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On the existence of periodic and homoclinic orbits for first order superquadratic Hamiltonian systems

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Abstract. In this paper we consider the following Hamiltonian system

$$J\dot{u} + B(t)u + \nabla W(t,u) = 0. \tag{HS}$$

Under a new superquadratic assumption on the potential, we prove that (HS) has a sequence of subharmonics. This will be done using a minimax result in critical point theory. Also, we study the asymptotic behavior of these subharmonics and we establish the existence of a homoclinic orbit for (HS). Previous results in the topic, mainly those due to Rabinowitz and Tanaka, are significantly improved.

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1. Introduction and main results

This paper deals with the existence of subharmonic solutions (i.e. kT-periodic solutions, $k \in \mathbb{N}$) and their asymptotic behavior for the following first order Hamiltonian system

$$J\dot{u} + B(t)u + \nabla W(t,u) = 0, \tag{HS}$$

where $W \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is T-periodic in the t-variable, B is a continuous T-periodic and symmetric $2N \times 2N$-matrix function, and J is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}.$$

Recall that a solution u of (HS) is said to be *homoclinic to* 0 if $u \not\equiv 0$ and $u(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$.

The problem of finding subharmonic and homoclinic solutions for (HS) has been the object of many works under different assumptions on the growth of W at infinity (see [4–9,12,14,15,17] and references therein). Most of them treat the superquadratic case using the so-called Ambrosetti–Rabinowitz condition, that is, there exists $\mu > 2$ such that

$$0 < \mu W(t,x) \leq (\nabla W(t,x), x), \quad \textit{for all} \quad t \in \mathbb{R} \textit{ and } x \in \mathbb{R}^{2N} \backslash \{0\}. \quad \text{(AR)}$$

Here, the superquadraticity condition (AR) will be replaced by a new weaker one firstly introduced in [1] to establish the existence of a sequence of subharmonics for (HS). Also, the existence of a homoclinic orbit for (HS) is proved when the matrix B satisfies

$$B \textit{ is independent of } t \text{ and } \quad \sigma(JB) \cap i\mathbb{R} = \emptyset, \tag{1.1}$$

where $\sigma(B)$ denotes the spectrum of B. This work is motivated by the results of [12,15] mainly.

Definition 1.1. A vector field V defined on $\mathbb{R}^{2N}$ is called positive if $(Vx, x) > 0$, for all $x \in \mathbb{R}^{2N} \setminus \{0\}$. We call V is a normalized positive vector field if V is positive, linear and satisfies the following conditions:
(V_1) $JV = VJ$
(V_2) $(Vx, x) = (x, x), \quad \forall x \in \mathbb{R}^{2N}$.

Throughout this paper $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^{2N}$ and $|\cdot|$ is the induced norm.

Definition 1.2. Let V a normalized positive vector field on $\mathbb{R}^{2N}$ and $\{\phi_s\}$ its flow, $\mu > 0$ be a constant and $h \in C(\mathbb{R}^{2N}, \mathbb{R})$. We call h a positive homogenous function of degree μ with respect to V if $h(x) > 0$ for all $x \in \mathbb{R}^{2N} \backslash \{0\}$ and

$$h(\phi_s x) = e^{s\mu} h(x) \qquad \forall\ x \in \mathbb{R}^{2N}, \forall\ s \in \mathbb{R}.$$

If $Vx = x$ then $\phi_s x = e^s x$ and we obtain the classical definition of homogenous function of degree μ.

Theorem 1.3. *We assume that W satisfies the following assumptions:*

There exist a normalized positive vector field V, constants
(H_1) *$\mu > 2$ and $R > 1$ such that*

$$0 < \mu W(t,x) \leq (\nabla W(t,x), Vx), \quad \forall\ |x| \geq R,$$

(H_2) *$W(t,x) = o(|x|^2)$as $x \longrightarrow 0$ uniformly in t*
(H_3) *$W(t,x) \geq 0,\ \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^{2N}$,*
there exists
(H_4) *$c > 0$ such that*

$$|\nabla W(t,x)| \leq c(\nabla W(t,x), Vx), \quad \forall\ |x| \geq R.$$

Then there is a sequence $(k_j) \subset \mathbb{N}, k_j \longrightarrow \infty$, and corresponding distinct $k_j T$ periodic solutions of the system (HS).

The assumption (H_4) is new, as far as the author is aware, and it can be replaced by the following one firstly introduced by Xu in [16].

$$(H_5) \qquad \liminf_{|x|\to\infty} \frac{W_t(t,x)}{|x|^\mu W(t,x)} = 0 \ , \ or \quad \limsup_{|x|\to\infty} \frac{W_t(t,x)}{|x|^\mu W(t,x)} = 0.$$

Precisely, assume that B is of class C^1, we have

Theorem 1.4. *Under the assumptions (H_1)–(H_3) and (H_5), the conclusion of Theorem 1.3 holds.*

In the case where W is independent of t, the assumption (H_5) is naturally satisfied, so, we obtain

Theorem 1.5. *Suppose W is independent of t and satisfies (H_1)–(H_3), then the conclusion of Theorem 1.3 holds.*

Corollary 1.6. [2, Theorem 1.3]. *Suppose W is independent of t, satisfies (H_1)–(H_3) and $B = 0$, then the system (HS) possesses a nonconstant T-periodic solution.*

In recent years many authors studied the existence of homoclinic orbits for Hamiltonian systems. Most of them treat the second order systems. The first order systems (HS) is studied by Coti-Zelati et al. in [5] under an assumption of strict convexity on $W(t,x)$ with respect to x and using a dual variational formulation. Also, the problem was considered by Hofer and Wysocki in [9], where they made the following assumptions essentially

(H_3') there exist $\alpha \geq \mu, k_1 > 0$ such that

$$W(t,x) \geq k_1|x|^\alpha, \qquad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^{2N},$$

(H_4') there exists $c > 0$ such that

$$|\nabla W(t,x)| \leq c|x|^{\mu-1}, \qquad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

In their approach, they used first order elliptic system and nonlinear Fredholm operator theory. Later, Tanaka in [15], kept the assumption (H_3') and changed (H_4') by a weaker condition similar to (H_4). He used a minimax result in critical point theory to establish the existence of a sequence of subharmonics which converges to a nontrivial homoclinic orbit. In all above works, the assumption (AR) is fundamental. In this paper, we use a globally version of the assumption (H_1) which is weaker than (AR), (see Remark 1.9), we drop the restrictive assumption (H_3') and we assume that all the eigenvalues of the constant matrix JB are not purely imaginary. Furthermore, the following technical condition is needed:

(H_6) *There exists a normalized positive vector field V such that*

$$(Bx, Vx) \geq (Bx, x), \quad \forall\ x \in \mathbb{R}^{2N}.$$

The existence of a nontrivial homoclinic solution for (HS) will be proved as it was done in [15]. Precisely we have:

Theorem 1.7. *Suppose B satisfies (1.1) and (H_6) and W satisfies*

(H_1') *there exists a constant $\mu > 2$ such that*

$$0 < \mu W(t,x) \leq (\nabla W(t,x), Vx), \quad \forall\ x \in \mathbb{R}^{2N}\backslash\{0\},$$

(H_2') $\nabla W(t,x) = o(|x|)$ *as $x \longrightarrow 0$ uniformly in t, and (H_4) or (H_5). Then the system (HS) possesses a nontrivial homoclinic solution emanating from* 0.

An interesting class of Hamiltonians satisfying the assumption (H_1) are the homogenous ones in the sense of Definition 1.2. This class was studied in [3] in the autonomous case, where a result of existence of periodic solution for (HS) was proved [3, Theorem 1.5]. The following theorem completes and improves the result in [3].

Theorem 1.8. *Suppose W is a positive homogenous function of degree $\mu > 2$ with respect to a normalized positive vector field V, then there exist a sequence $(k_j) \subset \mathbb{N}, k_j \longrightarrow \infty$, and corresponding distinct k_jT periodic solutions of the system (HS). Moreover, if B satisfies the assumptions (1.1) and (H_6), then the system (HS) possesses a nontrivial homoclinic solution emanating from* 0.

Remark 1.9. It is obvious that if $Vx = x$, then (H_1) becomes the classical condition (AR) and (H_6) is naturally satisfied. Example 1.4 in [2] shows that (H_1) is weaker than (AR) essentially. Moreover, let $N = 1$,

$$V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A straightforward computation shows that V and B satisfy the assumptions (1.1) and (H_6). Therefore the above theorems improve the results in [12,15,17] mainly.

Remark 1.10. In [15], it is shown that condition (1.1) implies that, in an appropriate function space, the operator $-J\frac{d}{dt} - B$ is invertible , i.e., $\sigma(-J\frac{d}{dt} - B) \cap (-\alpha, \alpha) = \emptyset$ for some $\alpha > 0$. In a recent work, Ding and Willem [7] relaxed the above condition. They allowed the matrix B to be t-dependent, periodic and such that $\sigma(-J\frac{d}{dt} - B) \cap (0, \alpha) = \emptyset$ for some $\alpha > 0$ and they established the existence of a homoclinic orbit for (HS). We note that our approach can be modified so that it includes systems with the more general linear term like in [8]. As it is mentioned in [17], this needs to change our functional setting. So, in order to minimize technicalities, we restrict our attention to the case of systems with invertible linear part.

2. Functional framework and variational formulation

Let $S_T = \mathbb{R}/(T\mathbb{Z})$ and $E_T = H^{1/2}(S_T, \mathbb{R}^{2N})$ be the Sobolev space of all T-periodic $\mathbb{R}^{2N}$-valued functions u in $L^2(S_T, \mathbb{R}^{2N})$ whose Fourier series

$$u(t) = \sum_{j=-\infty}^{j=+\infty} \exp\left(\frac{2j\pi tJ}{T}\right) a_j, \quad a_j \in \mathbb{R}^{2N}$$

satisfies

$$||u||_{E_T}^2 := T|a_0|^2 + T\sum_{j=-\infty}^{j=+\infty} |j|\cdot|a_j|^2 < +\infty.$$

The inner product on E_T is defined by

$$(u,v)_{E_T} = T(a_0,b_0) + T\sum_{j=-\infty}^{j=+\infty} |j|(a_j,b_j)$$

where

$$v(t) = \sum_{j=-\infty}^{j=+\infty} \exp\left(\frac{2j\pi tJ}{T}\right) b_j, \quad b_j \in \mathbb{R}^{2N}.$$

It is well known that E_T is compactly embedded in $L^\gamma(S_T, \mathbb{R}^{2N})$ for every $\gamma \in [1,+\infty)$ and there exists a constant c_γ such that

$$\|u\|_{L^\gamma} \le c_\gamma \|u\|_{E_T}, \quad \forall\, u \in E_T. \tag{2.1}$$

Define two self-adjoint operators $\mathcal{A},\mathcal{B} \in L(E_T)$ by extending the bilinear forms

$$(\mathcal{A}u,v) = \int_0^T (-J\dot{u},v)\,dt, \quad (\mathcal{B}u,v) = \int_0^T (B(t)u,v)\,dt, \quad \forall\, u,v \in E_T.$$

By [10] and the standard spectral theory, $\mathcal{B}$ is compact on E_T. Denote the eigenvalues of $\mathcal{A}-\mathcal{B}$ on E by

$$\cdots \le \lambda_{-2} \le \lambda_{-1} < 0 = (\lambda_0) < \lambda_1 \le \lambda_2 \cdots$$

where when dim ker $(\mathcal{A}-\mathcal{B}) = 0, \lambda_0 \notin \sigma(\mathcal{A}-\mathcal{B})$. Let $\{e_{\pm j}\}_{j\in\mathbb{N}}$ be the eigenvectors of $\mathcal{A}-\mathcal{B}$ corresponding to $\{\lambda_{\pm j}\}$, respectively. Define $E^+ =$ span $\{e_1,e_2,\ldots\}, E^- =$ span $\{e_{-1},e_{-2},\ldots\}, E^0 = \ker(\mathcal{A}-\mathcal{B})$. Hence there exist an orthogonal decomposition $E_T = E^+ \oplus E^- \oplus E^0$ with dim $E^0 < \infty$, dim $E^+ =$ dim $E^- = \infty$ and an equivalent inner product in E_T, denoted by $(\cdot,\cdot)$ and defined by

$$(u,v) = ((\mathcal{A}-\mathcal{B})u^+,v^+)_{E_T} - ((\mathcal{A}-\mathcal{B})u^-,v^-)_{E_T} + (u^0,v^0)_{E_T}.$$

for all $u = u^+ + u^- + u^0, v = v^+ + v^- + v^0$ in $E^+ \oplus E^- \oplus E^0$. Hence, we have

$$\int_0^T (-J\dot{u} - B(t)u,u)\,dt = ((\mathcal{A}-\mathcal{B})u,u)_{E_T} = ||u^+||^2 - ||u^-||^2$$

where $||\cdot||$ is the norm induced by $(\cdot,\cdot)$. Let V be the normalized vector field in (H_1), then V is an invertible endomorphism on $\mathbb{R}^{2N}$ and there exist constants $a,b > 0$ such that

$$a|x| \le |Vx| \le b|x|, \quad \forall\, x \in \mathbb{R}^{2N}. \tag{2.2}$$

Define a vector field $\widetilde{V}$ on E_T by $(\widetilde{V}u)(t) = (Vu)(t)$. Using conditions $(V_1),(V_2)$ and the Fourier series, a direct computation gives

Lemma 2.1. [1] *For all* $u \in E_T$, *we have*

i) $(\mathcal{A}u, \widetilde{V}u)_{E_T} = (\mathcal{A}u, u)_{E_T}$,
ii) $a||u||_{E_T} \le ||\widetilde{V}u||_{E_T} \le b||u||_{E_T}$.

Since the growth rate of the the function W at ∞ is not restricted, we introduce modification of W as follows. Let $K \ge 1$ and $\chi_K \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi_K(s) = 1$ for $s \le K, \chi_K(s) = 0$ for $s \ge K+1$, and $\chi'_K(s) \le 0$. Define

$$W_K(t,x) = \chi_K(|x|)W(t,x) + (1 - \chi_K(|x|))r_K|x|^\mu,$$

where

$$r_K = \max_{K \le |x| \le K+1} \frac{W(t,x)}{|x|^\mu}.$$

Lemma 2.2. *W_K satisfies all assumption satisfied by W. Furthermore, by (H_1), there are constants $a_1, a_2 > 0$ independent of K such that*

$$W_K(t,x) \ge a_1|x|^\mu - a_2, \qquad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^{2N}. \tag{2.3}$$

Proof. The properties $(H_2), (H'_2), (H_3)$ and (H_5) are easy to prove. For the proof of (H_1) and (2.3) see [2]. It remains only to prove the new condition (H_4). Indeed, by the definition of χ_K and r_K we have $\chi'_K(|x|)(W(t,x) - r_K|x|^\mu) \ge 0$ and therefore

$$\begin{aligned}|\nabla W_K(t,x)| \le \chi'_K(|x|)(W(t,x) - r_K|x|^\mu) + \chi_K(|x|)|\nabla W(t,x)| \\ + \mu(1 - \chi_K(|x|))r_K|x|^{\mu-1}.\end{aligned}$$

On the other hand, for $|x| \ge R$,

$$\begin{aligned}(\nabla W_K(t,x), Vx) \ge \chi'_K(|x|)(W(t,x) - r_K|x|^\mu) + \chi_K(|x|)(\nabla W(t,x), Vx) \\ +\mu(1 - \chi_K(|x|))r_K|x|^{\mu-1}.\end{aligned}$$

So, by (H_4), we obtain

$$|\nabla W_K(t,x)| \le \widetilde{c}(\nabla W_K(t,x), Vx),$$

where $\widetilde{c} = c + 1$. □

Now, to prove Theorem 1.3, it suffices to find a nontrivial solution for

$$J\dot{u} + B(t)u + \nabla W_K(t,u) = 0, \qquad (HS_K)$$

with $||u||_{L^\infty} \le K$. Define the functional $I_K : E_T \longrightarrow \mathbb{R}$, by

$$\begin{aligned}I_K(u) &= \frac{1}{2}\int_0^T (-J\dot{u} - B(t)u, u)\,dt - \int_0^T W_K(t,u)\,dt \\ &= \frac{1}{2}(||u^+||^2 - ||u^-||^2) - \int_0^T W_K(t,u)\,dt.\end{aligned}$$

By the form of W_K at infinity, we know that $I_K \in C^1(E_T, \mathbb{R})$, and the critical points of the functional I_K are the T-periodic solutions of $(HS)_K$ (see [11,13]).

Moreover,

$$I'_K(u)(v) = \int_0^T (-J\dot{u} - B(t)u, v)dt - \int_0^T (\nabla W_K(t,u), v)dt,$$
$$= (u^+ - u^-, v) - \int_0^T (\nabla W_K(t,u), v)dt, \quad \forall u, v \in E_T. \quad (2.4)$$

Using the ideas of [16], we have the following two Lemmas on the C^0 bound of the periodic solutions of $(HS)_K$.

Lemma 2.3. *Suppose W satisfies (H_1) and (H_4). If u_K is a T-periodic solution of $(HS)_K$ such that*

$$\int_0^T W_K(t,u_K)dt \leq M, \int_0^T (\nabla W_K(t,u_K), \widetilde{V} u_k)dt \leq M \quad (2.5)$$

then there is a constant L independent of K and dependent on M only such that

$$||u_K||_{L^\infty} \leq L.$$

Proof. Let u_K a T-periodic solution of $(HS)_K$ satisfying (2.5). By (2.3), we have

$$M \geq \int_0^T W_K(t,u_K)dt \geq a_1 \int_0^T |u_K(t)|^\mu dt - a_2 T \geq a_1 T \Big(\min_{t \in S_T} |u_K(t)|\Big)^\mu - a_2 T,$$

which implies

$$\min_{t \in S_T} |u_K(t)| \leq C_0, \quad (2.6)$$

where C_0 is independent of K. We claim that there exists a constant L independent of K and depends on M only verifying $||u_K||_{L^\infty} \leq L$. If not, without loss of generality, using (2.6), we may assume that there are $K_n, L_n \in \mathbb{R}, t_n \in S_T$ such that

$$\begin{cases} L_n \longrightarrow +\infty \text{ as } n \longrightarrow +\infty, \\ |u_n(t)| \geq R, \ \forall\, t \ \in [0, t_n), \\ |u_n(0)| = R, \ |u_n(t_n)| = L_n, \end{cases}$$

where $u_n := u_{K_n}$.

Now, since u_n is a solution of $(HS)_{K_n}$ we have

$$|u_n(t_n)| - |u_n(0)| = \int_0^{t_n} \frac{d}{ds}|u_n(s)|\, ds$$
$$= \int_0^{t_n} \frac{(u_n(s), \dot{u}_n(s))}{|u_n(s)|}\, ds$$
$$\leq \int_0^{t_n} |\dot{u}_n(s)|\, ds$$
$$\leq \int_0^{t_n} (|B(s)u_n(s)| + |\nabla W_{K_n}(s, u_n(s))|)\, ds. \quad (2.7)$$

and by (H_4) we get

$$L_n \leq ||B||_{L^\infty} \int_0^{t_n} |u_n(s)|^\mu \, ds + c \int_0^{t_n} (\nabla W_{K_n}(s, u_n(s)), \widetilde{V}(u_n(s))) \, ds + R. \tag{2.8}$$

From the proof of Lemma 2.2, we know that

$$W_K(t, x) \geq a_1 |x|^\mu, \ \forall \ |x| \geq R. \tag{2.9}$$

So, by (2.5), (2.8) and (2.9), we obtain

$$\begin{aligned} L_n &\leq \frac{||B||_{L^\infty}}{a_1} \int_0^{t_n} W_{K_n}(s, u_n(s)) \, ds + c \int_0^{t_n} (\nabla W_{K_n}(s, u_n(s)), \widetilde{V}(u_n(s))) \, ds + R, \\ &\leq \frac{||B||_{L^\infty}}{a_1} M + cM + R. \end{aligned}$$

but this contradicts the fact $L_n \longrightarrow \infty$ as $n \longrightarrow \infty$. □

Lemma 2.4. *Suppose W satisfies (H_1) and (H_5). If u_K is a T-periodic solution of $(HS)_K$ verifying (2.5), then there is a constant L independent of K and dependent on M only such that*

$$||u_K||_{L^\infty} \leq L.$$

Proof. Define

$$\sigma(r) = \sup_{|x| \geq r, t \in S_T} \frac{W_t(t, x)}{|x|^\mu W(t, x)}, \quad \delta(r) = \inf_{|x| \geq r, t \in S_T} \frac{W_t(t, x)}{|x|^\mu W(t, x)}.$$

By (H_5), we have

$$\lim_{r \longrightarrow +\infty} \sigma(r) = 0 \quad \text{or} \quad \lim_{r \longrightarrow +\infty} \delta(r) = 0.$$

Case I: Suppose that $\lim_{r \longrightarrow +\infty} \sigma(r) = 0$. By definition, $\sigma(r)$ decreases to 0. Fix $r_0 > R$ large enough such that $a_1 - \sigma(r_0)M > 0$. Let u_K a T-periodic solution of $(HS)_K$ verifying (2.5). Note that u_K satisfies (2.6). We claim that there exists L independent of K and depend on M only such that $||u_K|| \leq L$. If not, there exist $K_n, L_n, a_n, b_n \in \mathbb{R}$ such that $L_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$,

$$(a_n, b_n) \subset \{t \in S_T; r_0 < |u_n(t)| < L_n\},$$

$|u_n(a_n)| = r_0$ and $|u_n(b_n)| = L_n$ for all n large enough, where $u_n := u_{K_n}$. Denote

$$H_K(t, x) = \frac{1}{2}(B(t)x, x) + W_K(t, x), \ H(t, x) = \frac{1}{2}(B(t)x, x) + W(t, x).$$

In the following, B_t and $W_{K,t}$ denote respectively the derivative of B and W_K with respect to the t-variable. Since u_n is a solution of $(HS)_{K_n}$, we have

$$\begin{aligned} H_{K_n}(b_n, u_n(b_n)) - H_{K_n}(a_n, u_n(a_n)) &= \int_{a_n}^{b_n} \frac{d}{dt} H_{K_n}(t, u_n(t))\, dt \\ &= \int_{a_n}^{b_n} [(\nabla H_{K_n}(t, u_n(t)), \dot{u}_n(t)) + H_{K_n,t}(t, u_n(t)]\, dt \\ &= \int_{a_n}^{b_n} \left[\frac{1}{2}(B_t(t)u_n(t), u_n(t)) + W_{K_n,t}(t, u_n(t))\right] dt \\ &\le \frac{1}{2}||B_t||_{L^\infty} \int_{a_n}^{b_n} |u_n(t)|^2\, dt + \int_{a_n}^{b_n} \sigma(|u_n(t)|)|u_n(t)|^\mu W_{K_n}(t, u_n(t))\, dt \\ &\le d \int_{a_n}^{b_n} W_{K_n}(t, u_n(t))\, dt + \sigma(r_0) L_n^\mu \int_{a_n}^{b_n} W_{K_n}(t, u_n(t))\, dt \\ &\le dM + \sigma(r_0) M L_n^\mu, \end{aligned} \tag{2.10}$$

where d is a positive constant.On the other hand, by (2.3), we get

$$\begin{aligned} H_{K_n}(b_n, u_n(b_n)) - H_{K_n}(a_n, u_n(a_n)) &- \max_{|x|\le r_0, t\in S_T} |H_{K_n}(t, x)| \\ &\ge a_1 |u_n(b_n)|^\mu - a_2 - \frac{1}{2}||B||_{L^\infty} |u_n(b_n)|^2 \\ &\ge a_1 L_n^\mu - a_2 - \frac{1}{2}||B||_{L^\infty} L_n^2 - \max_{|x|\le r_0, t\in S_T} |H(t, x)|. \end{aligned} \tag{2.11}$$

Combine the inequalities (2.10) and (2.11), we obtain

$$\Big(a_1 - \sigma(r_0)M\Big) L_n^\mu - \frac{1}{2}||B||_{L^\infty} L_n^2 \le a_2 + cM + \max_{|x|\le r_0, t\in S_T} |H(t, x)|.$$

Since $a_1 - \sigma(r_0)M > 0, \mu > 2$ and $L_n \longrightarrow \infty$ as $\longrightarrow +\infty$, the last inequality leads to a contradiction and our claim is true.

Case II: Suppose that $\lim_{r\longrightarrow+\infty} \delta(r) = 0$. We modify slightly the proof of Case I to obtain the following inequality

$$(a_1 + \delta(r_0)M) L_n^\mu - \frac{1}{2}||B||_{L^\infty} L_n^2 \le a_2 + cM + \max_{|x|\le r_0, t\in S_T} |H(t, x)|.$$

Using the same argument as in the above, we obtain a contradiction. □

In order to prove our main results, we state the following critical point theorem.

Theorem 2.5. [13] *Let E be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^\perp$. Suppose $f \in C^1(E, \mathbb{R})$, satisfies (PS) condition, and*

- (f_1) *$f(x) = \frac{1}{2}(Lx, x)_E + \phi(x)$, where $L = L_1P_1 + L_2P_2$ and $L_i : E_i \longrightarrow E_i$ is bounded and self-adjoint, $i = 1, 2$,*
- (f_2) *$\phi\,'$ is compact, and*
- (f_3) *thereexistasubsequence$\widetilde{E} \subset E$, sets $S \subset E, Q \subset \widetilde{E}$ and constants $\alpha > \omega$ such that*

(i) $S \subset E_1$ *and* $f|_S \geq \alpha$,
(ii) Q *is bounded and* $f|_{\partial Q} \leq \omega$,
(iii) S *and* ∂Q *link.*

Then

$$c = \inf_{h \in \Gamma} \sup_{x \in Q} f(h(1,x))$$

is a critical value of f *and* $c \geq \alpha$, *where* Γ *is defined by* $\Gamma = \{h \in C([0,1] \times E, E) | h(0,x) = x, h(1,x)|_{\partial Q} = x, h(t,x) = e^{\theta(t,x)L} + K(t,x)\}$.
Here, $\theta \in C([0,1] \times E, E)$ *and* K *is compact.*

3. Proof of main results

To find a nontrivial critical point of I_K, we use the above theorem with $E_1 = E^+, E_2 = E^- \oplus E^0$. As shown in [13], the functional I_K satisfies (f_1) and (f_2).

Lemma 3.1. *I_K satisfies the (PS) condition, i.e., if $\{u_n\} \subset E$, with $I'_K(u_n) \longrightarrow 0$ and $|I_K(u_n)| \leq M$, for some constant $M > 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. In this proof $C_k, k = 1, \ldots, 10$, are positive constants. By the equivalence of norms on E_T, for n large enough, we have

$$M + C_1\|u_n\| \geq M + \|\widetilde{V}u_n\|_E \geq I_K(u_n) - \frac{1}{2}I'_K(u_n)(\widetilde{V}u_n).$$

From Lemma 2.1, we receive

$$\int_0^T (-J\dot{u}_n, u_n)\, dt = \int_0^T (-J\dot{u}_n, \widetilde{V}u_n)\, dt.$$

Then, we obtain

$$M + C_1\|u_n\| \geq \frac{1}{2}\int_0^T (B(t)u_n, \widetilde{V}u_n - u_n)\, dt - \int_0^T W_K(t,u_n)\, dt$$
$$+\frac{1}{2}\int_0^T (\nabla W_K(t,u_n), \widetilde{V}u_n)\, dt.$$

By (H_1) and (2.3), we get

$$\begin{aligned} M + C_1\|u_n\| &\geq \frac{1}{2}\int_0^T (B(t)u_n, \widetilde{V}u_n - u_n)\, dt + \left(\frac{\mu}{2} - 1\right)\int_0^T W_K(t,u_n)dt - C_2 \\ &\geq \frac{1}{2}\int_0^T (B(t)u_n, \widetilde{V}u_n - u_n)\, dt + \left(\frac{\mu}{2} - 1\right)a_1\int_0^T |u_n|^\mu\, dt - C_3. \end{aligned} \tag{3.1}$$

On the other hand, by (2.2) and Hölder inequality, there is a constant $C_4 > 0$ such that

$$\left|\int_0^T (B(t)u_n, \widetilde{V}u_n - u_n)\, dt\right| \leq C_4\|u_n\|^2_{L^\mu}. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$M + C_1\|u_n\| \geq -\frac{C_4}{2}\|u_n\|^2_{L^\mu} + \left(\frac{\mu}{2} - 1\right) a_1\|u_n\|^\mu_{L^\mu} - C_3,$$

which implies

$$\|u_n\|^\mu_{L^\mu} \leq C_5(1 + \|u_n\|). \tag{3.3}$$

Writing $u_n = u_n^+ + u_n^- + u_n^0 \in E^+ \oplus E^- \oplus E^0$, since E^0 is a finite-dimensional space, it follows from (3.3) that

$$\|u_n^0\| \leq C_6(1 + \|u_n\|^{1/\mu}). \tag{3.4}$$

Now, taking $v = u_n^+$ in (2.4), yields

$$I'_K(u_n)(u_n^+) = \|u_n^+\|^2 - \int_0^T (\nabla W_K(t, u_n), u_n^+)dt.$$

For n large enough, by Hölder inequality and the form of ∇W_K at infinity, we get

$$\begin{aligned}
\|u_n^+\|^2 &\leq \|u_n^+\| + \left(\int_0^T |\nabla W_K(t, u_n)|^{\frac{\mu}{\mu-1}} dt\right)^{\frac{\mu-1}{\mu}} \left(\int_0^T |u_n^+|^\mu dt\right)^{\frac{1}{\mu}} \\
&\leq \|u_n^+\| + C_7\left(1 + \|u_n\|^{\mu-1}_{L^\mu}\right)\|u_n^+\|_{L^\mu} \\
&\leq C_8\left(1 + \|u_n\|^{\mu-1}_{L^\mu}\right)\|u_n^+\|.
\end{aligned}$$

Thus, by (3.3)

$$\|u_n^+\| \leq C_8\left(1 + \|u_n\|^{\mu-1}_{L^\mu}\right) \leq C_9\left(1 + \|u_n\|^{\frac{\mu-1}{\mu}}\right). \tag{3.5}$$

Analogously, with $v = u_n^-$ in (2.4), yields

$$\|u_n^-\| \leq C_9\Big(1 + \|u_n\|^{\frac{\mu-1}{\mu}}\Big). \tag{3.6}$$

The inequalities (3.4)–(3.6) imply

$$\|u_n\| \leq \|u_n^+\| + \|u_n^-\| + \|u_n^0\| \leq C_{10}\left(1 + \|u_n\|^{\frac{\mu-1}{\mu}} + \|u_n\|^{1/\mu}\right).$$

This shows that the sequence $\{u_n\}$ is bounded in E_T. Then by a standard argument (see [13]), $\{u_n\}$ has a convergent subsequence. □

Lemma 3.2. *I_K satisfies the condition (f_3).*

Proof. By (H_2) and the form of W_K, for any $\epsilon > 0$, there exists $M > 0$ such that

$$W_K(t, x) \leq \epsilon|x|^2 + M|x|^\mu, \quad \forall\, x \in \mathbb{R}^{2N}. \tag{3.7}$$

Using the norm $\|\cdot\|$ instead of $\|\cdot\|_E$, (2.1) and (3.7), we get for $u \in E_1$,

$$\begin{aligned}
I_K(u) = \frac{1}{2}\|u\|^2 - \int_0^T W_K(t, u)dt &\geq \frac{1}{2}\|u\|^2 - (\epsilon c_2\|u\|^2 + Mc_\mu\|u\|^\mu) \\
&\geq \frac{1}{2}\|u\|^2 - (\epsilon c_2 + Mc_\mu\|u\|^{\mu-2})\|u\|^2.
\end{aligned}$$

Choose $\epsilon = \frac{1}{6c_2}, \rho = (\frac{1}{6Mc_\mu})^{\frac{1}{\mu-2}}$ and denote by B_ρ the closed ball in E_T of radius ρ centered at origin. Let $S = \partial B_\rho \cap E_1$, then $I_K(u) \geq \alpha = \frac{\rho^2}{6}$ for all $u \in S$, and (i) of (f_3) holds.

The proof of (ii) of (f_3) is slightly different from that in [13]. Indeed, to obtain Q with r_1 and r_2 independent of K, we let $e \in \partial B_1 \cap E_1$ and $u = u^0 + u^- \in E_2$, then

$$I_K(u + re) = \frac{1}{2}(r^2 - \|u^-\|^2) - \int_0^T W_K(t, u + re)dt. \tag{3.8}$$

By (H_3), it is obvious that $I_K(u) \leq 0$ on E_2. Since E^0 is finite dimensional, there exists $a_3 > 0$ such that

$$\|e\| \leq a_3\|e\|_{L^2}, \quad \|u^0\| \leq a_3\|u^0\|_{L^2} \tag{3.9}$$

for all $u^0 \in E^0$. Moreover, by (2.3), there is $a_4 > 0$ independent of K such that

$$W_K(t, x) \geq a_3^2|x|^2 - a_4, \ \forall\ (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}. \tag{3.10}$$

The orthogonality of u^0, u^- and e in L^2 and (3.10) imply

$$\begin{aligned}\int_0^T W_K(t, u + re)dt &\geq a_3^2\|u + re\|_{L^2}^2 - a_4T \\ &\geq a_3^2(\|u^-\|_{L^2}^2 + \|u^0\|_{L^2}^2 + r^2\|e\|_{L^2}^2) - a_4T. \end{aligned} \tag{3.11}$$

By (3.8), (3.9) and (3.11), we get

$$\begin{aligned} I_K(u + re) &\leq \frac{1}{2}(r^2 - \|u^-\|^2) - (\|u^0\|^2 + r^2) + a_4T \\ &\leq -\frac{1}{2}r^2 - \frac{1}{2}\|u\|^2 + a_4T \end{aligned} \tag{3.12}$$

for all $r > 0$ and $u \in E_2$. Let $r_1 = r_2 = \sqrt{2a_4T}$, then $I_K(u + re) \leq 0$ either $r \geq r_1$ or $\|u\| \geq r_2$. Consequently $I_K \leq 0 \equiv \omega$ on ∂Q, where $Q = \{re\ ; r \in [0, r_1]\} \oplus (B_{r_2} \cap E_2)$. By Lemma 6.27 of [13], S and ∂Q link and (ii) and (iii) of (f_3) hold. □

Proof of Theorem 1.3. The functional I_K satisfies the assumptions of Theorem 2.5, so, it possesses a critical value $c_K > 0$ and corresponding nontrivial critical point u_K. It remains to prove that u_K satisfies (HS) for appropriately chosen K. Since $h(t, x) \equiv x$ belongs to Γ, one has

$$\begin{aligned} c_K = I_K(u_K) &\leq \sup_{x \in Q} I_K(x) \\ &\leq \sup_{0 \leq r \leq r_1, \|u^- + u^0\| \leq r_2} \frac{1}{2}(r^2 - \|u^-\|^2) - \int_0^T W_K(t, u^0 + u^- + re)\, dt \\ &\leq \frac{1}{2}r_1^2. \end{aligned} \tag{3.13}$$

Note that r_1 and r_2 are independent of K. Arguing as in (3.1)– (3.2), we get

$$\frac{1}{2}r_1^2 \geq c_K = I_K(u_K) - \frac{1}{2}I_K'(u_K)(\widetilde{V}u_k)$$
$$\geq -\frac{C_4}{2}\|u_K\|_{L^\mu}^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T (\nabla W_K(t,u_K), \widetilde{V}u_k)\,dt - C_2 \quad (3.14)$$
$$\geq -\frac{C_4}{2}\|u_K\|_{L^\mu}^2 + \left(\frac{\mu}{2}-1\right)\int_0^T W_K(t,u_K)\,dt - C_2 \quad (3.15)$$
$$\geq -\frac{C_4}{2}\|u_K\|_{L^\mu}^2 + \left(\frac{\mu}{2}-1\right)a_1\|u_K\|_{L^\mu}^\mu - C_3.$$

Since C_3 and C_4 are independent of K and $\mu > 2$, the last inequality gives

$$\|u_K\|_{L^\mu} \leq M_1. \quad (3.16)$$

Also, by (3.14) and (H_4) for W_K, we get

$$\int_0^T |\nabla W_K(t,u_K)|\,dt \leq M_2. \quad (3.17)$$

The inequalities (3.16) and (3.17) yield

$$\|\dot{u}_K\|_{L^1} \leq \|B(t)u_K\|_{L^1} + \|\nabla W(t,u_K)\|_{L^1} \leq M_3. \quad (3.18)$$

Denote $\bar{u}_K = \frac{1}{T}\int_0^T u_K(t)\,dt$, $\widetilde{u}_K = u_K - \bar{u}_K$. The inequality $\|u\|_{L^\infty} \leq c\|\dot{u}\|_{L^1}$ (see [11, Proposition 1.1]) for $\widetilde{u}_K$, (3.16) and (3.18) provide K-independent L^∞ bounds for $\bar{u}_K$ and $\widetilde{u}_K$ respectively. Hence, for some $K_0 > 0$, $\|u_K\|_{L^\infty} \leq K_0$. Taking $K > K_0$, it follows that u_K is a T-periodic solution of (HS).

Now, we show that in fact there are infinitely many distinct solutions for (HS). Substituting kT for T in the above, it is easy to see that the problem (HS) possesses a sequence $(u_k)_{k\in\mathbb{N}}$ of kT-periodic solutions. Since any T-periodic solution is also kT-periodic, an additional argument is required to distinguish them. Let $c_k = I_k(u_k)$, where I_k is the functional defined on E_{kT} by

$$I_k(u) = \frac{1}{2}\int_0^{kT}(-J\dot{u} - B(t)u, u)dt - \int_0^{kT} W_K(t,u)\,dt \quad (3.19)$$

where K depends on k. Next we prove that c_k is bounded from above independently of k. Indeed, by (3.13), the corresponding critical point c_k satisfies

$$c_k \leq \frac{1}{2}r_1^2(k), \quad (3.20)$$

where $r_1(k)$ is determined by the condition

$$I_k(u+re) = \frac{1}{2}(r^2 - \|u^-\|^2) - \int_0^{kT} W_K(t,u+re)dt \leq 0 \quad (3.21)$$

for all $r \geq r_1(k)$ with u and e adequately chosen. Let $E_{kT} = E_k^+ \oplus E_k^- \oplus E_k^0$ be an orthogonal decomposition defined as above for E_T. Take $u = u^0 + u^- \in E_k^0 \oplus E_k^-$ and $e \in E_k^+$ be constructed as follows: For $\phi \in E_1^+ = E^+$ with $||\phi|| = 1$, let

$$e(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [0,T) \\ 0, & \text{if} \quad t \in [T,kT) \end{cases}$$

and regard e as kT-periodic function defined on $\mathbb{R}$. Then $e \in E_k^+, ||e|| = 1$ and particulary $e \not\equiv 0$. Moreover, by (H_3) and (2.3), we have

$$\int_0^{kT} W_K(t, u + re)\, dt \geq \int_0^T W_K(t, u + re)\, dt$$
$$\geq a_1 \int_0^T |u(t) + re(t)|^\mu\, dt - a_2 T. \qquad (3.22)$$

On the other hand, by Hölder's inequality

$$\int_0^T |u(t) + re(t)|^2\, dt \leq \left[\int_0^T |u(t) + re(t)|^\mu\, dt\right]^{2/\mu} (T)^{\frac{\mu-2}{\mu}}. \qquad (3.23)$$

Using the orthogonality of u and e in L^2 and combining (3.21)–(3.23), we get

$$I_k(u + re) \leq \frac{1}{2}(r^2 - \|u^-\|^2) - a_1 T^{\frac{2-\mu}{2}} r^\mu \left(\int_0^T |e(t)|^2 dt\right)^{\mu/2} + a_2 T. \qquad (3.24)$$

Since $\mu > 2$, it follows from (3.24) that there exists r_0 independent of k such that

$$I_k(u + re) \leq 0, \quad \text{for all } r \geq r_0.$$

Then $r_1(k) \leq r_0$ and it follows from (3.20) that

$$c_k \leq \frac{r_0^2}{2}. \qquad (3.25)$$

Now, if for some $k > m$, $u_k(t) \equiv u(t) = u_m(t)$. So u is kT-periodic, denote $\frac{kT}{l_k}$ its minimal period. By a simple change of variables (see (2.2) in [12]), we obtain

$$c_k = I_k(u_k) = l_k I_{\frac{k}{l_k}}(u).$$

Also, u is mT-periodic, then we have $\frac{k}{l_k} = \frac{m}{l_m}$ for some positive integer l_m and

$$c_m = I_m(u_m) = l_m I_{\frac{m}{l_m}}(u),$$

thus

$$c_k = \frac{k}{m} c_m.$$

Since $c_m > 0$ and the sequence (c_k) is bounded, it follows that there can be at most finitely many $k > m$ such that $u_k = u_m$ for any given $m \in \mathbb{N}$. □

Proof of Theorem 1.4. In the proof of Theorem 1.3 the assumption (H_4) is used only in (3.17) to show that u_K is bounded in $C^0(\mathbb{R}, \mathbb{R}^{2n})$ independently of K. Now, by combining (3.14)–(3.16), we obtain the assumption (2.5). Hence, replacing (H_4) by (H_5) in the proof of Theorem 1.3 and using Lemma 2.4, we obtain the same result.

Proof of Theorem 1.7. Let $\mathfrak{L} := -(J\frac{d}{dt} + B)$ be the self-adjoint operator acting on $L^2(S_{kT}, \mathbb{R}), k \in \mathbb{N}$, under periodic boundary conditions and $X = D(|\mathfrak{L}|^{1/2})$ with the norm

$$||u||_X = |||\mathfrak{L}|^{1/2} u||_{L^2}, \quad \forall\, u \in X.$$

In [15], it is shown that the condition (1.1) implies $X = E_{kT}$, the norms $||\cdot||_X$ and $||\cdot||_{E_{kT}}$ are equivalents and X has an orthogonal decomposition $X = X^+ \oplus X^-$ where the quadratic form $u \longmapsto (\mathfrak{L}u, u)_{L^2}$ is positive (resp. negative) definite on X^+ (resp. X^-). Moreover,

$$(\mathfrak{L}u, u)_{L^2} = ||u^+||^2 - ||u^-||^2 \quad \text{for all } u = u^+ + u^- \in X = X^+ \oplus X^-.$$

Recall that $c_k = I_k(u_k)$ is a critical value of the functional I_k defined in (3.19) and u_k is the corresponding critical point.

Lemma 3.3. *There is a constant $C_1 > 0$ independent of K and k such that*

$$||u_k||_{C^1} \leq C_1$$

for all $k \in \mathbb{N}$ and $K \geq 1$.

Proof. By (3.25) we know that c_k is bounded independently of K and k. Moreover, using (H_6) and (H_1'), there is a constant C such that

$$\begin{aligned} C \geq I_k(u_k) &= I_k(u_k) - \frac{1}{2} {I_k}'(u_k)(\widetilde{V} u_k) \\ &= \int_0^{kT} (Bu_k, \widetilde{V} u_k - u_k)\, dt - \int_0^{kT} W_K(t, u_k) dt \\ &\quad + \frac{1}{2} \int_0^{kT} (\nabla W_K(t, u_k), \widetilde{V} u_k)\, dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_0^{kT} (\nabla W_K(t, u_k), \widetilde{V} u_k)\, dt \\ &\geq \left(\frac{\mu}{2} - 1\right) \int_0^{kT} W_K(t, u_k)\, dt. \end{aligned}$$

which implies for certain constant $M > 0$

$$\int_0^{kT} (\nabla W_K(t, u_k), \widetilde{V} u_k) dt \leq M, \qquad \int_0^{kT} W_K(t, u_k) dt \leq M. \tag{3.26}$$

Note that since $W_K \geq 0$, by (2.3) and (3.26), we have

$$\begin{aligned} M &\geq \int_0^{kT} W_K(t, u_k) dt \\ &\geq \int_0^{T} W_K(t, u_k) dt \\ &\geq a_1 \int_0^{T} |u_k(t)|^{\mu} dt - a_2 T \\ &\geq a_1 T \Big(\min_{t \in S_T} |u_k(t)| \Big)^{\mu} - a_2 T \\ &\geq a_1 T \Big(\min_{t \in S_{kT}} |u_k(t)| \Big)^{\mu} - a_2 T. \end{aligned}$$

So, inequality (2.6) holds independently of K and k and the proof of Lemma 2.3 goes through with kT replacing T. From Lemmas 2.3 and 2.4, we know that u_k must be bounded in C^0 independently of K and k. Then, for a sufficiently large K_0 such that $||u_k||_{L^\infty} < K_0$, u_k is a C^1 solution of the original problem (HS) and therefore is bounded in $||\cdot||_{C^1}$. □

Lemma 3.4. *There is a constant $\delta > 0$ independent of k such that*

$$||u_k||_{L^\infty} \geq \delta \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. From the above we may assume that u_k is a critical point of the functional defined on X by

$$J_k(u) = \frac{1}{2}\int_0^{kT} (-J\dot{u} - Bu, u)\,dt - \int_0^{kT} W_{K_0}(t,u)\,dt.$$

Let $\varepsilon > 0$, by the assumption (H_2'), there exists $\delta_\varepsilon > 0$ such that

$$|\nabla W_{K_0}(t,x)| \leq \varepsilon |x| \qquad \text{for } |x| \leq \delta_\varepsilon. \tag{3.27}$$

Suppose that $||u_k||_{L^\infty} < \delta_\varepsilon$. Writing $u_k = u_k^+ + u_k^- \in X^+ \oplus X^-$ and using the definition of u_k, we obtain

$$0 = J_k'(u_k)(u_k^+ - u_k^-) = ||u_k||^2 - \int_0^{kT} (\nabla W_{K_0}(t,u_k), u_k^+ - u_k^-)dt.$$

Therefore, by (3.26), we get

$$\begin{aligned} ||u_k||^2 &= \int_0^{kT} (\nabla W_{K_0}(t,u_k), u_k^+ - u_k^-)\,dt. \\ &\leq \varepsilon \int_0^{kT} |u_k||u_k^+ - u_k^-|\,dt \\ &\leq \varepsilon ||u_k||_{L^2}^2. \end{aligned}$$

By (2.1) and the equivalence of norms, there exists a constant denoted also by c_2 such that

$$||u_k||^2 \leq \varepsilon c_2 ||u_k||^2.$$

Choosing $\varepsilon < 1/c_2$, we get $u_k = 0$. But this contradicts the fact $J_k(u_k) > 0$. Then we have $||u_k||_{L^\infty} \geq \delta_\varepsilon$. □

Now, we can find a sequence $(j_k)_{k\in\mathbb{N}}$ of integers such that

$$\max_{t\in[0,T]} |u_k(t + j_k T)| = \max_{t\in\mathbb{R}} |u_k(t)| \in [\delta, C_1].$$

Let $\widetilde{u}_k(t) := u_k(t + j_k T)$. It is easy to see that $\widetilde{u}_k$ is a solution of (HS) and $J_k(\widetilde{u}_k) = J_k(u_k)$. By Lemma 3.3, we can extract a subsequence from any given sequence of integers $k_n \longrightarrow \infty$ such that

$$\widetilde{u}_{k_n} \longrightarrow u_0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{in} \quad C^1_{loc}(\mathbb{R}, \mathbb{R}^{2N}),$$

where $u_0 \in C^1(\mathbb{R}, \mathbb{R}^{2N})$ is a solution of (HS). To conclude the proof of Theorem 1.7, we use the following [15, Lemma 2.9].

Lemma 3.5. [15] u_0 *satisfies the following*

(a) $u_0 \not\equiv 0$.
(b) $u_0 \in L^p(\mathbb{R}, \mathbb{R}^{2N})$ *for all* $p \in [2, \infty]$.
(c) $u_0 \longrightarrow 0$ *as* $t \longrightarrow \pm\infty$.

Proof of Theorem 1.8. The proof is based on the following [3, Lemma 1.3].

Lemma 3.6. [3] *Suppose* $h \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ *a positive homogenous function of degree* μ *with respect to a normalized positive vector field* V*, then for any* $x \in \mathbb{R}^{2N}$,

a) $\nabla h(\phi_s x) = e^{(\mu-2)s}\phi_s \nabla h(x)$,
b) $(\nabla h(x), Vx) = \mu h(x)$,
c) $h(x) \geq a_1|x|^\mu$,
d) $|\nabla h(x)| \leq a_2|x|^{\mu-1}$,

where a_1, a_2 *are positive constants.*

Let W a positive homogenous function of degree $\mu > 2$ with respect to a normalized positive vector field V, then, W satisfies (H_1') via property b) and (H_2') via property d) of Lemma 3.6. Also, by combining properties b), c) and d), it is easy to see that W satisfies (H_4). On the other hand, the assumptions (H_1') and (H_2') imply (H_1) and (H_2) respectively. So, all the hypothesis of Theorem 1.3 are satisfied. Furthermore, if B satisfies the assumptions (1.1) and (H_6), the conclusion of Theorem 1.7 holds.

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