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# Critical exponent for semilinear wave equation with critical potential

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**Abstract.** We consider the Cauchy problem for the semilinear wave equation

$$u_{tt} - \Delta u + V(x)u_t = |u|^p.$$

When $V(x) = V_0(1 + |x|^2)^{-1/2},\ V_0 \geq n$, we prove that the critical exponent for the problem is $p_c(n) = \begin{cases} 1 + \frac{2}{n-1}, & n \geq 2, \\ +\infty, & n = 1. \end{cases}$

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## 1. Introduction

We consider the Cauchy problem for the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u + V(x)u_t = |u|^p, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \epsilon u_0(x),\ u_t(x,0) = \epsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\epsilon > 0,\ p$ satisfies

$$1 < p < +\infty\,(n = 1, 2), \quad 1 < p \leq \frac{n}{n-2}\,(n \geq 3),$$

$(u_0, u_1) \in H^1 \times L^2$ satisfy: there exists a constant $R > 0$, such that $\text{supp}\{u_0, u_1\} \subset B_R(0) = \{x \in \mathbb{R}^n \,|\, |x| < R\}$, and $V(x) \in C(\mathbb{R}^n)$ is a potential function which will be specified later.

We focus on the critical exponent $p_c(n)$ of the problem (1.1), which is a number defined by the following property:

If $p > p_c(n)$, then all solutions of (1.1) with small initial values are global; while if $1 < p \leq p_c(n)$, then all solutions of (1.1) with nonnegative initial values blow up in finite time regardless of the smallness of the initial values.

When the potential $V(x)$ is a constant ($V(x) \equiv 1$), Todorova and Yordanov [8] in 2001 obtained that the critical exponent of (1.1) is $1 + \frac{2}{n}$, which is the same as the Fujita exponent for the heat equation $v_t - \Delta v = v^p$ (see Fujita [1] and Weissler [10]). More precisely, they proved that:

If $1 + \frac{2}{n} < p \le \frac{n}{n-2}\,(n \ge 3),\ 1 + \frac{2}{n} < p < \infty\,(n = 1, 2)$, and the initial values with compact supports are sufficiently small, then the problem (1.1) admits a unique global solution

$$u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2);$$

While if $1 < p < 1 + \frac{2}{n},\ \int_{\mathbb{R}^n} u_0(x)\,dx > 0,\ \int_{\mathbb{R}^n} u_1(x)\,dx > 0$, then all the solutions to (1.1) blow up in finite time.
Later Zhang [11] proved that the critical exponent $1 + \frac{2}{n}$ belongs to the blow up region. In 2005, Ikehata and Tanizawa [2] extended the global existence result of [8] to initial values without compact supports.

When $V(x) \in C^1(\mathbb{R}^n)$ is a radially symmetric function,

$$V(x) \sim V_0(1 + |x|)^{-\alpha},\ |x| \to \infty, \quad V_0 > 0,\ \alpha \in (0,1),$$

Ikehata et al. [3] in 2009 obtained that the critical exponent for (1.1) is $1+\frac{2}{n-\alpha}$. More precisely, they proved that:

If $1 + \frac{2}{n-\alpha} < p < \frac{n+2}{n-2}\,(n \ge 3),\ 1 + \frac{2}{n-\alpha} < p < \infty\,(n = 1, 2)$, and the initial values are sufficiently small, then the problem (1.1) admits a unique global solution

$$u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2),$$
$$\int_{\mathbb{R}^n} (|u_t|^2 + |\nabla u|^2)\,dx \le C(1+t)^{-(\frac{n-\alpha}{2-\alpha}+1-\delta)},$$

where $\delta > 0$ is an arbitrarily small number; While if

$$1 < p \le 1 + \frac{2}{n-\alpha}, \quad \int_{\mathbb{R}^n} (u_1 + V(x)u_0)\,dx > 0,$$

then all the solutions to (1.1) do not exist globally for any $\epsilon > 0$.

In this paper we solve the critical exponent problem for (1.1) with critical potential $V(x)$. More precisely, we assume that $V(x) \in C(\mathbb{R}^n)$ satisfies:

$$V_0(1 + |x|^2)^{-1/2} \le V(x) \le V_1(1 + |x|^2)^{-1/2}, \quad V_0,\ V_1 > 0. \tag{1.2}$$

To get the critical exponent we need the decay estimates for the homogeneous problem

$$\begin{cases} u_{tt} - \Delta u + V(x)u_t = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $V(x)$ satisfies (1.2), and $(u_0, u_1) \in H^1 \times L^2,\ \text{supp}\{u_0, u_1\} \subset B_R(0)$. Matsumura [5], Mochizuki and Nakazawa [6], Uesaka [9] discussed the energy decay rate of the problem (1.3), and obtained

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\,dx \le C(1+t)^{-\min\{V_0,1\}}.$$

Recently, Ikehata et al. [4] obtained that the solution to (1.3) satisfies:
when $n = 1,\ 2$,

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\,dx \le \begin{cases} C_\delta I_0^2(1+t)^{-V_0+\delta}, & 0 < V_0 \le n, \\ C I_0^2 (1+t)^{-n}, & V_0 > n, \end{cases}$$

when $n \geq 3$,

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\, dx \leq \begin{cases} C_\delta I_0^2 (1+t)^{-V_0+\delta}, & 0 < V_0 \leq 1, \\ C I_0^2 (1+t)^{-V_0}, & 1 < V_0 < n, \\ C_\delta I_0^2 (1+t)^{-n+\delta}, & V_0 \geq n, \end{cases}$$

where $I_0 = \|u_0\|_{H^1} + \|u_1\|_2$, $\delta > 0$ is an arbitrarily small number. Using multiplier method, in another paper, we can prove that when $n = 2$,

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\, dx \leq C I_0^2 (1+t)^{-V_0}, \quad 1 < V_0 \leq 2.$$

Above all, the optimal estimates obtained for the solutions to (1.3) are as follows:
when $n = 1,\ 2$,

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\, dx \leq C I_0^2 (1+t)^{-\min\{V_0, n\}}, \tag{1.4}$$

when $n \geq 3$,

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2)\, dx \leq \begin{cases} C I_0^2 (1+t)^{-V_0}, & 0 < V_0 < n, \\ C_\delta I_0^2 (1+t)^{-n+\delta}, & V_0 \geq n, \end{cases} \tag{1.5}$$

where $\delta > 0$ is an arbitrarily small number.

Now we are ready to state our main results. The main result of this paper is that the critical exponent $p_c(n)$ for (1.1) $(V_0 \geq n)$ is $p_c(n) = 1 + \frac{2}{n-1}$.

Our global existence result is as follows.

**Theorem 1.1.** *Let $n \geq 2$, $V(x)$ satisfy (1.2), $V_0 \geq n$, $1 + \frac{2}{n-1} < p < +\infty\,(n = 2)$, and $1 + \frac{2}{n-1} < p \leq \frac{n}{n-2}\,(n \geq 3)$. Then there exists $\epsilon_0 > 0$ such that problem (1.1) admits a unique global solution*

$$u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2),$$

*for each $\epsilon < \epsilon_0$. Moreover, for any $\delta \in (0, \min\{\frac{1}{4}, \frac{(n-1)p-(n+1)}{2(p-1)}\}]$, the global solution satisfies*

$$\|Du\|_2 \leq C\epsilon(\|u_0\|_{H^1} + \|u_1\|_2)(1+t)^{-1}, \quad n = 2,$$
$$\|Du\|_2 \leq C\epsilon(\|u_0\|_{H^1} + \|u_1\|_2)(1+t)^{-\frac{n}{2}+\delta}, \quad n \geq 3,$$

*where $D = (\partial_t, \nabla_x)$, $C > 0$ is a constant, independent of $t$.*

The blow up result is as follows.

**Theorem 1.2.** *Let $V(x)$ satisfy (1.2), $1 < p < \infty\,(n = 1)$, and $1 < p \leq 1 + \frac{2}{n-1}\,(n \geq 2)$. If $V(x)u_1(x) + u_0(x) \geq 0$ and $V(x)u_1(x) + u_0(x) \not\equiv 0$, then the solution to (1.1) does not exist globally, for any $\epsilon > 0$.*

**Remark 1.3.** We did not find good methods to solve the critical exponent problem to (1.1) for $0 < V_0 < n$.

This paper is organized as follows. In Sect. 2 we prove Theorem 1.1 by dividing the proof into several lemmas. In Sect. 3 we prove Theorem 1.2.

## 2. Global existence result for small initial values

The following local result for the problem (1.1) is well-known, which can be obtained by a simply modification of the result in Strauss [7].

**Lemma 2.1.** *Let $V(x) \in C(\mathbb{R}^n)$ satisfy (1.2). Then the problem (1.1) admits a unique local solution $u \in C([0,T), H^1) \cap C^1([0,T), L^2)$ satisfying*

$$u(x,t) \equiv 0, \quad |x| \geq t + R,$$

*where $T > 0$ depending only on $\|Du(0)\|_2$. Moreover, the solution can be continued beyond the interval $[0,T)$ if $\sup\limits_{[0,T)} \|Du(t)\|_2 < +\infty$.*

In view of this local existence result, global existence of a solution follows the boundedness of the energy at all times. In the following, we prove Theorem 1.1 by the method of Todorova and Yordanov [8].

Choosing a weight function $\psi(x,t) = \frac{1+|x|}{1+t}$, we obtain the following weighted energy estimate.

**Lemma 2.2.** *Let $V(x)$ satisfy (1.2), $V_0 \geq 1$, $u(x,t)$ be a local solution to (1.1) on $[0,T)$, and $\psi(x,t) = \frac{1+|x|}{1+t}$. Then for any $t \in [0,T)$, the following estimate holds:*

$$\|e^{\psi} Du\|_2 \leq C\epsilon + C\left(\max_{[0,t]} (s+1)^{\eta_1} \|e^{\gamma_1 \psi(s)} u(s)\|_{p+1}\right)^{(p+1)/2},$$

*where $\eta_1 > 0$, $\gamma_1 > \frac{2}{p+1}$ are arbitrarily numbers, and $C > 0$ is a constant, independent of $\epsilon$ and $t$.*

*Proof.* Multiplying Eq. (1.1) by $e^{2\psi} u_t$ and rearranging the terms, we obtain

$$\begin{aligned} &\left(e^{2\psi} \frac{u_t^2 + |\nabla u|^2}{2}\right)_t - \nabla(e^{2\psi} u_t \nabla u) + \frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 \\ &\quad + \frac{e^{2\psi}}{-\psi_t}(\psi_t^2 - |\nabla \psi|^2 - V(x)\psi_t) u_t^2 = \left(\frac{e^{2\psi}|u|^p u}{p+1}\right)_t - \frac{2e^{2\psi}|u|^p u \psi_t}{p+1}. \end{aligned} \tag{2.1}$$

By the choice of $\psi(x,t)$ and $V_0 \geq 1$, we obtain

$$\begin{aligned} -\psi_t &= \frac{1+|x|}{(1+t)^2} > 0, \quad \psi_{x_i} = \frac{1}{1+t}\frac{x_i}{|x|}, \quad |\nabla \psi|^2 = \frac{1}{(1+t)^2}, \\ \psi_t^2 - |\nabla \psi|^2 - V(x)\psi_t &\geq \frac{(1+|x|)^2}{(1+t)^4} - \frac{1}{(1+t)^2} + \frac{1+|x|}{(1+t)^2}\frac{V_0}{\sqrt{1+|x|^2}} \geq 0. \end{aligned}$$

Thus, by the above arguments, (2.1) can be simplified to

$$\left(e^{2\psi} \frac{u_t^2 + |\nabla u|^2}{2}\right)_t - \nabla(e^{2\psi} u_t \nabla u) \leq \left(\frac{e^{2\psi}|u|^p u}{p+1}\right)_t - \frac{2e^{2\psi}|u|^p u \psi_t}{p+1}. \tag{2.2}$$

Integrating (2.2) over $[0,t]\times\mathbb{R}^n$, and letting $\epsilon\le 1$, we obtain

$$\begin{aligned}\|e^{\psi}Du\|_2^2 &\le C(\epsilon^2+\epsilon^{p+1})+C\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}\,dx+C\int_0^t\int_{\mathbb{R}^n} e^{2\psi}|u|^{p+1}|\psi_s|\,dxds\\ &\le C\epsilon^2+C\|e^{\frac{2\psi}{p+1}}u\|_{p+1}^{p+1}\\ &\quad +C\int_0^t \max_{\mathrm{supp}u(s)}\left(|\psi_s|e^{(2-\gamma_1(p+1))\psi(s)}\right)\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}^{p+1}\,ds,\end{aligned}$$

where $\gamma_1>\frac{2}{p+1}$, and $C>0$ is a constant, independent of $\epsilon$ and $t$. By the choice of $\psi(x,t)$, we have

$$\begin{aligned}\max_{\mathrm{supp}u(s)}|\psi_s|e^{(2-\gamma_1(p+1))\psi(s)} &= \max_{|x|\le s+R}\frac{1+|x|}{(1+s)^2}e^{(2-\gamma_1(p+1))\psi(s)}\\ &\le \frac{1+R+s}{(1+s)^2}\le\frac{C}{1+s}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|e^{\psi}Du\|_2^2 &\le C\epsilon^2+C\|e^{\frac{2\psi}{p+1}}u\|_{p+1}^{p+1}+C\int_0^t\frac{1}{1+s}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}^{p+1}\,ds\\ &\le C\epsilon^2+C\|e^{\frac{2\psi}{p+1}}u\|_{p+1}^{p+1}+C\left[\max_{[0,t]}(1+s)^{\eta_1}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}\right]^{p+1}\\ &\le C\epsilon^2+C\left[\max_{[0,t]}(1+s)^{\eta_1}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}\right]^{p+1},\end{aligned}$$

where $\eta_1>0$ is an arbitrarily number. We complete the proof of Lemma 2.2. □

**Lemma 2.3.** *Let $n\ge 2$, $V(x)$ satisfy (1.2), $V_0\ge n$, $u(x,t)$ be a local solution to (1.1) on $[0,T)$, and $\delta\in(0,\frac14]$ is any fixed number. Then for any $t\in[0,T)$, the following estimates hold:*
*when $n=2$,*

$$\|Du(t)\|_2\le C(1+t)^{-1}\left(\epsilon+\max_{[0,t]}\left[(1+s)^{\frac{1+\eta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^p\right),$$

*when $n\ge 3$,*

$$\|Du(t)\|_2\le C_\delta(1+t)^{-n/2+\delta}\left(\epsilon+\max_{[0,t]}\left[(1+s)^{\frac{n/2-\delta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^p\right),$$

*where $\eta>0$ and $\gamma>0$ are arbitrarily small numbers.*

*Proof.* Let $u_L(x,t)$ be the solution to the homogeneous problem

$$\begin{cases}u_{tt}-\Delta u+V(x)u_t=0, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=\epsilon u_0(x),\ u_t(x,0)=\epsilon u_1(x), & x\in\mathbb{R}^n,\end{cases}\tag{2.3}$$

and $S(t)*u_1(x)$ be the solution to the problem

$$\begin{cases}u_{tt}-\Delta u+V(x)u_t=0, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=0,\ u_t(x,0)=u_1(x), & x\in\mathbb{R}^n.\end{cases}$$

Then

$$u(x,t) = u_L(x,t) + \int_0^t S(t-\tau) * |u|^p(\tau)\, d\tau$$

is a solution to the problem (1.1). In the following, we split the proof of Lemma 2.3 into two cases.

**Case 1** ($n = 2$). By the energy estimate (1.4) for problem (2.3), the linear term $\|Du_L(t)\|_2$ is bounded by

$$\|Du_L(t)\|_2 \le C\epsilon(1+t)^{-1}(\|u_0\|_{H^1} + \|u_1\|_2) \le C\epsilon(1+t)^{-1}, \tag{2.4}$$

and the integral term is estimated as follows:

$$\begin{aligned} \left\| D\left(\int_0^t S(t-\tau) * |u|^p(\tau)\, d\tau\right)\right\|_2 &\le \int_0^t \|DS(t-\tau) * |u|^p(\tau)\|_2\, d\tau \\ \le C\int_0^t (1+t-\tau)^{-1}\|u^p(\tau)\|_2\, d\tau &= C\int_0^t (1+t-\tau)^{-1}\|u(\tau)\|_{2p}^p\, d\tau. \end{aligned} \tag{2.5}$$

Since $\psi(x,t) > 0$, for any $\gamma > 0$, we have

$$\|u(\tau)\|_{2p} \le \|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}. \tag{2.6}$$

Inserting (2.6) into (2.5) and splitting the integral into two parts, we have

$$\begin{aligned} &\left\| D\left(\int_0^t S(t-\tau) * |u|^p(\tau)\, d\tau\right)\right\|_2 \\ &\quad \le C\left(\int_0^{t/2} + \int_{t/2}^t\right)(1+t-\tau)^{-1}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\, d\tau = C(I_1 + I_2). \end{aligned} \tag{2.7}$$

For any $\eta > 0$,

$$\begin{aligned} I_1 &= \int_0^{t/2} (1+t-\tau)^{-1}\frac{1}{(1+\tau)^{1+\eta}}(1+\tau)^{1+\eta}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\, d\tau \\ &\le C(1+t)^{-1}\max_{[0,t/2]}\left[(1+\tau)^{(1+\eta)/p}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}\right]^p, \end{aligned} \tag{2.8}$$

$$\begin{aligned} I_2 &= \int_{t/2}^t (1+\tau)^{-1}(1+t-\tau)^{-(1+\eta)}(1+t-\tau)^{\eta}(1+\tau)\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\, d\tau \\ &\le C(1+t)^{-1}\max_{[t/2,t]}\left[(1+\tau)^{(1+\eta)/p}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}\right]^p. \end{aligned} \tag{2.9}$$

Combining (2.4), (2.7), (2.8) and (2.9), we obtain, when $n = 2$,

$$\|Du(t)\|_2 \le C(1+t)^{-1}\left(\epsilon + \max_{[0,t]}\left[(1+s)^{\frac{1+\eta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^p\right).$$

**Case 2** ($n \ge 3$). By the energy estimate (1.5) for problem (2.3), the linear term $\|Du_L(t)\|_2$ is bounded by

$$\|Du_L(t)\|_2 \le C_\delta\epsilon(\|u_0\|_{H^1} + \|u_1\|_2)(1+t)^{-n/2+\delta}, \tag{2.10}$$

and the integral term is estimated as follows:

$$\begin{aligned}\left\|D\left(\int_0^t S(t-\tau)*|u|^p(\tau)\,d\tau\right)\right\|_2 &\le \int_0^t \|DS(t-\tau)*|u|^p(\tau)\|_2\,d\tau\\ &\le C_\delta\int_0^t(1+t-\tau)^{-n/2+\delta}\|u^p(\tau)\|_2\,d\tau = C_\delta\int_0^t(1+t-\tau)^{-n/2+\delta}\|u(\tau)\|_{2p}^p\,d\tau\\ &\le C_\delta\left(\int_0^{t/2}+\int_{t/2}^t\right)(1+t-\tau)^{-n/2+\delta}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\,d\tau = C_\delta(I_3+I_4). \quad (2.11)\end{aligned}$$

For any $\eta > 0$,

$$\begin{aligned}I_3 &= \int_0^{t/2}(1+t-\tau)^{-n/2+\delta}\frac{1}{(1+\tau)^{1+\eta}}(1+\tau)^{1+\eta}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\,d\tau\\ &\le C(1+t)^{-n/2+\delta}\max_{[0,t/2]}\left[(1+\tau)^{(1+\eta)/p}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}\right]^p, \quad (2.12)\end{aligned}$$

$$\begin{aligned}I_4 &= \int_{t/2}^t(1+\tau)^{-n/2+\delta}(1+t-\tau)^{-n/2+\delta}(1+\tau)^{n/2-\delta}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}^p\,d\tau\\ &\le C(1+t)^{-n/2+\delta}\max_{[t/2,t]}\left[(1+\tau)^{(n/2-\delta)/p}\|e^{\gamma\psi(\tau)}u(\tau)\|_{2p}\right]^p. \quad (2.13)\end{aligned}$$

Combining (2.10)–(2.13), we obtain, when $n \ge 3$,

$$\|Du(t)\|_2 \le C_\delta(1+t)^{-n/2+\delta}\left(\epsilon + \max_{[0,t]}\left[(1+s)^{\frac{n/2-\delta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^p\right),$$

which completes the proof of Lemma 2.3. □

**Lemma 2.4.** *Let $\sigma \in (0,1]$, $2 \le q < \infty\,(n=2)$, $2 \le q \le \frac{2n}{n-2}\,(n \ge 3)$, $\psi(x,t) = \frac{1+|x|}{1+t}$, $u(x,t)$ be a local solution of (1.1) on $[0,T)$. Then for any $t \in [0,T)$, the following estimate holds:*

$$\|e^{\sigma\psi(t)}u\|_q \le C(1+t)^{1-\theta(q)}\|e^{\psi(t)}\nabla u\|_2^\sigma\|\nabla u\|_2^{1-\sigma},$$

*where $\theta(q) = n(\frac{1}{2}-\frac{1}{q})$, and $C > 0$ is a constant, independent of $t$.*

*Proof.* Applying the Gagliardo–Nirenberg inequality to $e^{\sigma\psi}u$, we have

$$\|e^{\sigma\psi}u\|_q \le \|e^{\sigma\psi}u\|_2^{1-\theta(q)}\|\nabla(e^{\sigma\psi}u)\|_2^{\theta(q)}. \quad (2.14)$$

On the other hand,

$$e^{\sigma\psi}\nabla u = \nabla(e^{\sigma\psi}u) - \sigma e^{\sigma\psi}u\nabla\psi.$$

Thus,

$$\begin{aligned}\|e^{\sigma\psi}\nabla u\|_2^2 &= \int_{\mathbb{R}^n}(|\nabla(e^{\sigma\psi}u)|^2 + \sigma^2e^{2\sigma\psi}|u|^2|\nabla\psi|^2 - 2\sigma e^{\sigma\psi}u\nabla(e^{\sigma\psi}u)\nabla\psi)\,dx\\ &= \int_{\mathbb{R}^n}(|\nabla(e^{\sigma\psi}u)|^2 + \sigma^2e^{2\sigma\psi}|u|^2|\nabla\psi|^2 + \sigma\Delta\psi(e^{\sigma\psi}u)^2)\,dx. \quad (2.15)\end{aligned}$$

By the choice of $\psi(x,t)$, we have

$$|\nabla\psi|^2 = \frac{1}{(1+t)^2}, \quad \Delta\psi = \frac{n-1}{1+t}\frac{1}{|x|}. \quad (2.16)$$

Inserting (2.16) into (2.15), we have

$$\|e^{\sigma\psi}\nabla u\|_2^2 \geq \|\nabla(e^{\sigma\psi}u)\|_2^2 + \frac{\sigma^2}{(1+t)^2}\|e^{\sigma\psi}u\|_2^2. \tag{2.17}$$

By (2.17), (2.14) can be simplified to

$$\|e^{\sigma\psi}u\|_q \leq C(\sigma)(1+t)^{1-\theta(q)}\|e^{\sigma\psi}\nabla u\|_2. \tag{2.18}$$

Using Hölder inequality,

$$\begin{aligned}\|e^{\sigma\psi}\nabla u\|_2^2 &= \int_{\mathbb{R}^n} e^{2\sigma\psi}|\nabla u|^{2\sigma}|\nabla u|^{2(1-\sigma)}\,dx\\ &\leq \left(\int_{\mathbb{R}^n} e^{2\psi}|\nabla u|^2\right)^{\sigma}\left(\int_{\mathbb{R}^n}|\nabla u|^2\right)^{1-\sigma}.\end{aligned}$$

Thus,

$$\|e^{\sigma\psi}\nabla u\|_2 \leq \|e^{\psi}\nabla u\|_2^{\sigma}\|\nabla u\|_2^{1-\sigma}. \tag{2.19}$$

Inserting (2.19) into (2.18), we complete the proof of Lemma 2.4. □

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* We split the proof into two cases.

**Case 1** ($n = 2$). We introduce the weighted energy functional

$$W(t) = \|e^{\psi}Du\|_2 + (1+t)\|Du\|_2.$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}W(t) \leq C\epsilon &+ C\left(\max_{[0,t]}(s+1)^{\eta_1}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}\right)^{(p+1)/2}\\ &+C\left[\max_{[0,t]}(1+s)^{\frac{1+\eta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^{p},\end{aligned} \tag{2.20}$$

where $\eta > 0,\ \gamma > 0,\ \eta_1 > 0,\ \gamma_1 > \frac{2}{p+1}$ are arbitrarily small numbers. By Lemma 2.4 and the definition of $W(t)$, we obtain

$$\begin{aligned}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1} &\leq C(1+s)^{1-\theta(p+1)}\|e^{\psi(s)}\nabla u(s)\|_2^{\gamma_1}\|\nabla u(s)\|_2^{1-\gamma_1}\\ &\leq C(1+s)^{1-\theta(p+1)-(1-\gamma_1)}W(s),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\|e^{\gamma\psi(s)}u(s)\|_{2p} &\leq C(1+s)^{1-\theta(2p)}\|e^{\psi(s)}\nabla u(s)\|_2^{\gamma}\|\nabla u(s)\|_2^{1-\gamma}\\ &\leq C(1+s)^{1-\theta(2p)-(1-\gamma)}W(s),\end{aligned} \tag{2.22}$$

where $\theta(p+1) = n(\frac{1}{2} - \frac{1}{p+1}),\ \theta(2p) = n(\frac{1}{2} - \frac{1}{2p})$. Using (2.21) and (2.22), we obtain from (2.20)

$$\begin{aligned}W(t) \leq C\epsilon &+ C\max_{[0,t]}(s+1)^{\frac{p+1}{2}\eta_1+\frac{p+1}{2}[1-\theta(p+1)-(1-\gamma_1)]}W(s)^{(p+1)/2}\\ &+C\max_{[0,t]}(1+s)^{1+\eta+p[1-\theta(2p)-(1-\gamma)]}W(s)^{p}.\end{aligned} \tag{2.23}$$

In the following we calculate the exponents of $(1+s)$ in (2.23). Set $\gamma_1 = \frac{2}{p+1} + \eta_2,\ \eta_2 > 0$, then

$$\frac{p+1}{2}\eta_1 + \frac{p+1}{2}[1-\theta(p+1)-(1-\gamma_1)] = \frac{p+1}{2}(\eta_1+\eta_2) + \frac{3-p}{2},$$
$$1+\eta+p[1-\theta(2p)-(1-\gamma)] = \eta + p\gamma + (2-p).$$

Since $p > 3$, we can choose $\eta > 0,\ \lambda > 0,\ \eta_1 > 0,\ \eta_2 > 0$ sufficiently small, such that the exponents of $(1+s)$ are negative. Thus we have from (2.23)

$$W(t) \leq C\epsilon + C\max_{[0,t]} W(s)^{\frac{p+1}{2}} + C\max_{[0,t]} W(s)^p. \tag{2.24}$$

Set $M(t) = \max_{[0,t]} W(s)$. From (2.24), we have

$$M(t) \leq C\epsilon,$$

for sufficiently small $\epsilon$, which completes the proof of Theorem 1.1 for $n = 2$.

**Case 2** ($n \geq 3$). For any fixed $\delta \in (0, \min\{\frac{1}{4}, \frac{(n-1)p-(n+1)}{2(p-1)}\}]$, we introduce the weighted energy functional

$$W_1(t) = \|e^{\psi} Du\|_2 + (1+t)^{\frac{n}{2}-\delta}\|Du\|_2.$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} W_1(t) \leq C\epsilon &+ C\left(\max_{[0,t]}(s+1)^{\eta_1}\|e^{\gamma_1\psi(s)}u(s)\|_{p+1}\right)^{(p+1)/2} \\ &+ C_\delta\left[\max_{[0,t]}(1+s)^{\frac{n/2-\delta}{p}}\|e^{\gamma\psi(s)}u(s)\|_{2p}\right]^p, \end{aligned} \tag{2.25}$$

where $\gamma > 0,\ \eta_1 > 0,\ \gamma_1 > \frac{2}{p+1}$ are arbitrarily small numbers. By Lemma 2.4 and the definition of $W_1(t)$, we obtain

$$\begin{aligned} \|e^{\gamma_1\psi(s)}u(s)\|_{p+1} &\leq C(1+s)^{1-\theta(p+1)}\|e^{\psi(s)}\nabla u(s)\|_2^{\gamma_1}\|\nabla u(s)\|_2^{1-\gamma_1} \\ &\leq C(1+s)^{1-\theta(p+1)-(n/2-\delta)(1-\gamma_1)}W_1(s), \end{aligned} \tag{2.26}$$

$$\begin{aligned} \|e^{\gamma\psi(s)}u(s)\|_{2p} &\leq C(1+s)^{1-\theta(2p)}\|e^{\psi(s)}\nabla u(s)\|_2^{\gamma}\|\nabla u(s)\|_2^{1-\gamma} \\ &\leq C(1+s)^{1-\theta(2p)-(n/2-\delta)(1-\gamma)}W_1(s), \end{aligned} \tag{2.27}$$

where $\theta(p+1) = n(\frac{1}{2} - \frac{1}{p+1}),\ \theta(2p) = n(\frac{1}{2} - \frac{1}{2p})$. Using (2.26) and (2.27), we obtain from (2.25)

$$\begin{aligned} W_1(t) \leq C\epsilon &+ C\max_{[0,t]}(s+1)^{\frac{p+1}{2}\eta_1 + \frac{p+1}{2}[1-\theta(p+1)-(n/2-\delta)(1-\gamma_1)]}W_1(s)^{(p+1)/2} \\ &+ C\max_{[0,t]}(1+s)^{n/2-\delta+p[1-\theta(2p)-(n/2-\delta)(1-\gamma)]}W_1(s)^p. \end{aligned} \tag{2.28}$$

In the following we calculate the exponents of $(1+s)$ in (2.28). Set $\gamma_1 = \frac{2}{p+1} + \eta_2,\ \eta_2 > 0$, then

$$\begin{aligned}
&\frac{p+1}{2}\eta_1 + \frac{p+1}{2}[1-\theta(p+1)-(n/2-\delta)(1-\gamma_1)] \\
&\quad = \frac{p+1}{2}(\eta_1+\eta_2(n/2-\delta)) + \frac{p-1}{2}\delta + \frac{1}{2}[n+1-(n-1)p], \\
&n/2-\delta+p[1-\theta(2p)-(n/2-\delta)(1-\gamma)] \\
&\quad = (n/2-\delta)\gamma p + (p-1)\delta + [n-(n-1)p].
\end{aligned}$$

Since $p > 1 + \frac{2}{n-1},\ \ \delta \in (0, \min\{\frac{1}{4}, \frac{(n-1)p-(n+1)}{2(p-1)}\}]$, we have

$$\frac{p-1}{2}\delta + \frac{1}{2}[n+1-(n-1)p] < 0, \quad (p-1)\delta + [n-(n-1)p] < 0.$$

Choose $\gamma > 0,\ \eta_1 > 0,\ \eta_2 > 0$ sufficiently small, such that the exponents of $(1+s)$ are negative. Thus we have from (2.28)

$$W_1(t) \le C\epsilon + C\max_{[0,t]} W_1(s)^{\frac{p+1}{2}} + C\max_{[0,t]} W_1(s)^p. \tag{2.29}$$

Set $M_1(t) = \max_{[0,t]} W_1(s)$. From (2.29), we have

$$M_1(t) \le C\epsilon,$$

for sufficiently small $\epsilon$, which completes the proof of Theorem 1.1 for $n \ge 3$. □

## 3. Blow up result

In this section we prove Theorem 1.2 by the method of test functions (see Zhang [11]).

*Proof of Theorem 1.2.* Choose a function $\phi(s) \in C^\infty([0,\infty))$ satisfying

$$\phi(s) = \begin{cases} 1, & s < 1, \\ 0, & s > 2, \end{cases} \qquad 0 \le \phi(s) \le 1,$$

$$|\phi'(s)| \le C\phi^{\frac{1}{p}}(s), \quad |\phi''(s)| \le C\phi^{\frac{1}{p}}(s),$$

where $C > 0$ is a constant. Set $\varphi(x,t) = \phi(|x|)\phi(t)$. For any $L > 0$, define

$$\varphi_L(x,t) = \varphi\left(\frac{x}{L}, \frac{t}{L}\right), \quad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

In the following, we prove Theorem 1.2 by contradiction. Suppose $u(x,t) \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$ is a global solution to (1.1). Multiplying equation (1.1) by $\varphi_L(x,t)$ and integrating the corresponding equality by parts over $\mathbb{R}^n \times [0,\infty)$, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} u(\partial_{tt}\varphi_L - \Delta\varphi_L - V(x)\partial_t\varphi_L)\, dxdt \\
&\qquad - \int_{\mathbb{R}^n} (u_1(x)\varphi_L(x,0) - u_0(x)\partial_t\varphi_L(x,0) + V(x)u_0(x)\varphi_L(x,0))\, dx \\
&\quad = \int_0^\infty \int_{\mathbb{R}^n} \varphi_L |u|^p\, dxdt.
\end{aligned} \tag{3.1}$$

By the choice of $\varphi_L(x,t)$, we have

$$\varphi_L(x,0)=\phi\left(\frac{|x|}{L}\right)\geq 0, \quad \partial_t\varphi_L(x,0)\equiv 0.$$

By the assumption $u_1(x)+V(x)u_0(x)\geq 0$, we obtain from (3.1)

$$\int_0^\infty\int_{\mathbb{R}^n}\varphi_L|u|^p\,dxdt\leq\int_0^\infty\int_{\mathbb{R}^n}u(\partial_{tt}\varphi_L-\Delta\varphi_L-V(x)\partial_t\varphi_L)\,dxdt. \quad (3.2)$$

Using Hölder inequality, we estimate the integral on the right in (3.2)

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}u(\partial_{tt}\varphi_L-\Delta\varphi_L-V(x)\partial_t\varphi_L)\,dxdt\\ &\quad\leq C\left(\int_L^{2L}\int_{B_{2L}(0)}|u|^p\varphi_L\,dxdt+\int_0^{2L}\int_{B_{2L}(0)\setminus B_L(0)}|u|^p\varphi_L\,dxdt\right)^{1/p}\\ &\qquad\times\left(\int_0^\infty\int_{\mathbb{R}^n}\varphi_L^{-p'/p}|\partial_{tt}\varphi_L-\Delta\varphi_L-V(x)\partial_t\varphi_L|^{p'}\,dxdt\right)^{1/p'},\end{aligned} \quad (3.3)$$

where $p'=p/(p-1)$. By the properties of $\varphi_L(x,t)$, we obtain

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}\varphi_L^{-p'/p}|\partial_{tt}\varphi_L-\Delta\varphi_L-V(x)\partial_t\varphi_L|^{p'}\,dxdt\\ &=\int_0^\infty\int_{\mathbb{R}^n}\varphi^{-p'/p}\left(\frac{x}{L},\frac{t}{L}\right)\left|\frac{1}{L^2}\varphi_{tt}\left(\frac{x}{L},\frac{t}{L}\right)-\frac{1}{L^2}\Delta\varphi\left(\frac{x}{L},\frac{t}{L}\right)\right.\\ &\qquad\left.-V(x)\frac{1}{L}\varphi_t\left(\frac{x}{L},\frac{t}{L}\right)\right|^{p'}\,dxdt\\ &\leq\frac{C}{L^{2p'}}\int_0^{2L}\int_{B_{2L}(0)}dxdt+\frac{C}{L^{p'}}\int_0^{2L}\int_{B_{2L}(0)}|V(x)|^{p'}\,dxdt\\ &\leq CL^{n+1-2p'}+CL^{1-p'}\int_0^{2L}\frac{(1+r)^{n-1}}{(1+r)^{p'}}\,dr\\ &\leq\begin{cases}CL^{n+1-2p'}, & n-p'>0,\\ CL^{1-p'}\ln L, & n-p'=0,\\ CL^{1-p'}, & n-p'<0.\end{cases}\end{aligned} \quad (3.4)$$

Inserting (3.3) and (3.4) into (3.2), we obtain

$$\begin{aligned}&\left(\int_0^\infty\int_{\mathbb{R}^n}\varphi_L|u|^p\,dxdt\right)^{p/(p-1)}\\ &\leq C\left(\int_L^{2L}\int_{B_{2L}(0)}|u|^p\varphi_L\,dxdt+\int_0^{2L}\int_{B_{2L}(0)\setminus B_L(0)}|u|^p\varphi_L\,dxdt\right)^{1/(p-1)}\\ &\quad\times\begin{cases}CL^{n+1-2p'}, & n-p'>0,\\ CL^{1-p'}\ln L, & n-p'=0,\\ CL^{1-p'}, & n-p'<0.\end{cases}\end{aligned} \quad (3.5)$$

Finally, we show that the above inequality can not hold as $L\to\infty$.

**Case 1**. When $1 < p < 1 + \frac{2}{n-1}\,(n \geq 2),\ 1 < p < \infty\,(n = 1)$, the exponents of $L$ in (3.5) are negative. In (3.5), letting $L \to \infty$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dxdt \leq 0,$$

which is impossible, since $u$ is a nontrivial solution.

**Case 2**. When $p = 1 + \frac{2}{n-1}\,(n \geq 2)$, the exponents of $L$ in (3.5) are nonpositive. In (3.5), letting $T \to \infty$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dxdt \leq C. \tag{3.6}$$

In (3.5), letting $T \to \infty$, and considering (3.6), we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dxdt \leq 0,$$

which is impossible, since $u$ is a nontrivial solution. By Case 1 and Case 2, we complete the proof of Theorem 1.2. □

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