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On the Fučik spectrum of non-local elliptic operators

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Abstract. In this article, we study the Fučik spectrum of the fractional Laplace operator which is defined as the set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\left.\begin{array}{ll}(-\Delta)^s u = \alpha u^+ - \beta u^- & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega.\end{array}\right\}$$

has a non-trivial solution u, where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $n > 2s$, $s \in (0, 1)$. The existence of a first nontrivial curve $\mathcal{C}$ of this spectrum, some properties of this curve $\mathcal{C}$, e.g. Lipschitz continuous, strictly decreasing and asymptotic behavior are studied in this article. A variational characterization of second eigenvalue of the fractional eigenvalue problem is also obtained. At the end, we study a nonresonance problem with respect to the Fučik spectrum.

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Keywords. Non-local operator · Fractional Laplacian · Fučik spectrum · Nonresonance.

1. Introduction

The Fučik spectrum of fractional Laplace operator is defined as the set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{cases}(-\Delta)^s u = \alpha u^+ - \beta u^- & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega.\end{cases}$$

has a non-trivial solution u, where $s \in (0, 1)$ and $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \quad \text{for all } x \in \mathbb{R}^n.$$

In general, we study the Fučik spectrum of an equation driven by the non-local operator $\mathcal{L}_K$ which is defined as

$$\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y) dy \quad \text{for all } x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ satisfies the following:

(i) $mK \in L^1(\mathbb{R}^n)$, where $\mathrm{m(x)} = \min\{|x|^2, 1\}$,

(ii) There exist $\lambda > 0$ and $s \in (0,1)$ such that $K(x) \geq \lambda |x|^{-(n+2s)}$,

(iii) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

In case $K(x) = |x|^{-(n+2s)}$, $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$.

The Fučik spectrum of the non-local operator $\mathcal{L}_K$ is defined as the set $\sum_K$ of $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\left.\begin{array}{ll} -\mathcal{L}_K u = \alpha u^+ - \beta u^- & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{array}\right\} \tag{1.1}$$

(1.1) has a nontrivial solution u. Here $u^{\pm} = \max(\pm u, 0)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipshitz boundary. For $\alpha = \beta$, the Fučik spectrum of (1.1) becomes the usual spectrum of

$$\left.\begin{array}{ll} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{array}\right\} \tag{1.2}$$

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the distinct eigenvalues of (1.2). It is proved in [22] that the first eigenvalue λ_1 of (1.2) is simple, isolated and can be characterized as follows

$$\lambda_1 = \inf_{u \in X_0} \left\{ \int_Q (u(x) - u(y))^2 K(x-y) dx dy : \int_\Omega u^2 = 1 \right\}.$$

The author also proved that the eigenfunctions corresponding to λ_1 are of constant sign. We observe that $\sum_K$ clearly contains (λ_k, λ_k) for each $k \in \mathbb{N}$ and two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. $\sum_K$ is symmetric with respect to the diagonal. In this paper we will prove that the two lines $\mathbb{R} \times \lambda_1$ and $\lambda_1 \times \mathbb{R}$ are isolated in $\sum_K$ and give a variational characterization of the second eigenvalue λ_2 of $-\mathcal{L}_K$.

When $s = 1$, the fractional Laplacian operator becomes the usual Laplace operator. The Fučik spectrum was introduced by Fučik (1976) who studied the negative Laplacian in one dimension with periodic boundary conditions, and by Dancer [6]. The Fučik spectrum in the case of Laplacian, p-Laplacian equation with Dirichlet, Neumann and Robin boundary condition has been studied by many authors [2–5,8,9,12,15,16,18–20]. A nonresonance problem with respect to Fučik spectrum is also discussed in many papers [4,14,15]. To the best of our knowledge, no work has been done to find the Fučik spectrum for non-local operators. Recently a lot of attention is given to the study of fractional and non-local operator of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc [21,23–25]. Here we use a similar approach to find the Fučik spectrum as the one used in [4].

In [21], Servadei and Valdinoci discussed the Dirichlet boundary value problem in case of fractional Laplacian using the Variational techniques. We also use similar variational techniques to find $\sum_K$. Due to non-localness of the fractional Laplacian, the space $(X_0, \|.\|_{X_0})$ is introduced by Servadei. We introduce this space as follows:

$$X = \Big\{ u|\ u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^2(\Omega) \textit{ and } (u(x) - u(y)) \\ \times \sqrt{K(x-y)} \in L^2(Q) \Big\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\int_Q |u(x) - u(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}}.$$

Then we define

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}}$$

is a Hilbert space. Note that the norm $\|.\|_{X_0}$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$.

Remark 1.1. (i) *$C_c^2(\Omega) \subseteq X_0$, $X \subseteq H^s(\Omega)$ and $X_0 \subseteq H^s(\mathbb{R}^n)$, where $H^s(\Omega)$ denotes the usual fractional Sobolev space endowed with the norm*

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2} + \left(\int_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

(ii) *The embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n) = L^{2^*}(\Omega)$ is continuous, where $2^* = \frac{2n}{n-2s}$. To see the details of these embeddings, one can refer [10, 21].*

Definition 1.2. A function $u \in X_0$ is a weak solution of (1.1), if for every $v \in X_0$, u satisfies

$$\int_Q (u(x) - u(y))(v(x) - v(y)) K(x-y) dxdy = \alpha \int_\Omega u^+ v dx - \beta \int_\Omega u^- v dx.$$

Weak solutions of (1.1) are exactly the critical points of the functional $J : X_0 \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{2} \int_Q (u(x) - u(y))^2 K(x-y) dxdy - \frac{\alpha}{2} \int_\Omega (u^+)^2 dx - \frac{\beta}{2} \int_\Omega (u^-)^2 dx.$$

Then J is Fréchet differentiable in X_0 and

$$\langle J'(u), \phi \rangle = \int_Q (u(x) - u(y))(\phi(x) - \phi(y)) K(x-y) dxdy \\ -\alpha \int_\Omega u^+ \phi dx - \beta \int_\Omega u^- \phi dx.$$

The paper is organized as follows: In Sect. 2 we construct a first nontrivial curve in $\sum_K$, described as $(p + c(p), c(p))$. In Sect. 3 we prove that the lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ are isolated in $\sum_K$, the curve that we obtained in Sect. 2 is the first nontrivial curve and give the variational characterization of second eigenvalue of $-\mathcal{L}_K$. In Sect. 4 we prove some properties of the first curve. A nonresonance problem with respect to the Fučik spectrum is also studied in Sect. 5.

We shall throughout use the function space X_0 with the norm $\|.\|_{X_0}$ and we use the standard $L^p(\Omega)$ space whose norms are denoted by $\|u\|_{L^p}$. Also ϕ_1 is the eigenfunction of $-\mathcal{L}_K$ corresponding to λ_1.

2. Fučik spectrum $\sum_K$ for $-\mathcal{L}_K$

In this section we study the existence of the first nontrivial curve in the Fučik spectrum $\sum_K$ of $-\mathcal{L}_K$. We find that the points in $\sum_K$ are associated with the critical value of some restricted functional.

For this we fix $p \in \mathbb{R}$ and for $p \geq 0$, consider the functional $J_p : X_0 \to \mathbb{R}$ by

$$J_p(u) = \int_Q (u(x) - u(y))^2 K(x-y)dxdy - p\int_\Omega (u^+)^2 dx.$$

Then $J_p \in C^1(X_0, \mathbb{R})$ and for any $\phi \in X_0$

$$\langle J_p'(u), \phi\rangle = 2\int_Q (u(x) - u(y))(\phi(x) - \phi(y))K(x-y)dxdy - 2p\int_\Omega u^+(x)\phi(x)dx.$$

Also $\tilde{J}_p := J_p|_{\mathcal{P}}$ is $C^1(X_0, \mathbb{R})$, where $\mathcal{P}$ is defined as

$$\mathcal{P} = \left\{u \in X_0 : I(u) := \int_\Omega u^2 dx = 1\right\}.$$

We first note that $u \in \mathcal{P}$ is a critical point of $\tilde{J}_p$ if and only if there exists $t \in \mathbb{R}$ such that

$$\int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy - p\int_\Omega u^+ v dx = t\int_\Omega uv dx, \quad (2.1)$$

for all $v \in X_0$. Hence $u \in \mathcal{P}$ is a nontrivial weak solution of the problem

$$\begin{aligned} -\mathcal{L}_K u &= (p+t)u^+ - tu^- \quad \text{in } \Omega;\\ u &= 0 \quad \text{on } \mathbb{R}^n \backslash \Omega, \end{aligned}$$

which exactly means $(p+t, t) \in \sum_K$. Putting $v = u$ in (2.1), we get $t = \tilde{J}_p(u)$. Thus we have the following result, which describes the relationship between the critical points of $\tilde{J}_p$ and the spectrum $\sum_K$.

Lemma 2.1. *For $p \geq 0$, $(p+t, t) \in \mathbb{R}^2$ belongs to the spectrum $\sum_K$ if and only if there exists a critical point $u \in \mathcal{P}$ of $\tilde{J}_p$ such that $t = \tilde{J}_p(u)$, a critical value.*

Now we look for the minimizers of $\tilde{J}_p$.

Proposition 2.2. *The first eigenfunction ϕ_1 is a global minimum for $\tilde{J}_p$ with $\tilde{J}_p(\phi_1) = \lambda_1 - p$. The corresponding point in $\sum_K$ is $(\lambda_1, \lambda_1 - p)$ which lies on the vertical line through (λ_1, λ_1).*

Proof. It is easy to see that $\tilde{J}_p(\phi_1) = \lambda_1 - p$ and

$$\begin{aligned}\tilde{J}_p(u) =& \int_Q (u(x) - u(y))^2 K(x-y)dxdy - p\int_\Omega (u^+)^2 dx\\ \geq& \lambda_1 \int_\Omega u^2 dx - p\int_\Omega (u^+)^2 dx \geq \lambda_1 - p.\end{aligned}$$

Thus ϕ_1 is a global minimum of $\tilde{J}_p$ with $\tilde{J}_p(\phi_1) = \lambda_1 - p$. □

Now we have a second critical point of $\tilde{J}_p$ at $-\phi_1$ corresponding to a strict local minimum.

Proposition 2.3. *The negative eigenfunction $-\phi_1$ is a strict local minimum for $\tilde{J}_p$ with $\tilde{J}_p(-\phi_1) = \lambda_1$. The corresponding point in $\sum_K$ is $(\lambda_1 + p, \lambda_1)$, which lies on the horizontal line through (λ_1, λ_1).*

Proof. Let us suppose by contradiction that there exists a sequence $u_k \in \mathcal{P}$, $u_k \neq -\phi_1$ with $\tilde{J}_p(u_k) \leq \lambda_1$, $u_k \to -\phi_1$ in X_0. Firstly, we show that u_k changes sign for sufficiently large k. Since $u_k \neq -\phi_1$, it must be ≤ 0 for some $x \in X_0$. If $u_k \leq 0$ for a.e $x \in \Omega$, then

$$\tilde{J}_p(u_k) = \int_Q (u_k(x) - u_k(y))^2 K(x-y)dxdy > \lambda_1,$$

since $u_k \neq \pm\phi_1$ and we get contradiction as $\tilde{J}_p(u_k) \leq \lambda_1$. So u_k changes sign for sufficiently large k. Define $w_k := \frac{u_k^+}{\|u_k^+\|_{L^2}}$ and

$$r_k := \int_Q (w_k(x) - w_k(y))^2 K(x-y)dxdy.$$

Now we claim that $r_k \to \infty$. Let us suppose by contradiction that r_k is bounded. Then there exists a subsequence of w_k still denoted by w_k and $w \in X_0$ such that $w_k \rightharpoonup w$ weakly in X_0 and $w_k \to w$ strongly in $L^2(\mathbb{R}^n)$. Therefore $\int_\Omega w^2 dx = 1$, $w \geq 0$ a.e. and so for some $\epsilon > 0$, $\delta = |\{x \in X_0 : w(x) \geq \epsilon\}| > 0$. As $u_k \to -\phi_1$ in X_0 and hence in $L^2(\Omega)$. Therefore $|\{x \in \Omega : u_k(x) \geq \epsilon\}| \to 0$ as $k \to \infty$ and so $|\{x \in \Omega : w_k(x) \geq \epsilon\}| \to 0$ as $k \to \infty$ which is a contradiction as $\delta > 0$. Hence the claim. Next,

$$\begin{aligned}(u_k(x) - u_k(y))^2 &= ((u_k^+(x) - u_k^+(y)) - (u_k^-(x) - u_k^-(y)))^2\\ &= (u_k^+(x) - u_k^+(y))^2 + (u_k^-(x) - u_k^-(y))^2 - 2(u_k^+(x) - u_k^+(y))(u_k^-(x) - u_k^-(y))\\ &= (u_k^+(x) - u_k^+(y))^2 + (u_k^-(x) - u_k^-(y))^2 + 2u_k^+(x)u_k^-(y) + 2u_k^-(x)u_k^+(y),\end{aligned}$$

where we have used $u_k^+(x)u_k^-(x) = 0$. Using $K(x) = K(-x)$ we have

$$\int_Q u_k^+(x)u_k^-(y)K(x-y)dxdy = \int_Q u_k^+(y)u_k^-(x)K(x-y)dxdy. \tag{2.2}$$

Then from above estimates, we get

$$\begin{aligned}\tilde{J}_p(u_k) &= \int_Q (u_k(x)-u_k(y))^2K(x-y)dxdy - p\int_\Omega (u_k^+)^2dx\\ &= \int_Q (u_k^+(x)-u_k^+(y))^2K(x-y)dxdy + \int_Q (u_k^-(x)-u_k^-(y))^2K(x-y)dxdy\\ &\quad + 4\int_Q u_k^+(x)u_k^-(y)K(x-y)dxdy - p\int_\Omega (u_k^+)^2dx\\ &\geq (r_k-p)\int_\Omega (u_k^+)^2dx + \lambda_1\int_\Omega (u_k^-)^2dx + 4\int_Q u_k^+(x)u_k^-(y)K(x-y)dxdy\\ &\geq (r_k-p)\int_\Omega (u_k^+)^2dx + \lambda_1\int_\Omega (u_k^-)^2dx.\end{aligned}$$

As $u_k \in \mathcal{P}$, we get

$$\tilde{J}_p(u_k) \leq \lambda_1 = \lambda_1\int_\Omega (u_k^+)^2dx + \lambda_1\int_\Omega (u_k^-)^2dx.$$

Combining both the inequalities we have,

$$(r_k-p)\int_\Omega (u_k^+)^2dx + \lambda_1\int_\Omega (u_k^-)^2dx \leq \lambda_1\int_\Omega (u_k^+)^2dx + \lambda_1\int_\Omega (u_k^-)^2dx.$$

This implies $(r_k - p - \lambda_1)\int_\Omega (u_k^+)^2dx \leq 0$, and hence $r_k - p \leq \lambda_1$, which contradicts the fact that $r_k \to +\infty$, as required. □

We will now find the third critical point based on the mountain pass theorem by Ambrosetti–Robinowitz. A norm of derivative of the restriction $\tilde{J}_p$ of J_p at $u \in \mathcal{P}$ is defined as

$$\|\tilde{J}_p(u)\|_* = \min\{\|J_p'(u) - tI'(u)\|_{X_0} : t \in \mathbb{R}\}.$$

Definition 2.4. We say that J_p satisfies the Palais–Smale [in short, (P.S)] condition on $\mathcal{P}$ if for any sequence $u_k \in \mathcal{P}$ such that $J_p(u_k)$ is bounded and $\|\tilde{J}_p'(u_k)\|_* \to 0$, then there exists a subsequence that converges strongly in X_0.

Now we state here the version of mountain pass theorem which will be used later:

Proposition 2.5. [1] *Let E be a Banach space, $g, f \in C^1(E,\mathbb{R})$, $M = \{u \in E \mid g(u) = 1\}$ and u_0, $u_1 \in M$. Let $\epsilon > 0$ such that $\|u_1 - u_0\| > \epsilon$ and*

$$\inf\{f(u) : u \in M \text{ and } \|u-u_0\|_E = \epsilon\} > \max\{f(u_0), f(u_1)\}.$$

Assume that f satisfies the $(P.S)$ condition on M and that

$$\Gamma = \{\gamma \in C([-1,1], M) : \gamma(-1) = u_0 \text{ and } \gamma(1) = u_1\}$$

is non empty. Then $c = \inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]} f(u)$ is a critical value of $f|_M$.

Lemma 2.6. *J_p satisfies the $(P.S)$ condition on $\mathcal{P}$.*

Proof. Let $\{u_k\}$ be a $(P.S)$ sequence. i.e., there exists $K > 0$ and t_k such that

$$|J_p(u_k)| \le K, \tag{2.3}$$

$$\int_Q (u_k(x) - u_k(y))(v(x) - v(y))K(x-y)dxdy - p\int_\Omega u_k^+ v \\ - t_k \int_\Omega u_k v = o_k(1)\|v\|_{X_0}. \tag{2.4}$$

From (2.3), we get u_k is bounded in X_0. So we may assume that up to a subsequence $u_k \rightharpoonup u_0$ weakly in X_0, and $u_k \to u_0$ strongly in $L^2(\Omega)$. Putting $v = u_k$ in (2.4), we get t_k is bounded and up to a subsequence t_k converges to t. We now claim that $u_k \to u_0$ strongly in X_0. As $u_k \rightharpoonup u_0$ weakly in X_0, we have

$$\int_Q (u_k(x) - u_k(y))(v(x) - v(y))K(x-y)dxdy \\ \to \int_Q (u_0(x) - u_0(y))(v(x) - v(y))K(x-y)dxdy \tag{2.5}$$

for all $v \in X_0$. Also $\tilde{J}_p'(u_k)(u_k - u_0) = o_k(1)$. Therefore we get

$$\left|\int_Q (u_k(x) - u_k(y))^2 \ K(x-y)dxdy - \int_Q (u_k(x) - u_k(y))(u_0(x) - u_0(y))K(x-y)dxdy\right| \\ \le o_k(1) + p\|u_k^+\|_{L^2}\|u_k - u_0\|_{L^2} + |t_k|\|u_k\|_{L^2}\|u_k - u_0\|_{L^2} \\ \to 0 \quad \text{as } k \to \infty.$$

Taking $v = u_0$ in (2.5), we get

$$\int_Q (u_k(x) - u_k(y))(u_0(x) - u_0(y))K(x-y)dxdy \\ \to \int_Q (u_0(x) - u_0(y))^2 K(x-y)dxdy.$$

From above two equations, we have

$$\int_Q (u_k(x) - u_k(y))^2 K(x-y)dxdy \to \int_Q (u_0(x) - u_0(y))^2 K(x-y)dxdy.$$

Thus $\|u_k\|_{X_0}^2 \to \|u_0\|_{X_0}^2$. Now using this and $v = u_0$ in (2.5), we get

$$\|u_k - u_0\|_{X_0}^2 = \|u_k\|_{X_0}^2 + \|u_k\|_{X_0}^2 - 2\int_Q (u_k(x) - u_k(y))(u_0(x) - u_0(y))K(x-y)dxdy \\ \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence $u_k \to u_0$ strongly in X_0. □

Lemma 2.7. *Let $\epsilon_0 > 0$ be such that*

$$\tilde{J}_p(u) > \tilde{J}_p(-\phi_1) \tag{2.6}$$

for all $u \in B(-\phi_1, \epsilon_0) \cap \mathcal{P}$ with $u \ne -\phi_1$, where the ball is taken in X_0. Then for any $0 < \epsilon < \epsilon_0$,

$$\inf\{\tilde{J}_p(u) : u \in \mathcal{P} \text{ and } \|u - (-\phi_1)\|_{X_0} = \epsilon\} > \tilde{J}_p(-\phi_1). \tag{2.7}$$

Proof. Assume by contradiction that the infimum in (2.7) is equal to $\tilde{J}_p(-\phi_1) = \lambda_1$ for some ϵ with $0 < \epsilon < \epsilon_0$. Then there exists a sequence $u_k \in \mathcal{P}$ with $\|u_k - (-\phi_1)\|_{X_0} = \epsilon$ such that

$$\tilde{J}_p(u_k) \leq \lambda_1 + \frac{1}{2k^2}.$$

Consider the set $C = \{u \in \mathcal{P} : \epsilon - \delta \leq \|u - (-\phi_1)\|_{X_0} \leq \epsilon + \delta\}$, where δ is chosen such that $\epsilon - \delta > 0$ and $\epsilon + \delta < \epsilon_0$. In view of our contradiction hypothesis and (2.6), it follows that $\inf\{\tilde{J}_p(u) : u \in C\} = \lambda_1$. Now for each k, we apply Ekeland's variational principle to the functional $\tilde{J}_p$ on C to get the existence of $v_k \in C$ such that

$$\begin{aligned} &\tilde{J}_p(v_k) \leq \tilde{J}_p(u_k), \quad \|v_k - u_k\|_{X_0} \leq \frac{1}{k}, \\ &\tilde{J}_p(v_k) \leq \tilde{J}_p(u) + \frac{1}{k}\|v - v_k\|_{X_0} \ \forall\, v \in C \end{aligned} \tag{2.8}$$

We claim that v_k is a (P.S) sequence for $\tilde{J}_p$ on $\mathcal{P}$ i.e. $\tilde{J}_p(v_k)$ is bounded and $\|\tilde{J}_p'(v_k)\|_* \to 0$. Once this is proved we get by Lemma 2.6, up to a subsequence $v_k \to v$ strongly in X_0. Clearly $v \in \mathcal{P}$ and satisfies $\|v - (-\phi_1)\|_{X_0} \leq \epsilon + \delta < \epsilon_0$ and $\tilde{J}_p(v) = \lambda_1$ which contradicts the given hypotheses. Clearly $\tilde{J}_p(v_k)$ is a bounded. So we only need to prove that $\|\tilde{J}_p'(v_k)\|_* \to 0$. Let $k > \frac{1}{\delta}$ and take $w \in X_0$ tangent to $\mathcal{P}$ at v_k i.e

$$\int_\Omega v_k w dx = 0,$$

and for $t \in \mathbb{R}$, define

$$u_t := \frac{v_k + tw}{\|v_k + tw\|_{L^2}}.$$

For sufficiently small t, $u_t \in \mathcal{C}$ and take $v = u_t$ in (2.8), we get

$$|\langle J_p'(v_k), w\rangle| \leq \frac{1}{k}\|w\|_{X_0}. \tag{2.9}$$

For complete details refer to Lemma 2.9 of [4].

Since w is arbitrary in X_0, we choose α_k such that $\int_\Omega v_k(w - \alpha_k v_k)dx = 0$. Replacing w by $w - \alpha_k v_k$ in (2.9), we have

$$\left|\langle J_p'(v_k), w\rangle - \alpha_k\langle J_p'(v_k), v_k\rangle\right| \leq \frac{1}{k}\|w - \alpha_k v_k\|_{X_0},$$

since $\|\alpha_k v_k\|_{X_0} \leq C\|w\|_{X_0}$, we get

$$\left|\langle J_p'(v_k), w\rangle - t_k \int_\Omega v_k w dx\right| \leq \frac{C}{k}\|w\|_{X_0}$$

where $t_k = \langle J_p'(v_k), v_k\rangle$. Hence

$$\|\tilde{J}_p'(v_k)\|_* \to 0 \quad \text{as } k \to \infty,$$

as we required. $\square$

Proposition 2.8. *Let $\epsilon > 0$ such that $\|\phi_1 - (-\phi_1)\|_{X_0} > \epsilon$ and*

$$\inf\{\tilde{J}_p(u) : u \in \mathcal{P} \text{ and } \|u - (-\phi_1)\|_{X_0} = \epsilon\} > \max\{\tilde{J}_p(-\phi_1), \tilde{J}_p(\phi_1)\}.$$

Then $\Gamma = \{\gamma \in C([-1,1], \mathcal{P}) : \gamma(-1) = -\phi_1$ and $\gamma(1) = \phi_1\}$ is non empty and

$$c(p) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} J_p(u) \tag{2.10}$$

is a critical value of $\tilde{J}_p$. Moreover $c(p) > \lambda_1$.

Proof. Let $\phi \in X_0$ be such that $\phi \notin \mathbb{R}\phi_1$ and consider the path $\gamma(t) = \frac{t\phi_1+(1-|t|)\phi}{\|t\phi_1+(1-|t|)\phi\|_{L^2}}$, then $\gamma(t) \in \Gamma$. Moreover by Lemmas 2.6 and 2.7, $\tilde{J}_p$ satisfies (P.S) condition and the geometric assumptions. Then by Proposition 2.5, $c(p)$ is a critical value of $\tilde{J}_p$. Using the definition of $c(p)$ we have $c(p) > \max\{\tilde{J}_p(-\phi_1), \tilde{J}_p(\phi_1)\} = \lambda_1$. □

Thus we have proved the following:

Theorem 2.9. *For each $p \geq 0$, the point $(p + c(p), c(p))$, where $c(p) > \lambda_1$ is defined by the minimax formula (2.10), then the point $(p + c(p), c(p))$ belongs to $\sum_K$.*

It is a trivial fact that $\sum_K$ is symmetric with respect to the diagonal. The whole curve, which we obtain using Theorem 2.9 and symmetrizing, is denoted by

$$\mathcal{C} := \{(p + c(p), c(p)), (c(p), p + c(p)) : p \geq 0\}.$$

3. First nontrivial curve

We start this section by establishing that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in $\sum_K$. Then we state some topological properties of the functional $\tilde{J}_p$ and finally we prove that the curve $\mathcal{C}$ constructed in the previous section is the first nontrivial curve in the spectrum $\sum_K$.

Proposition 3.1. *The lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ are isolated in $\sum_K$. In other words, there exists no sequence $(\alpha_k, \beta_k) \in \sum_K$ with $\alpha_k > \lambda_1$ and $\beta_k > \lambda_1$ such that $(\alpha_k, \beta_k) \to (\alpha, \beta)$ with $\alpha = \lambda_1$ or $\beta = \lambda_1$.*

Proof. Suppose by contradiction that there exists a sequence $(\alpha_k, \beta_k) \in \sum_K$ with α_k, $\beta_k > \lambda_1$ and $(\alpha_k, \beta_k) \to (\alpha, \beta)$ with α or $\beta = \lambda_1$. Let $u_k \in X_0$ be a solution of

$$-\mathcal{L}_K u_k = \alpha_k u_k^+ - \beta_k u_k^- \text{ in } \Omega, \quad u_k = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega \tag{3.1}$$

with $\|u_k\|_{L^2} = 1$. Then we have

$$\int_Q (u_k(x) - u_k(y))^2 K(x - y)dxdy = \alpha_k \int_\Omega (u_k^+)^2 dx - \beta_k \int_\Omega (u_k^-)^2 dx \leq \alpha_k,$$

which shows that u_k is bounded sequence in X_0. Therefore up to a subsequence $u_k \rightharpoonup u$ weakly in X_0 and $u_k \to u$ strongly in $L^2(\Omega)$. Then the limit u satisfies

$$\int_Q (u(x)-u(y))^2 K(x-y)dxdy = \lambda_1 \int_\Omega (u^+)^2 dx - \beta \int_\Omega (u^-)^2 dx,$$

since $u_k \rightharpoonup u$ weakly in X_0 and $\langle \tilde{J}_p'(u_k), u_k - u\rangle \to 0$ as $k \to \infty$. i.e u is a weak solution of

$$-\mathcal{L}_K u = \alpha u^+ - \beta u^- \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega \tag{3.2}$$

where we have considered the case $\alpha = \lambda_1$. Multiplying by u^+ in (3.2), integrate, using

$$(u(x)-u(y))(u^+(x)-u^+(y)) = (u^+(x)-u^+(y))^2 + u^+(x)u^-(y) + u^+(y)u^-(x)$$

and (2.2), we get

$$\int_Q (u^+(x)-u^+(y))^2 K(x-y)dxdy + 2\int_Q u^+(x)u^-(y)K(x-y)dxdy$$
$$= \lambda_1 \int_\Omega (u^+)^2 dx.$$

Using this we have,

$$\lambda_1 \int_\Omega (u^+)^2 dx \le \int_Q (u^+(x)-u^+(y))^2 K(x-y)dxdy \le \lambda_1 \int_\Omega (u^+)^2 dx$$

Thus

$$\int_Q (u^+(x)-u^+(y))^2 K(x-y)dxdy = \lambda_1 \int_\Omega (u^+)^2 dx,$$

so that either $u^+ \equiv 0$ or $u = \phi_1$. If $u^+ \equiv 0$ then $u \le 0$ and (3.2) implies that u is an eigenfunction with $u \le 0$ so that $u = -\phi_1$. So in any case u_k converges to either ϕ_1 or $-\phi_1$ in $L^2(\Omega)$. Thus for every $\epsilon > 0$

$$\text{either } |\{x \in \Omega : u_k(x) \le \epsilon\}| \to 0 \ \text{ or } |\{x \in \Omega : u_k(x) \ge \epsilon\}| \to 0. \tag{3.3}$$

On the other hand, taking u_k^+ as test function in (3.1), we get

$$\int_Q (u_k^+(x)-u_k^+(y))^2 K(x-y)dxdy + 2\int_Q u_k^+(x)u_k^-(y)K(x-y)dxdy$$
$$= \alpha_k \int_\Omega (u_k^+)^2 dx.$$

Using this, Hölders inequality and Sobolev embeddings we get

$$\int_Q (u_k^+(x)-u_k^+(y))^2 K(x-y)dxdy$$
$$\le \int_Q (u_k^+(x)-u_k^+(y))^2 K(x-y)dxdy + 2\int_Q u_k^+(x)u_k^-(y)K(x-y)dxdy$$
$$= \alpha_k \int_\Omega (u_k^+)^2 dx$$
$$\le \alpha_k C |\{x \in \Omega : u_k(x) > 0\}|^{1-\frac{2}{q}} \|u_k^+\|_{X_0}^2$$

with a constant $C > 0$, $2 < q \leq 2^* = \frac{2n}{n-2s}$. Then we have

$$|\{x \in \Omega : u_k(x) > 0\}|^{1-\frac{2}{q}} \geq \alpha_k^{-1} C^{-1}.$$

Similarly, one can show that

$$|\{x \in \Omega : u_k(x) < 0\}|^{1-\frac{2}{q}} \geq \beta_k^{-1} C^{-1}.$$

Since (α_k, β_k) does not belong to the trivial lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ of $\sum_K$, by using (3.1) we have that u_k changes sign. Hence, from the above inequalities, we get a contradiction with (3.3). Hence the trivial lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ are isolated in $\sum_K$. □

Lemma 3.2. [4] *Let* $\mathcal{P} = \{u \in X_0 : \int_\Omega u^2 = 1\}$ *then*

1. $\mathcal{P}$ *is locally arcwise connected.*
2. *Any open connected subset* $\mathcal{O}$ *of* $\mathcal{P}$ *is arcwise connected.*
3. *If* $\mathcal{O}^{'}$ *is any connected component of an open set* $\mathcal{O} \subset \mathcal{P}$, *then* $\partial\mathcal{O}' \cap \mathcal{O} = \emptyset$.

Lemma 3.3. *Let* $\mathcal{O} = \{u \in \mathcal{P} : \tilde{J}_p(u) < r\}$, *then any connected component of* $\mathcal{O}$ *contains a critical point of* $\tilde{J}_p$.

Proof. Proof follows in the same lines as Lemma 3.6 of [4] by replacing $\|.\|_{1,p}$ by $\|.\|_{X_0}$. □

Theorem 3.4. *Let* $p \geq 0$ *then the point* $(p + c(p), c(p))$ *is the first nontrivial point in the intersection between* $\sum_K$ *and the line* $(p, 0) + t(1, 1)$.

Proof. Assume by contradiction that there exists μ such that $\lambda_1 < \mu < c(p)$ and $(p+\mu, \mu) \in \sum_K$. Using the fact that $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in $\sum_K$ and $\sum_K$ is closed we can choose such a point with μ minimum. In other words $\tilde{J}_p$ has a critical value μ with $\lambda_1 < \mu < c(p)$, but there is no critical value in the open interval (λ_1, μ). If we construct a path connecting from $-\phi_1$ to ϕ_1 such that $\tilde{J}_p \leq \mu$, then we get a contradiction with the definition of $c(p)$, which completes the proof.

Let $u \in \mathcal{P}$ be a critical point of $\tilde{J}_p$ at level μ. Then u satisfies,

$$\int_Q (u(x) - u(y))(v(x) - v(y))K(x - y)dxdy = (p + \mu)\int_\Omega u^+ v dx - \mu\int_\Omega u^- v dx.$$

for all $v \in X_0$. Replacing v by u^+ and u^-, we have

$$\int_Q (u^+(x) - u^+(y))^2 K(x - y)dxdy + 2\int_Q u^+(x)u^-(y)K(x - y)dxdy$$
$$= (p + \mu)\int_\Omega (u^+)^2 dx,$$

and

$$\int_Q (u^-(x) - u^-(y))^2 K(x - y)dxdy + 2\int_Q u^+(x)u^-(y)K(x - y)dxdy$$
$$= \mu\int_\Omega (u^-)^2 dx.$$

Thus we obtain,

$$\tilde{J}_p(u) = \mu, \quad \tilde{J}_p\left(\frac{u^+}{\|u^+\|_{L^2}}\right) = \mu - \frac{2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|u^+\|_{L^2}^2},$$

$$\tilde{J}_p\left(\frac{u^-}{\|u^-\|_{L^2}}\right) = \mu - p - \frac{2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|u^-\|_{L^2}^2},$$

and

$$\tilde{J}_p\left(-\frac{u^-}{\|u^-\|_{L^2}}\right) = \mu - \frac{2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|u^-\|_{L^2}^2}.$$

Since u changes sign, the following paths are well-defined on $\mathcal{P}$:

$$u_1(t) = \frac{(1-t)u + tu^+}{\|(1-t)u + tu^+\|_{L^2}}, \quad u_2(t) = \frac{tu^- + (1-t)u^+}{\|tu^- + (1-t)u^+\|_{L^2}},$$

$$u_3(t) = \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_{L^2}}.$$

Then by using the above calculation one can easily get that for all $t \in [0,1]$,

$$\begin{aligned}\tilde{J}_p(u_1(t)) &= \frac{\int_Q [(u^+(x) - u^+(y))^2 + (1-t)^2(u^-(x) - u^-(y))^2]K(x-y)dxdy}{\|u^+ - (1-t)u^-\|_{L^2}^2}\\ &\quad + \frac{4(1-t)\int_Q u^-(x)u^-(y)K(x-y)dxdy - p\int_\Omega (u^+)^2dx}{\|u^+ - (1-t)u^-\|_{L^2}^2}\\ &= \mu - \frac{2t^2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|u^+ - (1-t)u^-\|_{L^2}^2}.\end{aligned}$$

$$\begin{aligned}\tilde{J}_p(u_2(t)) &= \frac{\int_Q [(1-t)^2(u^+(x) - u^+(y))^2 + t^2(u^-(x) - u^-(y))^2]K(x-y)dxdy}{\|(1-t)u^+ + tu^-\|_{L^2}^2}\\ &\quad - \frac{4t(1-t)\int_Q u^+(x)u^-(y)K(x-y)dxdy + p(1-t)^2\int_\Omega (u^+)^2 + pt^2\int_\Omega (u^-)^2}{\|(1-t)u^+ + tu^-\|_{L^2}^2}\\ &= \mu - \frac{2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|(1-t)u^+ + tu^-\|_{L^2}^2} - \frac{pt^2\int_\Omega (u^-)^2dx}{\|(1-t)u^+ + tu^-\|_{L^2}^2}.\end{aligned}$$

$$\begin{aligned}\tilde{J}_p(u_3(t)) &= \frac{\int_Q [(1-t)^2(u^+(x) - u^+(y))^2 + (u^-(x) - u^-(y))^2]K(x-y)dxdy}{\|(1-t)u^+ - u^-\|_{L^2}^2}\\ &\quad + \frac{4(1-t)\int_Q u^+(x)u^-(y)K(x-y)dxdy - p(1-t)^2\int_\Omega (u^+)^2dx}{\|(1-t)u^+ - u^-\|_{L^2}^2}\\ &= \mu - \frac{2t^2\int_Q u^+(x)u^-(y)K(x-y)dxdy}{\|(1-t)u^+ - u^-\|_{L^2}^2}.\end{aligned}$$

Let $\mathcal{O} = \{v \in \mathcal{P} : \tilde{J}_p(v) < \mu - p\}$. Then clearly $\phi_1 \in \mathcal{O}$, while $-\phi_1 \in \mathcal{O}$ if $\mu - p > \lambda_1$. Moreover ϕ_1 and $-\phi_1$ are the only possible critical points of $\tilde{J}_p$ in $\mathcal{O}$ because of the choice of μ.

We note that $\tilde{J}_p\left(\frac{u^-}{\|u^-\|_{L^2}}\right) \leq \mu - p$, $\frac{u^-}{\|u^-\|_{L^2}}$ does not change sign and vanishes on a set of positive measure, it is not a critical point of $\tilde{J}_p$. Therefore there exists a C^1 path $\eta : [-\epsilon, \epsilon] \to \mathcal{P}$ with $\eta(0) = \frac{u^-}{\|u^-\|_{L^2}}$ and $\frac{d}{dt}\tilde{J}_p(\eta(t))|_{t=0} \neq 0$. Using this path we can move from $\frac{u^-}{\|u^-\|_{L^2}}$ to a point v with $\tilde{J}_p(v) < \mu - p$. Taking a connected component of $\mathcal{O}$ containing v and applying Lemma 3.3 we have that either ϕ_1 or $-\phi_1$ is in this component. Let us assume that it is ϕ_1. So we continue by a path $u_4(t)$ from $\left(\frac{u^-}{\|u^-\|_{L^2}}\right)$ to ϕ_1 which is at level less than μ. Then the path $-u_4(t)$ connects $\left(-\frac{u^-}{\|u^-\|_{L^2}}\right)$ to $-\phi_1$. We observe that

$$|\tilde{J}_p(u) - \tilde{J}_p(-u)| \leq p.$$

Then it follows that

$$\tilde{J}_p(-u_4(t)) \leq \tilde{J}_p(u_4(t)) + p \leq \mu - p + p = \mu \,\forall\, t.$$

Connecting $u_1(t)$, $u_2(t)$ and $u_4(t)$, we get a path from u to ϕ_1 and joining $u_3(t)$ and $-u_4(t)$ we get a path from u to $-\phi_1$. These yields a path $\gamma(t)$ on $\mathcal{P}$ joining from $-\phi_1$ to ϕ_1 such that $\tilde{J}_p(\gamma(t)) \leq \mu$ for all t, which concludes the proof. □

Corollary 3.5. *The second eigenvalue λ_2 of* (1.2) *has the variational characterization given as*

$$\lambda_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \int_Q (u(x) - u(y))^2 K(x-y)dxdy,$$

where Γ is as in Proposition 2.8.

Proof. Take $p = 0$ in Theorem 3.4. Then we have $c(0) = \lambda_2$ and (2.10) concludes the proof. □

4. Properties of the curve

In this section we prove that the curve $\mathcal{C}$ is Lipschitz continuous, has a certain asymptotic behavior and is strictly decreasing.

Proposition 4.1. *The curve $p \to (p+c(p), c(p))$, $p \in \mathbb{R}^+$ is Lipschitz continuous.*

Proof. Proof follows as in Proposition 4.1 of [4]. For completeness we give details. Let $p_1 < p_2$ then $\tilde{J}_{p_1}(u) > \tilde{J}_{p_2}(u)$ for all $u \in \mathcal{P}$. So we have $c(p_1) \geq c(p_2)$. Now for every $\epsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_{p_2}(u) \leq c(p_2) + \epsilon,$$

and so

$$0 \leq c(p_1) - c(p_2) \leq \max_{u\in\gamma[-1,1]} \tilde{J}_{p_1}(u) - \max_{u\in\gamma[-1,1]} \tilde{J}_{p_2}(u) + \epsilon.$$

Let $u_0 \in \gamma[-1,1]$ such that

$$\max_{u\in\gamma[-1,1]} \tilde{J}_{p_1}(u) = \tilde{J}_{p_1}(u_0)$$

then

$$0 \le c(p_1) - c(p_2) \le \tilde{J}_{p_1}(u_0) - \tilde{J}_{p_2}(u_0) + \epsilon \le p_2 - p_1 + \epsilon,$$

as $\epsilon > 0$ is arbitrary so the curve $\mathcal{C}$ is Lipschitz continuous with constant ≤ 1. □

Lemma 4.2. *Let A, B be two bounded open sets in $\mathbb{R}^n$, with $A \subsetneq B$ and B is connected then $\lambda_1(A) > \lambda_1(B)$.*

Proof. From the Proposition 4 of [25] and Theorem 2 of [24], we see that ϕ_1 is continuous and is a solution of (1.2) in viscosity sense. Then from Lemma 12 of [13], $\phi_1 > 0$. Now from the variational characterization, we see that for $A \subset B$, $\lambda_1(A) \ge \lambda_1(B)$. Since $\phi_1(B) > 0$ in B, we get the strict inequality as claimed. □

Lemma 4.3. *Let $(\alpha, \beta) \in \mathcal{C}$, and let $\alpha(x)$, $\beta(x) \in L^\infty(\Omega)$ satisfying*

$$\lambda_1 \le \alpha(x) \le \alpha, \quad \lambda_1 \le \beta(x) \le \beta. \tag{4.1}$$

Assume that

$$\lambda_1 < \alpha(x) \text{ and } \lambda_1 < \beta(x) \quad \text{on subsets of positive measure.} \tag{4.2}$$

Then any non-trivial solution u of

$$-\mathcal{L}_K u = \alpha(x)u^+ - \beta(x)u^- \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \backslash \Omega. \tag{4.3}$$

changes sign in Ω and

$$\alpha(x) = \alpha \text{ a.e. on } \{x \in \Omega : u(x) > 0\}, \quad \beta(x) = \beta \text{ a.e. on } \{x \in \Omega : u(x) < 0\}.$$

Proof. Let u be a nontrivial solution of (4.3). Replacing u by $-u$ if necessary. we can assume that the point (α, β) in $\mathcal{C}$ is such that $\alpha \ge \beta$. We first claim that u changes sign in Ω. Suppose by contradiction that this is not true, we first consider the case $u \ge 0$ (the case $u \le 0$ can be prove similarly). Then u solves

$$-\mathcal{L}_k u = \alpha(x)u \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega.$$

This implies that the first eigenvalue of $-\mathcal{L}_K$ on X_0 with respect to weight $\alpha(x)$ is equal to 1. i.e

$$\inf\left\{\frac{\int_Q (v(x) - v(y))^2 K(x - y)dxdy}{\int_\Omega \alpha(x)v^2 dx} : v \in X_0, v \ne 0\right\} = 1. \tag{4.4}$$

We deduce from (4.1), (4.2) and (4.4) that

$$1 = \frac{\int_Q (\phi_1(x) - \phi_1(y))^2 K(x - y)dxdy}{\lambda_1} > \frac{\int_Q (\phi_1(x) - \phi_1(y))^2 K(x - y)dxdy}{\int_\Omega \alpha(x)\phi_1^2 dx} \ge 1,$$

a contradiction and hence the claim.

Again we assume by contradiction that either

$$|\{x \in X_0 : \alpha(x) < \alpha \text{ and } u(x) > 0\}| > 0 \tag{4.5}$$

or

$$|\{x \in X_0 : \beta(x) < \beta \text{ and } u(x) < 0\}| > 0. \tag{4.6}$$

Suppose that (4.5) holds [a similar argument will hold for (4.6)]. Put $\alpha - \beta = p \geq 0$. Then $\beta = c(p)$, where $c(p)$ is given by (2.10). We show that there exists a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \tilde{J}_p(u) < \beta, \tag{4.7}$$

which yields a contradiction with the definition of $c(p)$.

In order to construct γ we show that there exists a function $v \in X_0$ such that it changes sign and satisfies

$$\frac{\int_Q (v^+(x) - v^+(y))^2 K(x-y)dxdy}{\int_\Omega (v^+)^2 dx} < \alpha \quad \text{and} \quad \frac{\int_Q (v^-(x) - v^-(y))^2 K(x-y)dxdy}{\int_\Omega (v^-)^2 dx} < \beta. \tag{4.8}$$

For this let $\mathcal{O}_1$ be a component of $\{x \in \Omega : u(x) > 0\}$ satisfying

$$|x \in \mathcal{O}_1 : \alpha(x) < \alpha| > 0,$$

which is possible by (4.5). Let $\mathcal{O}_2$ be a component of $\{x \in \Omega : u(x) < 0\}$ satisfying

$$|x \in \mathcal{O}_1 : \beta(x) < \beta| > 0,$$

which is possible by (4.6). Then we claim that

$$\lambda_1(\mathcal{O}_1) < \alpha \quad \text{and} \quad \lambda_1(\mathcal{O}_2) \leq \beta, \tag{4.9}$$

where $\lambda_1(\mathcal{O}_i)$ denotes the first eigenvalue of $-\mathcal{L}_k$ on $X_0|_{\mathcal{O}_i} = \{u \in X|_{\mathcal{O}_i} : u = 0 \text{ on } \mathbb{R}^n \setminus \mathcal{O}_i\}$. Clearly $u|_{\mathcal{O}_i} \in X_0|_{\mathcal{O}_i}$ then we have

$$\frac{\int_{Q|_{\mathcal{O}_1}} (u(x) - u(y))^2 K(x-y)dxdy}{\int_{\mathcal{O}_1} u^2 dx} < \alpha \frac{\int_{Q|_{\mathcal{O}_1}} (u(x) - u(y))^2 K(x-y)dxdy}{\int_{\mathcal{O}_1} \alpha(x) u^2 dx} = \alpha$$

which implies that $\lambda_1(\mathcal{O}_1) < \alpha$. The other inequality in (4.9) is proved similarly. Now with some modification on the sets $\mathcal{O}_1$ and $\mathcal{O}_2$, we construct the sets $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ such that $\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_2 = \emptyset$ and $\lambda_1(\tilde{\mathcal{O}}_1) < \alpha$ and $\lambda_1(\tilde{\mathcal{O}}_2) < \beta$. For this, we consider for $\nu \geq 0$, $\mathcal{O}_1(\nu) = \{x \in \mathcal{O}_1 : dist(x, \mathcal{O}_1^c) > \nu\}$. Then clearly $\lambda_1(\mathcal{O}_1(\nu)) \geq \lambda_1(\mathcal{O}_1)$ and moreover $\lambda_1(\mathcal{O}_1(\nu)) \to \lambda_1(\mathcal{O}_1)$ as $\nu \to 0$. Then there exists $\nu_0 > 0$ such that

$$\lambda_1(\mathcal{O}_1(\nu)) < \alpha \ \text{ for all } \ 0 \leq \nu \leq \nu_0. \tag{4.10}$$

Let $x_0 \in \partial\mathcal{O}_2 \cap \Omega$ (is not empty as $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$) and choose $0 < \nu < \min\{\nu_0, dist(x_0, \Omega^c)\}$ and $\tilde{\mathcal{O}}_1 = \mathcal{O}_1(\nu)$ and $\tilde{\mathcal{O}}_2 = \mathcal{O}_2 \cup B(x_0, \frac{\nu}{2})$. Then $\tilde{\mathcal{O}}_1 \cap \tilde{\mathcal{O}}_2 = \emptyset$ and by (4.10), $\lambda_1(\tilde{\mathcal{O}}_1) < \alpha$. Since $\tilde{\mathcal{O}}_2$ is connected then by (4.9) and Lemma 4.2, we get $\lambda_1(\tilde{\mathcal{O}}_2) < \beta$. Now we define $v = v_1 - v_2$, where v_i are the extension by zero outside $\tilde{\mathcal{O}}_i$ of the eigenfunctions associated to $\lambda_i(\tilde{\mathcal{O}}_i)$. Then v satisfies

(4.8). Thus there exist $v \in X_0$ which changes sign and satisfies condition (4.8), and moreover we have

$$\tilde{J}_p\left(\frac{v}{\|v\|_{L^2}}\right) = \frac{\int_Q (v^+(x)-v^+(y))^2 K(x-y)dxdy}{\|v\|_{L^2}^2} + \frac{\int_Q (v^-(x)-v^-(y))^2 K(x-y)dxdy}{\|v\|_{L^2}^2}$$
$$- 2\frac{\int_Q v^+(x)v^-(y)K(x-y)dxdy}{\|v\|_{L^2}^2} - p\frac{\int_\Omega (v^+)^2dx}{\|v\|_{L^2}^2}$$
$$< (\alpha-p)\frac{\int_\Omega (v^+)^2dx}{\|v\|_{L^2}^2} + \beta\frac{\int_\Omega (v^-)^2dx}{\|v\|_{L^2}^2} - 2\frac{\int_Q v^+(x)v^-(y)K(x-y)dxdy}{\|v\|_{L^2}^2} < \beta.$$

$$\tilde{J}_p\left(\frac{v^+}{\|v^+\|_{L^2}}\right) < \alpha - p = \beta, \quad \tilde{J}_p\left(\frac{v^-}{\|v^-\|_{L^2}}\right) < \beta - p.$$

Using Lemma 3.3, we have that there exists a critical point in the connected component of the set $\mathcal{O} = \{u \in \mathcal{P} : \tilde{J}_p(u) < \beta - p\}$. As the point $(\alpha, \beta) \in \mathcal{C}$, the only possible critical point is ϕ_1, then we can construct a path from $-\phi_1$ to ϕ_1 exactly in the same manner as in Theorem 3.4 only by taking v in place of u. Thus we have constructed a path satisfying (4.7), and hence the result follows. □

Corollary 4.4. *Let $(\alpha, \beta) \in \mathcal{C}$ and let $\alpha(x), \beta(x) \in L^\infty(\Omega)$ satisfying $\lambda_1 \le \alpha(x) \le \alpha$ a.e, $\lambda_1 \le \beta(x) \le \beta$ a.e. Assume that $\lambda_1 < \alpha(x)$ and $\lambda_1 < \beta(x)$ on subsets of positive measure. If either $\alpha(x) < \alpha$ a.e in Ω or $\beta(x) < \beta$ a.e. in Ω, then (4.3) has only the trivial solution.*

Lemma 4.5. *The curve $p \to (p + c(p), c(p))$ is strictly decreasing, (in the sense that $p_1 < p_2$ implies $p_1 + c(p_1) < p_2 + c(p_2)$ and $c(p_1) > c(p_2)$).*

Proof. Let $p_1 < p_2$ and suppose by contradiction that either (i) $p_1 + c(p_1) \ge p_2 + c(p_2)$ or (ii) $c(p_1) \le c(p_2)$. In case (i) we deduce from $p_1 + c(p_1) \ge p_2 + c(p_2) > p_1 + c(p_2)$ that $c(p_1) \ge c(p_2)$. If we take $(\alpha, \beta) = (p_1 + c(p_1), c(p_1))$ and $(\alpha(x), \beta(x)) = (p_2 + c(p_2), c(p_2))$, then by Corollary 4.4, the only solution of (4.3) with $(\alpha(x), \beta(x))$ is the trivial solution which contradicts the fact that $(p_2 + c(p_2), c(p_2)) \in \sum_K$. If (ii) holds then $p_1 + c(p_1) \le p_1 + c(p_2) < p_2 + c(p_2)$, if we take $(\alpha, \beta) = (p_2 + c(p_2), c(p_2))$ and $(\alpha(x), \beta(x)) = (p_1 + c(p_1), c(p_1))$, then the only solution of (4.3) with $(\alpha(x), \beta(x))$ is the trivial one which contradicts the fact that $(p_1 + c(p_1), c(p_1)) \in \sum_K$ and hence the result follows. □

As $c(p)$ is decreasing and positive so the limit of $c(p)$ exists as $p \to \infty$. In the next Theorem we find the asymptotic behavior of the first nontrivial curve.

Theorem 4.6. *If $n \ge 2s$ then the limit of $c(p)$ as $p \to \infty$ is λ_1.*

Proof. For $n \ge 2s$, we can choose a function $\phi \in X_0$ such that there does not exist $r \in \mathbb{R}$ such that $\phi(x) \le r\phi_1(x)$ a.e. in Ω. For this it suffices to take $\phi \in X_0$ such that it is unbounded from above in a neighborhood of some point $x \in X_0$. Then by contradiction argument, one can similarly show $c(p) \to \lambda_1$ as $p \to \infty$ as in Proposition 4.4 of [4]. □

5. Nonresonance between (λ_1, λ_1) and $\mathcal{C}$

In this section we study the following problem

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.1}$$

where $f(x,u)/u$ lies asymptotically between (λ_1, λ_1) and $(\alpha, \beta) \in \mathcal{C}$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying $L^\infty(\Omega)$ Caratheodory conditions. Given a point $(\alpha, \beta) \in \mathcal{C}$, we assume following:

$$\gamma_\pm(x) \leq \liminf_{s \to \pm\infty} \frac{f(x,s)}{s} \leq \limsup_{s \to \pm\infty} \frac{f(x,s)}{s} \leq \Gamma_\pm(x) \tag{5.2}$$

hold uniformly with respect to x, where $\gamma_\pm(x)$ and $\Gamma_\pm(x)$ are L^∞ functions which satisfy

$$\begin{cases} \lambda_1 \leq \gamma_+(x) \leq \Gamma_+(x) \leq \alpha \text{ a.e.} & \text{in } \Omega \\ \lambda_1 \leq \gamma_-(x) \leq \Gamma_-(x) \leq \beta \text{ a.e.} & \text{in } \Omega. \end{cases} \tag{5.3}$$

Write $F(x,s) = \int_0^s f(x,t)dt$, we also assume the following inequalities:

$$\delta_\pm(x) \leq \liminf_{s \to \pm\infty} \frac{2F(x,s)}{s^2} \leq \limsup_{s \to \pm\infty} \frac{2F(x,s)}{s^2} \leq \Delta_\pm(x) \tag{5.4}$$

hold uniformly with respect to x, where $\delta_\pm(x)$ and $\Delta_\pm(x)$ are L^∞ functions which satisfy

$$\begin{cases} \lambda_1 \leq \delta_+(x) \leq \Delta_+(x) \leq \alpha \text{ a.e. in } \Omega \\ \lambda_1 \leq \delta_-(x) \leq \Delta_-(x) \leq \beta \text{ a.e. in } \Omega \\ \delta_+(x) > \lambda_1 \text{ and } \delta_-(x) > \lambda_1 \text{ on subsets of positive measure,} \\ \text{either } \Delta_+(x) < \alpha \text{ a.e. in } \Omega \text{ or } \Delta_-(x) < \beta \text{ a.e. in } \Omega. \end{cases} \tag{5.5}$$

Theorem 5.1. *Let (5.2), (5.3), (5.4), (5.5) hold and $(\alpha, \beta) \in \mathcal{C}$. Then problem (5.1) admits at least one solution u in X_0.*

Define the energy functional $\Psi : X_0 \to \mathbb{R}$ as

$$\Psi(u) = \frac{1}{2} \int_Q (u(x) - u(y))^2 K(x-y)dxdy - \int_\Omega F(x,u)dx$$

Then Ψ is a C^1 functional on X_0 and $\forall v \in X_0$

$$\langle \Psi'(u), v \rangle = \int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy - \int_\Omega f(x,u)vdx$$

and critical points of Ψ are exactly the weak solutions of (5.1).

Lemma 5.2. *Ψ satisfies the $(P.S)$ condition on X_0.*

Proof. Let u_k be a (P.S) sequence in X_0, i.e

$$\begin{aligned} |\Psi(u_k)| &\leq c, \\ |\langle \Psi'(u_k), v \rangle| &\leq \epsilon_k \|v\|_{X_0}, \ \forall\, v \in X_0, \end{aligned} \tag{5.6}$$

where c is a constant and $\epsilon_k \to 0$ as $k \to \infty$. It suffices to show that u_k is a bounded sequence in X_0. Assume by contradiction that u_k is not a bounded sequence. Then define $v_k = \frac{u_k}{\|u_k\|_{X_0}}$ which is a bounded sequence. Therefore there exists a subsequence v_k of v_k and $v_0 \in X_0$ such that $v_k \rightharpoonup v_0$ weakly in X_0, $v_k \to v_0$ strongly in $L^2(\Omega)$ and $v_k(x) \to v_0(x)$ a.e. in $\mathbb{R}^n$. Also by using (5.2) and (5.3), we have $f(x, u_k)/\|u_k\|_{X_0} \rightharpoonup f_0(x)$ weakly in $L^2(\Omega)$. Take $v = v_k - v_0$ in (5.6) and divide by $\|u_k\|_{X_0}$ we get $v_k \to v_0$. In particular $\|v_0\|_{X_0} = 1$. One can easily see from (5.6) that

$$\int_Q (v_0(x) - v_0(y))(v(x) - v(y))K(x-y)dxdy - \int_\Omega f_0(x)vdx = 0 \; \forall\, v \in X_0.$$

Now by a standard argument based on assumption (5.2), $f_0(x) = \alpha(x)v_0^+ - \beta(x)v_0^-$ for some L^∞ functions $\alpha(x)$, $\beta(x)$ satisfying (4.1). In the expression of $f_0(x)$, the value of $\alpha(x)$ (resp.$\beta(x)$) on $\{x : v_0(x) \le 0\}$ (resp. $\{x : v_0(x) \ge 0\}$) are irrelevant, and consequently we can assume that

$$\alpha(x) > \lambda_1 \quad \text{on } \{x : v_0(x) \le 0\} \quad \text{and} \quad \beta(x) > \lambda_1 \quad \text{on } \{x : v_0(x) \ge 0\}. \tag{5.7}$$

So v_0 is a nontrivial solution of Eq. (4.3). It then follows from Lemma 4.3 that either (i) $\alpha(x) = \lambda_1$ a.e in Ω or (ii) $\beta(x) = \lambda_1$ a.e in Ω, or (iii) v_0 is an eigenfunction associated to the point $(\alpha, \beta) \in \mathcal{C}$. We show that in each case we get a contradiction. If (i) holds then by (5.7), $v_0 > 0$ a.e. in Ω and (4.3) gives $\int_Q (v_0(x) - v_0(y))^2 K(x-y)dxdy = \lambda_1 \int_\Omega v_0^2$, which implies that v_0 is a multiple of ϕ_1. Dividing (5.6) by $\|u_k\|^2_{X_0}$ and taking limit we get,

$$\begin{aligned}\lambda_1 \int_\Omega v_0^2 &= \int_Q (v_0(x) - v_0(y))^2 K(x-y)dxdy \\ &= \lim_{k\to\infty} \int_\Omega \frac{2F(x,u_k)}{\|u_k\|^2_{X_0}} \ge \int_\Omega \delta_+(x)v_0^2 dx.\end{aligned}$$

This contradicts assumption (5.5). The case (ii) is treated similarly. Now if (iii) holds, we deduce from (5.4) that

$$\begin{aligned}\int_\Omega \alpha(v_0^+)^2 + \beta(v_0^-)^2 &= \int_Q (v_0(x) - v_0(y))^2 K(x-y)dxdy = \lim_{k\to\infty} \int_\Omega \frac{2F(x,u_k)}{\|u_k\|^2_{X_0}} \\ &\le \int_\Omega \Delta_+(x)(v_0^+)^2 + \Delta_-(x)(v_0^-)^2.\end{aligned}$$

This contradicts assumption (5.5), since v_0 changes sign. Hence u_k is a bounded sequence in X_0. $\square$

Now we study the geometry of Ψ.

Lemma 5.3. *There exists $R > 0$ such that*

$$\max\{\Psi(R\phi_1), \Psi(-R\phi_1)\} < \max_{u\in\gamma[-1,1]} \Psi(u) \tag{5.8}$$

for any $\gamma \in \Gamma_1 := \{\gamma \in C([-1,1], X_0) : \gamma(\pm 1) = \pm R\phi_1\}$.

Proof. From (5.4), we have for any $\epsilon > 0$ there exists $a_\epsilon(x) \in L^2(\Omega)$ such that for a.e x,

$$\begin{cases}(\delta_+(x) - \epsilon)\frac{s^2}{2} - a_\epsilon(x) \le F(x,s) \le (\Delta_+(x) + \epsilon)\frac{s^2}{2} + a_\epsilon(x) \;\forall\, s > 0 \\ (\delta_-(x) - \epsilon)\frac{s^2}{2} - a_\epsilon(x) \le F(x,s) \le (\Delta_-(x) + \epsilon)\frac{s^2}{2} + a_\epsilon(x) \;\forall\, s < 0.\end{cases} \tag{5.9}$$

Now consider the following functional associated to the functions $\Delta_{\pm}(x)$ as

$$\Phi(u) = \int_Q (u(x)-u(y))^2 K(x-y)dxdy - \int_\Omega \Delta_+(x)(u^+)^2 dx \\ - \int_\Omega \Delta_-(x)(u^-)^2 dx.$$

Then we claim that

$$d = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} \Psi(u) > 0 \tag{5.10}$$

where Γ is the set of all continuous paths from $-\phi_1$ to ϕ_1 in $\mathcal{P}$. Write $p = \alpha - \beta \geq 0$. we can choose $(\alpha, \beta) \in \mathcal{C}$ such that $\alpha \geq \beta$ (by replacing u by $-u$ if necessary), we have for any $\gamma \in \Gamma$

$$\max_{u\in\gamma[-1,1]} \tilde{J}_p(u) \geq c(p) = \beta.$$

i.e. $\max_{u\in\gamma[-1,1]} \left(\int_Q (u(x)-u(y))^2 \;\; K(x-y)dxdy - \int_\Omega \alpha(u^+)^2 dx - \int_\Omega \beta(u^-)^2 dx \right) \geq 0$

which implies

$$\max_{u\in\gamma[-1,1]} \Phi(u) \geq 0,$$

by (5.5). So $d \geq 0$. On the other hand, since $\delta_\pm(x) \leq \Delta_\pm(x)$,

$$\Phi(\pm\phi_1) \leq \int_\Omega (\lambda_1 - \delta_\pm(x))\phi_1^2 dx < 0$$

by (5.5). Thus we have a mountain pass geometry for the restriction $\tilde{\Phi}$ of Φ to $\mathcal{P}$,

$$\max\{\tilde{\Phi}(\phi_1), \tilde{\Phi}(-\phi_1)\} < 0 \leq \max_{u\in\gamma[-1,1]} \tilde{\Phi}(u)$$

for any path $\gamma \in \Gamma$ and moreover one can verify exactly as in Lemma 2.6 that Φ satisfies the $(P.S.)$ condition on X_0. Then d is a critical value of $\tilde{\Phi}$ i.e there exists $u \in \mathcal{P}$ and $\mu \in \mathbb{R}$ such that

$$\begin{cases} \Phi(u) = d \\ \langle \Phi'(u), v\rangle = \mu \langle I'(u), v\rangle \;\forall\; v \in X_0. \end{cases}$$

Assume by contradiction that $d = 0$. Taking $v = u$ in above, we get $\mu = 0$ so u is a nontrivial solution of

$$-\mathcal{L}_K u = \Delta_+(x)u^+ - \Delta_-(x)u^- \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

Using (5.5), we get a contradiction with Lemma 4.3 . This completes the proof of the claim.

Next we show that (5.8) holds. From the left hand side of inequality (5.9), we have for $R > 0$ and $\eta > 0$,

$$\Psi(\pm R\phi_1) \leq \frac{R^2}{2}\int_\Omega (\lambda_1 - \delta_\pm(x))\phi_1^2 + \frac{\eta R^2}{2} + \|a_\eta\|_{L^1},$$

Then, $\Psi(\pm R\phi_1) \to -\infty$ as $R \to +\infty$, by (5.5) and letting η to be sufficiently small. Fix ϵ with $0 < \epsilon < d$. We can choose $R = R(\epsilon)$ so that

$$\Psi(\pm R\phi_1) < -\|a_\epsilon\|_{L^1}, \tag{5.11}$$

where a_ϵ is associated to ϵ using (5.9). Consider a path $\gamma \in \Gamma_1$. Then if $0 \in \gamma[-1,1]$, then by (5.11),

$$\Psi(\pm R\phi_1) < -\|a_\epsilon\|_{L^1} \le 0 = \Psi(0) \le \max_{u\in\gamma[-1,1]} \Psi(u),$$

so the Lemma is proved in this case. If $0 \notin \gamma[-1,1]$, then we take the normalized path $\tilde\gamma(t) = \frac{\gamma(t)}{\|\gamma(t)\|_{L^2}}$ belonging to Γ. Since by (5.9) we have

$$\Psi(u) \ge \frac{\Phi(u) - \epsilon\|u\|_{L^2}^2}{2} - \|a_\epsilon\|_{L^1}.$$

Then we obtain

$$\max_{\gamma\in[-1,1]} \frac{2\Psi(u) + \epsilon\|u\|_{L^2}^2 + 2\|a_\epsilon\|_{L^1}}{\|u\|_{L^2}^2} \ge \max_{\tilde\gamma\in[-1,1]} \Phi(v) \ge d,$$

and consequently, by the choice of ϵ, we have

$$\max_{\gamma\in[-1,1]} \frac{2\Psi(u) + 2\|a_\epsilon\|_{L^1}}{\|u\|_{L^2}^2} \ge d - \epsilon > 0.$$

This implies that

$$\max_{u\in\gamma[-1,1]} \Psi(u) > -\|a_\epsilon\|_{L^1} > \Psi(\pm R\phi_1),$$

by (5.11) and hence the Lemma. □

Proof of Theorem 5.1: Lemmas 5.2 and 5.3 complete the proof.

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