm 1,\end{aligned}$$

which is a contradiction. Thus if $a < f(b)$ and near $f(b)$, (1.6) admits at least two positive solutions. Then it follows from Theorem (5.3) that there exists $\varepsilon \in (0, f(b) - \rho_1)$ such that $\text{Prj}_a \Sigma = [f(b) - \varepsilon, \infty)$ (see Fig. 3), where Σ is the global bifurcation got in Theorem 5.3, and the conclusion follows. □

Remark 5.6. *Let's make some Comments to Theorem 5.5.*

1. *Since Φ and Ψ are both independent of c, then*

$$\begin{aligned}\lim_{c \to \infty} \frac{a_1}{c} &= \lim_{c \to \infty} \int_\Omega \left[\left(\frac{1}{c} - \frac{\lambda \theta_b}{\mu + 1 + d\theta_b} \right) \Phi^2 + \frac{\lambda(\mu+1)^2}{c(\mu + 1 + d\theta_b)^2} \Phi^2 \Psi \right] dx \\ &= -\int_\Omega \frac{\lambda \theta_b}{\mu + 1 + d\theta_b} \Phi^2 dx < 0.\end{aligned}$$

   So, $a_1 < 0$ can be achieved if we fix $\lambda, \mu, d > 0$, $b > \rho_1$, and take c large enough.
2. *At the beginning of this section, we proposed a question, i.e., what happens if $(a, b) \in G_3$. Theorem 5.5 gives answer to this question in some extent. That is problem (1.6) possesses at leat two positive solution when $(a, b) \in G_3$ and (a, b) near the curve $\{(a, b) : a = f(b), b > \rho_1\}$ under the condition $a_1 < 0$, where G_3 is defined in (4.7).*

6. Uniqueness of positive solution

By Theorem 5.5, we obtain (1.6) has multiple solutions if $a_1 < 0$. A natural question is that if (1.6) has unique positive solution when $a_1 > 0$ and $(a, b) \in G_1$, where G_1 is defined in (4.7). In this section, we will answer this question in some extent. Note that a_1, defined in (5.5), is positive if

$(C_{\text{uni}})$$c$ is small enough; or λ is small enough; or d is large enough.

In this section, we will show the uniqueness of positive solution to (1.6) under the condition (C_{uni}). To this end, we first introduce some lemmas.

Consider the following problem:

$$\begin{cases} -\Delta u + \frac{\lambda v}{1+dv} u = u(a-u), & x \in \Omega, \\ -\Delta v = v\left(b - v - \frac{\mu r v}{u+r}\right), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

Lemma 6.1. *Assume $b > \rho_1$ and $a \geq \rho_1(\frac{\lambda\theta_b}{1+d\theta_b})$. Then (6.1) has a unique positive solution, which is denoted by (u^*, v^*).*

Proof. First we consider the following problem for $v \in C^1(\overline{\Omega})$:

$$-\Delta u + \frac{\lambda v}{1+dv} u = u\,(a-u),\ x \in \Omega,\ \ u = 0,\ x \in \partial\Omega. \tag{6.2}$$

Then it follows from Theorem 8.5 that (6.2) has a unique positive solution if $a > \rho_1\left(\frac{\lambda v}{1+dv}\right)$. So for an arbitrary function $v \in C^1(\overline{\Omega})$, we define $u(v)$ as a function on Ω by

$$u(v) = \begin{cases} 0, & \text{if } a \leq \rho_1\left(\frac{\lambda v}{1+dv}\right); \\ \text{unique positive solution of problem (6.2)}, & \text{if } a > \rho_1\left(\frac{\lambda v}{1+dv}\right). \end{cases} \tag{6.3}$$

By [6] or [34, page 360], the following conclusions hold.

1. The mapping $v \mapsto u(v)$ defined by (6.3) considered as a function from $C^1(\overline{\Omega})$ to $C^1(\Omega)$ is continuous;
2. if $v_1 \geq v_2$ in $\overline{\Omega}$, then $u(v_1) \leq u(v_2)$ in $\overline{\Omega}$.

Next, we consider the following problem

$$-\Delta v = v\left(b - v - \frac{\mu r v}{u(v)+r}\right),\ x \in \Omega,\ \ u = 0,\ x \in \partial\Omega. \tag{6.4}$$

Then it follows from Theorem 4.3 that (6.4) has a unique positive solution by the properties of $u(v)$ and the fact that $b > \rho_1$, which we denote by v^*. Furthermore, it is easy to see that $v^* < \theta_b$. Since

$$a \geq \rho_1\left(\frac{\lambda\theta_b}{1+d\theta_b}\right) > \left(\frac{\lambda v^*}{1+dv^*}\right),$$

$u^* := u(v^*)$ is the unique positive solution of (6.2). Finally, the above analysis shows that (u^*, v^*) is the unique positive solution of (6.1). □

Fixed $a > \rho_1$, let's consider the following problem:

$$-\Delta v = v\left(b - v - \frac{\mu r v}{\theta_a + r}\right),\ x \in \Omega,\ \ v = 0,\ x \in \partial\Omega. \tag{6.5}$$

By Theorem 8.5, the following result holds.

Lemma 6.2. *Assume $a, b > \rho_1$. Then (6.5) has a unique positive solution, which is denoted by v_{θ_a}.*

Based on above preparations, we can consider the asymptotic behavior of positive solutions of (1.6), when c or μ or λ or d changes.

Lemma 6.3. *The following results hold.*

(i) *Assume $b > \rho_1$ and $a \geq \rho_1(\frac{\lambda\theta_b}{1+d\theta_b})$. Suppose (u_i, v_i) is a positive solution of (1.6) with $c = c_i$ and $c_i \to 0$ as $i \to \infty$, then (u_i, v_i) converges to (u^*, v^*) uniformly as $i \to \infty$, where (u^*, v^*) is got in Lemma 6.1. Furthermore, if*

$$\lambda^2 a \leq 1 \ \ or \ \ \frac{\mu^2 b^3}{r^2} \leq \frac{\mu r}{a + r} + 1, \tag{6.6}$$

then there exists a positive constant τ_1 such that any positive solution of (1.6) is non-degenerate and linearly stable if $c < \tau_1$.

(ii) *Assume $a, b > \rho_1$. Suppose (u_i, v_i) is a positive solution of (1.6) with $\lambda = \lambda_i$ and $\lambda_i \to 0$ as $i \to \infty$, then (u_i, v_i) converges to (θ_a, v_{θ_a}) uniformly as $i \to \infty$, where v_{θ_a} is got in Lemma 6.2. Furthermore, there exists a positive constant τ_3 such that any positive solution of (1.6) is non-degenerate and linearly stable if $\lambda < \tau_3$.*

(iii) *Assume $a, b > \rho_1$. Suppose (u_i, v_i) is a positive solution of (1.6) with $d = d_i$ and $d_i \to \infty$ as $i \to \infty$, then (u_i, v_i) converges to (θ_a, v_{θ_a}) uniformly as $i \to \infty$, where v_{θ_a} is got in Lemma 6.2. Furthermore, there exists a positive constant τ_4 such that any positive solution of (1.6) is non-degenerate and linearly stable if $d > \tau_4$.*

Proof. We only give the proof of (i) since the proofs for others are similar. Assume $c_i \to 0$ as $i \to \infty$, and (u_i, v_i) is a positive solution of (1.6) with $c = c_i$. By Lemma 3.2, $\|u_i\|_{L^\infty(\Omega)} \leq a$ and $\|v_i\|_{L^\infty(\Omega)} \leq b$ and the upper bounds are both independent of i (thus independent of c_i), then

$$\|(u_i, v_i)\|_{C^{2+\alpha}(\overline{\Omega})\times C^{2+\alpha}(\overline{\Omega})} := \|u_i\|_{C^{2+\alpha}(\overline{\Omega})} + \|v_i\|_{C^{2+\alpha}(\overline{\Omega})},\ \ i = 1, 2, \ldots$$

are uniformly bounded by regularity theory of elliptic equations [23] and the fact that c_i are uniformly bounded. Then there exits a subsequence of $\{(u_i, v_i)\}_{i=1}^{\infty}$, relabeled by itself, and two nonnegative functions $u, v \in C^{1+\beta}(\overline{\Omega})$ with $0 < \beta < \alpha$ such that $(u_i, v_i) \to (u, v)$ in $C^{2+\beta}(\overline{\Omega}) \times C^{2+\beta}(\overline{\Omega})$ as $i \to \infty$. Then (u, v) must be a solution of (6.1). From strong maximum principle, we know that each of u and v is either > 0 in Ω or $\equiv 0$ in Ω. So if we can show that $u, v > 0$ in Ω, then the convergence result follows as the positive solution of (6.1) is unique hence it must be the limit of the sequence $\{(u_i, v_i)\}_{i=1}^{\infty}$.

To the contrary, we assume $u \equiv 0$, $\|u_i\|_{L^\infty(\Omega)} \to 0$ as $i \to \infty$, then $v = 0$ or $v = \theta_b/(\mu+1)$. Let $\overline{u}_i = u_i/\|u_i\|_{L^\infty(\Omega)}$, then $\overline{u}_i$ satisfies

$$\begin{cases} -\Delta \overline{u}_i + \frac{\lambda v_i}{(1+c_i u_i)(1+dv_i)} \overline{u}_i = \overline{u}_i(a - u_i), & x \in \Omega, \\ \overline{u}_i = 0, & x \in \partial\Omega. \end{cases}$$

Similar to the arguments as above, $\|\overline{u}_i\|_{C^{2+\alpha}(\Omega)}$ are uniformly bounded, thus there exists a subsequence of $\{\overline{u}_i\}_{i=1}^\infty$, relabeled by itself, and a nonnegative function $\overline{u} \in C^{2+\beta}(\overline{\Omega})$ with $0 < \beta < \alpha$ such that $\overline{u}_i \to \overline{u}$ in $C^{2+\beta}(\overline{\Omega})$. Obviously, $\|\overline{u}\|_{L^\infty(\overline{\Omega})} = 1$ and $\overline{u}$ satisfies

$$\begin{cases} -\Delta \overline{u} + \frac{\lambda v}{1+dv} \overline{u} = a\overline{u}, & x \in \Omega, \\ \overline{u} = 0, & x \in \partial\Omega. \end{cases}$$

Therefore, it follows from Krein–Rutman Theorem that

$$a = \rho_1\left(\frac{\lambda v}{1+dv}\right) = \begin{cases} \rho_1, & \text{if } v = 0; \\ \rho_1\left(\frac{\lambda\theta_b}{\mu+1+d\theta_b}\right), & \text{if } v = \theta_b/(\mu+1), \end{cases}$$

which again contradicts with the assumption that $a \geq \rho_1(\frac{\lambda\theta_b}{1+d\theta_b})$.

On the other hand, if we assume that $v \equiv 0$, then $u = 0$ or $u = \theta_a$. The same arguments as above show that there exits a nonnegative function $\overline{v} \in C^{1+\beta}(\overline{\Omega})$ such that $\overline{v}_i := v_i/\|v_i\|_{L^\infty(\Omega)} \to \overline{v}$ in $C^{2+\beta}(\overline{\Omega})$ as $i \to \infty$. Furthermore, $\|\overline{v}\|_{L^\infty(\overline{\Omega})} = 1$ satisfies

$$-\Delta\overline{v} = b\overline{v},\ x \in \Omega,\ \ v = 0,\ x \in \partial\Omega.$$

Therefore $b = \rho_1$, which again contradicts with the assumption that $b > \rho_1$. Hence the limit $(u, v) > (0, 0)$ must be the unique positive solution of (6.1).

Next we will study the stability result. To the contrary, we assume that there exists a sequence $\{c_i\}_{i=1}^\infty$ such that $c_i \to 0$ as $i \to \infty$, ρ_i with $\mathrm{Re}\rho_i \leq 0$ and (ξ_i, η_i) with $\|\xi_i\|_{L^2(\Omega)} + \|\eta_i\|_{L^2(\Omega)} = 1$ satisfying

$$\begin{cases} -\Delta\xi_i - \left(a - 2u_i - \frac{\lambda v_i}{(1+c_i u_i)^2(1+dv_i)}\right)\xi_i + \frac{\lambda u_i}{(1+c_i u_i)(1+dv_i)^2}\eta_i = \rho_i\xi_i & x \in \Omega, \\ -\Delta\eta_i - \frac{\mu r v_i^2}{(u_i+r)^2}\xi_i - \left(b - 2v_i - \frac{2\mu r v_i}{u_i+r}\right)\eta_i = \rho_i\eta_i, & x \in \Omega, \\ \xi_i = \eta_i = 0, & x \in \partial\Omega, \end{cases} \tag{6.7}$$

where (u_i, v_i) is a positive solution of (1.6) with $c = c_i$. Multiplying $(6.7)_1$ by $\overline{\xi}_i$ and $(6.7)_2$ by $\overline{\eta}_i$, integrating the results over Ω, and then adding the results, we obtain

$$\begin{aligned} \rho_i \ &= \underbrace{\int_\Omega (|\nabla\xi_i|^2 + |\nabla\eta_i|^2)dx}_{I_1} + \underbrace{\int_\Omega \left(\frac{\lambda v_i}{(1+c_i u_i)^2(1+dv_i)} + 2u_i - a\right)|\xi_i|^2 dx}_{I_2} \\ &\quad + \underbrace{\int_\Omega \left(\frac{\lambda u_i}{(1+c_i u_i)(1+dv_i)^2}\eta_i\overline{\xi}_i - \frac{\mu r v_i^2}{(u_i+r)^2}\xi_i\overline{\eta}_i\right)dx}_{I_3} \qquad (6.8) \\ &\quad + \underbrace{\int_\Omega \left(\frac{2\mu r v_i}{u_i+r} + 2v_i - b\right)|\eta_i|^2 dx}_{I_4}. \qquad (6.9) \end{aligned}$$

Then by the fact that $\{(u_i,v_i)\}_{i=1}^{\infty}$ is uniformly bounded and $\mathrm{Re}\rho_i \le 0$, we obtain

$$I_1 = \mathrm{Re}\rho_i - I_2 - I_4 \le -I_2 - I_4 \tag{6.10}$$

is uniformly bounded. So it follows from (6.8) and (6.10) that there exists a constant C independent of i such that

$$|\rho_i|^2 = (\mathrm{Re}\rho_i)^2 + (\mathrm{Im}\rho_i)^2 = (I_1+I_2+I_4)^2 + (I_3)^2 \le 2\sum_{j=1}^{4}(I_j)^2 \le C.$$

So there exists a subsequence of $\{\rho_i\}_{i=1}^{\infty}$, doted by itself, such that $\lim_{i\to\infty}\rho_i = \rho$ with $\mathrm{Re}\rho \le 0$. Using the boundedness of $\{\rho_i\}_{i=1}^{\infty}$ and L^p-theory of elliptic equations (see [23]), we get that form any $p > N$, $\{\xi_i\}_{i=1}^{\infty}$ and $\{\eta_i\}_{i=1}^{\infty}$ are uniformly bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is embedded in $C^1(\overline{\Omega})$ compactly, there exist subsequences of $\{x1_i\}_{i=1}^{\infty}$ and $\{\eta_i\}_{i=1}^{\infty}$, denoted by themselves, such that $\lim_{i\to\infty}\xi_i = \xi$ and $\lim_{i\to\infty}\eta_i = \eta$ in $C^1(\overline{\Omega})$ and $\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} = 1$. Letting $i \to \infty$ in (6.7), we obtain (ρ,ξ,η) satisfies the following equation in the sense of distribution

$$\begin{cases} -\Delta\xi - \left(a - 2u^* - \frac{\lambda v^*}{1+dv^*}\right)\xi + \frac{\lambda u^*}{(1+dv^*)^2}\eta = \rho\xi & x\in\Omega, \\ -\Delta\eta - \frac{\mu r(v^*)^2}{(u^*+r)^2}\xi - \left(b - 2v^* - \frac{2\mu r v^*}{u^*+r}\right)\eta = \rho\eta, & x\in\Omega, \\ \xi = \eta = 0, & x\in\partial\Omega. \end{cases} \tag{6.11}$$

Since $\xi,\eta\in C^1(\overline{\Omega})$, by the regularity theory of elliptic equations, we get $\xi,\eta\in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ and (ρ,ξ,η) satisfies (6.11) in classical sense.

Since (u^*,v^*) is a positive solution of (6.1), it follows from Krein–Rutman Theorem

$$a = \rho_1\left(\frac{\lambda v^*}{1+dv^*} + u^*\right) \text{ and } b = \rho_1\left(v^* + \frac{\mu r v^*}{u^*+r}\right). \tag{6.12}$$

If $\xi \not\equiv 0$ and $\eta \equiv 0$, by the first equation of (6.11) we have

$$\mathrm{Re}\rho \ge \rho_1\left(\frac{\lambda v^*}{1+dv^*} + 2u^* - a\right) > \rho_1\left(\frac{\lambda v^*}{1+dv^*} + u^*\right) - a = 0,$$

which contradicts to $\mathrm{Re}\rho \le 0$.

If $\xi \not\equiv 0$, $\eta \not\equiv 0$ and $\lambda^2 a \le 1$, by variational characterization of the principal eigenvalue and the first equation of (6.11), we have

$$\mathrm{Re}\rho \ge \inf_{\phi\in S}\frac{\int_\Omega\left(|\nabla\phi|^2 + \left(\frac{\lambda v^*}{1+dv^*} + 2u^* - a\right)\phi^2 + \frac{\lambda u^*}{(1+dv^*)^2}\eta\phi\right)dx}{\int_\Omega \phi^2 dx}, \tag{6.13}$$

where

$$S = \left\{\phi\in H_0^1(\Omega) : \int_\Omega\left(\frac{\lambda u^*}{(1+dv^*)^2}\phi\right)^2 = 1\right\}.$$

Since $\|\eta\|_{L^2(\Omega)} \leq 1$ and $\phi \in S$, we obtain

$$\begin{aligned}\int_\Omega \frac{\lambda u^*}{(1+dv^*)^2}\eta\phi dx &\geq -\left[\int_\Omega \eta^2 dx\right]^{1/2}\left[\int_\Omega \left(\frac{\lambda u^*}{(1+dv^*)^2}\phi\right)^2 dx\right]^{1/2}\\ &\geq -\left[\int_\Omega \left(\frac{\lambda u^*}{(1+dv^*)^2}\phi\right)^2 dx\right]^{1/2}\\ &= -\int_\Omega \left(\frac{\lambda u^*}{(1+dv^*)^2}\phi\right)^2 dx. \end{aligned}\tag{6.14}$$

So, it follows (6.13), (6.14), $\lambda^2 a \leq 1$, and $u^* < a$ that

$$\begin{aligned}\mathrm{Re}\rho &\geq \rho_1\left(u^*\left(1-\frac{\lambda^2 u^*}{(1+dv^*)^4}\right)+\left(\frac{\lambda v^*}{1+dv^*}+u^*-a\right)\right)\\ &> \rho_1\left(u^*\left(1-\lambda^2 a\right)+\left(\frac{\lambda v^*}{1+dv^*}+u^*-a\right)\right)\\ &\geq \rho_1\left(\frac{\lambda v^*}{1+dv^*}+u^*\right)-a=0,\end{aligned}$$

which contradicts to $\mathrm{Re}\rho \leq 0$.

If $\xi \not\equiv 0$, $\eta \not\equiv 0$ and $\frac{\mu r}{a+r}+1 \geq \frac{\mu^2 b^3}{r^2}$, then it follows from the second equation of (6.11) and the similar analysis as above, one can get

$$\begin{aligned}\mathrm{Re}\rho &\geq \rho_1\left(v^*\left(\frac{\mu r}{u^*+r}+1-\frac{\mu^2 r^2 (v^*)^3}{(u^*+r)^4}\right)+\left(\frac{\mu r v^*}{u^*+r}+v^*-b\right)\right)\\ &> \rho_1\left(u^*\left(\frac{\mu r}{a+r}+1-\frac{\mu^2 b^3}{r^2}\right)+\left(\frac{\mu r v^*}{u^*+r}+v^*-b\right)\right)\\ &\geq \rho_1\left(\frac{\mu r v^*}{u^*+r}+v^*\right)-b=0,\end{aligned}$$

which contradicts to $\mathrm{Re}\rho \leq 0$.

If $\xi \equiv 0$ and $\eta \not\equiv 0$, by the second equation of (6.11) we have

$$\mathrm{Re}\rho \geq \rho_1\left(\frac{2\mu r v^*}{u^*+r}+2v^*-b\right) > \rho_1\left(\frac{\mu r v^*}{u^*+r}+v^*\right)-b=0,$$

which contradicts to $\mathrm{Re}\rho \leq 0$.

The above analysis shows (i) holds. □

Base on Lemma 6.3, we can state our first uniqueness result as follows.

Theorem 6.4. *The following results hold.*

(i) *Suppose $b > \rho_1$ and $a \geq \rho_1(\frac{\lambda\theta_b}{1+d\theta_b})$ are fixed parameters such that (6.6) holds, then there exists a constant $\delta_1 > 0$ such that when $0 < c < \delta_1$, the problem (1.6) has a unique positive solution which is locally asymptotically stable.*

(ii) *Suppose $b > \rho_1$ and $a > \rho_1$ are fixed parameters, then there exists a constant $\delta_3 > 0$ such that when $0 < \lambda < \delta_3$, the problem (1.6) has a unique positive solution which is locally asymptotically stable.*

(iii) *Suppose* $b > \rho_1$ *and* $a > \rho_1$ *are fixed parameters, then there exists a constant* $\delta_4 > 0$ *such that when* $d > \delta_4$*, the problem* (1.6) *has a unique positive solution which is locally asymptotically stable.*

Proof. Note that

1. $\rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b}) < \rho_1(\frac{\lambda\theta_b}{1+d\theta_b})$ and
2. $\rho_1(\frac{\lambda\theta_b}{\mu+1+d\theta_b}) \to \rho_1$ when $\lambda \to 0$ or $d \to \infty$.

Then existence of a positive solution easily follows from Theorem 4.3. Hence we only need to show the uniqueness and local stability. Recall that A is the operator defined in (4.1) and D is the region that positive solutions lie in. By the compactness, G has at most finitely many positive fixed points in the region D. We denote all the positive fixed points of A in D by (u_i, v_i) for $i = 1, 2, \ldots, \ell$. From Lemma 6.3, we have $\mathrm{index}_{\mathcal{W}}(A,(u_i, v_i)) = 1$. According to the additive property of Leray–Schauder degree, we get

$$\begin{aligned} 1 &= \deg_{\mathcal{W}}(I - A, \mathrm{int}D) \\ &= \mathrm{index}_{\mathcal{W}}(A,(0,0)) + \mathrm{index}_{\mathcal{W}}(A,(\theta_a,0)) + \mathrm{index}_{\mathcal{W}}(A,(0,\theta_b^*)) \\ &\quad + \sum_{i=1}^{\ell} \mathrm{index}_{\mathcal{W}}(A,(u_i,v_i)) = 0+0+0+\ell. \end{aligned}$$

Hence $\ell \equiv 1$, which asserts the uniqueness. The local stability has been proved in Lemma 6.3. □

Next, we will give some specific conditions to ensure the uniqueness of positive solution of (1.6) for $N = 1$. Consider the following problem:

$$\begin{cases} -u'' = u\left(a - u - \frac{\lambda v}{(1+cu)(1+dv)}\right), & 0 < x < L, \\ -v'' = v\left(b - v - \frac{\mu r v}{u+r}\right), & 0 < x < L, \\ u(0) = u(L) = v(0) = v(L) = 0, \end{cases} \tag{P_λ}$$

where L is a positive constant and $'' := d/dx^2 = \Delta$.

Firstly, we will introduce some lemmas to get the uniqueness result.

Lemma 6.5. *Assume*

$$[(\lambda c - d)b - 1](a + r) \le \mu r. \tag{6.15}$$

Let (u_0, v_0) *be an arbitrary positive solution of* (P_λ) *then the linearized system of* (P_λ) *at* (u_0, v_0) *has only the trivial solution* (P_λ). *Hence any positive solution of* (P_λ) *is not degenerate.*

Proof. The linearized system of (P_λ) at (u_0, v_0) is

$$\begin{cases} -\phi'' + \mathfrak{L}_1\phi = \mathcal{A}\psi, & 0 < x < L, \\ -\psi'' + \mathfrak{L}_2\psi = \mathcal{B}\phi, & 0 < x < L, \\ \phi(0) = \phi(L) = \psi(0) = \psi(L) = 0, \end{cases}$$

where

$$\mathfrak{L}_1 = 2u_0 + \frac{\lambda v_0}{(1+cu_0)^2(1+dv_0)} - a, \quad \mathfrak{L}_2 = 2v_0 + \frac{2\mu r v_0}{u_0+r} - b,$$
$$\mathcal{A} = -\frac{\lambda u_0}{(1+cu_0)(1+dv_0)^2} < 0, \quad \mathcal{B} = \frac{\mu r v_0^2}{(u_0+r)^2} > 0.$$

Since (u_0, v_0) is a positive solution of (1.6), it follows from the Krein–Rutman Theorem that

$$\rho_1\left(u_0 + \frac{\lambda v_0}{(1+cu_0)(1+dv_0)} - a\right) = 0 \text{ and } \rho_1\left(v_0 + \frac{\mu r v_0}{u_0+r} - b\right) = 0. \tag{6.16}$$

By the second equation of (P_λ), one can show $v_0 < (a+r)b/(a+r+\mu r)$, which together with (6.15) leads to

$$2u_0 + \frac{\lambda v_0}{(1+cu_0)^2(1+dv_0)} - a > u_0 + \frac{\lambda v_0}{(1+cu_0)(1+dv_0)} - a.$$

Then it follows from (6.16) and Theorem 8.3 that $\rho_1(\mathfrak{L}_i) > 0, \ \ i = 1, 2$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be the Banach spaces defined in (5.4), and define a linear operator $\triangle = (\triangle_1, \triangle_2) : \mathbf{X} \mapsto \mathbf{Y}$, where $\triangle_i = -'' + \mathfrak{L}_i, \ \ i = 1, 2$. Let $\mathbf{P}$ be the set defined in (5.6), and let $\triangle_i^{-1}$ be the inverse operator of $\triangle_i$. It is obvious that $\triangle_i^{-1}$ is compact and strictly order-preserving operators with respect to $\mathbf{P}$. Moreover $\triangle_i^{-1}(\mathbf{P}\backslash\{0\}) \subset \mathrm{int}\mathbf{P}$. In this settings, we can show that $\phi = \psi = 0$ using a similar proof as in [10,39], which completes the proof. $\square$

A perturbation argument can be used to show that if (P_λ) has exactly one positive solution, which is assumed to be non-degenerate, then ($P_{\lambda+\epsilon}$) also has exactly one positive solution provided ϵ is small enough. For that purpose, we state the following lemma. Since the proof is basically same as [10, Lemma 5.4], we omit its proof.

Lemma 6.6. *Assume* (6.15) *holds and* (P_λ) *has exactly one positive solution* (u_0, v_0)*, which is not degenerate. Then there exists* $\epsilon_0 = \epsilon_0(\lambda) > 0$ *such that for every* $\epsilon \in (-\epsilon_0, \epsilon_0)$*, problem* ($P_{\lambda+\epsilon}$) *has exactly one positive solution* $(u(\epsilon), v(\epsilon))$*. Moreover* $(u(0), v(0)) = (u_0, v_0)$ *and the mapping* $\epsilon \mapsto (u(\epsilon), v(\epsilon))$*, from* $(-\epsilon_0, \epsilon_0)$ *to* $\mathbf{P} \times \mathbf{P}$*, is of class* C^1*, where* $\mathbf{P}$ *is defined in* (5.6).

Next let's make some analysis of the condition (6.15). It is obvious that (6.15) holds if $\lambda c \le d$ or $b \le b_0$ and $\lambda c > d$, where $b_0 := 1/(\lambda c - d)$. For $b > b_0$ and $\lambda c > d$, we define a function $a = h(b)$ by

$$a = h(b) = \frac{\mu r}{(\lambda c - d)b - 1} - r; \tag{6.17}$$

then (6.15) holds if and only if $a \le h(b)$. Obviously, $h(b)$ is a smooth strictly decreasing function such that

$$\lim_{b\to b_0} h(b) = \infty, \ \lim_{b\to\infty} h(b) = -\infty, \ h(b_1) = 0, \tag{6.18}$$

where $b_1 = (\mu + 1)/(\lambda c - d)$.

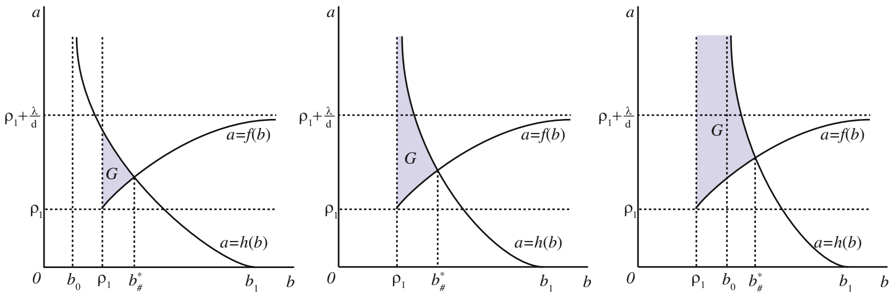

FIGURE 4. Illusions of the sets in Theorem 6.7 when $\lambda c > d$ and $b > b_0$. *Left* is the case $b_0 < \rho_1$ and (6.19) holds; *middle* is the case $b_0 = \rho_1$; *right* is the case $b_0 > \rho_1$

In order to state the uniqueness result, we make some discussions on $f(b)$ and $h(b)$, where $f(b)$ is defined in (2.6). It is easy to see $f(b)$ and $h(b)$ intersect at only one point $(a, b) = (h(b^*_{\#}), b^*_{\#})$ for $b^*_{\#} \in (\rho_1, \infty)$ if and only if $b_0 \geq \rho_1$ or $b_0 < \rho_1$ and $h(\rho_1) > \rho_1$. It is easy to show that $h(\rho_1) > \rho_1$ is equivalent to

$$\mu > [(\lambda c - d)\rho_1 - 1]\left(\frac{\rho_1}{r} + 1\right). \tag{6.19}$$

Let $G = \{(a, b) : a, b \text{ satisfy } (6.15)\} \cap G_1$, where G_1 is defined in (4.7). With the notations introduced above and the analysis given above, we can see that (see Figs. 4, 5)

$$G = \begin{cases} G_1, & \text{if } \lambda c \leq d; \\ G_1, & \text{if } \lambda c > d \text{ and } b \leq b_0; \\ \{(a, b) : f(b) < a \leq h(b),\ \rho_1 < b < b^*_{\#}\}, & \text{if } \lambda c > d, b_0 < \rho_1 \text{ and (6.19) holds;} \\ \{(a, b) : f(b) < a \leq h(b),\ \rho_1 < b < b^*_{\#}\}, & \text{if } \lambda c > d \text{ and } b_0 = \rho_1; \\ \{(a, b) : a > f(b),\ \rho_1 < b \leq b_0\} & \\ \cup\{(a, b) : f(b) < a \leq h(b),\ b_0 < b < b^*_{\#}\}, & \text{if } \lambda c > d \text{ and } b_0 > \rho_1; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{6.20}$$

By Lemma 6.2, (P_0) has exactly one positive solution (θ_a, v_{θ_a}) if $a > \rho_1$ and $b > \rho_1$. Then it follows from Lemmas 6.5 and 6.6, we can state the following uniqueness result. Again the proof is similar to [10, Theorem 5.1], thus we omit its proof.

Theorem 6.7. *Let G be the set defined in (6.20). Then problem (P_λ) has a unique positive solution if $G \neq \emptyset$ and $(a, b) \in G$.*

Compared with Theorem 6.4, Theorem 6.7 has some specific inequalities to ensure the existence of a unique positive solution. We remark that $G \neq \emptyset$, when c is small or λ is small or d is large, which is consistent with the conditions in Theorem 6.4.

7. Conclusions

In this paper, we study the existence, multiplicity and uniqueness results for the steady state equations (1.6) of a modified Leslie–Gower predator model

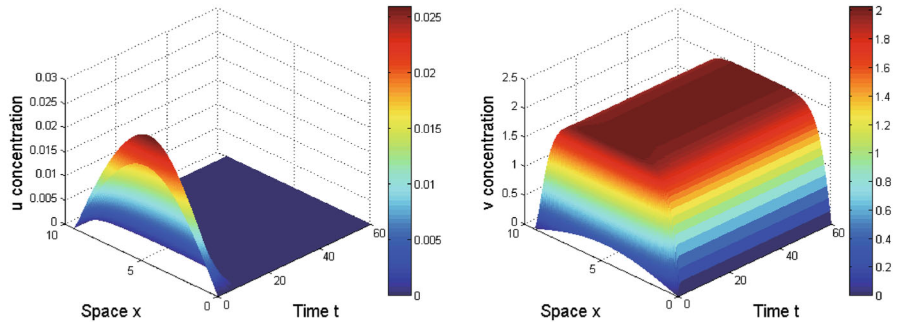

Figure 5. Numerical simulation of the system (2.3) with $N = 1$, $\Omega = (0, 3\pi)$, parameters $a = 0.7$, $b = 4$, $r = \lambda = \mu = 1$, $c = 100$ and $d = 0.001$, and initial values $u_0(x) = 0.02\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$. As $t \to \infty$, the solution converges to the semi-trivial steady state solution $(0, \theta_b^*(x))$

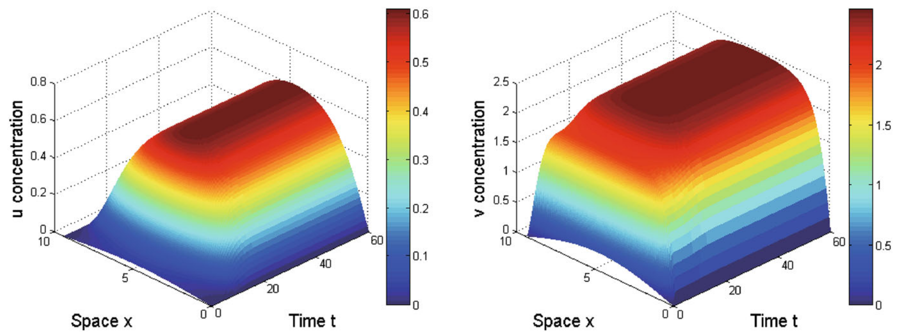

Figure 6. Numerical simulation of the system (2.3) with $N = 1$, $\Omega = (0, 3\pi)$, parameters $a = 0.7$, $b = 4$, $r = \lambda = \mu = 1$, $c = 100$ and $d = 0.001$, and initial values $u_0(x) = 0.03\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$. As $t \to \infty$, the solution converges to a positive steady state solution

with Crowley–Martin functional responses. To illustrate the multiplicity results shown in Sect. 6, we show an example of numerical simulation in Figs. 5 and 6. We consider the one-dimension spatial domain $\Omega = (0, 3\pi)$, so the principal eigenvalue $\rho_1 = 1/9 \approx 0.11$. For parameters we choose $a = 0.7, b = 4, r = \lambda = \mu = 1, c = 100$ and $d = 0.001$, so $a > \rho_1$ and $b > \rho_1$ are satisfied, and following the Remark 5.6 and uniqueness results in Sect. 7, we choose c, λ and μ much larger and d much smaller, one can indeed show that $a_1 < 0$. In Fig. 5, we use initial condition $u_0(x) = 0.02\sin(x/3)$, $v_0(x) = 0.3\sin(x/3)$, and in Fig. 6, we use initial values $u_0(x) = 0.03\sin(x/3)$ and the same v_0. Then one can see that a small difference in the initial prey populations triggers a drastic difference of asymptotic fates, and it indeed shows that there exits two stable

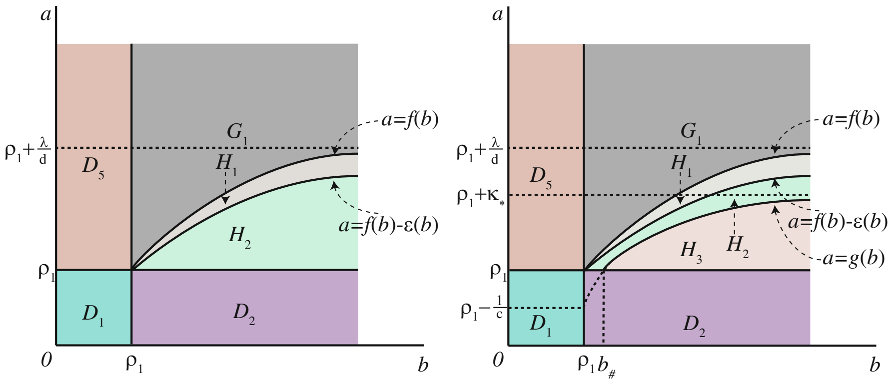

Figure 7. Illusions of the sets in the conclusions for the case $a_1 < 0$ where a_1 is defined in (5.5). *Left* is the case $d \geq 4\lambda c$; *right* is the case $d < 4\lambda c$. Here, $f(b)$ is defined in Lemma 2.2, $g(b)$ is defined in Lemma 3.3, $\varepsilon(b)$ is got in Theorem 5.5, $b_\#$ is defined in Lemma 3.3 and $\kappa_* = 2\sqrt{\frac{\lambda}{cd}} - \frac{1}{c}$

asymptotic states in this case. The existence of at least two stable steady state has profound impact on the ecological conservation, as a sudden collapse of the ecosystem is more likely to occur in such systems (see [26,45,47]).

Next we summarize the results got in this paper. First we state the results about stability, existence, non-existence and multiplicity (see Fig. 7). Let D_1, D_2 and D_5 be the sets defined in (2.18), G_1 be the set defined in (4.7), and define sets $H_i, i = 1, 2, 3$, as follows:

$$H_1 := \begin{cases} \{(a,b) : b > \rho_1, f(b) - \varepsilon(b) < a < f(b)\}, & \text{if } a_1 < 0; \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$H_2 := \begin{cases} \{(a,b) : b > \rho_1, \rho_1 < a \leq f(b) - \varepsilon(b)\}, & \text{if } d \geq 4\lambda c \text{ and } a_1 < 0; \\ \{(a,b) : b > \rho_1, \rho_1 < a < f(b), & \text{if } d \geq 4\lambda c \text{ and } a_1 \geq 0; \\ \{(a,b) : \max\{\rho_1,\ g(b)\} < a \leq f(b) - \varepsilon(b)\}, & \text{if } d < 4\lambda c \text{ and } a_1 < 0; \\ \{(a,b) : \max\{\rho_1,\ g(b)\} < a < f(b)\}, & \text{if } d \geq 4\lambda c \text{ and } a_1 \geq 0, \end{cases}$$

$$H_3 := \begin{cases} \emptyset, & \text{if } d \geq 4\lambda c; \\ \{(a,b) : b > b_\#, \rho_1 < a \leq g(b)\}, & \text{if } d < 4\lambda c, \end{cases} \tag{7.1}$$

where $f(b)$ and $g(b)$ are defined in (2.2) and (3.3), respectively; $\varepsilon(b)$ is the constant got in Theorem 5.5 when $a_1 < 0$ and a_1 is defined in (5.5); $b_\#$ is defined in Lemma 3.3. Then (see Corollaries 2.3, 3.4, 4.8, and Theorem 5.5),

1. $(0,0)$ is globally asymptotically stable if $(a,b) \in D_1$, while it is unstable if $(a,b) \in (0,\infty) \times (0,\infty) \backslash \overline{D_1}$.
2. $(0,\theta_b^*)$ is asymptotically stable if $(a,b) \in D_2 \cup H_1 \cup H_2 \cup H_3$; $(0,\theta_b^*)$ is globally asymptotically stable if $(a,b) \in D_2$, while $(0,\theta_b^*)$ is unstable if $(a,b) \in G_1$.
3. $(\theta_a,0)$ is globally asymptotically stable if $(a,b) \in D_5$, while it is unstable if $(a,b) \in H_1 \cup H_2 \cup H_3 \cup G_1 \cup \{(a,b) : a = f(b), b > \rho_1\}$.

4. Problem (1.6) has no positive solution if $(a, b) \in D_1 \cup D_2 \cup D_5 \cup H_3$.
5. Problem (1.6) has at least one positive solution if $(a, b) \in D_1 \cup \overline{H_1}$, and (1.6) has at least two positive solutions if $(a, b) \in H_1$.

In all, we divide the region $\{(a, b) : a > 0,\ b > 0\}$ into different subregions, and discuss the existence and multiplicity results for the steady state equations (1.6). We have studied the existence and non-existence positive solutions for (1.6) in all subregions but H_2. So there is an **open question**:

Does problem (1.6) has a positive solution if $(a, b) \in H_2$?

The uniqueness result has been discussed when $(a, b) \in G_1$ in Sect. 7. The result for $N \geq 1$ can be found in Theorem 6.4; while for $N = 1$, we can character the result in (a, b)-plane: problem (1.6) with $N = 1$ has a unique positive solution if $(a, b) \in G$, where G is defined in (6.20). Compared with the existence result, we have only discuss a little about the uniqueness even for $N = 1$. So there is another **open question**:

Does problem (1.6) has a unique positive solution if $N = 1$, $\lambda c > d, b > b_0$, and $(a, b) \in G_1 \setminus G$? Here $b_0 = 1/(\lambda c - d)$.

8. Appendix

In this part we list the well-known facts which are used in this paper.

First we recall some well-known abstract bifurcation theorems. Consider an abstract equation

$$\mathfrak{F}(u, a) = 0,$$

where $\mathfrak{F} : \mathbf{X} \times \mathbb{R} \to \mathbf{Y}$ is a nonlinear differential mapping, and $\mathbf{X}, \mathbf{Y}$ are Banach spaces such that $\mathbf{X}$ is continuously embedding in $\mathbf{Y}$. The following bifurcation and stability theorems were obtained in [12,13,44] (see also [48,49]).

Theorem 8.1. *Let U be a neighborhood of (u_0, a_0) in $\mathbf{X} \times \mathbb{R}$, and let $\mathfrak{F} : U \to \mathbf{Y}$ be a twice continuously differentiable mapping. Assume that $\mathfrak{F}(u_0, a) = 0$ for all $(u_0, a) \in U$. At (u_0, a_0), $\mathfrak{F}$ satisfies*

$$\dim \mathbf{N}(\mathfrak{F}_u(u_0, a_0)) = \operatorname{codim} \mathbf{R}(\mathfrak{F}_u(u_0, a_0)) = 1.$$

and

$$\mathfrak{F}_{au}(u_0, a_0)[w_0] \notin \mathbf{R}(\mathfrak{F}_u(u_0, a_0)).$$

Here $\mathbf{N}(\mathfrak{F}_u(u_0, a_0)) = \operatorname{span}\{w_0\}$. Let $\mathbf{Z}$ be the complement of $\operatorname{span}\{w_0\}$ in $\mathbf{X}$. Then the solution set of $\mathfrak{F}(u, a) = 0$ near (u_0, a_0) consists precisely of the curves $u = u_0$ and $\Gamma := \{(u(s), a(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $a : I \to \mathbb{R}$, $z : I \to \mathbf{Z}$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s), a(0) = a_0, z(0) = 0$, and

$$a'(0) = -\frac{\langle \ell, \mathfrak{F}_{uu}(u_0, a_0)[w_0, w_0] \rangle}{2\langle \ell, \mathfrak{F}_{au}(u_0, a_0)[w_0] \rangle},$$

where $\ell \in \mathbf{Y}^*$ satisfies $\mathbf{R}(\mathfrak{F}_u(u_0, a_0)) = \{\phi \in \mathbf{Y} : \langle \ell, \phi \rangle = 0\}$. Moreover if in addition, $\mathfrak{F}_u(u, a)$ is a Fredholm operator for all $(u, a) \in U$, then the bifurcation curve Γ is contained in Σ, which is a connected component of $\overline{S}$, *where* $\overline{S} := \{(u,a) \in U : \mathfrak{F}(u,a) = 0, u \neq u_0\}$*; and either* Σ *is not compact in* U*, or* Σ *contains a point* (u_0, a_*) *with* $a_* \neq a_0$*.*

Theorem 8.2. *Assume that all assumptions in Theorem 8.1 are satisfied, and let* $\{u(s), a(s)\}$ *be the solution curve in Theorem 8.1. Then there exists* C^2 *function* $m : (a_0 - \epsilon, a_0 + \epsilon) \to \mathbb{R}$*,* $z : (a_0 - \epsilon, a_0 + \epsilon) \to \mathbf{X}$*,* $\varrho : (-\delta, \delta) \to \mathbb{R}$*, and* $w : (-\delta, \delta) \to \mathbf{X}$ *such that*

$$\mathfrak{F}_u(u_0, a)z(a) = m(a)z(a), \quad a \in (a_0 - \epsilon, a_0 + \epsilon),$$
$$\mathfrak{F}_u(u(s), a(s))w(s) = \varrho(s)w(s), \quad s \in (-\delta, \delta),$$

where $m(a_0) = \varrho(0) = 0$*,* $z(a_0) = w(0) = w_0$*. Moreover, near* $s = 0$ *the functions* $\varrho(s)$ *and* $-sa'(s)m'(a_0)$ *have the same zeros and, whenever* $\varrho(s) \neq 0$*, the same sign. More precisely,*

$$\lim_{s \to 0} \frac{-sa'(s)m'(a_0)}{\varrho(s)} = 1.$$

The principal eigenvalue $\rho_1(q)$ defined in (2.2) has some useful properties as follows (see [31, Proposition A.1] or [53, Proposition 1.1]).

Theorem 8.3. *The following conclusions hold.*

(i) *If* $q_i \in C(\overline{\Omega})$ $(i = 1, 2)$ *satisfy* $q_1 \geq q_2$ *and* $q_1 \not\equiv q_2$*, then* $\rho_1(q_1) > \rho_1(q_2)$*.*
(ii) *For* $q_n \in C(\overline{\Omega})$ *and* $q \in C(\overline{\Omega})$*, let* $\phi_n \in H_0^1(\Omega)$ *and* $\phi \in H_0^1(\Omega)$ *be the corresponding eigenfunctions of* (2.1) *satisfying* $\|\phi_n\|_{L^2(\Omega)} = \|\phi\|_{L^2(\Omega)} = 1$*, where* $n \in \mathbb{N}$*. If* $\lim_{n\to\infty} \|q_n - q\|_{L^\infty(\Omega)} = 0$*, then* $\lim_{n\to\infty} \rho_1(q_n) = \rho_1(q)$ *and* $\lim_{n\to\infty} \phi_n = \phi$ *strongly in* $H_0^1(\Omega)$*.*
(iii) *Let* (k, ℓ) *be an open interval and assume that a mapping* $\alpha \to q_\alpha$ *is continuously differentiable from* (k, ℓ) *to* $C(\overline{\Omega})$ *with respect to supremum norm. If* $\phi_\alpha \in H_0^1(\Omega)$ *with* $\|\phi_\alpha\|_{L^2(\Omega)} = 1$ *is the unique positive eigenfunction corresponding to* $\rho_1(q_\alpha)$*, then* $\alpha \to \rho_1(q_\alpha)$ *is continuously differentiable from* (k, ℓ) *to* $\mathbb{R}$ *and*

$$\frac{d}{d\alpha}\rho_1(q_\alpha) = \int_\Omega \frac{\partial q_\alpha}{\partial \alpha}\phi_\alpha^2 dx.$$

Let p be a sufficiently large constant such that $p - q(x) > 0$ for any $x \in \overline{\Omega}$. Define a bounded linear operator $T : C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ by $u = Tv = (-\Delta + pI)^{-1}(p - q(x))v$, where $u \in C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ is the unique solution of the following problem

$$\begin{cases} -\Delta u + pu = (p - q(x))v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{8.1}$$

Denote $r(T)$ be the spectral radius of T. Then the relationship between $\rho_1(q)$ and $r(T)$ can be given as follows (see [16, Proposition 1] or [36, Lemmas 2.1 and 2.3]).

Theorem 8.4. *Let* $q \in C(\overline{\Omega})$ *and let* p *be a sufficiently large number such that* $p > q(x)$ *for any* $x \in \overline{\Omega}$*. Then we have*

(i) $\rho_1(q) > 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) < 1$;
(ii) $\rho_1(q) < 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) > 1$;

(iii) $\rho_1(q) = 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) = 1$.

Consider the following equation

$$- \Delta u + q(x)u = uf(u),\ x \in \Omega,\ \ u = 0,\ x \in \partial\Omega, \tag{8.2}$$

where $q(x) \in C(\overline{\Omega})$, $f : [0, \infty) \mapsto \mathbb{R}$ is a strictly decreasing function of class C^1 and there exists a positive constant C such that $f(u) < 0$ for $u \in [C, \infty)$. Then the following results hold (see [7]).

Theorem 8.5. *Problem* (8.2) *has no positive solution if* $f(0) \le \rho_1(q)$, *while it has a unique positive solution if* $f(0) > \rho_1(q)$.

In particular, we consider the following steady-state problem for logistic equation with linear diffusion

$$\begin{cases} -\Delta u = u(a - u), & x \in \Omega, \\ u \ge 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{8.3}$$

where a is a positive constant and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$. Then the following results are well known (see [16, Lemma 1] and [24, Propositions 6.1-6.4]).

Theorem 8.6. *Let* $\rho_1 := \rho_1(0)$ *be the constant defined in* (2.2). *Then the following conclusions hold.*

(i) *If* $a \le \rho_1$, *then* (8.3) *has no nontrivial solutions.*

(ii) *If* $a > \rho_1$, *then there exists a unique positive solution* θ_a *of* (8.3) *such that* θ_a *is strictly increasing with respect to* a *and* $0 < \theta_a < a$ *for every* $x \in \Omega$.

(iii) $\lim_{a \to \rho_1^+} \theta_a = 0$ *uniformly in* Ω. *More precisely,*

$$\theta_a = \left(\int_\Omega \phi_1^3 dx \right)^{-1} (a - \rho_1)\phi_1 + o(a - \rho_1) \quad \text{as } a \to \rho_1^+.$$

(iv) $\lim_{a\to\infty} \theta_a = \infty$ *and* $\lim_{a\to\infty} \theta_a/a = 1$ *uniformly in* K, *where* K *is any compact subset of* Ω.

(v) $a \to \theta_a$ *is a* C^1*-mapping from* (ρ_1, ∞) *to* $C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ *and* $\frac{\partial\theta_a}{\partial a} > 0$ *in* Ω. *More precisely,*

$$\frac{\partial \theta_a}{\partial a} = (-\Delta + (2\theta_a - a)I)^{-1}\theta_a,$$

where $(-\Delta + (2\theta_a - a)I)^{-1}$ *denotes the inverse operator of* $-\Delta + (2\theta_a - a)I$ *with zero Dirichlet boundary condition.*

Next, we consider the following semi-linear parabolic equation related to (8.3):

$$\begin{cases} u_t - \Delta u = u(a - bu), & x \in \Omega,\ t > t_0 \ge 0 \\ u = 0, & x \in \partial\Omega\ t > t_0, \\ u(x, t_0) \ge 0, \not\equiv 0, & x \in \Omega, \end{cases} \tag{8.4}$$

where a and b are positive constants. It is well-known that (8.4) has a unique nonnegative solution, which we denoted by $u_a^b(x,t)$. Obviously, $u_a^b(x,t) > 0$ in $\Omega \times (t_0, \infty)$ and the following conclusions hold (see [7]).

Theorem 8.7. *$u_a^b(x)$ converges to 0 uniformly on $\overline{\Omega}$ if $a \leq \rho_1$, while it converges to $\frac{\theta_a}{b}(x)$ uniformly on $\overline{\Omega}$ if $a > \rho_1$.*

Finally we summarize the index theory on a positive cone, which has been developed by Amann [3] and Dancer [15,16] to study positive solutions for nonlinear elliptic equations.

Let E be a real Banach space and let $\mathcal{W}$ be a closed convex set in E. We use the notation following the paper of Dancer [15]. Let y be any element of $\mathcal{W}$ and define $\mathcal{W}_y$ by

$$\mathcal{W}_y := \{x \in E : y + \gamma x \in \mathcal{W} \text{ for some } \gamma > 0\},$$

which is also a convex set. Define

$$S_y = \overline{\mathcal{W}_y} \cap (-\overline{\mathcal{W}_y}) \quad \text{for } y \in \mathcal{W},$$

which is also a closed subspace of E. Assume that $T : E \to E$ is a compact linear operator such that

$$T(\overline{\mathcal{W}_y}) \subset \overline{\mathcal{W}_y}.$$

It is easy to say S_y is invariant under T. This fact implies that T induces a compact linear mapping $\widetilde{T}$ from $E \backslash S_y$ into itself. We denote by $\widetilde{\mathcal{W}_y}$ an image of $\overline{\mathcal{W}_y}$ under the quotient mapping $E \to E \backslash S_y$. Since $T(\overline{\mathcal{W}_y}) \subset \overline{\mathcal{W}_y}$, it follows that $\widetilde{T}\widetilde{\mathcal{W}_y} \subset \widetilde{\mathcal{W}_y}$.

Let $A : \mathcal{W} \to \mathcal{W}$ be a compact and Fréchet differentiable mapping. Denote by $A'(x)$ the Fréchet derivative of A at $x \in \mathcal{W}$. Let $y \in \mathcal{W}$ be any fixed point of A, and assume that $A'(y)$ is compact. By [15, §2,lemma 1], $A'(y)$ maps $\overline{\mathcal{W}_y}$ into itself. Then one can define Leray-Schauder degree $\deg_{\mathcal{W}}(I - A, U, 0)$ for any open subset U in $\mathcal{W}$ if A has no fixed points on ∂U. For each isolated fixed $y \in \mathcal{W}$, $\text{index}_{\mathcal{W}}(A, y)$ means $\deg_{\mathcal{W}}(I - A, N(y), 0)$, where $N(y)$ is a suitable neighborhood of y in $\mathcal{W}$. Moreover, it is known that, if $\deg_{\mathcal{W}}(I - A, 0) \neq 0$, then A has at least one fixed point in U.

We know given an index formula which is essentially due to Dancer [15].

Theorem 8.8. *Let $y \in \mathcal{W}$ be a fixed point of A. If $(I - A'(y))x \neq 0$ for every $x \in \overline{\mathcal{W}_y} \backslash \{0\}$, then*

(i) $\text{index}_{\mathcal{W}}(A, y) = 0$ *if* $r(\widetilde{A'(y)}) > 1$;
(ii) $\text{index}_{\mathcal{W}}(A, y) = 1$ *if* $r(A'(y)) < 1$.

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