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Multiple positive solutions of singular and critical elliptic problem in $\mathbb{R}^2$ with discontinuous nonlinearities

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. We consider the following singular and critical elliptic problem with discontinuous nonlinearity:

$$(P_\lambda)\begin{cases} -\Delta u = \lambda\left(\frac{m(x,u)e^{\alpha u^2}}{|x|^\beta} + u^q g(u-a)\right), & u>0 \quad \text{in} \quad \Omega \\ u \qquad = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where $0 \le q < 1, 0 < \alpha \le 4\pi$ and $\beta \in [0,2)$ such that $\frac{\beta}{2} + \frac{\alpha}{4\pi} \le 1$ and $g(t-a) = \begin{cases} 1, \; t \le a \\ 0, \; t > a. \end{cases}$ Under the suitable assumptions on $m(x,t)$ we show the existence and multiplicity of solutions for maximal interval for λ.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Consider the following elliptic problem with the discontinuous nonlinearity g:

$$(P_\lambda)\begin{cases} -\Delta u = \lambda\left(\frac{m(x,u)e^{\alpha u^2}}{|x|^\beta} + u^q g(u-a)\right), & u>0 \quad \text{in}\ \Omega \\ u \qquad = 0 \text{ on } \partial\Omega \end{cases}$$

where $0 \le q < 1, 0 < \alpha \le 4\pi$ and $\beta \in [0,2)$ such that $\frac{\beta}{2} + \frac{\alpha}{4\pi} \le 1$. The discontinuous function $g(t-a)$ is defined as $g(t-a) = \begin{cases} 1, \; t \le a \\ 0, \; t > a \end{cases}$ and $m(x,t) \in C^1(\bar{\Omega} \times \mathbb{R})$ satisfies the following assumptions:

(H1) $m(x,0)=0, m(x,t)>0$ for $t>0$ and is nondecreasing in t.

(H2) $\lim\limits_{t\to\infty}\sup\limits_{x\in\bar\Omega} m(x,t)e^{-\epsilon t^2}=0, \lim\limits_{t\to\infty}\inf\limits_{x\in\bar\Omega} m(x,t)e^{\epsilon t^2}=\infty$, for $\epsilon>0$.

(H3) $\lim\limits_{t\to\infty}\inf\limits_{x\in\bar\Omega} m(x,t)te^{\epsilon t}=\infty$, for $\epsilon>0$.

(H4) There exists $M_1>0$ such that for $t>0$, $F(x,t)=\int_0^t f(x,s)ds<(f(x,t)+1)M_1$ where $f(x,t)=\frac{m(x,t)e^{\alpha t^2}}{|x|^\beta}$.

(H5) For $\epsilon>0$, $\lim\limits_{t\to\infty}\sup\limits_{x\in\bar\Omega}\dfrac{\partial f}{\partial t}e^{-\alpha(1+\epsilon)t^2}=0$ and there exists $M_2,\ K_0>0$ such that $\dfrac{\partial f}{\partial t}\geq -M_2+K_0 f(x,t)$ for all $x\in\Omega,\ t>0$.

Problems of the type (P_λ) have drawn the attention of many authors in recent years as they occur in various branches of physics. Due to the discontinuity in the nonlinearity, the solutions are realized as zeroes of the generalized gradient given by Clarke [9]. In [5] authors have discussed the existence and multiplicity for Laplace equation with discontinuous nonlinearities with critical exponents in $\mathbb{R}^n$ for $n\geq 3$. The existence of unique solution with sublinear nonlinearity has been proved in [14]. Existence results for semilinear equation with continuous and exponential nonlinearity motivated from Moser–Trudinger inequality has been extensively studied starting from [1–3, 10]. The combined effects of concave and convex nonlinearities are studied in the beautiful work of Ambrosetti et al. [4] for critical exponent problems and these results are discussed for the exponential nonlinearities in [12].

In this work we study the combined effect of exponential and discontinuous sublinear nonlinearity by combining the generalized gradient approach and usual variational and sub-super solutions methods. Let $h(t)=g(t-a)t_+^q$ and $H(s)=\int_0^s h(t)dt$ where $t_+=\max\{t,0\}$. Consider the following functional associated to (P_λ):

$$I_\lambda(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda\int_\Omega F(x,u)-\lambda\int_\Omega H(u).$$

The idea is to characterize the solutions of (P_λ) as critical points in the sense of Chang [8] and Clarke [9] for the functional I_λ on $H_0^1(\Omega)$. To obtain the existence of solution we minimize I_λ over the closed convex set $M_\lambda=\{u\in H_0^1(\Omega):\underline{u}\leq u\leq\bar u\}$ for suitably chosen sub and super solutions $\underline{u}$ and $\bar u$ of (P_λ), respectively. Then using the generalized gradients theory we show that this minimizer is a critical point of I_λ. Next we show that the first solution thus obtained is also a local minimum for I_λ in $H_0^1(\Omega)$. Due to the presence of singularity $\frac{1}{|x|^\beta}$, the solutions are no more C^1. So the classical result of Brezis and Nirenberg [7], cannot be applied to this case. Here we adopt the approach as in [13]. Due to the presence of discontinuity in the sublinear term in (P_λ), it is interesting to investigate the $H_0^1(\Omega)$ local minimizer of I_λ for all $a>0$ and maximal range of λ. The important feature of it is the appropriate use of the Hopf maximum principle to handle the discontinuity(see Theorem 3.1). The study of Palais–Smale level is also delicate due to the effect of discontinuous nature of the sublinear term. We use sequence of Moser functions with variable support as in [12] to obtain the Palais–Smale sequence below the critical level. Here we would like to mention that the results obtained are new even for the case $\beta = 0$.

We consider the following preliminary notions. Let X be a real Banach space, X^* be it's dual space and $\langle , \rangle$ denotes it's duality action. Then for $x, v \in X$ and a locally Lipschitz continuous function $f \in X^*$, the generalized directional derivative $f^0(x, v)$ (of f at x along v) is defined as follows:

$$f^0(x,v) = \lim_{h\to 0} \sup_{\lambda \downarrow 0} \frac{1}{\lambda}(f(x+h+\lambda v) - f(x+h)).$$

The generalized gradient of f at x, denoted by $\partial f(x)$, is defined as follows:

$$\partial f(x) \subseteq X^*; w \in \partial f(x) \text{ if and only if } \langle w, v\rangle \le f^0(x,v), \quad \text{for all } v \in X.$$

Also $\partial(kf)(x) = k\partial(f)(x)$ for all $k \in \mathbb{R}$. We define $x \in X$ to be a critical point for f if $0 \in \partial f(x)$. Thus in view of Theorem 2.1 and Theorem 2.2 in [8], $w \in \partial I_\lambda(u)$ if and only if there exists a function $\bar{w} \in L^\infty(\Omega)$ such that

$$\langle w, v\rangle = \int_\Omega \nabla u \nabla v - \lambda \int_\Omega f(x,u)v - \lambda \int_\Omega u^q \bar{w} v, \text{ for all } v \in H_0^1(\Omega)$$

and

$$\bar{w}(x) \in [\underline{g}(u(x)-a), \bar{g}(u(x)-a)] \quad \text{for a.e. } x \in \Omega$$

where $\underline{g}(u-a) = \liminf\limits_{s\to u} \; g(s-a)$ and $\bar{g}(u-a) = \limsup\limits_{s\to u} \; g(s-a)$. Hence u defines a critical point for I_λ if and only if there exists a function $\bar{w} \in L^\infty(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v - \lambda \int_\Omega f(x,u)v - \lambda \int_\Omega u^q \bar{w} v = 0, \quad \text{for all } v \in H_0^1(\Omega) \tag{1.1}$$

with

$$\bar{w}(x) \in [\underline{g}(u(x)-a), \bar{g}(u(x)-a)] \text{ for a.e. } x \in \Omega. \tag{1.2}$$

Definition 1.1. *A solution of* (P_λ) *is a function* $u \in H_0^1(\Omega)$ *such that* $u > 0$ *in* Ω *and* u *satisfies Eqs.* (1.1) *and* (1.2).

By elliptic regularity, such a solution $u \in C^{1,\gamma}(\Omega \backslash \{0\})$ for some $\gamma \in (0,1)$ and hence $|\Omega_a| = 0$ where $\Omega_a = \{x \in \Omega : u(x) = a\}$ and $|\Omega_a|$ is the $N-$dimensional Lebesgue measure of Ω_a. The exponential nature of the singular nonlinearity $f(x,u)$ is motivated by the following singular version of Moser–Trudinger inequality studied in [3], where the existence of positive solution for singular nonlinearity of Hardy's type is studied.

Theorem 1.1. [3] Theorem 2.1 *Let* $\Omega \subset \mathbb{R}^n$, $n \ge 2$ *be a bounded domain containing the origin and* $u \in W_0^{1,n}(\Omega)$. *Then for every* $\alpha > 0$ *and* $\beta \in [0, n)$

$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} < \infty. \tag{1.3}$$

Moreover,

$$\sup_{\|\nabla u\|_{L^2}\le 1}\int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta} < \infty \quad \textit{if} \quad \textit{and} \quad \textit{only if} \quad \frac{\alpha}{4\pi}+\frac{\beta}{2}\le 1. \tag{1.4}$$

The imbedding in (1.4) is compact if $\frac{\alpha}{4\pi}+\frac{\beta}{2}<1$. The non-compactness of the imbedding in the limiting cases $(\frac{\alpha}{4\pi}+\frac{\beta}{2}=1)$ is obtained using the Moser functions as in [15,17]. The presence of singularity $\frac{1}{|x|^\beta}$ in the nonlinearity implies the unboundedness of positive solutions of (P_λ) when $\beta>1$. The asymptotic behaviour of solutions can be obtained using the comparison principles and clever choice of test functions [16]. With this introduction, we state the main result of this paper:

Theorem 1.2. *Assume* (H1)–(H5)*. Then there exists* $\Lambda>0$ *such that*

(i) *for all* $a>0$ *and* $\lambda\in(0,\Lambda]$, (P_λ) *admits a solution.*
(ii) *for* $\lambda>\Lambda$, (P_λ) *has no solution.*
(iii) *For* $\lambda\in(0,\Lambda)$ *and* $a>0$, (P_λ) *admits two distinct ordered solutions* $u_\lambda(x)\le v_\lambda(x)$, $x\in\Omega$.

2. Existence of a solution

In this section we show the existence of a positive solution of (P_λ) for λ in the maximal interval $(0,\Lambda]$ and for all $a>0$. We consider the following problem.

$$(Q_\lambda)\ \begin{cases} -\Delta u = \lambda u^q g(u-a),\ u>0 \ \text{ in } \ \Omega,\\ u \quad\ = 0 \ \text{ on } \ \partial\Omega.\end{cases}$$

Lemma 2.1. *For all* $\lambda>0$ *and* $a>0$, (Q_λ) *admits unique solution.*

Proof. We adopt the approach as in [14] to the bounded domain set up. The associated functional $E_\lambda(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda\int_\Omega H(u)$ is bounded below on $H_0^1(\Omega)$. Indeed, by Sobolev imbedding, $E_\lambda(u)\ge\frac{1}{2}\|u\|^2-C\frac{\lambda}{q+1}\|u\|^{q+1}>-\infty$. Also we note that for $\phi\in C_0^\infty(\Omega)$ and $t>0$ small,

$$E_\lambda(t\phi)=\frac{t^2}{2}\|\phi\|^2-\lambda\frac{t^{q+1}}{q+1}\int_\Omega|\phi|^{q+1}<0.$$

Let $\{u_n\}$ be the minimizing sequence for the minimization problem $E_a=\min\limits_{\|u\|\le R}E_\lambda(u)$ for $R>0$ sufficiently small. Then by the Ekeland Variational principle, there exists v_n such that $E_\lambda(v_n)\le E_\lambda(u_n)\le E_a+\frac{1}{n}$, $\|v_n-u_n\|\le\frac{1}{n}$ and $E_\lambda(v_n)\le E_\lambda(v)+\frac{1}{n}\|v-v_n\|$ for all $v\in H_0^1(\Omega)$. Taking $v=v_n+t\psi,\psi\in H_0^1(\Omega)$, we get

$$\frac{E_\lambda(v_n+t\psi)-E_\lambda(v_n)}{t}\ge-\frac{\|\psi\|}{n}.$$

Letting $t\to 0^+$, we get $E_\lambda^0(v_n,\psi)\ge-\frac{\|\psi\|}{n}$, for all $\psi\in H_0^1(\Omega)$. Therefore by Theorem 2.1 and Theorem 2.2 in [8], there exists $w_n(x)\in[\underline{g}(v_n(x)-a),\overline{g}(v_n(x)-a)]$ such that

$$\int_\Omega \nabla v_n \cdot \nabla \psi - \lambda \int_\Omega h(v_n) w_n \psi \geq -\frac{\|\psi\|}{n}, \text{ for all } \psi \in H_0^1(\Omega) \tag{2.5}$$

Now by the compact imbedding of $H_0^1(\Omega)$ into $L^{q+1}(\Omega)$ and the boundedness of $\{v_n\}$, we get a subsequence $\{v_n\}$ and $v_\lambda \in H_0^1(\Omega)$ such that $\int_\Omega h(v_n) w_n \psi \to \int_\Omega h(v_\lambda) w \psi$ where w is the weak* limit of w_n in $L^\infty(\Omega)$ and $\int_\Omega \nabla v_n \cdot \nabla \psi \to \int_\Omega \nabla v_\lambda \cdot \nabla \psi$. Therefore, taking limit $n \to \infty$ in (2.5), we get

$$\int_\Omega \nabla v_\lambda \cdot \nabla \psi - \lambda \int_\Omega h(v_\lambda) w \psi \geq 0, \text{ for all } \psi \in H_0^1(\Omega)$$

Hence $E_\lambda^0(v_\lambda, \psi) \geq 0$ and v_λ is a solution of (Q_λ). Using the weak maximum principle it is easy to check that $v_\lambda > 0$. Now uniqueness follows as in Lemma 3.3 of [4]. □

Lemma 2.2. *Assume* $(H1)$ *and* $(H2)$*. Then there exists* $\overline{\Lambda} > 0$ (*small enough*) *such that* (P_λ) *admits a solution for all* $\lambda \in (0, \overline{\Lambda})$.

Proof. Let $\underline{u}$ be the unique solution of (Q_λ). Then $\underline{u}$ is a sub solution to (P_λ). Also, consider the minimal solution $\bar{u}$ of

$$\begin{aligned} -\Delta u &= \lambda \left(\tfrac{m(x,u) e^{\alpha u^2}}{|x|^\beta} + u^q \right), \quad u > 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $\lambda \in (0, \bar{\Lambda}]$ for some $\bar{\Lambda} > 0$. Existence of $\bar{u}$ follows by minimizing the corresponding functional over the ball $B_r = \{u \in H_0^1(\Omega) : \|u\| \leq r\}$ for $r > 0$ small as in [6]. Thus $\bar{u}$ is a super solution to (P_λ). Also as $-\Delta \underline{u} \leq \lambda \underline{u}^q$ and $-\Delta \bar{u} \geq \lambda \bar{u}^q$ in Ω with $\underline{u}, \bar{u} = 0$ on $\partial\Omega$, by the comparison principle (Lemma 3.3 of [4]), we get $\underline{u} \leq \bar{u}$. Let $\lambda \in (0, \bar{\Lambda}]$ and $M_\lambda = \{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \bar{u} \text{ a.e. in } \Omega\}$. Then M_λ is a closed convex subset of $H_0^1(\Omega)$ and I_λ is weakly lower semi-continuous on M_λ. Indeed, if $u_n \in M_\lambda$, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then from $(H2)$ and $(H4)$ we have $F(x, u_n) \leq c_1 \dfrac{e^{(\alpha+\epsilon)\bar{u}^2}}{|x|^\beta} + c_2$. Therefore by applying the dominated convergence theorem and (1.4) we obtain $\displaystyle\int_\Omega F(x, u_n) \to \int_\Omega F(x, u)$. It is easy to see that $\displaystyle\int_\Omega H(u_n) \to \int_\Omega H(u)$, thanks to compact imbedding of $H_0^1(\Omega)$ into $L^{q+1}(\Omega)$. Since I_λ is coercive on M_λ, there exists $u_\lambda \in M_\lambda$ such that $I_\lambda(u_\lambda) = \inf\limits_{u \in M_\lambda} I_\lambda(u)$. Here we use Perron's method to show that u_λ solves problem (P_λ). For $\phi \in C_0^\infty(\Omega)$, define $\phi^\epsilon = (u_\lambda + \epsilon\phi - \bar{u})^+$, $\phi_\epsilon = (u_\lambda + \epsilon\phi - \underline{u})^-$ and $v_\epsilon = u_\lambda + \epsilon\phi - \phi^\epsilon + \phi_\epsilon$. Then $v_\epsilon \in M_\lambda$ and it is easy to check that $u_\lambda + t(v_\epsilon - u_\lambda) \in M_\lambda$ for $0 < t < 1$. Since u_λ is minimum of I_λ over M_λ, we have

$$\frac{I_\lambda(u_\lambda + t(v_\epsilon - u_\lambda)) - I_\lambda(u_\lambda)}{t} \geq 0.$$

Taking the limit $t \to 0^+$, we get $I_\lambda^0(u_\lambda, v_\epsilon - u_\lambda) \geq 0$. Hence, by Theorem 2.1 and Theorem 2.2 in [8], there exists $w_\epsilon \in L^\infty(\Omega)$ such that $w_\epsilon(x) \in [\underline{g}(u_\lambda(x) - a), \bar{g}(u_\lambda(x) - a)]$ and

$$\int_\Omega \nabla u_\lambda \nabla(v_\epsilon - u_\lambda) - \lambda \int_\Omega f(x,u_\lambda)(v_\epsilon - u_\lambda) - \lambda \int_\Omega u_\lambda^q w_\epsilon (v_\epsilon - u_\lambda) \geq 0.$$

So,

$$\int_\Omega \nabla u_\lambda \nabla \phi - \lambda \int_\Omega f(x,u_\lambda)\phi - \lambda \int_\Omega u_\lambda^q w_\epsilon \phi \geq \frac{1}{\epsilon}(E^\epsilon - E_\epsilon)$$

where

$$E^\epsilon = \int_\Omega \nabla u_\lambda \nabla \phi^\epsilon - \lambda \int_\Omega f(x,u_\lambda)\phi^\epsilon - \lambda \int_\Omega u_\lambda^q w_\epsilon \phi^\epsilon$$

and

$$E_\epsilon = \int_\Omega \nabla u_\lambda \nabla \phi_\epsilon - \lambda \int_\Omega f(x,u_\lambda)\phi_\epsilon - \lambda \int_\Omega u_\lambda^q w_\epsilon \phi_\epsilon.$$

Further, as $f(x,t)$ is nondecreasing in $t > 0$ and $\bar{u}$ is a supersolution for (P_λ), we have

$$\begin{aligned}
\frac{1}{\epsilon}E^\epsilon &= \frac{1}{\epsilon}\left(\int_{\Omega^\epsilon} \nabla u_\lambda \nabla(u_\lambda + \epsilon\phi - \bar{u}) - \lambda \int_\Omega f(x,u_\lambda)\phi^\epsilon - \lambda \int_\Omega u_\lambda^q w_\epsilon \phi^\epsilon\right) \\
&\geq \frac{1}{\epsilon}\int_{\Omega^\epsilon} |\nabla(u_\lambda - \bar{u})|^2 + \int_{\Omega^\epsilon} \nabla(u_\lambda - \bar{u})\nabla\phi + \frac{\lambda}{\epsilon}\int_\Omega (f(x,\bar{u}) - f(x,u_\lambda))\phi^\epsilon \\
&\quad + \frac{\lambda}{\epsilon}\int_\Omega (\bar{u}^q - u_\lambda^q w_\epsilon)\phi^\epsilon \\
&\geq \int_{\Omega^\epsilon} \nabla(u_\lambda - \bar{u})\nabla\phi + \frac{\lambda}{\epsilon}\int_\Omega (\bar{u}^q - u_\lambda^q w_\epsilon)\phi^\epsilon \\
&\geq \int_{\Omega^\epsilon} \nabla(u_\lambda - \bar{u})\nabla\phi = o_\epsilon(1) \text{ as } \epsilon \to 0
\end{aligned}$$

where $\Omega^\epsilon = \{x \in \Omega : (u_\lambda + \epsilon\phi)(x) \geq \bar{u}(x) \geq u_\lambda(x)\}$ and the last equality holds since $|\Omega^\epsilon| \to 0$ as $\epsilon \to 0$. Similarly, $\frac{1}{\epsilon}E_\epsilon \leq o_\epsilon(1)$. Hence,

$$\int_\Omega \nabla u_\lambda \nabla \phi - \lambda \int_\Omega f(x,u_\lambda)\phi - \lambda \int_\Omega u_\lambda^q w_\epsilon \phi \geq o_\epsilon(1), \text{ for all } \phi \in C_0^\infty(\Omega).$$

Letting $\epsilon \to 0$, we get

$$\int_\Omega \nabla u_\lambda \nabla \phi - \lambda \int_\Omega f(x,u_\lambda)\phi - \lambda \int_\Omega u_\lambda^q w_\lambda \phi \geq 0, \text{ for all } \phi \in C_0^\infty(\Omega)$$

where $w_\lambda(x) \in [\underline{g}(u_\lambda(x) - a), \bar{g}(u_\lambda(x) - a)]$ is the weak* limit of $\{w_\epsilon\}$ in $L^\infty(\Omega)$. This implies that $I_\lambda^0(u_\lambda, \phi) \geq 0$, for all $\phi \in C_0^\infty(\Omega)$. Thus u_λ solves (P_λ). □

Lemma 2.3. *Let $\Lambda = \sup\{\lambda > 0 : (P_\lambda)$ has at least one solution$\}$. Then $\Lambda < \infty$.*

Proof. Multiplying the equation in (P_λ) by the first eigen function ϕ_1 of $-\Delta$ and integrating by parts, we get

$$\lambda_1 \int_\Omega u\phi_1 = \lambda \int_\Omega \frac{m(x,u)e^{\alpha u^2}}{|x|^\beta}\phi_1 + \lambda \int_\Omega u^q g(u-a)\phi_1$$

Now, from (H2), for any $\epsilon > 0$ small and $t > 0$, there exists $K = K(a) > 0$ such that $\frac{m(x,t)e^{\alpha t^2}}{|x|^\beta} + t^q g(t-a) > Kt$. From this it follows that $\Lambda < \infty$. □

Lemma 2.4. *For all $\lambda \in (0, \Lambda]$, (P_λ) admits a solution.*

Proof. The existence of solution to (P_λ) for $\lambda \in (0, \bar{\Lambda}]$ follows from Lemma 2.2. Let $\bar{\Lambda} \leq \lambda \leq \Lambda$. Then by the definition of Λ, there exists $\bar{\lambda} \in (\lambda, \Lambda)$ such that $(P_{\bar{\lambda}})$ admits a solution, say $\bar{u}$. Then $\bar{u}$ is a supersolution for (P_λ), and $\bar{u} \geq \underline{u}$ where $\underline{u}$ is the unique solution of (Q_λ). This comparison can be obtained using the comparison theorem in Lemma 3.3 of [4] by taking $\underline{u}$ and $\bar{u}$ as sub and super solutions of $-\Delta u = \lambda u^q g(u-a)$ in Ω and $u = 0$ on $\partial\Omega$, respectively. Now considering the set $M_\lambda = \{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \bar{u}\}$ and proceeding as in Lemma 2.2 we obtain a solution of (P_λ). To show the existence of solution to (P_Λ), we take $\{\lambda_n\} \subset (0, \Lambda)$ such that $\lambda_n \to \Lambda$ as $n \to \infty$. As there exists solution u_{λ_n} of (P_{λ_n}) such that $u_{\lambda_n} \in C^{1,\gamma}(\Omega \backslash \{0\})$ for some $\gamma \in (0,1)$ and hence the $N-$ dimensional Lebesgue measure of the set $\{x \in \Omega : u_{\lambda_n}(x) = a\} = 0$, we have

$$\int_\Omega \nabla u_{\lambda_n} \nabla \phi = \lambda_n \left(\int_\Omega f(x, u_{\lambda_n})\phi + \int_\Omega g(u_{\lambda_n} - a) u_{\lambda_n}^q \phi \right) \text{ for all } \quad \phi \in H_0^1(\Omega). \tag{2.6}$$

Taking $\phi = u_n$ in (2.6),

$$\int_\Omega |\nabla u_{\lambda_n}|^2 = \lambda_n \left(\int_\Omega f(x, u_{\lambda_n}) u_{\lambda_n} + \int_\Omega g(u_{\lambda_n} - a) u_{\lambda_n}^{q+1} \right). \tag{2.7}$$

Moreover, by the definition of u_{λ_n} we have $I_{\lambda_n}(u_{\lambda_n}) = \min_{u \in M_{\lambda_n}} I_{\lambda_n}(u) \leq I_{\lambda_n}(\underline{u})$ $< \frac{1}{2}\|\underline{u}\|^2$. This implies,

$$\frac{1}{2}\|u_{\lambda_n}\|^2 < \frac{1}{2}\|\underline{u}\|^2 + \lambda_n \int_\Omega F(x, u_{\lambda_n}) + \lambda_n \int_\Omega H(u_{\lambda_n}). \tag{2.8}$$

From (2.7) and (2.8) we get

$$\begin{aligned} \frac{1}{2}\int_\Omega f(x, u_{\lambda_n}) u_{\lambda_n} - \int_\Omega F(x, u_{\lambda_n}) &< \frac{1}{2\lambda_n}\|\underline{u}\|^2 \\ &+ \int_\Omega H(u_{\lambda_n}) - \frac{1}{2}\int_\Omega g(u_{\lambda_n} - a) u_{\lambda_n}^{q+1} \end{aligned} \tag{2.9}$$

Also, from (H4), for any $K > 0$, there exists $C > 0$ such that

$$\frac{1}{2} f(x,t)t - F(x,t) \geq \left(\frac{1}{2} - \frac{1}{K}\right) f(x,t)t - C, \text{ for all } t > 0. \tag{2.10}$$

Thus, from (2.9) and (2.10), for $K > 2$ we have

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{K}\right) \int_\Omega f(x, u_{\lambda_n}) u_{\lambda_n} &< \frac{1}{2\lambda_n}\|\underline{u}\|^2 \\ &+ \int_\Omega H(u_{\lambda_n}) - \frac{1}{2}\int_\Omega g(u_{\lambda_n} - a) u_{\lambda_n}^{q+1} + C. \end{aligned} \tag{2.11}$$

Using (2.7), (2.11) and Sobolev inequality we estimate $\|u_{\lambda_n}\|$ as

$$\begin{aligned}\|u_{\lambda_n}\|^2 &< K_1\|\underline{u}\|^2 + K_2\int_\Omega u_{\lambda_n}^{q+1} + C\\ &\leq K_1\|\underline{u}\|^2 + K_2 S\|u_{\lambda_n}\|^{q+1} + C \end{aligned} \tag{2.12}$$

where S is the best constant in Sobolev imbedding. As $0 < q < 1$, $\{u_{\lambda_n}\}$ is bounded in $H_0^1(\Omega)$. Thus $u_{\lambda_n} \rightharpoonup u_\Lambda$ weakly for some $u_\Lambda \in H_0^1(\Omega)$. Also from (2.11) and (2.12) we get $\tilde{\mathrm{C}}= \sup_n \int_\Omega f(x,u_{\lambda_n})u_{\lambda_n} < \infty$. Given $\epsilon > 0$, define $\mu_\epsilon = \sup_{x\in\bar{\Omega},\ |s|\leq \frac{2\tilde{\mathrm{C}}}{\epsilon}} |f(x,s)|$. Then for any $A \subset \Omega$ with $|A| < \frac{\epsilon}{2\mu_\epsilon}$ we get,

$$\begin{aligned}\int_A |f(x,u_{\lambda_n})| &\leq \int_{A\cap\{|u_{\lambda_n}|>\frac{2\tilde{\mathrm{C}}}{\epsilon}\}} \frac{|f(x,u_{\lambda_n})u_{\lambda_n}|}{|u_{\lambda_n}|}\\ &\quad + \int_{A\cap\{|u_{\lambda_n}|\leq\frac{2\tilde{\mathrm{C}}}{\epsilon}\}} |f(x,u_{\lambda_n})| < \epsilon.\end{aligned}$$

Thus by Vitali's convergence theorem we have $\int_\Omega f(x,u_{\lambda_n}) \to \int_\Omega f(x,u_\Lambda)$ which in turn implies $\int_\Omega f(x,u_{\lambda_n})\phi \to \int_\Omega f(x,u_\Lambda)\phi$ for all $\phi \in C_0^\infty(\Omega)$. Also using dominated convergence theorem it is easy to check that $\int_\Omega g(u_{\lambda_n} - a)u_{\lambda_n}^q\phi \to \int_\Omega g(u_\Lambda - a)u_\Lambda^q\phi$ for all $\phi \in C_0^\infty(\Omega)$. Hence by passing through the limit in (2.6), we get u_Λ as a solution to (P_Λ). Also by using the comparison theorem of Lemma 3.3 in [4] we have $\underline{u} \leq u_\Lambda$ and thus $u_\Lambda \not\equiv 0$. □

3. $H_0^1(\Omega)$ local minimum of I_λ for $\lambda \in (0,\Lambda)$

In this section we prove that for all $\lambda \in (0,\Lambda)$ and $a > 0$, u_λ (the solution of (P_λ) obtained in Sect. 2) is also a local minimum for I_λ in $H_0^1(\Omega)$ topology.

Theorem 3.1. *For $\lambda \in (0,\Lambda)$ and $a > 0$, u_λ is a local minimum for I_λ in $H_0^1(\Omega)$.*

Proof. Here we adopt the approach as in [13]. Let us suppose by contradiction that there exists $\lambda \in (0,\Lambda)$ and a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$ and $I_\lambda(u_n) < I_\lambda(u_\lambda)$. Let $\underline{u}$, $\bar{u}$ and u_λ be the subsolution, supersolution and solution of (P_λ), respectively as in Lemma 2.4. Define $v_n = \max\{\underline{u}, \min\{u_n,\bar{u}\}\}$, $\bar{w_n} = (u_n - \bar{u})^+$, $\underline{w_n} = (u_n - \underline{u})^-$, $\bar{S_n} =$ support $(\bar{w_n})$ and $\underline{S_n} =$ support $(\underline{w_n})$. Then $u_n = v_n - \underline{w}_n + \bar{w}_n$, $v_n \in M = \{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \bar{u}\}$ and $I_\lambda(u_n) = I_\lambda(v_n) + A_n + B_n \geq I_\lambda(u_\lambda) + A_n + B_n$ where,

$$\begin{aligned}A_n = \frac{1}{2}\int_{\bar{S}_n}(|\nabla u_n|^2 - |\nabla \bar{u}|^2) &- \lambda\int_{\bar{S}_n}(F(x,u_n) - F(x,\bar{u}))\\ &-\lambda\int_{\bar{S}_n}(H(u_n) - H(\bar{u})).\end{aligned}$$

$$B_n = \frac{1}{2}\int_{\underline{S}_n} (|\nabla u_n|^2 - |\nabla \underline{u}|^2) - \lambda \int_{\underline{S}_n} (F(x,u_n) - F(x,\underline{u}))$$
$$-\lambda \int_{\underline{S}_n} (H(u_n) - H(\underline{u})).$$

Now we claim that $A_n,\ B_n \geq 0$ for large n which is a contradiction to our assumption that $I_\lambda(u_n) < I_\lambda(u_\lambda)$.

Since $\bar{u}$ is supersolution for (P_λ) we estimate A_n as follows.

$$\begin{aligned} A_n &\geq \frac{1}{2}\|\bar{w}_n\|^2 + \lambda \int_{\bar{S}_n} f(x,\bar{u})\bar{w}_n + \lambda \int_{\bar{S}_n} h(\bar{u})\bar{w}_n \\ &\quad -\lambda \int_{\bar{S}_n} (F(x,\bar{u}+\bar{w}_n) - F(x,\bar{u})) \\ &\quad -\lambda \int_{\bar{S}_n} (H(\bar{u}+\bar{w}_n) - H(\bar{u})) \\ &= \frac{1}{2}\|\bar{w}_n\|^2 + \lambda \int_{\bar{S}_n} (f(x,\bar{u}) - f(x,\bar{u}+\theta\bar{w}_n))\bar{w}_n \\ &\quad +\lambda \int_{\bar{S}_n} h(\bar{u})\bar{w}_n - (H(\bar{u}+\bar{w}_n) - H(\bar{u})) \end{aligned}$$

As $\lim\limits_{t\to\infty}\sup\limits_{x\in\bar{\Omega}} \dfrac{\partial f}{\partial t} e^{-\alpha(1+\epsilon)t^2} = 0$, we can estimate the critical term as in [2] to get,

$$\int_{\bar{S}_n} (f(x,\bar{u}+\theta\bar{w}_n) - f(x,\bar{u}))\bar{w}_n = C_1|\bar{S}_n|^{1/4}\|\bar{w}_n\|^2.$$

Now to estimate the discontinuous term we divide the domain $\bar{S_n}$ into three subdomains, viz., $\Omega_1 = \bar{S_n} \cap \{x \in \Omega : \bar{u}(x) > a\}$, $\Omega_2 = \bar{S_n} \cap \{x \in \Omega : \bar{u}(x) \leq a < (\bar{u}+\bar{w}_n)(x)\}$ and $\Omega_3 = \bar{S_n} \cap \{x \in \Omega : (\bar{u}+\bar{w}_n)(x) \leq a\}$. Since in $\Omega_1,\ h(\bar{u}) = 0, H(\bar{u}) = H(\bar{u}+\bar{w}_n) = \dfrac{a^{q+1}}{q+1}$ we get $\displaystyle\int_{\Omega_1} [h(\bar{u})\bar{w}_n - (H(\bar{u}+\bar{w}_n) - H(\bar{u}))] = 0$. In Ω_2, we have $h(\bar{u}) = \bar{u}^q$, $H(\bar{u}) = \dfrac{\bar{u}^{q+1}}{q+1}$ and $H(\bar{u}+\bar{w}_n) = \dfrac{a^{q+1}}{q+1}$ and thus estimate the corresponding integral as,

$$\begin{aligned} \int_{\Omega_2} [h(\bar{u})\bar{w}_n - (H(\bar{u}+\bar{w}_n) - H(\bar{u}))] &= \int_{\Omega_2} \bar{u}^q\bar{w}_n - \left(\frac{a^{q+1}}{q+1} - \frac{\bar{u}^{q+1}}{q+1}\right) \\ &= \int_{\Omega_2} \bar{u}^q\bar{w}_n - (a-\bar{u})(\bar{u}+\theta(a-\bar{u}))^q, \\ &\qquad \theta \in (0,1) \\ &\geq \int_{\Omega_2} (\bar{u}^q - (\bar{u}+\theta(a-\bar{u}))^q)\bar{w}_n \geq -q\int_{\Omega} \frac{\bar{w}_n{}^2}{\bar{u}^{1-q}} \end{aligned} \tag{3.13}$$

In Ω_3, $h(\bar{u}) = \bar{u}^q$, $H(\bar{u}) = \dfrac{\bar{u}^{q+1}}{q+1}$ and $H(\bar{u}+\bar{w}_n) = \dfrac{(\bar{u}+\bar{w}_n)^{q+1}}{q+1}$ and hence the corresponding integral over Ω_3 is estimated from below as

$$\begin{aligned}\int_{\Omega_3}[h(\bar{u})\bar{w}_n - (H(\bar{u}+\bar{w}_n) - H(\bar{u}))] &= \textstyle\int_{\Omega_3} \bar{u}^q\bar{w}_n - \frac{1}{q+1}((\bar{u}+\bar{w}_n)^{q+1} - \bar{u}^{q+1})\\ &= -\textstyle\int_{\Omega_3}((\bar{u}+\theta\bar{w}_n)^q - \bar{u}^q)\bar{w}_n,\ \theta\in(0,1)\end{aligned} \tag{3.14}$$

Since $\bar{u}$ is a supersolution of (P_λ), by Hopf maximum principle, there exists $C' > 0$ such that $\bar{u} \geq C' d(x,\partial\Omega)$ where $d(x,\partial\Omega)$ =distance of x from $\partial\Omega$. Hence, noting that $d^\gamma(x,\partial\Omega)\in L^1(\Omega)$ for $\gamma > -1$ and using Hölder's inequality with $r > 1$ such that $(1-q)r < 1$ and Sobolev inequality we get

$$\begin{aligned}\int_\Omega \frac{\bar{w_n}^2}{\bar{u}^{1-q}} \leq C\int_\Omega \frac{\bar{w_n}^2}{(d(x,\partial\Omega))^{1-q}} &\leq C\left(\int_\Omega \frac{1}{(d(x,\partial\Omega))^{r(1-q)}}\right)^{1/r}\|\bar{w_n}\|^2_{L^{2r/(r-1)}}\\ &\leq C_2\|\bar{w}_n\|^2.\end{aligned}$$

Using this in (3.13) and (3.14), we have the following estimation for A_n.

$$A_n \geq \tfrac{1}{2}\|\bar{w}_n\|^2 - C_1\ |\bar{S}_n|^{1/4}\|\bar{w}_n\|^2 - C_2\|\bar{w}_n\|^2,$$

Similarly, we get $B_n \geq \frac{1}{2}\|\underline{w}_n\|^2 - C_1\ |\underline{S}_n|^{1/4}\|\underline{w}_n\|^2 - C_2\|\underline{w}_n\|^2$.

Now we prove that $|\bar{S}_n|, |\underline{S}_n|$, $\|\bar{w}_n\|$ and $\|\underline{w}_n\| \to 0$ as $n\to\infty$. To obtain this first we claim that there exist a constant $C > 0$ such that $\bar{u}(x) - u_\lambda(x) > Cd(x,\partial\Omega)$ and $\underline{u} - u_\lambda < -Cd(x,\partial\Omega)$ in Ω. Since $\bar{u}$ is not a solution of (P_λ), we can choose a ball $B\subset\Omega$ such that $\bar{u}\geq u_\lambda + 2\delta$ in B for some $\delta > 0$. Now consider the following problem

$$\begin{aligned}-\Delta v &= \lambda u_\lambda^q\Psi(v - (\bar{u} - u_\lambda)) \quad \text{in} \quad \Omega\backslash B,\\ v &= \delta \quad \text{on } \partial B,\\ v &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where $\Psi(s) = 1$ if $s\leq 0, \Psi(s) = -1$ if $s > 0$. Then by the standard elliptic theory there exists $v\in W^{2,p}(\Omega\backslash B)\cap C^{1,\gamma}(\overline{\Omega\backslash B})$ for some $\gamma\in(0,1)$ and for all $p\geq 1$. Also taking v^- as the test function in the above problem and noting that $\bar{u}\geq u_\lambda$ in $\Omega\backslash B$, we have $v\geq 0$ in $\Omega\backslash B$. Furthermore, as $m(x,t)$ is nondecreasing in t, we get

$$-\Delta(\bar{u} - u_\lambda) \geq \lambda(\bar{u}^q g(\bar{u}-a) - u_\lambda^q g(u_\lambda - a)) \geq -\lambda u_\lambda^q \text{ in } \Omega,$$

and $\bar{u} - u_\lambda \geq 2\delta$ on ∂B, $\bar{u} - u_\lambda = 0$ on $\partial\Omega$. Thus,

$$\begin{aligned}-\Delta(\bar{u} - u_\lambda - v) &\geq \lambda(-u_\lambda^q - u_\lambda^q\Psi(v - (\bar{u} - u_\lambda)) \text{ in } \Omega\backslash B,\\ \bar{u} - u_\lambda - v &\geq \delta \quad \text{on } \partial B,\\ \bar{u} - u_\lambda - v &= 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{3.15}$$

Taking $(\bar{u} - u_\lambda - v)^-$ as the test function in (3.15) and integrating over $\Omega\backslash B$, we have

$$-\int_{\Omega\backslash B}|\nabla(\bar{u} - u_\lambda - v)^-|^2 \geq \lambda\int_{\Omega\backslash B}[-u_\lambda^q - u_\lambda^q\Psi(v - (\bar{u} - u_\lambda))](\bar{u} - u_\lambda - v)^-$$

Now for $(\bar{u} - u_\lambda - v)^- > 0$, $\Psi(v - (\bar{u} - u_\lambda)) = -1$ and thus the right hand side in the above inequality is zero. This implies $(\bar{u} - u_\lambda - v)^- \equiv 0$ in $\Omega\backslash B$, i.e., $v \le \bar{u} - u_\lambda$ in $\Omega\backslash B$. Therefore, $-\Delta v = \lambda u_\lambda^q$ in $\Omega\backslash B$, $v \in C^{1,\gamma}(\Omega\backslash B)$, $v > 0$ in $\Omega\backslash B$ and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$ where ν is the outward unit normal at $\partial\Omega$. Thus we can find $C_1 > 0$ such that $v(x) \ge C_1 d(x, \partial\Omega)$ for all $x \in \Omega\backslash B$ and hence $\bar{u} - u_\lambda \ge Cd(x, \partial\Omega)$ for all $x \in \Omega$ and $C > 0$ sufficiently small.

Similarly, as $\underline{u}$ is a subsolution for (P_λ), $\underline{u} - u_\lambda \le -2\delta$ in B for some ball $B \subset \Omega$ and $\delta > 0$. Then considering a solution v of the problem

$$\begin{aligned} -\Delta v = \lambda u_\lambda^q \Psi(v - (\underline{u} - u_\lambda)) &\quad \text{in } \Omega\backslash B,\\ v = -\delta &\quad \text{on } \partial B,\\ v = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

and using the similar arguments as above we can find a $C_2 > 0$ such that $v(x) \le -C_2 d(x, \partial\Omega)$ for $x \in \Omega\backslash B$ and $\underline{u} - u_\lambda \le v$ in $\Omega\backslash B$. Thus $\underline{u} - u_\lambda \le -Cd(x, \partial\Omega)$ in Ω for $C > 0$ sufficiently small. This proves the claim.

Now to estimate $|\bar{S}_n|$, we take $\Omega_\sigma = \{x \in \Omega : d(x, \partial\Omega) > \sigma\}$ and note that for any $\epsilon > 0$ there exists a $\sigma > 0$ such that $|\Omega \setminus \Omega_\sigma| < \frac{\epsilon}{2}$. Since $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$, we have

$$\begin{aligned} |\bar{S}_n| &\le |\Omega \setminus \Omega_\sigma| + |\bar{S}_n \cap \Omega_\sigma|\\ &< \frac{\epsilon}{2} + \frac{1}{C^2\sigma^2} \int_{\bar{S}_n \cap \Omega_\sigma} (u_n - u_\lambda)^2 < \epsilon. \end{aligned}$$

Therefore,

$$\|\bar{w}_n\|^2 = \int_{\bar{S}_n} |\nabla(u_n - \bar{u})|^2 \le 2\left(\|(u_n - u_\lambda)\|^2 + \int_{\bar{S}_n} |\nabla(u_\lambda - \bar{u})|^2\right) \to 0 \text{ as } n \to \infty.$$

Also, using the same approach as for $|\bar{S}_n|$ and $\|\bar{w}_n\|$, we get $|\underline{S}_n|$ and $\|\underline{w}_n\| \to 0$ as $n \to \infty$ and thus $B_n \ge 0$ for n large. This completes the proof of the Theorem 3.1. □

4. Existence of second solution

Now we prove the existence of the second critical point of I_λ using Ekeland's variational principle on the set $\Lambda_\lambda = \{u \in H_0^1(\Omega) : u \ge u_\lambda \text{ a.e. in } \Omega\}$. Since u_λ is a local minimum for I_λ in $H_0^1(\Omega)$, there exists $\rho_0 > 0$ such that $I_\lambda(u) \ge I_\lambda(u_\lambda)$ for $\{u \in H_0^1(\Omega) : \|u - u_\lambda\| \le \rho_0\}$. Furthermore one of the following holds:

(m_1) $\inf\{I_\lambda(u) : u \in \Lambda_\lambda, \|u - u_\lambda\| = \rho\} = I_\lambda(u_\lambda)$, for all $\rho \in (0, \rho_0)$.

(m_2) There exists $\rho_1 \in (0, \rho_0)$ such that $\inf\{I_\lambda(u) : u \in \Lambda_\lambda, \|u - u_\lambda\| = \rho_1\} > I_\lambda(u_\lambda)$.

Here in case of (m_1) we use the generalized Mountain pass lemma of Ghoussoub and Preiss [11] and in case of (m_2) we use the classical Mountain pass lemma.

Lemma 4.1. *Let (m_1) holds. Then for each $\rho \in (0, \rho_0)$ there exists $v_\lambda \in \Lambda_\lambda$ such that $\|v_\lambda - u_\lambda\| = \rho$ and v_λ is a critical point of I_λ.*

Proof. From (m_1), for any $\rho \in (0, \rho_0)$ there exists a sequence $\{u_n\} \subset \Lambda_\lambda$ such that

$$\|u_n - u_\lambda\| = \rho \text{ and } I_\lambda(u_n) \leq I_\lambda(u_\lambda) + \frac{1}{n}.$$

For $0 < \delta < \min\{\rho_0 - \rho, \rho\}$, define the closed subset $R = \{u \in \Lambda_\lambda : \rho - \delta \leq \|u - u_\lambda\| \leq \rho + \delta\}$. Then we have $\inf\{I_\lambda(u) : u \in R\} = I_\lambda(u_\lambda)$. Since I_λ is lower semicontinuous on R, by Ekeland's variational principle there exists a sequence $\{v_n\} \subset R$ such that $I_\lambda(v_n) \leq I_\lambda(u_n) \leq I_\lambda(u_\lambda) + \frac{1}{n}$, $\|u_n - v_n\| \leq \frac{1}{n}$ and $I_\lambda(v_n) \leq I_\lambda(v) + \frac{1}{n}\|v - v_n\|$ for all $v \in R$. For any $w \in \Lambda_\lambda$ choose $\epsilon > 0$ sufficiently small such that $v_n + \epsilon(w - v_n) \in R$ and thus we get

$$\frac{I_\lambda(v_n + \epsilon(w - v_n)) - I_\lambda(v_n)}{\epsilon} \geq -\frac{1}{n}\|w - v_n\|,$$

and letting $\epsilon \to 0^+$ we get

$$I_\lambda^0(v_n, w - v_n) \geq -\frac{1}{n}\|w - v_n\|. \tag{4.16}$$

Since v_n is bounded in $H_0^1(\Omega)$, up to a subsequence, $v_n \rightharpoonup v_\lambda \in \Lambda_\lambda$ weakly in $H_0^1(\Omega)$. We claim that v_λ is a critical point of I_λ. To prove this, for $\phi \in C_0^\infty(\Omega)$, set $\varphi_{n,\epsilon} = (v_n + \epsilon\phi - u_\lambda)^-$. Then choosing $w = (v_n + \epsilon\phi + \varphi_{n,\epsilon}) \in \Lambda_\lambda$, from (4.16) we get

$$I_\lambda^0(v_n, \epsilon\phi + \varphi_{n,\epsilon}) \geq -\frac{1}{n}\|\epsilon\phi + \varphi_{n,\epsilon}\|.$$

Thus from the results on generalized gradients in [8], for each $n \in \mathbb{N}$, there exists $w_{n,\epsilon} \in L^\infty(\Omega)$ with $w_{n,\epsilon}(x) \in [\underline{g}(v_n(x) - a), \bar{g}(v_n(x) - a)]$ and

$$\begin{aligned}\int_\Omega \nabla v_n \nabla(\epsilon\phi + \varphi_{n,\epsilon}) - \lambda \int_\Omega f(x, v_n)(\epsilon\phi + \varphi_{n,\epsilon}) - \lambda \int_\Omega v_n^q w_{n,\epsilon}(\epsilon\phi + \varphi_{n,\epsilon})& \\ \geq -\frac{1}{n}\|\epsilon\phi + \varphi_{n,\epsilon}\|.&\end{aligned} \tag{4.17}$$

As $\|w_{n,\epsilon}\|_{L^\infty(\Omega)} \leq 1$ for all n, we can assume that $w_{n,\epsilon} \to w_\epsilon$ in $L^\infty(\Omega)$ as $n \to \infty$ with respect to the weak* topology and $w_\epsilon(x) \in [\underline{g}(v_\lambda(x) - a), \bar{g}(v_\lambda(x) - a)]$. Now following the approach as in [5], we define $\Omega_{n,\epsilon} = \{x \in \Omega : v_n(x) + \epsilon\phi(x) \leq u_\lambda(x)\}$, $\Omega_\epsilon = \{x \in \Omega : v_\lambda(x) + \epsilon\phi(x) \leq u_\lambda(x)\}$ and $\Omega_0 = \{x \in \Omega : v_\lambda(x) = u_\lambda(x)\}$.

Since $0 \leq \varphi_{n,\epsilon} \leq u_\lambda + \epsilon|\phi|$ and $v_n \leq u_\lambda - \epsilon\phi$ in $\Omega_{n,\epsilon}$, by dominated convergence theorem we get $\varphi_{n,\epsilon} \to \varphi_\epsilon$ in $L^2(\Omega)$ and $\int_\Omega f(x, v_n)(\epsilon\phi + \varphi_{n,\epsilon}) \to \int_\Omega f(x, v_\lambda)(\epsilon\phi + \varphi_\epsilon)$ as $n \to \infty$ where $\varphi_\epsilon = (v_\lambda + \epsilon\phi - u_\lambda)^-$. Also $\int_\Omega w_{n,\epsilon}\varphi_{n,\epsilon}v_n^q \to \int_\Omega w_\epsilon\varphi_\epsilon v_\lambda^q$ as $n \to \infty$. Note that $|\Omega_{n,\epsilon} \backslash \Omega_\epsilon|$, $|\Omega_\epsilon \backslash \Omega_{n,\epsilon}| \to 0$ as $n \to \infty$ and $|\Omega_\epsilon \backslash \Omega_0| \to 0$ as $\epsilon \to 0$. The first term in (4.17) can be estimated as

$$\begin{aligned}\int_\Omega \nabla v_n \nabla \varphi_{n,\epsilon} &= \int_{\Omega_{n,\epsilon}} \nabla v_n \nabla (u_\lambda - \epsilon\phi - v_\lambda) + \int_{\Omega_{n,\epsilon}} \nabla v_n \nabla (v_\lambda - v_n)\\ &= \int_{\Omega_\epsilon} \nabla v_n \nabla \varphi_\epsilon - \int_{\Omega_{n,\epsilon}} |\nabla (v_n - v_\lambda)|^2 + \int_{\Omega_{n,\epsilon}} \nabla v_\lambda \nabla (v_\lambda - v_n) + o_n(1)\\ &\leq \int_\Omega \nabla v_n \nabla \varphi_\epsilon + \int_{\Omega_\epsilon} \nabla v_\lambda \nabla (v_\lambda - v_n) + o_n(1) = \int_\Omega \nabla v_\lambda \nabla \varphi_\epsilon + o_n(1)\end{aligned}$$

Now as $\|\varphi_{n,\epsilon}\| \leq C$ for all $n \in \mathbb{N}$ and $\epsilon \in (0,1)$, by passing to the limit in (4.17) as $n \to \infty$ we get,

$$\int_\Omega \nabla v_\lambda \nabla (\epsilon\phi + \varphi_\epsilon) - \lambda \int_\Omega f(x, v_\lambda)(\epsilon\phi + \varphi_\epsilon) - \lambda \int_\Omega v_\lambda^q w_\epsilon (\epsilon\phi + \varphi_\epsilon) \geq 0. \quad (4.18)$$

Using the facts that $f(x,t)$ is nondecreasing in t, $\varphi_\epsilon \in H_0^1(\Omega)$, u_λ is a critical point for I_λ and $|\{x \in \Omega : u_\lambda(x) = a\}| = 0$, from (4.18) we have the following estimate.

$$\begin{aligned}&\epsilon \left(\int_\Omega \nabla v_\lambda \nabla \phi - \lambda \int_\Omega f(x, v_\lambda)\phi - \lambda \int_\Omega w_\epsilon \phi v_\lambda^q \right)\\ &\geq - \int_\Omega \nabla (v_\lambda - u_\lambda) \nabla \varphi_\epsilon + \lambda \int_\Omega (f(x, v_\lambda) - f(x, u_\lambda))\varphi_\epsilon\\ &\quad + \lambda \int_\Omega (w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q)\varphi_\epsilon\\ &\geq - \int_\Omega \nabla (v_\lambda - u_\lambda) \nabla \varphi_\epsilon + \lambda \int_\Omega (w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q)\varphi_\epsilon \quad (4.19)\end{aligned}$$

Now as $|\Omega_\epsilon \backslash \Omega_0| \to 0$ for $\epsilon \to 0$ and $v_\lambda = u_\lambda$ on Ω_0, the first term on the right-hand side of (4.19) can be estimated as follows:

$$\begin{aligned}- \int_\Omega \nabla (v_\lambda - u_\lambda) \nabla \varphi_\epsilon &= \int_{\Omega_\epsilon} |\nabla (u_\lambda - v_\lambda)|^2 + \epsilon \int_{\Omega_\epsilon} \nabla (v_\lambda - u_\lambda) \nabla \varphi\\ &\geq \epsilon \int_{\Omega_\epsilon} \nabla (v_\lambda - u_\lambda) \nabla \varphi\\ &= \epsilon \int_{\Omega_\epsilon \backslash \Omega_0} \nabla (v_\lambda - u_\lambda) \nabla \varphi = o_\epsilon(\epsilon)\end{aligned}$$

Also as $(w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q) = 0$ a.e. in Ω_0, we estimate the second term on the right-hand side of (4.19) as

$$\begin{aligned}\int_\Omega (w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q)\varphi_\epsilon &= \int_{\Omega_\epsilon \cap \Omega_0} (w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q)\varphi_\epsilon\\ &\quad + \int_{\Omega_\epsilon \backslash \Omega_0} (w_\epsilon v_\lambda^q - g(u_\lambda - a) u_\lambda^q)\varphi_\epsilon\\ &\geq - \int_{\Omega_\epsilon \backslash \Omega_0} u_\lambda^q (u_\lambda - \epsilon\phi - v_\lambda)\\ &\geq -C\epsilon \int_{\Omega_\epsilon \backslash \Omega_0} u_\lambda^q |\phi| = o_\epsilon(\epsilon)\end{aligned}$$

Therefore, from (4.19) it follows that

$$\int_\Omega \nabla v_\lambda \nabla \phi - \lambda \int_\Omega f(x, v_\lambda)\phi - \lambda \int_\Omega w_\epsilon \phi v_\lambda^q \geq \frac{1}{\epsilon} o_\epsilon(\epsilon).$$

Thus as $\epsilon \to 0$ we get $I_\lambda^0(v_\lambda, \phi) \geq 0,$ for all $\phi \in C_0^\infty(\Omega)$. That is, v_λ is a critical point for I_λ. Now we prove that $v_\lambda \not\equiv u_\lambda$. In view of the definition of R, it suffices to prove that $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. Taking $w = v_\lambda$ in (4.16) we get $w_n \in L^\infty(\Omega)$ with $w_n(x) \in [\underline{g}(v_n(x) - a), \bar{g}(v_n(x) - a)]$ such that

$$\int_\Omega \nabla v_n \nabla (v_\lambda - v_n) - \lambda \int_\Omega f(x, v_n)(v_\lambda - v_n) - \lambda \int_\Omega v_n^q w_n (v_\lambda - v_n) \geq -\frac{1}{n}\|v_\lambda - v_n\|.$$

This implies,

$$\int_\Omega \nabla v_\lambda \nabla (v_\lambda - v_n) - \lambda \int_\Omega f(x, v_n)(v_\lambda - v_n) - \lambda \int_\Omega v_n^q w_n (v_\lambda - v_n) + \frac{1}{n}\|v_\lambda - v_n\| \geq \|v_\lambda - v_n\|^2. \tag{4.20}$$

We claim that $\int_\Omega f(x, v_n)v_n \to \int_\Omega f(x, v_\lambda)v_\lambda$. As in [2], from $(H2)$, for $\epsilon > 0$ small, we have

$$\begin{aligned}\int_{v_n \geq k} f(x, v_n)v_n &= \int_{v_n \geq k} \frac{m(x, v_n)e^{\alpha v_n^2} v_n}{|x|^\beta} \\ &= O\left(e^{-3\pi k^2} \int_\Omega \frac{e^{8\pi v_n^2}}{|x|^\beta}\right) = O\left(e^{-3\pi k^2} \int_\Omega \frac{e^{16\pi u_\lambda^2 + 16\pi(v_n - u_\lambda)^2}}{|x|^\beta}\right) \\ &= O(e^{-3\pi k^2}),\end{aligned}$$

where the last equality follows from the following estimate.

$$\begin{aligned}\int_\Omega \frac{e^{16\pi u_\lambda^2 + 16\pi(v_n - u_\lambda)^2}}{|x|^{\beta p}} &\leq \left(\int_\Omega \frac{e^{16\pi p' u_\lambda^2}}{|x|^{\beta p'}}\right)^{\frac{1}{p'}} \left(\int_\Omega \frac{e^{16\pi p (v_n - u_\lambda)^2}}{|x|^{\beta p}}\right)^{\frac{1}{p}} \\ &\leq C \left(\int_\Omega \frac{e^{16\pi p \|v_n - u_\lambda\|^2 \frac{(v_n - u_\lambda)^2}{\|v_n - u_\lambda\|^2}}}{|x|^{\beta p}}\right)^{\frac{1}{p}}\end{aligned}$$

with $p > 1$ such that $\beta p < 2$ and $p' = \frac{p}{p-1}$. Note that as $v_n \in R$ and hence $\|v_n - u_\lambda\| \leq \rho + \delta$, we can choose ρ and δ small such that the right hand side of the above inequality is finite for all n, thanks to (1.4). Hence, by the dominated convergence theorem, letting $n \to \infty$ and $k \to \infty$, we get $\int_\Omega f(x, v_n)v_n \to \int_\Omega f(x, v_\lambda)v_\lambda$. Thus, by passing through the limit in (4.20), we conclude that $\|v_n - v_\lambda\| \to 0$ as $n \to \infty$. □

To obtain the second solution in case of (m_2), we need to study the compactness of the Palais Smale sequence. First, we recall the following Moser functions ϕ_n

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}} & |x| \le \frac{1}{n}, \\ \frac{\log(\frac{1}{r})}{(\log n)^{\frac{1}{2}}} & \frac{1}{n} \le |x| \le 1, \\ 0 & |x| \ge 1. \end{cases}$$

Define $\phi_n^{d_n}(x) = \phi_n(\frac{x-x_0}{d_n})$ where $d_n = (\log n)^{-1/2}$ and x_0 is choosen such that $B_{d_n}(x_0) \subset \Omega$ as $n \to \infty$. It is easy to check that $\|\phi_n^{d_n}\| = 1$.

Lemma 4.2. *Assume* $(H1) - (H5)$*. Then we have the following.*

(i) $I_\lambda(u_\lambda + t\phi_n^{d_n}) \to -\infty$ *as* $t \to \infty$.

(ii) $\sup\limits_{t>0} I_\lambda(u_\lambda + t\phi_n^{d_n}) < I_\lambda(u_\lambda) + \dfrac{\pi}{\alpha}(2-\beta)$.

Proof. Using $(H2)$, we can find positive constants C_1 and C_2, such that for $t \ge 0$ and $0 < \epsilon < \alpha$, $F(x,t) \ge C_1 \dfrac{e^{(\alpha-\epsilon)t^2}}{|x|^\beta} - C_2$. Thus, we have

$$\begin{aligned} I_\lambda(u_\lambda + t\phi_n^{d_n}) &\le O(t^2 + t^{1+q} + 1) - C \int_{B(x_0, \frac{d_n}{n})} \frac{e^{(\alpha-\epsilon)t^2 |\phi_n^{d_n}|^2}}{|x|^\beta} \\ &\le O(t^2 + t^{1+q} + 1) - C d_n^2 n^{(\alpha-\epsilon)t^2} \to -\infty \text{ as } t \to \infty. \end{aligned}$$

This proves (i). Now contrary to (ii), suppose there exists a subsequence, say $\{\phi_n^{d_n}\}$, such that

$$\max_{t \ge 0} I_\lambda(u_\lambda + t\phi_n^{d_n}) \ge I_\lambda(u_\lambda) + \frac{\pi}{\alpha}(2-\beta) \tag{4.21}$$

where

$$\begin{aligned} I_\lambda(u_\lambda + t\phi_n^{d_n}) &= \frac{1}{2}\int_\Omega |\nabla(u_\lambda + t\phi_n^{d_n})|^2 - \lambda \int_\Omega F(x, u_\lambda + t\phi_n^{d_n}) - \lambda \int_\Omega H(u_\lambda + t\phi_n^{d_n}) \\ &= I_\lambda(u_\lambda) + \frac{t^2}{2} + \lambda \int_\Omega F(x,u_\lambda) - F(x, u_\lambda + t\phi_n^{d_n}) + t\phi_n^{d_n} f(x,u_\lambda) \\ &\quad + \lambda \int_\Omega H(u_\lambda) - H(u_\lambda + t\phi_n^{d_n}) + t\phi_n^{d_n} u_\lambda^q g(u_\lambda - a) \end{aligned} \tag{4.22}$$

Also from (H5), for some $\theta_n \in (0,1)$, we get

$$\begin{aligned} \int_\Omega F(x,u_\lambda) - F(x,u_\lambda + t\phi_n^{d_n}) + t\phi_n^{d_n} f(x,u_\lambda) &= -\lambda \int_\Omega \frac{\partial f}{\partial t}(u_\lambda + t\theta_n \phi_n^{d_n}) t^2 |\phi_n^{d_n}|^2 \\ &\le C t^2 \int_\Omega |\phi_n^{d_n}|^2 = t^2\, O\left(\frac{1}{\log n}\right) \end{aligned} \tag{4.23}$$

We estimate the last term on the right hand side of (4.22) as follows.

$$\begin{aligned} \int_\Omega \big(H(u_\lambda) - H(u_\lambda + t\phi_n^{d_n}) + t\phi_n^{d_n} u_\lambda^q g(u_\lambda - a)\big) &\le C \int_\Omega t\phi_n^{d_n} \\ &= t\, O\left(\frac{d_n}{(\log n)^{\frac{1}{2}}}\right) \end{aligned} \tag{4.24}$$

Now by (i) of Lemma 4.2 and (4.21), there exists $t_n \in (0, \infty)$ with $t_n \leq K$, for some $K > 0$ such that

$$\max_{t \geq 0} I_\lambda(u_\lambda + t\phi_n^{d_n}) = I_\lambda(u_\lambda + t_n\phi_n^{d_n}) \tag{4.25}$$

Thus from (4.21)–(4.25) we get,

$$t_n^2 \geq \frac{2\pi}{\alpha}(2-\beta) - O\left(\frac{d_n}{(\log n)^{\frac{1}{2}}}\right). \tag{4.26}$$

Also from (4.25) we have $I_\lambda(u_\lambda + t_n\phi_n^{d_n}) \geq I_\lambda(u_\lambda + t_n\phi_n^{d_n} + t\phi_n^{d_n})$ for $t > 0$. This implies

$$\limsup_{t \downarrow 0} \frac{I_\lambda(u_\lambda + t_n\phi_n^{d_n}) - I_\lambda(u_\lambda + t_n\phi_n^{d_n} + t\phi_n^{d_n})}{t} \geq 0.$$

That is, $(-I_\lambda)^0(u_\lambda + t_n\phi_n^{d_n}, \phi_n^{d_n}) \geq 0$. Thus $0 \in \partial(-I_\lambda)(u_\lambda + t_n\phi_n^{d_n})$ and hence $0 \in \partial I_\lambda(u_\lambda + t_n\phi_n^{d_n})$. So there exists $w \in L^\infty(\Omega)$ such that

$$w(x) \in \left[\underline{g}\left((u_\lambda + t_n\phi_n^{d_n})(x) - a\right), \bar{g}\left((u_\lambda + t_n\phi_n^{d_n})(x) - a\right)\right]$$

and

$$\int_\Omega \nabla(u_\lambda + t_n\phi_n^{d_n})\nabla v - \lambda \int_\Omega f(x, u_\lambda + t_n\phi_n^{d_n})v$$
$$- \lambda \int_\Omega (u_\lambda + t_n\phi_n^{d_n})^q wv = 0 \text{ for all } v \in H_0^1(\Omega).$$

Now taking $v = t_n\phi_n^{d_n}$ in the above equation we get

$$t_n^2 + t_n \int_\Omega \nabla u_\lambda \nabla\phi_n^{d_n} - \lambda \int_\Omega (u_\lambda + t_n\phi_n^{d_n})^q w t_n\phi_n^{d_n} = \lambda \int_\Omega f(x, u_\lambda + t_n\phi_n^{d_n})t_n\phi_n^{d_n}. \tag{4.27}$$

Now we estimate the right hand side of (4.27). Define $C_n = \min_{|x - x_0| \leq \frac{d_n}{n}} u_\lambda(x)$ where $C_n \to u_\lambda(x_0)$ as $n \to \infty$. Then,

$$\begin{aligned}
&\lambda \int_\Omega f(x, u_\lambda + t_n\phi_n^{d_n})t_n\phi_n^{d_n}\\
&\geq \lambda \int_{B(x_0, \frac{d_n}{n})} t_n\phi_n^{d_n}\ m(x, C_n + t_n\phi_n^{d_n}) \frac{e^{\alpha(C_n + t_n\phi_n^{d_n})^2}}{|x|^\beta}\\
&\geq K_1 m(t_n\phi_n^{d_n}(x_0))t_n\phi_n^{d_n}(x_0)e^{2\alpha C_n t_n\phi_n^{d_n}(x_0) + \alpha(t_n\phi_n^{d_n}(x_0))^2} \int_{B(x_0, \frac{d_n}{n})} |x|^{-\beta}dx\\
&\geq K_1 m(t_n\phi_n^{d_n}(x_0))t_n\phi_n^{d_n}(x_0)d_n^{(2-\beta)}e^{\left(K_2 t_n(\log n)^{1/2} + (\frac{\alpha}{2\pi}t_n^2 - (2-\beta))\log n\right)}
\end{aligned}$$

Taking $d_n = (\log n)^{-1/2}$ and using (4.26), it is easy to check that

$$K_2 t_n(\log n)^{1/2} + \left(\frac{\alpha}{2\pi}t_n^2 - (2-\beta)\right)\log n = K_3(\log n)^{1/2}$$

Thus we get

$$\lambda \int_\Omega f(x, u_\lambda + t_n\phi_n^{d_n})t_n\phi_n^{d_n} \geq K_1 m(t_n\phi_n^{d_n}(x_0))t_n\phi_n^{d_n}(x_0)\frac{e^{K_3(\log n)^{1/2}}}{(\log n)^{(2-\beta)/2}}$$

where the right hand side of the above inequality tends to ∞ as $n \to \infty$. But this is a contradiction as left-hand side of (4.27) is bounded. This completes the proof of (ii). $\square$

Lemma 4.3. *Assume* (H1)–(H5). *Suppose that* (m_2) *holds. Then there exists a second solution* v_λ *of* (P_λ) *such that* $0 < u_\lambda < v_\lambda$.

Proof. Here we follow the approach as in [5]. *Define a complete metric space* (X, d) *as*

$$X = \{\eta \in C([0,1], \Lambda_\lambda) : \eta(0) = u_\lambda, \|\eta(1) - u_\lambda\| > \rho_1, I_\lambda(\eta(1)) < I_\lambda(u_\lambda)\},$$

where $d(\eta, \chi) = \max\limits_{s\in[0,1]} \|\eta(s) - \chi(s)\|$.

Since $I_\lambda(u_\lambda + t\phi_n^{d_n}) \to -\infty$ *as* $t \to \infty$, *we have* $\eta(s) = u_\lambda + st_0\phi_n^{d_n} \in X$ *for* t_0 *sufficiently large. Thus* X *defines a nonempty complete metric space. Also set* $\gamma_0 = \inf\limits_{\eta\in X} \max\limits_{s\in[0,1]} I_\lambda(\eta(s))$.

Then (ii) *of Lemma* 4.2 *and* (m_2) *imply*

$$I_\lambda(u_\lambda) < \gamma_0 < I_\lambda(u_\lambda) + \frac{\pi}{\alpha}(2 - \beta).$$

Now applying Ekeland's variational principle to the functional $\psi : X \to \mathbb{R}$ *defined as*

$$\psi(\eta) = \max_{s\in[0,1]} I_\lambda(\eta(s))$$

we get a sequence $\{\eta_k\} \subseteq X$ *such that*

(i) $\max\limits_{s\in[0,1]} I_\lambda(\eta_k(s)) \leq \gamma_0 + \dfrac{1}{k}$.

(ii) $\max\limits_{s\in[0,1]} I_\lambda(\eta_k(s)) \leq \max\limits_{s\in[0,1]} I_\lambda(\eta(s)) + \dfrac{1}{k} \max\limits_{s\in[0,1]} \|\eta(s) - \eta_k(s)\|$ *for all* $\eta \in X$.

Now as in [5] *there exists* $t_k \in \Gamma_k$ *where* $\Gamma_k = \{t \in (0,1) : I_\lambda(\eta_k(t)) = \max\limits_{s\in[0,1]} I_\lambda(\eta_k(s))\}$ *such that if we set* $v_k = \eta_k(t_k)$ *then*

$$I_\lambda^0\left(v_k, \frac{w - v_k}{max\,\{1, \|w - v_k\|\}}\right) \geq -\frac{1}{k} \text{ for all } w \in \Lambda_\lambda \tag{4.28}$$

and

$$I_\lambda(v_k) \to \gamma_0 \text{ as } k \to \infty. \tag{4.29}$$

Now we claim that $\|v_k\| \leq C$ *for all* $k \in \mathbb{N}$. *Putting* $w = 2v_k$ *in* (4.28) *we find* $\bar{w}_k \in L^\infty(\Omega)$, $\bar{w}_k(x) \in [\underline{g}(v_k(x) - a), \bar{g}(v_k(x) - a)]$ *such that*

$$\|v_k\|^2 - \lambda \int_\Omega f(x, v_k)v_k - \lambda \int_\Omega \bar{w}_k v_k^{q+1} \geq -\frac{1}{k} max\,\{1, \|v_k\|\}. \tag{4.30}$$

Also from (4.29) *we get*

$$\frac{1}{2}\|v_k\|^2 - \lambda \int_\Omega F(x, v_k) - \lambda \int_\Omega H(v_k) \leq \gamma_0 + o_k(1). \tag{4.31}$$

From (H4) we note that $F(x,t) \leq \epsilon f(x,t)t + M$ *for all* $t > 0$ *and* $\epsilon > 0$. *Substituting this in* (4.31) *we get*

$$\frac{1}{2}\|v_k\|^2 - \epsilon\lambda \int_\Omega f(x,v_k)v_k - \lambda \int_\Omega H(v_k) \leq \gamma_0 + C + o_k(1), \tag{4.32}$$

Now from (4.30) *and* (4.32) *we get*

$$(\frac{1}{2} - \epsilon)\|v_k\|^2 \leq \frac{\epsilon}{k} max\ \{1, \|v_k\|\} + \gamma_0 + C + o_k(1)$$

Thus $\{v_k\}$ *is bounded in* $H_0^1(\Omega)$. *Using this in* (4.28) *we conclude that*

$$I_\lambda^0(v_k, w - v_k) \geq -\frac{C}{k}(1 + \|w\|) \ \ \textit{for all } w \in \Lambda_\lambda.$$

By taking a subsequence if necessary, we find $v_\lambda \in \Lambda_\lambda$ *such that* $v_k \rightharpoonup v_\lambda$ *weakly in* $H_0^1(\Omega)$ *and pointwise a.e. in* Ω. *Now we proceed as in case of* (m_1) *to derive that* v_λ *is a critical point of* I_λ.
Now we claim that $v_\lambda \not\equiv u_\lambda$. *Since* $\gamma_0 \neq I_\lambda(u_\lambda)$, *in view of* (4.29) *it suffices to prove that* $v_k \to v_\lambda$ *strongly in* $H_0^1(\Omega)$ *as* $k \to \infty$. *Note that from* (4.30) *and the fact that* $\|v_k\| \leq C$ *we have* $\tilde{C} = \lim_{k\to+\infty}\sup \int_\Omega f(x,v_k)v_k < \infty$. *First we claim that* $\{f(.,v_k)\}$ *is an equi-integrable family in* $L^1(\Omega)$. *For this, define* $\mu_\epsilon = \max_{x\in\bar{\Omega}, |s|\leq \frac{2\tilde{C}}{\epsilon}} |f(x,s)|$. *Then, for any* $A \subset \Omega$ *with* $|A| \leq \frac{\epsilon}{2\mu_\epsilon}$, *we get*

$$\begin{aligned}\int_A |f(x,v_k)| &\leq \int_{A\cap\{|v_k|\geq \frac{2\tilde{C}}{\epsilon}\}} \frac{|f(x,v_k)v_k|}{|v_k|} + \int_{A\cap\{|v_k|< \frac{2\tilde{C}}{\epsilon}\}} |f(x,v_k)| \\ &\leq \frac{\epsilon}{2} + \mu_\epsilon|A| \leq \epsilon.\end{aligned}$$

Now by applying Vitali's convergence theorem, we get

$$\lim_{k\to+\infty} \int_\Omega f(x,v_k) = \int_\Omega f(x,v_\lambda). \tag{4.33}$$

Therefore, from (H4) *and the dominated convergence theorem, we have*

$$\lim_{k\to\infty} \int_\Omega F(x,v_k) = \int_\Omega F(x,v_\lambda) \ \ \textit{and} \ \ \lim_{k\to\infty} \int_\Omega H(v_k) = \int_\Omega H(v_\lambda). \tag{4.34}$$

Thus we get

$$I_\lambda(v_\lambda) \leq \lim_{k\to+\infty} \inf I_\lambda(v_k). \tag{4.35}$$

If $\{v_k\}$ *does not converge to* v_λ *in* $H_0^1(\Omega)$ *strongly, then from* (4.34) *and* (4.35), $I_\lambda(v_\lambda) < \gamma_0$. *Also as* $I_\lambda(u_\lambda) \leq I_\lambda(v_\lambda)$ *we take* $\epsilon > 0$ *sufficiently small such that the following equation holds.*

$$(\gamma_0 - I_\lambda(v_\lambda))(1+\epsilon) \leq (\gamma_0 - I_\lambda(u_\lambda))(1+\epsilon) < \frac{\pi}{\alpha}(2-\beta) \tag{4.36}$$

Now let $\delta_0 = \lambda(\int_\Omega (F(x, v_\lambda) + \int_\Omega H(v_\lambda))$. *Then , we have*

$$\lim_{k\to+\infty} \|v_k\|^2 = 2 \lim_{k\to+\infty} \left(I_\lambda(v_k) + \lambda \int_\Omega F(x, v_k) + \lambda \int_\Omega H(v_k) \right) = 2(\gamma_0 + \delta_0). \tag{4.37}$$

From (4.36) *and* (4.37),

$$(1+\epsilon)\|v_k\|^2 < \frac{2\pi(2-\beta)(\gamma_0+\delta_0)}{\alpha(\gamma_0 - I_\lambda(v_\lambda))} = \frac{2\pi(2-\beta)}{\alpha}\left(1 - \frac{\|v_\lambda\|^2}{2(\gamma_0+\delta_0)}\right)^{-1}$$

Choose p *such that*

$$\frac{\alpha(1+\epsilon)\|v_k\|^2}{2\pi(2-\beta)} \le p < \left(1 - \frac{\|v_\lambda\|^2}{2(\gamma_0+\delta_0)}\right)^{-1} \tag{4.38}$$

By the Lion's type Lemma in [3] (Theorem 2.3), (H2) *and* (4.38) *we obtain,*

$$\sup_k \int_\Omega \frac{e^{\alpha(1+\epsilon)v_k^2}}{|x|^\beta} \le \sup_k \int_\Omega \frac{e^{2\pi p(2-\beta)\frac{v_k^2}{\|v_k\|^2}}}{|x|^\beta} < \infty. \tag{4.39}$$

Now for $M > 0$, $\int_\Omega f(x, v_k)v_k = \int_{v_k\le M} f(x, v_k)v_k + \int_{v_k>M} f(x, v_k)v_k$ *and by* (4.39) *it is easy to check that*

$$\int_{v_k>M} f(x, v_k)v_k = O\left(\int_{v_k>M} \frac{e^{\alpha(1+\frac{\epsilon}{2})v_k^2}}{|x|^\beta}\right) = O\left(e^{\frac{-\alpha\epsilon}{2}N^2} \int_{v_k>M} \frac{e^{\alpha(1+\epsilon)v_k^2}}{|x|^\beta}\right).$$

Hence $\int_\Omega f(x, v_k)v_k \to \int_\Omega f(x, v_\lambda)v_\lambda$ *as* $k \to \infty$, *thanks to dominated convergence theorem. Now proceeding as in case of* (m_1), *we get* $v_k \to v_\lambda$ *strongly in* $H_0^1(\Omega)$. $\square$

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