

Quasilinear elliptic and parabolic Robin problems on Lipschitz domains

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Abstract. We prove Hölder continuity up to the boundary for solutions of quasi-linear degenerate elliptic problems in divergence form, not necessarily of variational type, on Lipschitz domains with Neumann and Robin boundary conditions. This includes the p -Laplace operator for all $p \in (1, \infty)$, but also operators with unbounded coefficients. Based on the elliptic result we show that the corresponding parabolic problem is well-posed in the space $C(\overline{\Omega})$ provided that the coefficients satisfy a mild monotonicity condition. More precisely, we show that the realization of the elliptic operator in $C(\overline{\Omega})$ is m -accretive and densely defined. Thus it generates a non-linear strongly continuous contraction semigroup on $C(\overline{\Omega})$.

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1. Introduction

Given a bounded Lipschitz domain Ω in \mathbb{R}^N , we show that all weak solutions of certain degenerate quasi-linear elliptic problems are Hölder-continuous up to the boundary of Ω , which generalizes the results in [27] to non-linear equations. More precisely, this is true for equations of the form

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu + h(x, u) = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

where $A: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $B: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions such that there exist constants $1 < p < \infty$ and $0 < \mu_1 \leq \mu_2$ and non-negative functions ψ_1 , ψ_2 and ψ_3 satisfying

$$\begin{aligned} zA(x, u, z) &\geq \mu_1|z|^p - \psi_1(x)|u|^p - \psi_1(x) \\ |A(x, u, z)| &\leq \mu_2|z|^{p-1} + \psi_2(x)|u|^{p-1} + \psi_2(x) \end{aligned}$$

$$\begin{aligned}
 |B(x, u, z)| &\leq \psi_3(x)|z|^{p-1} + \psi_1(x)|u|^{p-1} + \psi_1(x) \\
 |h(x, u)| &\leq \psi_4(x)|u|^{p-1} + \psi_4(x)
 \end{aligned}
 \tag{1.2}$$

for all $x \in \Omega$, $u \in \mathbb{R}$ and $z \in \mathbb{R}^N$, and such that

$$\begin{aligned}
 \psi_1 \in L^{\frac{N}{p-\varepsilon}}(\Omega), \quad \psi_2 \in L^{\frac{N}{p-1}}(\Omega), \quad \psi_3 \in L^{\frac{N}{1-\varepsilon}}(\Omega), \quad \psi_4 \in L^{\frac{N-1}{p-1-\varepsilon}}(\partial\Omega) \quad \text{if } p < N, \\
 \psi_1 \in L^{\frac{N}{N-\varepsilon}}(\Omega), \quad \psi_2 \in L^{\frac{N}{N-1-\varepsilon}}(\Omega), \quad \psi_3 \in L^{\frac{N}{1-\varepsilon}}(\Omega), \quad \psi_4 \in L^{\frac{N-1}{N-1-\varepsilon}}(\partial\Omega) \quad \text{if } p = N, \\
 \psi_1 \in L^1(\Omega), \quad \psi_2 \in L^{\frac{p}{p-1}}(\Omega), \quad \psi_3 \in L^p(\Omega), \quad \psi_4 \in L^1(\partial\Omega) \quad \text{if } p > N
 \end{aligned}$$

with some $\varepsilon \in (0, 1)$. In particular, the Δ_p -equation with Robin boundary conditions is included for all $p \in (1, \infty)$, together with a large variety of lower order perturbations, including unbounded coefficients. Also we do not require a variational structure of the equation. We refer to [21] and references therein for a short account on applications for the Δ_p -operator. The assumptions on the coefficients are optimal in that the regularity assumptions on A and B are the right ones within the class of L^p -functions for results about interior regularity [29, 25], see also [32] or [23, Theorem 4.1.1] for bounded coefficients.

The other main result of this article, which crucially depends on the elliptic regularity of the previous sections, gives conditions on the coefficients which ensure that for every $\tau > 0$ the unique solution $u \in W^{1,\infty}(0, \tau; L^2(\Omega)) \cap L^\infty(0, \tau; W^{1,p}(\Omega))$

$$\begin{cases}
 u_t(t, x) - \operatorname{div} a(x, \nabla u(t, x)) + b(x, u(t, x)) = 0 & 0 < t < \tau, \quad x \in \Omega \\
 a(x, \nabla u(t, x)) \cdot \nu + h(x, u(x)) = 0 & 0 < t < \tau, \quad x \in \partial\Omega \\
 u(0, x) = u_0(x) & x \in \Omega
 \end{cases}
 \tag{1.3}$$

is continuous on the parabolic cylinder $[0, \tau] \times \bar{\Omega}$ whenever $u_0 \in C(\bar{\Omega})$. More precisely, we show that the corresponding elliptic operator is m -accretive on $C(\bar{\Omega})$ and thus generates a non-linear contraction C_0 -semigroup on $C(\bar{\Omega})$. In order to obtain this result we have to assume that a , b and h are Carathéodory functions that such $A(x, u, z) := a(x, z)$, $B(x, u, z) := b(x, u)$ and h satisfy the above conditions (1.2). Moreover, we assume the following mild monotonicity assumptions,

$$\begin{cases}
 (z_1 - z_2) (a(x, z_1) - a(x, z_2)) \geq 0, \\
 (u_1 - u_2) (b(x, u_1) - b(x, u_2)) \geq 0, \\
 (u_1 - u_2) (h(x, u_1) - h(x, u_2)) \geq 0,
 \end{cases}
 \tag{1.4}$$

which are much weaker than the standard monotonicity assumptions as considered for example in [26]. In particular, (1.4) includes the Δ_p -operator for every $p \in (1, \infty)$. Thus for all $p \in (1, \infty)$ the problem

$$\begin{cases}
 u_t(t, x) - \Delta_p u(t, x) + b_0(x)|u|^{p-2}u = 0 & t > 0, \quad x \in \Omega \\
 |\nabla u(t, x)|^{p-2} \frac{\partial u(t, x)}{\partial \nu} + h_0(x)|u|^{p-2}u = 0 & t > 0, \quad x \in \partial\Omega \\
 u(0, x) = u_0(x) & x \in \Omega
 \end{cases}$$

is well-posed in $C(\bar{\Omega})$ if $b_0 \in L^\infty(\Omega)$ and $h_0 \in L^\infty(\partial\Omega)$ are nonnegative. We also obtain a similar result for Wentzell–Robin boundary conditions. All of these results are based on the author’s PhD thesis [28].

There are several good reasons to study elliptic and parabolic equations in $C(\bar{\Omega})$. On the one hand, for Dirichlet boundary conditions it is the natural space to formulate the boundary conditions, cf. [3–5], and it is the natural space for maximum principles. But also for Neumann or Robin boundary conditions the space $C(\bar{\Omega})$ is nicer in some respects than the L^q -spaces with $q \in [1, \infty)$. For instance, the composition operator $f \mapsto g \circ f$ is locally Lipschitz continuous in $C(\bar{\Omega})$ whenever g is locally Lipschitz continuous, but in general fails to map $L^q(\Omega)$ into $L^q(\Omega)$, so rapidly growing non-linear perturbations can more easily be handled in $C(\bar{\Omega})$ than in $L^q(\Omega)$.

The results of this article are new regarding several aspects. Our results are valid for open sets whose closure is a Lipschitz submanifold with boundary in \mathbb{R}^N , which for simplicity we call *Lipschitz domains* for the purpose of this article. Since the class of all Lipschitz domains contains the class of *strong Lipschitz domains*, i.e., the domains that are locally the epigraph of a Lipschitz continuous functions, there is no harm in using the ambiguous, but convenient terminology “Lipschitz domain”. On the other hand, there are Lipschitz domains that are not strongly Lipschitz, for example the topologically regularized union of two crossing beams, which is physically relevant [18, Section 7.3]. This explains that the class of Lipschitz domains has drawn some interest lately, see for example [13, 18, 19, 22].

Hölder continuity of solutions of linear equations with Robin boundary conditions on Lipschitz domains has been extensively studied, sometimes only in special cases, see for example [6, 11, 14, 27, 33]. The main elliptic result of this article, Theorem 4.4, seems to be new in the non-linear case even for smooth domains, but compare [17, 24] for corresponding results under more restrictive assumptions on the coefficients, which are obtained by different methods. The linear parabolic problem has been studied in [27, 33] in terms of semigroups. Our non-linear results seem to be new, but one can find similar regularity results under stronger assumptions on the coefficients in the literature [8, 15]. Our result seems to be particularly interesting because we neither assume that the corresponding elliptic operator has a variational structure nor that it is strongly monotone.

The article is structured as follows. After introducing some notational conventions and basic properties of Lipschitz domains in Sect. 2, we show in Sect. 3 that every solution of (1.1) for $h = 0$ is Hölder continuous, thus proving the main elliptic result for Neumann boundary conditions. The proof is based on a reflection argument that the author has used already for the linear case [27]. The general idea is much older, compare for example [31, Section 2.4.3], but has apparently not been exploited to this extent before.

In Sect. 4 we obtain a priori estimates for the Robin problem. We use Moser’s iteration in a similar manner as in [9]. Our result is more general than those in [9] in that we allow general quasi-linear operators, but less general in that we restrict ourselves to Lipschitz domains. Combining these a priori estimates with our main result for problems with Neumann boundary conditions we extend the regularity result to general h .

Finally, in Sect. 5 we make use of the elliptic theory in order to show that the parabolic problem with Robin boundary conditions is well-posed in the space of continuous functions. The result is based on non-linear semi-group theory. Following the ideas in [2], we are able to apply our methods also to equations with Wentzell–Robin boundary conditions. We do not have to assume that the L^2 -realization of the operator is a subdifferential, i.e., we do not assume that the corresponding elliptic problem has a variational formulation.

2. Notation and preliminaries

Throughout the article we follow the convention that constants denoted by c are allowed to vary from one line to the next and may depend freely on the parameters N , Ω , p , ε and upper bounds for $\|\psi_1\|$, $\|\psi_2\|$, $\|\psi_3\|$ and $\|\psi_4\|$ in their respective spaces, as introduced in (1.2). Any additional dependencies are explicitly indicated by subscripts.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We say that Ω is a *Lipschitz domain* or, more precisely, that $\bar{\Omega}$ is a *Lipschitz submanifold with boundary* in \mathbb{R}^N if for every $x \in \partial\Omega$ there exists an \mathbb{R}^N -neighborhood V of x and a bi-Lipschitz mapping ψ from V onto $(-1, 1)^N$ such that $\psi(V \cap \Omega) = (-1, 1)^{N-1} \times (0, 1)$, i.e., ψ is invertible and both ψ and ψ^{-1} are Lipschitz continuous.

Remark 2.2 ([16, Section 1.2.1]). Every domain with Lipschitz boundary is a Lipschitz domain, so there is no harm in confusion between these notions in the present context.

If Ω is a Lipschitz domain, then it is an extension domain [12, Theorem 7.25] which implies that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $p \in [1, \infty)$ and that the usual Sobolev embeddings hold. The natural measure on the boundary of a Lipschitz domain is the $(N - 1)$ -dimensional Hausdorff measure. It is the unique measure for which the divergence theorem holds. This fact is easily transported from the reference domain $(-1, 1)^{N-1} \times (0, 1)$ to Ω , see also [10, Section 5.8]. We agree that integrals over the boundary of a Lipschitz domain are always to be understood to be taken with respect to the $(N - 1)$ -dimensional Hausdorff measure, which we denote by σ if the need arises.

Let $1 \leq q \leq \infty$. If we write $\|u\|_{L^q(\Omega)}$, where u is a measurable function on Ω , we regard that expression to equal infinity if $u \notin L^q(\Omega)$. For convenience we use the notation

$$\|u\|_{L^q(\partial\Omega)} := \|u|_{\partial\Omega}\|_{L^q(\partial\Omega)}$$

for functions $u \in W^{1,p}(\Omega)$, which admit a trace $u|_{\partial\Omega} \in L^p(\partial\Omega)$, and again we define this expression to equal infinity if $u|_{\partial\Omega} \notin L^q(\partial\Omega)$.

We will need a change of variables formula for boundary integrals. In order to prove it, we start with a few facts about the derivative of bi-Lipschitz mappings.

Lemma 2.3. *Let $U \subset \mathbb{R}^n$ be open and let $\psi: U \rightarrow \mathbb{R}^m$ be a bi-Lipschitz mapping, $m \geq n$. Then ψ is differentiable almost everywhere and the Jacobian $J\psi(x) := (\det(\psi'(x)^T \psi'(x)))^{1/2}$ of ψ satisfies $\alpha \leq J\psi \leq \beta$ almost everywhere, where the constants α and β depend only on n, m and the Lipschitz constants of ψ and ψ^{-1} . If $m = n$, then ψ' is invertible almost everywhere with uniformly bounded inverse and $\alpha \leq |\det \psi'| \leq \beta$.*

Proof. Rademacher’s theorem asserts that ψ is differentiable almost everywhere. More precisely, the entries of ψ' are essentially bounded by the Lipschitz constant of ψ , which proves the upper estimate for $J\psi$. As for the lower estimate, let $L > 0$ be the Lipschitz constant of ψ^{-1} , so that $|\psi(y) - \psi(x)| \geq L^{-1}|x - y|$ for all $x, y \in U$. If x is a point of differentiability of ψ , then

$$L^{-1}|tv| \leq |\psi(x + tv) - \psi(x)| = |t\psi'(x)v + o(t)|.$$

For $t \rightarrow 0$ we obtain that $|\psi'(x)v| \geq L^{-1}|v|$ almost everywhere for all $v \in \mathbb{R}^n$, hence

$$|v| |\psi'(x)^T \psi'(x)v| \geq (\psi'(x)^T \psi'(x)v | v) = |\psi'(x)v|^2 \geq L^{-2}|v|^2.$$

Thus the eigenvalues of $\psi'(x)^T \psi'(x)$ can be bounded from below by L^{-2} , which shows that $J\psi \geq L^{-n}$ holds almost everywhere.

Finally, if $m = n$, then the chain rule [34, Theorem 2.2.2] implies that

$$(\psi^{-1})'(\psi(x)) = (\psi'(x))^{-1}.$$

Since the entries of $(\psi^{-1})'$ are essentially bounded, this proves that $(\psi')^{-1}$ is uniformly bounded outside a set of measure zero. The estimate for the determinant follows from $J\psi = |\det \psi'|$. □

We can now prove the following change of boundary integrals.

Lemma 2.4. *Let Ω_1 and Ω_2 be Lipschitz domains and $\psi: \Omega_2 \rightarrow \Omega_1$ a bi-Lipschitz function. Then ψ has a unique extension to $\bar{\Omega}_1$, and $\psi(\partial\Omega_2) = \partial\Omega_1$, where we identify ψ with its extension. In this situation, there exists a measurable function $m: \Omega_2 \rightarrow (0, \infty)$, which is unique up to nullsets, such that*

$$\int_{\partial\Omega_1} g = \int_{\partial\Omega_2} (g \circ \psi)m$$

for all positive measurable functions g on $\partial\Omega_1$ and hence for all integrable functions. Moreover, $0 < \alpha \leq m \leq \beta$ almost everywhere with constants α and β that depend only on ψ, Ω_1 and Ω_2 .

Proof. The assertions about ψ and the uniqueness of m are clear. In order to show the existence of m , fix $y \in \partial\Omega_2$ and define $x := \psi(y) \in \partial\Omega_1$. Fix neighborhoods V_2 of y and V_1 of x such that there exist bi-Lipschitz transformations $\psi_1: V_1 \rightarrow (-1, 1)^N$ and $\psi_2: V_2 \rightarrow (-1, 1)^N$ as in Definition 2.1. Without loss of generality we pick V_2 so small that $\psi(V_2) \subset V_1$. Write

$$\varphi_i := \psi_i|_{\partial\Omega_i \cap V_i}: \partial\Omega_i \cap V_i \rightarrow H := (-1, 1)^{N-1} \times \{0\}.$$

Then the bi-Lipschitz function $\varphi := \varphi_1 \circ \psi \circ \varphi_2^{-1}: H \rightarrow H$ is the local representation of $\psi|_{\partial\Omega_2}$, and we regard H as an open subset of \mathbb{R}^{N-1} . Then for every positive measurable function f on H we obtain that

$$\int_H f = \int_H (f \circ \varphi) |\det \varphi'|$$

by the change of variables formula for Lipschitz functions [10, Section 3.3.3]. Also by the change of variables formula we obtain that

$$\int_H f J\varphi_i^{-1} = \int_{\partial\Omega_i} f \circ \varphi_i$$

for $i = 1, 2$, where $J\varphi_i^{-1}$ denotes the Jacobian of φ_i^{-1} as a mapping from H into \mathbb{R}^N . Combining these formulas, we see that

$$\begin{aligned} \int_{\partial\Omega_1} g &= \int_H (g \circ \varphi_1^{-1}) J\varphi_1^{-1} = \int_H (g \circ \varphi_1^{-1} \circ \varphi) (J\varphi_1^{-1} \circ \varphi) |\det \varphi'| \\ &= \int_{\partial\Omega_2} (g \circ \varphi_1^{-1} \circ \varphi \circ \varphi_2) \frac{J\varphi_1^{-1} \circ \varphi \circ \varphi_2}{J\varphi_2^{-1} \circ \varphi_2} |\det \varphi' \circ \varphi_2| \end{aligned}$$

for all positive measurable functions g on $\partial\Omega_1$. By Lemma 2.3, this implies the claim. □

3. Elliptic Neumann problems

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $p \in (1, \infty)$. We prove that all weak solutions $u \in W^{1,p}(\Omega)$ of

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = f(x) - (\operatorname{div} F)(x) & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu = g(x) + (F \cdot \nu)(x) & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

are Hölder continuous, provided A satisfies the structure conditions (1.2) and f, F and g are in appropriate Lebesgue spaces, namely

$$\begin{cases} f \in L^{\frac{N}{p-\varepsilon}}(\Omega), F \in L^{\frac{N}{p-1-\varepsilon}}(\Omega; \mathbb{R}^N), g \in L^{\frac{N-1}{p-1-\varepsilon}}(\partial\Omega) & \text{if } p \leq N, \\ f \in L^1(\Omega), F \in L^{\frac{p}{p-1}}(\Omega; \mathbb{R}^N), g \in L^1(\partial\Omega) & \text{if } p > N. \end{cases} \quad (3.2)$$

We could absorb f and F (but not g) into the coefficients, but it turns out to be more convenient to have an explicit right hand side. By convention, $\|f\|, \|F\|$ and $\|g\|$ will always refer to the norms of f, F and g in the spaces indicated in (3.2).

Definition 3.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of (3.1) if

$$\int_{\Omega} \nabla \eta A(x, u, \nabla u) + \int_{\Omega} \eta B(x, u, \nabla u) = \int_{\Omega} \eta f + \int_{\Omega} \nabla \eta F + \int_{\partial\Omega} \eta g \quad (3.3)$$

for all $\eta \in C^\infty(\overline{\Omega})$. If (3.3) holds merely for all $\eta \in C_c^\infty(\Omega)$, we say that u is a solution of the equation given by the first line of (3.1), without any boundary conditions.

Remark 3.2. A function $u \in W^{1,p}(\Omega)$ is a weak solution of (3.1) if and only if (3.3) holds for all $\eta \in W^{1,p}(\Omega)$ since $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ and all expressions in (3.3) are continuous as η varies in $W^{1,p}(\Omega)$, compare Proposition 5.1, where an even stronger assertion is proved. In what follows, we will use this fact frequently.

We deduce boundary regularity from the following interior regularity result, which is an immediate consequence of results due to Serrin.

Theorem 3.3. ([29, Section 1.1, Section 1.4, Section 1.5]) *Let A and B satisfy the structure conditions (1.2) and let f and F be as in (3.2). Then there exists $\alpha \in (0, 1)$ such that every weak solution $u \in W^{1,p}(\Omega)$ of*

$$-\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = f(x) - (\operatorname{div} F)(x) \text{ on } \Omega \tag{3.4}$$

is in $C_{\text{loc}}^{0,\alpha}(\Omega)$. Moreover, for every relatively compact subdomain $\omega \subset \Omega$ there exists $c_{\alpha,\omega} \geq 0$ such that

$$\|u\|_{C^{0,\alpha}(\omega)} \leq c_{\alpha,\omega} \left(\|f\|^{\frac{1}{p-1}} + \|F\|^{\frac{1}{p-1}} + \|u\|_{L^p(\Omega)} \right) + c_{\alpha,\omega} \tag{3.5}$$

holds for all weak solutions $u \in W^{1,p}(\Omega)$ of (3.4).

In order to apply Theorem 3.3, we extend the solutions of (3.1) locally along the boundary of Ω and show that the extension satisfies an elliptic equation on the larger domain. Then interior regularity of the extended function implies boundary regularity of the original function.

It is convenient to carry over the function to the reference domain $(-1, 1)^{N-1} \times (0, 1)$ and to extend the resulting function on that domain. As a first step, we show that the structural properties of the equation are preserved under bi-Lipschitz transformations.

Proposition 3.4. *Let Ω_1 and Ω_2 be Lipschitz domains in \mathbb{R}^N and let $\psi: \Omega_2 \rightarrow \Omega_1$ be a bi-Lipschitz bijection. Let $u \in W^{1,p}(\Omega_1)$ and $v := u \circ \psi$. Given functions A, B, f, F and g as in (1.2) and (3.2), define*

$$\begin{aligned} \hat{A}(x, u, z) &:= (\psi'(x))^{-1} A(\psi(x), u, z \psi'(x)^{-1}) |\det \psi'(x)| \\ \hat{B}(x, u, z) &:= B(\psi(x), u, z \psi'(x)^{-1}) |\det \psi'(x)| \end{aligned}$$

for $x \in \Omega_2, u \in \mathbb{R}$ and row vectors $z \in \mathbb{R}^N$. Moreover, let $\hat{f} := (f \circ \psi) |\det \psi'|, \hat{F} := (\psi')^{-1} (F \circ \psi) |\det \psi'|$ and $\hat{g} := (g \circ \psi)m$ with m as in Lemma 2.4.

- (a) *The function v is in $W^{1,p}(\Omega_2)$ with $\nabla v = (\nabla u \circ \psi) \psi'$ almost everywhere, and the functions \hat{f}, \hat{F} and \hat{g} are in Lebesgue spaces with the same exponent as f, F and g , respectively. More precisely, $\|\hat{f}\| \leq c_\psi \|f\|, \|\hat{F}\| \leq c_\psi \|F\|$ and $\|\hat{g}\| \leq c_\psi \|g\|$.*
- (b) *The functions \hat{A} and \hat{B} satisfy the structure conditions (1.2) on Ω_2 , where the parameters depend only on ψ and the parameters for A and B .*
- (c) *If $u \in W^{1,p}(\Omega_1)$ satisfies*

$$\int_{\Omega_1} \nabla \eta A(x, u, \nabla u) + \int_{\Omega_1} \eta B(x, u, \nabla u) = \int_{\Omega_1} \eta f + \int_{\Omega_1} \nabla \eta F + \int_{\partial \Omega_1} \eta g \tag{3.6}$$

for all $\eta \in C_c^\infty(\Omega_1 \cup \Gamma)$ with some relatively open set $\Gamma \subset \partial\Omega_1$ then $v \in W^{1,p}(\Omega_2)$ satisfies

$$\int_{\Omega_2} \nabla \tilde{\eta} \hat{A}(x, v, \nabla v) + \int_{\Omega_2} \tilde{\eta} \hat{B}(x, v, \nabla v) = \int_{\Omega_2} \tilde{\eta} \hat{f} + \int_{\Omega_2} \nabla \tilde{\eta} \hat{F} + \int_{\partial\Omega_2} \tilde{\eta} \hat{g} \tag{3.7}$$

for all $\tilde{\eta} \in C_c^\infty(\Omega_2 \cup \psi^{-1}(\Gamma))$.

Proof. The assertions in (a) follow from the chain rule for Sobolev functions [34, Theorem 2.2.2], the change of coordinates formula for Lipschitz transformations [10, Section 3.3.3] and Lemma 2.4. We have also used that $|\det \psi'|$ is bounded from below and above, see Lemma 2.3.

In order to check (b) we fix $u \in \mathbb{R}$ and a row vector $z \in \mathbb{R}^N$. Then

$$\begin{aligned} z \hat{A}(x, u, z) &= z(\psi'(x))^{-1} A(\psi(x), u, z(\psi'(x))^{-1}) |\det \psi'(x)| \\ &\geq \left(\mu_1 |z \psi'(x)^{-1}|^p - \psi_1(\psi(x)) |u|^p - \psi_1(\psi(x)) \right) |\det \psi'(x)| \\ &\geq \operatorname{ess\,inf}_{\Omega_2} |\det \psi'| \cdot \left(\frac{\mu_1}{\operatorname{ess\,sup}_{\Omega_2} \|\psi'\|^p} |z|^p - \hat{\psi}_1 |u|^p - \hat{\psi}_1 \right) \end{aligned}$$

with the function $\hat{\psi}_1 := \psi_1 \circ \psi$ possessing the same degree of integrability as ψ_1 . Similarly,

$$\begin{aligned} |\hat{A}(x, u, z)| &\leq \operatorname{ess\,sup}_{\Omega_2} |\det \psi'| s (\mu_2 s^{p-1} |z|^{p-1} + \hat{\psi}_2 |u|^{p-1} + \hat{\psi}_2) \\ |\hat{B}(x, u, z)| &\leq \operatorname{ess\,sup}_{\Omega_2} |\det \psi'| (s^{p-1} \hat{\psi}_3 |z|^{p-1} + \hat{\psi}_1 |u|^{p-1} + \hat{\psi}_1) \end{aligned}$$

where $s := \operatorname{ess\,sup}_{\Omega_2} \|(\psi')^{-1}\|$, $\hat{\psi}_2 := \psi_2 \circ \psi$ and $\hat{\psi}_3 := \psi_3 \circ \psi$. Hence \hat{A} and \hat{B} satisfy (1.2) on Ω_2 .

As for (c), let $\Gamma \subset \partial\Omega_1$ be relatively open and assume that u satisfies (3.6) for all $\eta \in C_c^\infty(\Omega_1 \cup \Gamma)$. Then by denseness (3.6) is fulfilled for all $\eta \in W_0^{1,p}(\Omega_1 \cup \Gamma)$, the closure of $C_c^\infty(\Omega_1 \cup \Gamma)$ in $W^{1,p}(\Omega)$. Let $\tilde{\eta}$ be in $C_c^\infty(\Omega_2 \cup \psi^{-1}(\Gamma))$ and write $\eta := \tilde{\eta} \circ \psi^{-1}$. Then by a standard smoothing argument we obtain that $\eta \in W_0^{1,p}(\Omega_1 \cup \Gamma)$. Moreover,

$$\nabla \eta = (\nabla \tilde{\eta} \circ \psi^{-1})(\psi^{-1})' = (\nabla \tilde{\eta} \circ \psi^{-1})(\psi' \circ \psi^{-1})^{-1}$$

by the chain rule. Hence

$$\begin{aligned} &\int_{\Omega_2} \nabla \tilde{\eta}(x) \hat{A}(x, v(x), \nabla v(x)) + \int_{\Omega_2} \tilde{\eta}(x) \hat{B}(x, v(x), \nabla v(x)) \\ &= \int_{\Omega_2} \nabla \eta(\psi(x)) A(\psi(x), u(\psi(x)), \nabla u(\psi(x))) |\det \psi'(x)| \\ &\quad + \int_{\Omega_2} \eta(\psi(x)) B(\psi(x), u(\psi(x)), \nabla u(\psi(x))) |\det \psi'(x)| \\ &= \int_{\Omega_1} \nabla \eta(x) A(x, u(x), \nabla u(x)) + \int_{\Omega_1} \eta(x) B(x, u(x), \nabla u(x)) \\ &= \int_{\Omega_1} \eta f + \int_{\Omega_1} \nabla \eta F + \int_{\partial\Omega_1} \eta g \end{aligned}$$

$$\begin{aligned} &= \int_{\Omega_2} \tilde{\eta} (f \circ \psi) |\det \psi'| + \int_{\Omega_2} \nabla \tilde{\eta} (\psi')^{-1} (F \circ \psi) |\det \psi'| + \int_{\partial\Omega_2} \tilde{\eta} (g \circ \psi) m \\ &= \int_{\Omega_2} \tilde{\eta} \hat{f} + \int_{\Omega_2} \nabla \tilde{\eta} \hat{F} + \int_{\partial\Omega_2} \tilde{\eta} \hat{g}, \end{aligned}$$

where by the change of variables formula for integrals over the domain as well as over its boundary, see Lemma 2.4. □

Remark 3.5. In part (c) of Proposition 3.4, the values of g on $\partial\Omega_1 \setminus \Gamma$ do not appear in the assertions. Hence we can apply the result also if g is given only on Γ and is left undefined on the remaining part of $\partial\Omega_1$ by artificially defining $g := 0$ on $\partial\Omega_1 \setminus \Gamma$.

We will also use the following representation of boundary integrals as integrals over the domain itself.

Lemma 3.6. *Let $\Omega' := (-1, 1)^N$ and $H := (-1, 1)^{N-1} \times \{0\}$, and let $g \in L^q(H)$, $q \in [1, \infty)$. There exist functions k and K in the spaces*

$$\begin{cases} k \in L^{\frac{qN}{N-1}}(\Omega'), & K \in L^{\frac{qN}{N-1}}(\Omega'; \mathbb{R}^N) & \text{if } q > 1 \\ k \in L^{\frac{N}{N-1+\varepsilon}}(\Omega'), & K \in L^{\frac{N}{N-1+\varepsilon}}(\Omega'; \mathbb{R}^N) & \text{if } q = 1, \end{cases}$$

where $\varepsilon \in (0, 1)$ is arbitrary, such that k and K satisfy

$$\int_H \eta g = \int_{\Omega'} \eta k + \int_{\Omega'} \nabla \eta K$$

for all $\eta \in C_c^\infty(\Omega')$. Moreover, $\|k\| + \|K\| \leq c_{q,\varepsilon} \|g\|_{L^q(H)}$.

Proof. Define the linear functional $\varphi: C_c^\infty(\Omega') \rightarrow \mathbb{R}$ by $\varphi(\eta) := \int_H \eta g$. If $q > 1$, then

$$|\varphi(\eta)| \leq \|\eta\|_{L^{\frac{q}{q-1}}(H)} \|g\|_{L^q(H)} \leq c_q \|\eta\|_{W^{1, \frac{qN}{qN-N+1}}(\Omega')} \|g\|_{L^q(H)}$$

by the Sobolev embedding theorems. Similarly, if $q = 1$, then

$$|\varphi(\eta)| \leq \|\eta\|_{L^\infty(H)} \|g\|_{L^1(H)} \leq c_\varepsilon \|\eta\|_{W^{1, \frac{N}{1-\varepsilon}}(\Omega')} \|g\|_{L^1(H)}$$

for every $\varepsilon > 0$. Hence φ extends to a continuous linear functional on $W_0^{1, \frac{qN}{qN-N+1}}(\Omega')$ or $W_0^{1, \frac{N}{1-\varepsilon}}(\Omega')$, respectively, which implies the claim, see [34, Section 4.3]. □

We now prove the main result of this section: every weak solution of (3.1) is Hölder continuous up to the boundary of Ω .

Theorem 3.7. *Let Ω be a Lipschitz domain and assume (1.2) and (3.2). Then there exist $\alpha \in (0, 1)$ and $c_\alpha \geq 0$ such that every weak solution $u \in W^{1,p}(\Omega)$ of (3.1) is in $C^{0,\alpha}(\Omega)$ and satisfies*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq c_\alpha \left(\|f\|^{\frac{1}{p-1}} + \|F\|^{\frac{1}{p-1}} + \|g\|^{\frac{1}{p-1}} + \|u\|_{L^p(\Omega)} \right) + c_\alpha. \tag{3.8}$$

Proof. Let $u \in W^{1,p}(\Omega)$ be a weak solution of (3.1). Let z be in $\partial\Omega$ and fix V and $\psi: V \rightarrow (-1, 1)^N$ as in Definition 2.1. We will show that u is Hölder continuous in a neighborhood $V_z \subset V$ of z . Since u solves (3.1) we know in particular that

$$\int_{\Omega} \nabla\eta A(x, u, \nabla u) + \int_{\Omega} \eta B(x, u, \nabla u) = \int_{\Omega} \eta f + \int_{\Omega} \nabla\eta F + \int_{\partial\Omega} \eta g$$

for all $\eta \in C_c^\infty(\Omega \cup (V \cap \partial\Omega))$. Write $\Omega_1 := \psi(\Omega \cap V) = (-1, 1)^{N-1} \times (0, 1)$. Then by Proposition 3.4, see also Remark 3.5, the function $v_1 := u \circ \psi^{-1}$ satisfies

$$\int_{\Omega_1} \nabla\eta \hat{A}_1(x, v_1, \nabla v_1) + \int_{\Omega_1} \eta \hat{B}_1(x, v_1, \nabla v_1) = \int_{\Omega_1} \eta \hat{f}_1 + \int_{\Omega_1} \nabla\eta \hat{F}_1 + \int_{\partial\Omega_1} \eta \hat{g}_1$$

for all $\eta \in C_c^\infty((-1, 1)^{N-1} \times [0, 1))$ with functions $\hat{A}_1, \hat{B}_1, \hat{f}_1, \hat{F}_1$ and \hat{g}_1 that satisfy the conditions (1.2) and (3.2),

Define the reflection $\psi_0: \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$\psi_0(x_1, \dots, x_{N-1}, x_N) := (x_1, \dots, x_{N-1}, -x_N).$$

Then $v_2 := v_1 \circ \psi_0 \in W^{1,p}(\Omega_2)$ satisfies

$$\int_{\Omega_2} \nabla\eta \hat{A}_2(x, v_2, \nabla v_2) + \int_{\Omega_2} \eta \hat{B}_2(x, v_2, \nabla v_2) = \int_{\Omega_2} \eta \hat{f}_2 + \int_{\Omega_2} \nabla\eta \hat{F}_2 + \int_{\partial\Omega_2} \eta \hat{g}_2$$

for all $\eta \in C_c^\infty((-1, 1)^{N-1} \times (-1, 0])$ by Proposition 3.4 with functions $\hat{A}_2, \hat{B}_2, \hat{f}_2, \hat{F}_2$ and \hat{g}_2 that satisfy the conditions (1.2) and (3.2).

Define $\Omega_0 := (-1, 1)^N$ and regard

$$\begin{aligned} v &:= v_1 \mathbf{1}_{\Omega_1} + v_2 \mathbf{1}_{\Omega_2}, \\ \hat{A} &:= \hat{A}_1 \mathbf{1}_{\Omega_1} + \hat{A}_2 \mathbf{1}_{\Omega_2}, & \hat{B} &:= \hat{B}_1 \mathbf{1}_{\Omega_1} + \hat{B}_2 \mathbf{1}_{\Omega_2}, \\ \hat{f} &:= \hat{f}_1 \mathbf{1}_{\Omega_1} + \hat{f}_2 \mathbf{1}_{\Omega_2}, & \hat{F} &:= \hat{F}_1 \mathbf{1}_{\Omega_1} + \hat{F}_2 \mathbf{1}_{\Omega_2} \end{aligned}$$

as functions on Ω_0 . Then $\hat{A}, \hat{B}, \hat{f}, \hat{F}$ and \hat{g} satisfy the conditions (1.2) and (3.2). Moreover, $v \in W^{1,p}(\Omega_0)$ and $\nabla v = \nabla v_1 \mathbf{1}_{\Omega_1} + \nabla v_2 \mathbf{1}_{\Omega_2}$. In fact, the Gauss-Green theorem [10, Section 5.8] shows that for all $\eta \in C_c^\infty(\Omega_0)$ we have

$$\begin{aligned} \int_{\Omega_0} \nabla\eta v &= \int_{\Omega_1} \nabla\eta v_1 + \int_{\Omega_2} \nabla\eta v_2 \\ &= \int_{\partial\Omega_1} \eta v_1 \nu_{\Omega_1} - \int_{\Omega_1} \eta \nabla v_1 + \int_{\partial\Omega_2} \eta v_2 \nu_{\Omega_2} - \int_{\Omega_2} \eta \nabla v_2 \\ &= - \int_{\Omega_1} \eta \nabla v_1 - \int_{\Omega_2} \eta \nabla v_2 \end{aligned}$$

since $v_1 = v_2$ on the intersection of the boundaries and the outer normals equal $\nu_{\Omega_1} = -e_N$ and $\nu_{\Omega_2} = e_N$ on that set, where e_N denotes the N^{th} unit vector in \mathbb{R}^N . In addition, η vanishes on the remaining parts of $\partial\Omega_1$ and $\partial\Omega_2$ by assumption.

Using in addition Lemma 3.6 we thus obtain that

$$\begin{aligned} & \int_{\Omega_0} \nabla \eta \hat{A}(x, v, \nabla v) + \int_{\Omega_0} \eta \hat{B}(x, v, \nabla v) \\ &= \int_{\Omega_1} \eta \hat{f}_1 + \int_{\Omega_1} \nabla \eta \hat{F}_1 + \int_{\partial \Omega_1} \eta \hat{g}_1 + \int_{\Omega_2} \eta \hat{f}_2 + \int_{\Omega_2} \nabla \eta \hat{F}_2 + \int_{\partial \Omega_2} \eta \hat{g}_2 \\ &= \int_{\Omega_0} \eta (\hat{f} + k) + \int_{\Omega_0} \nabla \eta (\hat{F} + K), \end{aligned}$$

for all $\eta \in C_c^\infty(\Omega_0)$ with functions $k \in L^q(\Omega_0)$ and $K \in L^q(\Omega_0; \mathbb{R}^N)$, where $q > \frac{N}{p-1}$ for $p \leq N$ and $q = \frac{p}{p-1}$ for $p > N$. Thus $v \in W^{1,p}(\Omega_0)$ is a weak solution of

$$-\operatorname{div} \hat{A}(x, v, \nabla v) + \hat{B}(x, v, \nabla v) = (\hat{f} + k) - \operatorname{div}(\hat{F} + K) \text{ on } \Omega_0,$$

where the coefficients \hat{A} and \hat{B} and the right hand side $\hat{f} + k$ and $\hat{F} + K$ satisfy the assumptions of Theorem 3.3. Consequently, there exists $\alpha \in (0, 1)$ such that

$$\begin{aligned} \|v_1\|_{C^{0,\alpha}((-\frac{1}{2}, \frac{1}{2})^{N-1} \times (0, \frac{1}{2}))} &\leq \|v\|_{C^{0,\alpha}((-\frac{1}{2}, \frac{1}{2})^N)} \\ &\leq c_\alpha (\|\hat{f} + k\|^{\frac{1}{p-1}} + \|F + K\|^{\frac{1}{p-1}} + \|v\|_{L^p(\Omega_0)}) + c_\alpha \\ &\leq c_\alpha (\|f\|^{\frac{1}{p-1}} + \|F\|^{\frac{1}{p-1}} + \|g\|^{\frac{1}{p-1}} + \|u\|_{L^p(\Omega)}) + c_\alpha, \end{aligned}$$

see also part (a) of Proposition 3.4. Since $u = v_1 \circ \psi$ and ψ is Lipschitz continuous on V , we have shown that there exists a neighborhood $V_z \subset V$ of z such that $u|_{V_z} \in C^{0,\alpha}(V_z)$ and

$$\|u\|_{C^{0,\alpha}(V_z)} \leq c_\alpha (\|f\|^{\frac{1}{p-1}} + \|F\|^{\frac{1}{p-1}} + \|g\|^{\frac{1}{p-1}} + \|u\|_{L^p(\Omega)}) + c_\alpha.$$

Since $\partial\Omega$ is compact, there exist finitely many $z_i \in \partial\Omega$ such that $\partial\Omega \subset \bigcup_{i=1}^m V_{z_i}$. Set $\omega_i := V_{z_i} \cap \Omega$ and $\omega_0 := \Omega \setminus \bigcup_{i=1}^m \omega_i$. Then $u|_{\omega_0} \in C^{0,\alpha_0}(\omega_0)$ by Theorem 3.3 and $u|_{\omega_i} \in C^{0,\alpha_i}(\omega_i)$ for $i = 1, \dots, m$ by what we have just shown. Thus we have proved that $u \in C^{0,\alpha}(\Omega)$ for $\alpha := \min_{i=1, \dots, N} \alpha_i$, and more precisely we have shown that (3.8) holds. \square

Remark 3.8. (Existence of solutions) Theorem 3.7 is only concerned with the regularity of solutions, not with existence or uniqueness. Nevertheless, under the additional assumption that $(u, z) \mapsto A(x, u, z)$ and $(u, z) \mapsto B(x, u, z)$ are continuous for every $x \in \Omega$ and

$$(z_1 - z_2)(A(x, u, z_1) - A(x, u, z_2)) \geq 0$$

for all $x \in \Omega$, $u \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^N$ it can be shown that the operator $\mathcal{A}: W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)'$ given by

$$\langle \mathcal{A}u, \eta \rangle := \int_{\Omega} A(x, u, \nabla u) \nabla \eta + \int_{\Omega} B(x, u, \nabla u) \eta$$

is pseudo-monotone [30, Proof of Theorem II.6.1]. Moreover, one can check that there is some $\omega_0 \in \mathbb{R}$ such that

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \rightarrow \infty} \frac{\langle \mathcal{A}u, u \rangle + \omega \int_{\Omega} |u|^p}{\|u\|_{W^{1,p}(\Omega)}} = \infty \tag{3.9}$$

if $\omega \geq \omega_0$. Since by theorem due to Brézis [30, Theorem II.2.3] every pseudo-monotone operator that satisfies (3.9) is surjective, we see that under the above additional continuity and monotonicity conditions for every $\varphi \in W^{1,p}(\Omega)'$ there exists a function $u \in W^{1,p}(\Omega)$ satisfying

$$\langle Au, \eta \rangle + \omega \int_{\Omega} |u|^{p-2} u \eta = \varphi(\eta)$$

for all $\eta \in W^{1,p}(\Omega)$ if $\omega \geq \omega_0$. Combining this observation with Theorem 3.7 we see that

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + \omega |u|^{p-2} u = f(x) - (\operatorname{div} F)(x) & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu = g(x) + (F \cdot \nu)(x) & \text{on } \partial\Omega \end{cases}$$

has a weak solution u in $W^{1,p}(\Omega) \cap C^{0,\alpha}(\Omega)$ whenever $\omega \geq \omega_0$ provided that f, g and F are as in (3.2).

This reasoning for existence of solutions easily extends to Robin boundary conditions as considered in the next section. One can for example invoke a perturbation argument as in [28, Section 4.1] to show pseudo-monotonicity of the operator that includes the boundary term.

For $\omega = 0$, on the other hand, we do not expect existence of weak solutions for all right hand sides. In fact, the equation

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

is a special case of (3.1) and it is well-known that this equation has no weak solution $u \in W^{1,2}(\Omega)$ if $\int_{\Omega} f \neq 0$.

4. Elliptic Robin problems

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. In this section we prove that all weak solutions $u \in W^{1,p}(\Omega)$ of

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + \omega u = f(x) - (\operatorname{div} F)(x) & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu + h(x, u) = g(x) + (F \cdot \nu)(x) & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

are Hölder continuous, provided A, B and h satisfy the structure conditions (1.2), f, F and g are as in (3.2) and ω is a nonnegative constant. Like for Neumann boundary conditions, we say that $u \in W^{1,p}(\Omega)$ is a weak solution of (4.1) if

$$\begin{aligned} & \int_{\Omega} \nabla \eta A(x, u, \nabla u) + \int_{\Omega} \eta B(x, u, \nabla u) + \int_{\partial\Omega} \eta h(x, u) \\ & = \int_{\Omega} \eta f + \int_{\Omega} \nabla \eta F + \int_{\partial\Omega} \eta g \end{aligned}$$

holds for all $\eta \in C^\infty(\bar{\Omega})$ or, equivalently, for all $\eta \in W^{1,p}(\Omega) \cap L^2(\Omega)$.

We start by proving L^q -bounds for solutions of the slightly simpler equation

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + \omega u = 0 & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu + h(x, u) = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

via Moser’s iteration technique. If we assume some strong monotonicity of the coefficients, the a priori estimates could be obtained via an elegant interpolation argument, see [28]. But in order to cover the general case we have to use the iteration procedure instead.

We need the following variant of Ehrling’s lemma.

Lemma 4.1. *Let X, Y and Z be Banach spaces. Assume that X is reflexive. Let $T: X \rightarrow Y$ be a compact linear operator and let $S: X \rightarrow Z$ be an injective bounded linear operator. Then for every $\delta > 0$ there exists $c_\delta > 0$ such that*

$$\|Tx\|_Y \leq \delta \|x\|_X + c_\delta \|Sx\|_Z$$

holds for all $x \in X$.

Proof. Assume to the contrary that there exists $\delta_0 > 0$ and a sequence (x_n) in X such that

$$\|Tx_n\|_Y > \delta_0 \|x_n\|_X + n \|Sx_n\|_Z.$$

After rescaling (x_n) we can assume that $\|Tx_n\|_Y = 1$ for all $n \in \mathbb{N}$. Then (x_n) is bounded in X , and after passing to a subsequence we thus can assume that $x_n \rightharpoonup x$ in X . Then $Sx_n \rightarrow 0$ by the above estimate and $Sx_n \rightharpoonup Sx$ by weak continuity, so we obtain that $Sx = 0$, which implies $x = 0$. Since T is compact this implies $Tx_n \rightarrow 0$ in Y , contradicting $\|Tx_n\|_Y = 1$. \square

Proposition 4.2. *If $u \in W^{1,p}(\Omega)$ solves (4.2), then*

$$\|u\|_{W^{1,p}(\Omega)} \leq c \|u\|_{L^p(\Omega)} + c \tag{4.3}$$

and

$$\|u\|_{L^q(\Omega)} + \|u\|_{L^q(\partial\Omega)} \leq c_q \|u\|_{L^p(\Omega)} + c_q \tag{4.4}$$

for every $q \in [p, \infty)$.

Proof. Let $u \in W^{1,p}(\Omega)$ be a weak solution of (4.2), and let $q \in [p, \infty)$ be arbitrary. Fix $\alpha \geq 1$ and define

$$\begin{aligned} v_\alpha := & ((|u| + 1)^{q-p+1} - 1) \operatorname{sgn}(u) \mathbf{1}_{\{|u| \leq \alpha\}} \\ & + \frac{(\alpha + 1)^{q-p+1} - 1}{\alpha + 1} (|u| + 1) \operatorname{sgn}(u) \mathbf{1}_{\{|u| > \alpha\}} \end{aligned}$$

and

$$w_\alpha := (|u| + 1)^{\frac{q}{p}} \mathbf{1}_{\{|u| \leq \alpha\}} + (\alpha + 1)^{\frac{q-p}{p}} (|u| + 1) \mathbf{1}_{\{|u| > \alpha\}}.$$

Then by the chain rule [34, Theorem 2.1.11] the functions v_α and w_α are in $W^{1,p}(\Omega)$ with weak derivatives

$$\begin{aligned} \nabla v_\alpha = & (q - p + 1) (|u| + 1)^{q-p} \nabla u \mathbf{1}_{\{|u| \leq \alpha\}} \\ & + \frac{(\alpha + 1)^{q-p+1} - 1}{\alpha + 1} \nabla u \mathbf{1}_{\{|u| > \alpha\}} \end{aligned} \tag{4.5}$$

and

$$|\nabla w_\alpha| = \frac{q}{p}(|u| + 1)^{\frac{q-p}{p}} |\nabla u| \mathbb{1}_{\{|u| \leq \alpha\}} + (\alpha + 1)^{\frac{q-p}{p}} |\nabla u| \mathbb{1}_{\{|u| > \alpha\}},$$

hence

$$\left(\frac{p}{q}\right)^p |\nabla w_\alpha|^p \leq (|u| + 1)^{q-p} |\nabla u|^p \mathbb{1}_{\{|u| \leq \alpha\}} + (\alpha + 1)^{q-p} |\nabla u|^p \mathbb{1}_{\{|u| > \alpha\}}. \tag{4.6}$$

We will also need that

$$\frac{1}{2}(\alpha + 1)^{q-p} \leq \frac{(\alpha + 1)^{q-p+1} - 1}{\alpha + 1} \leq (\alpha + 1)^{q-p}, \tag{4.7}$$

which follows from the fact that $(\alpha + 1)^{q-p+1} \geq 2$.

From (4.5), (4.6), (4.7) and (1.2) we obtain that

$$\begin{aligned} & \int_{\Omega} \nabla v_\alpha A(x, u, \nabla u) \\ & \geq (q - p + 1) \int_{\{|u| \leq \alpha\}} (|u| + 1)^{q-p} \left(\mu_1 |\nabla u|^p - \psi_1 |u|^p - \psi_1 \right) \\ & \quad + \frac{(\alpha + 1)^{q-p+1} - 1}{\alpha + 1} \int_{\{|u| > \alpha\}} \left(\mu_1 |\nabla u|^p - \psi_1 |u|^p - \psi_1 \right) \\ & \geq \mu_1 \int_{\{|u| \leq \alpha\}} (|u| + 1)^{q-p} |\nabla u|^p - 2(q - p + 1) \int_{\{|u| \leq \alpha\}} \psi_1 (|u| + 1)^q \\ & \quad + \frac{\mu_1}{2} \int_{\{|u| > \alpha\}} (\alpha + 1)^{q-p} |\nabla u|^p - 2(q - p + 1) \int_{\{|u| > \alpha\}} \psi_1 (\alpha + 1)^{q-p} (|u| + 1)^p \\ & \geq \frac{\mu_1}{2} \left(\frac{p}{q}\right)^p \int_{\Omega} |\nabla w_\alpha|^p - 2(q - p + 1) \int_{\Omega} \psi_1 w_\alpha^p. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} & \left| \int_{\Omega} v_\alpha B(x, u, \nabla u) \right| \\ & \leq \int_{\{|u| \leq \alpha\}} (|u| + 1)^{q-p+1} \left(\psi_3 |\nabla u|^{p-1} + \psi_1 |u|^{p-1} + \psi_1 \right) \\ & \quad + \int_{\{|u| > \alpha\}} (\alpha + 1)^{q-p} (|u| + 1) \left(\psi_3 |\nabla u|^{p-1} + \psi_1 |u|^{p-1} + \psi_1 \right) \\ & \leq \int_{\{|u| \leq \alpha\}} \psi_3 \left(\frac{q}{p}\right)^{p-1} (|u| + 1)^{\frac{q-p}{p}(p-1)} |\nabla u|^{p-1} (|u| + 1)^{\frac{q}{p}} \\ & \quad + 2 \int_{\{|u| \leq \alpha\}} \psi_1 (|u| + 1)^q + \int_{\{|u| > \alpha\}} \psi_3 (\alpha + 1)^{\frac{q-p}{p}(p-1)} |\nabla u|^{p-1} \\ & \quad \times (\alpha + 1)^{\frac{q-p}{p}} (|u| + 1) + 2 \int_{\{|u| > \alpha\}} \psi_1 (\alpha + 1)^{q-p} (|u| + 1)^p \\ & = \int_{\Omega} \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha + 2 \int_{\Omega} \psi_1 w_\alpha^p \end{aligned}$$

and

$$\left| \int_{\partial\Omega} v_\alpha h(x, u) \right| \leq 2 \int_{\partial\Omega} |v_\alpha| \psi_4 (|u| + 1)^{p-1} \leq 2 \int_{\partial\Omega} \psi_4 w_\alpha^p.$$

Using in addition that u solves (4.2), we have shown that

$$\begin{aligned} 0 &= \int_{\Omega} \nabla v_\alpha A(x, u, \nabla u) + \int_{\Omega} v_\alpha B(x, u, \nabla u) + \int_{\partial\Omega} v_\alpha h(x, u) + \omega \int_{\Omega} v_\alpha u \\ &\geq \frac{\mu_1}{2} \left(\frac{p}{q}\right)^p \int_{\Omega} |\nabla w_\alpha|^p - 2(q - p + 2) \int_{\Omega} \psi_1 w_\alpha^p - \int_{\Omega} \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha \\ &\quad - 2 \int_{\partial\Omega} \psi_4 w_\alpha^p, \end{aligned}$$

i.e.,

$$\|w_\alpha\|_{W^{1,p}(\Omega)}^p \leq c_q \int_{\Omega} \psi_1 w_\alpha^p + c_q \int_{\Omega} \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha + c_q \int_{\partial\Omega} \psi_4 w_\alpha^p, \tag{4.8}$$

where for simplicity we have assumed that $\psi_1 \geq 1$ almost everywhere, which constitutes no loss of generality.

We now distinguish between the cases $p < N$, $p = N$ and $p > N$.

(a) Assume $p > N$. Then by [1, Theorem 5.8] there exists $\theta \in (0, 1)$ such that

$$\|w_\alpha\|_{L^\infty(\partial\Omega)} \leq \|w_\alpha\|_{L^\infty(\Omega)} \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^\theta \|w_\alpha\|_{L^p(\Omega)}^{1-\theta}.$$

Hence

$$\begin{aligned} &\int_{\Omega} \psi_1 w_\alpha^p + \int_{\Omega} \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha + \int_{\partial\Omega} \psi_4 w_\alpha^p \\ &\leq \|\psi_1\|_{L^1(\Omega)} \|w_\alpha\|_{L^\infty(\Omega)}^p + \|\psi_3\|_{L^p(\Omega)} \|\nabla w_\alpha\|_{L^{\frac{p}{p-1}}(\Omega)}^{p-1} \|w_\alpha\|_{L^\infty(\Omega)} \\ &\quad + \|\psi_4\|_{L^1(\partial\Omega)} \|w_\alpha\|_{L^\infty(\partial\Omega)}^p \\ &\leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^{\theta p} \|w_\alpha\|_{L^p(\Omega)}^{(1-\theta)p} + c \|w_\alpha\|_{W^{1,p}(\Omega)}^{p-1+\theta} \|w_\alpha\|_{L^p(\Omega)}^{1-\theta} \\ &\leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)}^p + c_\delta \|w_\alpha\|_{L^p(\Omega)}^p \end{aligned}$$

for every $\delta > 0$ by Young's inequality. Picking $\delta > 0$ small enough we obtain from this estimate and (4.8) that

$$\|w_\alpha\|_{W^{1,p}(\Omega)}^p \leq c_q \|w_\alpha\|_{L^p(\Omega)}^p.$$

This proves (4.3) since for $p = q$ we have $w_\alpha = |u| + 1$ and $|\nabla w_\alpha| = |\nabla u|$, thus

$$\|u\|_{W^{1,p}(\Omega)}^p \leq \|w_\alpha\|_{W^{1,p}(\Omega)}^p \leq c \|w_\alpha\|_{L^p(\Omega)}^p \leq c \|u\|_{L^p(\Omega)}^p + c$$

Finally, estimate (4.3) implies (4.4), in this particular case even for $q = \infty$, by the Sobolev embedding theorems.

(b) Now assume $p = N$. Then by the interpolation inequality for Lebesgue spaces and the Sobolev embedding theorems we find for every $r \in [1, \infty)$ an exponent $\theta_r \in (0, 1)$ such that

$$\|w_\alpha\|_{L^r(\Omega)} \leq \|w_\alpha\|_{L^{2r}(\Omega)}^{\theta_r} \|w_\alpha\|_{L^1(\Omega)}^{1-\theta_r} \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^{\theta_r} \|w_\alpha\|_{L^p(\Omega)}^{1-\theta_r}.$$

Moreover, since the trace operator is compact from $W^{1,p}(\Omega)$ to $L^r(\partial\Omega)$ for every $r \in [1, \infty)$, we can estimate

$$\|w_\alpha\|_{L^r(\partial\Omega)} \leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)} + c_\delta \|w_\alpha\|_{L^p(\Omega)}$$

for every $\delta > 0$ by Lemma 4.1. Using these two estimates and Young’s inequality, we obtain with certain exponents θ and θ' in $(0, 1)$ that

$$\begin{aligned} & \int_\Omega \psi_1 w_\alpha^p + \int_\Omega \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha + \int_{\partial\Omega} \psi_4 w_\alpha^p \\ & \leq \|\psi_1\|_{L^{\frac{p}{p-\varepsilon}}(\Omega)} \|w_\alpha^p\|_{L^{\frac{p}{\varepsilon}}(\Omega)} + \|\psi_3\|_{L^{\frac{p}{1-\varepsilon}}(\Omega)} \|\nabla w_\alpha|^{p-1}\|_{L^{\frac{p}{p-1}}(\Omega)} \|w_\alpha\|_{L^{\frac{p}{\varepsilon}}(\Omega)} \\ & \quad + \|\psi_4\|_{L^{\frac{p-1}{p-1-\varepsilon}}(\partial\Omega)} \|w_\alpha^p\|_{L^{\frac{p-1}{\varepsilon}}(\partial\Omega)} \\ & \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^\theta \|w_\alpha\|_{L^p(\Omega)}^{p(1-\theta)} + c \|w_\alpha\|_{W^{1,p}(\Omega)}^{p-1+\theta'} \|w_\alpha\|_{L^p(\Omega)}^{1-\theta'} + c \|w_\alpha\|_{L^{\frac{p(p-1)}{\varepsilon}}(\partial\Omega)}^p \\ & \leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)}^p + c_\delta \|w_\alpha\|_{L^p(\Omega)}^p \end{aligned}$$

for every $\delta > 0$. As in the previous case $p > N$ this implies (4.3) and hence (4.4) by the Sobolev embedding theorems.

(c) Finally, assume $p < N$. Then

$$\begin{aligned} \int_\Omega \psi_1 w_\alpha^p & \leq \|\psi_1\|_{L^{\frac{N}{p-\varepsilon}}(\Omega)} \|w_\alpha^{p-\varepsilon}\|_{L^{\frac{Np}{(N-p)(p-\varepsilon)}(\Omega)}} \|w_\alpha^\varepsilon\|_{L^{\frac{p}{\varepsilon}}(\Omega)} \\ & = c \|w_\alpha\|_{L^{\frac{Np}{N-p}}(\Omega)}^{p-\varepsilon} \|w_\alpha\|_{L^p(\Omega)}^\varepsilon \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^{p-\varepsilon} \|w_\alpha\|_{L^p(\Omega)}^\varepsilon \\ & \leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)}^p + c_\delta \|w_\alpha\|_{L^p(\Omega)}^p \end{aligned}$$

for every $\delta > 0$ by Young’s inequality. Similarly,

$$\begin{aligned} & \int_\Omega \psi_3 |\nabla w_\alpha|^{p-1} w_\alpha \\ & \leq \|\psi\|_{L^{\frac{N}{1-\varepsilon}}(\Omega)} \|\nabla w_\alpha|^{p-1}\|_{L^{\frac{p}{p-1}}(\Omega)} \|w_\alpha^{1-\varepsilon}\|_{L^{\frac{Np}{(N-p)(1-\varepsilon)}(\Omega)}} \|w_\alpha^\varepsilon\|_{L^{\frac{p}{\varepsilon}}(\Omega)} \\ & \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^{p-1} \|w_\alpha\|_{L^{\frac{Np}{N-p}}(\Omega)}^{1-\varepsilon} \|w_\alpha\|_{L^p(\Omega)}^\varepsilon \leq c \|w_\alpha\|_{W^{1,p}(\Omega)}^{p-\varepsilon} \|w_\alpha\|_{L^p(\Omega)}^\varepsilon \\ & \leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)}^p + c_\delta \|w_\alpha\|_{L^p(\Omega)}^p. \end{aligned}$$

Moreover, since the trace operator is compact from $W^{1,p}(\Omega)$ to $L^{\frac{(N-1)p}{N-p+\varepsilon}}(\partial\Omega)$, we can estimate

$$\|w_\alpha\|_{L^{\frac{(N-1)p}{N-p+\varepsilon}}(\partial\Omega)} \leq \delta \|w_\alpha\|_{W^{1,p}(\Omega)} + c_\delta \|w_\alpha\|_{L^p(\Omega)}$$

for every $\delta > 0$ by Lemma 4.1. Thus, picking $\delta > 0$ small enough in the previous estimates, inequality (4.8) yields

$$\|w_\alpha\|_{W^{1,p}(\Omega)}^p \leq c_q \|w_\alpha\|_{L^p(\Omega)}^p. \tag{4.9}$$

As in the previous cases this implies (4.3).

Moreover, from (4.9) and the Sobolev embedding theorems we deduce that

$$\|w_\alpha\|_{L^{\frac{Np}{N-p}}(\Omega)}^p + \|w_\alpha\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)}^p \leq c\|w_\alpha\|_{W^{1,p}(\Omega)}^p \leq c_q\|w_\alpha\|_{L^p(\Omega)}^p.$$

Since w_α approaches $(|u| + 1)^{q/p}$ from below as $\alpha \rightarrow \infty$, the monotone convergence theorem implies that

$$\|(|u| + 1)^{q/p}\|_{L^{\frac{Np}{N-p}}(\Omega)}^p + \|(|u| + 1)^{q/p}\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)}^p \leq c_q\|(|u| + 1)^{q/p}\|_{L^p(\Omega)}^p,$$

from which we can deduce that

$$\|u\|_{L^{\frac{Nq}{N-p}}(\Omega)} + \|u\|_{L^{\frac{(N-1)q}{N-p}}(\partial\Omega)} \leq c_q\|u\|_{L^q(\Omega)} + c_q. \tag{4.10}$$

Since this estimate holds for all $q \in [1, \infty)$, defining

$$q_n := \left(\frac{N}{N-p}\right)^n p$$

and iterating (4.10) we obtain that

$$\begin{aligned} \|u\|_{L^{q_n}(\Omega)} + \|u\|_{L^{\frac{N-1}{N}q_n}(\partial\Omega)} &= \|u\|_{L^{\frac{Nq_{n-1}}{N-p}}(\Omega)} + \|u\|_{L^{\frac{(N-1)q_{n-1}}{N-p}}(\partial\Omega)} \\ &\leq c_{q_{n-1}}\|u\|_{L^{q_{n-1}}(\Omega)} + c_{q_{n-1}} \\ &\leq c_{q_{n-1}}c_{q_{n-2}}\|u\|_{L^{q_{n-2}}(\Omega)} + c_{q_{n-1}}c_{q_{n-2}} + c_{q_{n-1}} \\ &\leq \dots \leq c_n\|u\|_{L^{q_0}(\Omega)} + c_n = c_n\|u\|_{L^p(\Omega)} + c_n. \end{aligned}$$

Since $q_n \rightarrow \infty$ as $n \rightarrow \infty$, this proves (4.4). □

Remark 4.3. In the proof of Proposition 4.2 we silently passed over a subtlety that arises when deriving (4.8). A priori we can test the Eq. (4.2) only against functions in $W^{1,p}(\Omega) \cap L^2(\Omega)$. However, we tested the equation against $v_\alpha \in W^{1,p}(\Omega)$ without knowing that $v_\alpha \in L^2(\Omega)$. Still, since $\text{sgn}(v_\alpha) = \text{sgn}(u)$ we can pick a sequence (η_n) of functions in $W^{1,p}(\Omega) \cap L^2(\Omega)$ that converges to v_α in $W^{1,p}(\Omega)$ such that the sequence $(\eta_n u)$ is pointwise monotonically increasing. Then it follows from the monotone convergence theorem that

$$\begin{aligned} 0 &= \int_\Omega \nabla \eta_n A(x, u, \nabla u) + \int_\Omega \eta_n B(x, u, \nabla u) + \omega \int_\Omega \eta_n u + \int_{\partial\Omega} \eta_n h(x, u) \\ &\rightarrow \int_\Omega \nabla v_\alpha A(x, u, \nabla u) + \int_\Omega v_\alpha B(x, u, \nabla u) + \omega \int_\Omega v_\alpha u + \int_{\partial\Omega} v_\alpha h(x, u) \end{aligned}$$

as $n \rightarrow \infty$.

Now that we have L^q -bounds at our disposal, it is easy to deduce the Hölder continuity of solutions from the corresponding result concerning Neumann boundary conditions.

Theorem 4.4. *Let Ω be a Lipschitz domain and assume (1.2) and (3.2). Then there exists $\alpha \in (0, 1)$ such that every weak solution $u \in W^{1,p}(\Omega)$ of (4.1) with $\omega \geq 0$ is in $C^{0,\alpha}(\Omega)$. Moreover, if f, F and g range over norm-bounded sets in their respective spaces, then the $C^{0,\alpha}$ -norms of the corresponding solutions u remain bounded provided their norms in $L^p(\Omega)$ remain bounded.*

Proof. Let $u \in W^{1,p}(\Omega)$ be a weak solution of (4.1). Then u is a weak solution of

$$\begin{cases} -\operatorname{div} \check{A}(x, u, \nabla u) + \check{B}(x, u, \nabla u) + \omega u = 0 & \text{on } \Omega \\ \check{A}(x, u, \nabla u) \cdot \nu + \check{h}(x, u) = 0 & \text{on } \partial\Omega \end{cases} \tag{4.11}$$

for $\check{A}(x, u, z) := A(x, u, z) - F(x)$, $\check{B}(x, u, z) := B(x, u, z) - f(x)$ and $\check{h}(x, u) := h(x, u) - g(x)$. Assumption (3.2) ensures \check{A} , \check{B} and \check{h} satisfy (1.2).

If $p > N$, then $u \in C^{0,\alpha}(\Omega)$ by a Sobolev embedding theorem. The boundedness assertion about u follows from Proposition 4.2 applied to the Eq. (4.11) since the constants in (4.3) depend only on upper bounds for the structure parameters of \check{A} , \check{B} and \check{h} , which in turn depend only on A, B, h and upper bound for f, F and g .

Now assume $p \leq N$. Pick $q \in [p, \infty)$ such that $q \geq \frac{2(N-1)(p-1)}{\varepsilon}$ and $q \geq \frac{N}{p-\varepsilon}$ with $\varepsilon > 0$ as in the structure condition (1.2). We obtain from Proposition 4.2 applied to Eq. (4.11) that $u \in L^q(\Omega)$ and $u|_{\partial\Omega} \in L^q(\partial\Omega)$ with a bound that depends only on bounds for the structure parameters of \check{A} , \check{B} and \check{h} . Thus

$$h_0 := h(\cdot, u(\cdot)) \in L^{\frac{N-1}{p-1-\varepsilon/2}}(\partial\Omega)$$

by the structure assumption (1.2). Since u is a weak solution of the Neumann problem

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = f(x) - \omega u(x) - (\operatorname{div} F)(x) & \text{on } \Omega \\ A(x, u, \nabla u) \cdot \nu = g(x) - h_0(x) + (F \cdot \nu)(x) & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

we obtain that $u \in C^{0,\alpha}(\Omega)$ with the same $\alpha \in (0, 1)$ as in Theorem 3.7. The boundedness assertion about u in $C^{0,\alpha}(\Omega)$ follows from (3.8) and the observation that the right hand side of (4.12) can be estimated in terms of A, B, h, ω and upper bounds for f, F and h . \square

Remark 4.5. Working with the full Eq. (4.1) instead of (4.2) in Proposition 4.2, we could have found an estimate like (3.8) also in the situation of Theorem (4.4). However, for our purposes nothing is gained by this more precise estimate, so for the sake of simplicity we avoided this additional effort.

Sometimes it is convenient to replace the functions on the right hand side of (4.1) by an arbitrary bounded linear functional on $W^{1,p}(\Omega)$. Therefore we formulate the regularity result of Theorem 4.4 also for such equations.

Corollary 4.6. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and assume (1.2). Let $q \in [1, \infty)$ satisfy*

$$\begin{cases} q \leq \frac{N}{N-p+1} & \text{if } p < N \\ q < N & \text{if } p = N \\ q \leq p & \text{if } p > N \end{cases}$$

and fix $\psi \in W^{1,q}(\Omega)'$. Then every function $u \in W^{1,p}(\Omega)$ that satisfies

$$\int_{\Omega} \nabla \eta A(x, u, \nabla u) + \int_{\Omega} \eta B(x, u, \nabla u) + \int_{\partial\Omega} \eta h(x, u) = \psi(\eta)$$

for all $\eta \in C^\infty(\overline{\Omega})$ is Hölder continuous on Ω .

Proof. This is a direct consequence of the representation formula for functionals on Sobolev spaces [34, Section 4.3] and Theorem 4.4. \square

5. Parabolic problems

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. We show that the parabolic problem

$$\begin{cases} u_t(t, x) - \operatorname{div} a(x, \nabla u(t, x)) + b(x, u(t, x)) = 0 & t > 0, x \in \Omega \\ a(x, \nabla u(t, x)) \cdot \nu + h(x, u(x)) = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega \end{cases} \quad (5.1)$$

with Robin boundary conditions is well-posed in the space $C(\overline{\Omega})$ under suitable conditions on a , b and h . More precisely, we assume that $a: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $b: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable and continuous in the second argument. We also require that there exists $p \in (1, \infty)$ such that

$$\begin{cases} z a(x, z) \geq \mu_1 |z|^p - \psi_1(x), & |a(x, z)| \leq \mu_2 |z|^{p-1} + \psi_2(x), \\ |b(x, u)| \leq \psi_3(x) |u|^{p-1} + \psi_3(x), & |h(x, u)| \leq \psi_4(x) |u|^{p-1} + \psi_4(x) \end{cases} \quad (5.2)$$

for all $x \in \Omega$, $u \in \mathbb{R}$ and $z \in \mathbb{R}^N$ with ψ_1, ψ_2 and ψ_4 as in (1.2), i.e., the functions $A(x, u, z) := a(x, z)$, $B(x, u, z) := b(x, z)$ and h satisfy (1.2). Moreover, we assume the monotonicity conditions

$$\begin{cases} (z_1 - z_2) (a(x, z_1) - a(x, z_2)) \geq 0, \\ (u_1 - u_2) (b(x, u_1) - b(x, u_2)) \geq 0, \\ (u_1 - u_2) (h(x, u_1) - h(x, u_2)) \geq 0 \end{cases} \quad (5.3)$$

for all $x \in \Omega$, $u_1, u_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^N$.

In order to prove well-posedness in $C(\overline{\Omega})$ we are going to show that the operator which is naturally associated with (5.1) is m-accretive on $C(\overline{\Omega})$ and thus generates a semigroup on that space.

It is convenient to first introduce a version \mathcal{A} of the operator associated with (5.1) acting from $V := W^{1,p}(\Omega) \cap L^2(\Omega)$ into its V' and to study its properties. Later on we will turn our attention to its part in $C(\overline{\Omega})$ in order to obtain the main result.

Proposition 5.1. *The space $V := W^{1,p}(\Omega) \cap L^2(\Omega)$ is a separable, reflexive Banach space for the norm $\|u\|_V := \|u\|_{W^{1,p}(\Omega)} + \|u\|_{L^2(\Omega)}$. Under the assumptions (5.2) and (5.3), via*

$$\langle \mathcal{A}u, v \rangle := \int_{\Omega} \nabla v a(x, \nabla u) + \int_{\Omega} v b(x, u) + \int_{\partial\Omega} v h(x, u). \quad (5.4)$$

we have defined a bounded, continuous, monotone operator $\mathcal{A}: V \rightarrow V'$.

Proof. We prove the assertions only for $p < N$; the case $p \geq N$ is similar. Identifying V with a closed subspace of the direct sum $W^{1,p}(\Omega) \oplus L^2(\Omega)$ we see that V is a separable, reflexive Banach space. For u and v in $W^{1,p}(\Omega)$ we have by (5.2) that

$$\begin{aligned} \int_{\Omega} |\nabla v a(x, \nabla u)| &\leq \|\nabla v\|_{L^p(\Omega)} \left(\int_{\Omega} (\mu_2 |\nabla u|^{p-1} + \psi_2)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq c \|v\|_{W^{1,p}(\Omega)} \left(\|\nabla u\|_{L^p(\Omega)}^{p-1} + \|\psi_2\|_{L^{\frac{p}{p-1}}(\Omega)} \right) \\ &\leq (c \|u\|_V^{p-1} + c) \|v\|_V. \end{aligned}$$

Similarly, using in addition the Sobolev embedding theorems, we obtain that

$$\begin{aligned} \int_{\Omega} |v b(x, u)| &\leq \|v\|_{L^{\frac{Np}{N-p}}(\Omega)} \|\psi_1\|_{L^{\frac{N}{p}}(\Omega)} \left(\|u\|^{p-1} + 1 \right)_{L^{\frac{Np}{(N-p)(p-1)}}(\Omega)} \\ &\leq c \|v\|_{W^{1,p}(\Omega)} (c \|u\|_{W^{1,p}(\Omega)}^{p-1} + c) \leq (c \|u\|_V^{p-1} + c) \|v\|_V \end{aligned}$$

and

$$\begin{aligned} \int_{\partial\Omega} |v h(x, u)| &\leq \|v\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|\psi_4\|_{L^{\frac{N-1}{p-1}}(\partial\Omega)} \left(\|u\|^{p-1} + 1 \right)_{L^{\frac{(N-1)p}{(N-p)(p-1)}}(\partial\Omega)} \\ &\leq c \|v\|_{W^{1,p}(\Omega)} (c \|u\|_{W^{1,p}(\Omega)}^{p-1} + c) \leq (c \|u\|_V^{p-1} + c) \|v\|_V. \end{aligned}$$

Thus \mathcal{A} is well-defined and $\|\mathcal{A}u\|_{V'} \leq c \|u\|_V^{p-1} + c$ for all $u \in V$, proving that the operator \mathcal{A} is bounded, i.e., that \mathcal{A} maps bounded sets into bounded sets.

Now we show that \mathcal{A} is continuous. To this end, let (u_n) be a sequence in V that converges to $u \in V$. Passing to a subsequence, we can assume that $\nabla u_n \rightarrow \nabla u$ pointwise and that $|\nabla u_n| \leq m$ for some function $m \in L^p(\Omega)$. Then $a(x, \nabla u_n(x)) \rightarrow a(x, \nabla u(x))$ for almost every $x \in \Omega$ and

$$|a(x, \nabla u_n)| \leq \mu_2 m^{p-1} + \psi_2 \in L^{\frac{p}{p-1}}(\Omega).$$

Hence $a(\cdot, \nabla u_n) \rightarrow a(\cdot, \nabla u)$ in $L^{\frac{p}{p-1}}(\Omega)$ by the dominated convergence theorem. An analogous reasoning yields that

$$b(\cdot, u_n) \rightarrow b(\cdot, u) \text{ in } L^{\frac{Np}{Np-N+p}}(\Omega) \quad \text{and} \quad h(\cdot, u_n) \rightarrow h(\cdot, u) \text{ in } L^{\frac{(N-1)p}{N(p-1)}}(\partial\Omega).$$

Thus, given $\varepsilon > 0$, we have

$$\begin{aligned} |\langle \mathcal{A}u_n - \mathcal{A}u, v \rangle| &\leq \|\nabla v\|_{L^p(\Omega)} \|a(\cdot, \nabla u_n) - a(\cdot, \nabla u)\|_{L^{\frac{p}{p-1}}(\Omega)} \\ &\quad + \|v\|_{L^{\frac{Np}{N-p}}(\Omega)} \|b(\cdot, u_n) - b(\cdot, u)\|_{L^{\frac{Np}{Np-N+p}}(\Omega)} \\ &\quad + \|v\|_{L^{\frac{(N-1)p}{N-p}}(\partial\Omega)} \|h(\cdot, u_n) - h(\cdot, u)\|_{L^{\frac{(N-1)p}{N(p-1)}}(\partial\Omega)} \\ &\leq \varepsilon \|v\|_{W^{1,p}(\Omega)} \end{aligned}$$

for $n \geq n_0(\varepsilon)$, which shows that $\mathcal{A}u_n \rightarrow \mathcal{A}u$ in V' .

Finally, the monotonicity of \mathcal{A} , i.e., $\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \geq 0$ for all $u, v \in V$, is a trivial consequence of (5.3). \square

Next we show that $I + \alpha\mathcal{A}$ is bijective for all $\alpha > 0$.

Proposition 5.2. *Under the assumptions of Proposition 5.1, for every $\varphi \in V'$ and every $\alpha > 0$ there exists a unique function $u \in V$ such that*

$$(u \mid v)_{L^2(\Omega)} + \alpha \langle \mathcal{A}u, v \rangle = \varphi(v) \tag{5.5}$$

for all $v \in V$.

Proof. Define the operator $\mathcal{A}_\alpha: V \rightarrow V'$ by

$$\langle \mathcal{A}_\alpha u, v \rangle := (u \mid v)_{L^2(\Omega)} + \alpha \langle \mathcal{A}u, v \rangle.$$

From Proposition 5.1 we obtain that \mathcal{A}_α is bounded, continuous and monotone. From (5.2) we obtain that

$$\int_\Omega \nabla u a(x, \nabla u) \geq \mu_1 \int_\Omega |\nabla u|^p - \int_\Omega \psi_1 = \mu_1 \|\nabla u\|_{L^p(\Omega)}^p - c.$$

for all $u \in V$. Moreover, (5.2) and (5.3) yield

$$\begin{aligned} \int_\Omega u b(x, u) &= \int_\Omega (u - 0) (b(x, u) - b(x, 0)) + \int_\Omega u b(x, 0) \geq - \int_\Omega \psi_1 |u| \\ &\geq - \|\psi_1\|_{L^{\frac{N}{p}}(\Omega)} \|u\|_{L^{\frac{N}{N-p}}(\Omega)} \geq -c \|u\|_{W^{1,p}(\Omega)} \end{aligned}$$

for all $u \in V$, and analogously we see that $\int_\Omega u h(x, u) \geq -c \|u\|_{W^{1,p}(\Omega)}$. Combining the latter three estimates we have shown that

$$\begin{aligned} \langle \mathcal{A}_\alpha u, u \rangle &\geq \|u\|_{L^2(\Omega)}^2 + \alpha \mu_1 \|\nabla u\|_{L^p(\Omega)}^p - c \|u\|_{W^{1,p}(\Omega)} - c \\ &\geq \beta (\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(\Omega)})^q - c \|u\|_V - c \end{aligned} \tag{5.6}$$

for $q := \min\{2, p\} > 1$ and some $\beta > 0$.

From Lemma 4.1 we obtain that

$$\|u\|_{L^p(\Omega)} \leq \delta \|u\|_V + c_\delta \|u\|_{L^2(\Omega)} \leq c\delta \|\nabla u\|_{L^p(\Omega)} + c\delta \|u\|_{L^p(\Omega)} + c_\delta \|u\|_{L^2(\Omega)}$$

for all $u \in V$ with an arbitrary $\delta > 0$. Picking $\delta > 0$ small enough we deduce that

$$\|u\|_{L^p(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} + c \|u\|_{L^2(\Omega)}.$$

Hence $\|u\|_{2,p} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm on V . Thus from (5.6) we obtain that

$$\liminf_{\|u\|_V \rightarrow \infty} \frac{\langle \mathcal{A}_\alpha u, u \rangle}{\|u\|_V} \geq \liminf_{\|u\|_{2,p} \rightarrow \infty} \frac{\beta \|u\|_{2,p}^q - c \|u\|_{2,p} - c}{\|u\|_{2,p}} = \infty \tag{5.7}$$

and call (5.6) the coercivity of \mathcal{A}_α . Since \mathcal{A}_α is also bounded, continuous and monotone, the operator is surjective by the Minty-Browder theorem [30, Section II.2]. This means that (5.5) has a solution for every $\varphi \in V'$. Moreover, if $\mathcal{A}_\alpha u_1 = \mathcal{A}_\alpha u_2$ for two functions u_1 and u_2 in V , then by monotonicity of \mathcal{A} we obtain that

$$0 = \langle \mathcal{A}_\alpha u_1 - \mathcal{A}_\alpha u_2, u_1 - u_2 \rangle \geq (u_1 - u_2 \mid u_1 - u_2)_{L^2(\Omega)} = \|u_1 - u_2\|_{L^2(\Omega)}^2,$$

implying that $u_1 = u_2$. Hence the solution of (5.5) is unique. □

Our next step is to show that the part \mathcal{A} in $L^q(\Omega)$ is accretive, i.e., that $(I + \alpha\mathcal{A})^{-1}$ is a contraction with respect to the norm of $L^q(\Omega)$ for each $q \in [2, \infty]$. Setting $q = \infty$ we thus obtain in particular that the part of \mathcal{A} in $C(\bar{\Omega})$ is accretive.

Proposition 5.3. *Let (5.2) and (5.3) be satisfied. Let f_1 and f_2 be in $L^q(\Omega) \subset L^2(\Omega)$, $q \in [2, \infty]$ and define $\varphi_i(v) := (f_i | v)_{L^2(\Omega)}$ so that $\varphi_i \in V'$. Then the unique solutions u_i of (5.5) for the right hand sides φ_i satisfy $\|u_1 - u_2\|_{L^q(\Omega)} \leq \|f_1 - f_2\|_{L^q(\Omega)}$. Moreover, if $q > \frac{N}{p}$, then u_1 and u_2 are in $C(\bar{\Omega})$.*

Proof. Fix $q \in [2, \infty)$ and $k \geq 1$. Then by the chain rule [34, Theorem 2.2.2] the function

$$v_k := (|u_1 - u_2| \wedge k)^{q-2}(u_1 - u_2),$$

where $x \wedge y$ denotes the minimum of x and y , is in V with weak derivative

$$\begin{aligned} \nabla v_k &= (q - 1)|u_1 - u_2|^{q-2}(\nabla u_1 - \nabla u_2)\mathbb{1}_{\{|u_1 - u_2| \leq k\}} \\ &\quad + k^{q-2}(\nabla u_1 - \nabla u_2)\mathbb{1}_{\{|u_1 - u_2| > k\}}. \end{aligned}$$

Since at each point v_k is a positive multiple of $u_1 - u_2$ and ∇v_k is a positive multiple of $\nabla u_1 - \nabla u_2$, we deduce from (5.3) that

$$\langle \mathcal{A}u_1 - \mathcal{A}u_2, v_k \rangle \geq 0.$$

Hence with \mathcal{A}_α as in the proof of Proposition 5.2 we obtain that

$$\begin{aligned} \int_{\Omega} (|u_1 - u_2| \wedge k)^{q-2}|u_1 - u_2|^2 &= (u_1 - u_2 | v_k)_{L^2(\Omega)} \leq \langle \mathcal{A}_\alpha u_1 - \mathcal{A}_\alpha u_2, v_k \rangle \\ &= \varphi_1(v_k) - \varphi_2(v_k) \leq \|f_1 - f_2\|_{L^q(\Omega)} \|v_k\|_{L^{\frac{q}{q-1}}(\Omega)} \\ &\leq \|f_1 - f_2\|_{L^q(\Omega)} \left(\int_{\Omega} (|u_1 - u_2| \wedge k)^{q-2}|u_1 - u_2|^2 \right)^{\frac{q-1}{q}} \end{aligned}$$

Dividing in this equation and afterwards letting k tend to infinity, we obtain from the monotone convergence theorem that $\|u_1 - u_2\|_{L^q(\Omega)} \leq \|f_1 - f_2\|_{L^q(\Omega)}$ for every $q \in [2, \infty)$. If f_1 and f_2 are in $L^\infty(\Omega)$, we pass to the limit $q \rightarrow \infty$ and obtain that $\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \|f_1 - f_2\|_{L^\infty(\Omega)}$.

Now assume $q > \frac{N}{p}$. By definition, the function u_i is a weak solution of

$$\begin{cases} -\operatorname{div} a(x, \nabla u) + b(x, u) + \alpha^{-1}u_i = \alpha^{-1}f_i & \text{on } \Omega \\ a(x, \nabla u) \cdot \nu + h(x, u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $u_i \in C^{0,\alpha}(\Omega) \subset C(\bar{\Omega})$ by Theorem 4.4. □

Regard $C(\bar{\Omega})$ as a subspace of V' by identifying a function $f \in C(\bar{\Omega})$ with the linear functional $v \mapsto (f | v)_{L^2(\Omega)}$. Then the part \mathcal{A}^c of \mathcal{A} in $C(\bar{\Omega})$ is the restriction of \mathcal{A} to

$$D(\mathcal{A}^c) := \{u \in V \cap C(\bar{\Omega}) : \mathcal{A}u \in C(\bar{\Omega})\}.$$

We regard \mathcal{A}^c as a non-linear (and single-valued) operator on $C(\bar{\Omega})$. Proposition 5.3 implies that \mathcal{A}^c is m-accretive, i.e., for all $\alpha > 0$ the operator

$I + \alpha \mathcal{A}^c: D(\mathcal{A}^c) \rightarrow C(\overline{\Omega})$ is bijective and $(I + \alpha \mathcal{A}^c)^{-1}$ is a contraction on $C(\overline{\Omega})$. We now show that \mathcal{A}^c is densely defined.

Proposition 5.4. *Under assumptions (5.2) and (5.3) the set $D(\mathcal{A}^c)$ is dense in $C(\overline{\Omega})$.*

Proof. We give the proof only for $p < N$ and only mention that the case $p \geq N$ can be treated analogously. First assume that $u \in C^\infty(\overline{\Omega})$, so in particular $u \in V$. Then $\varphi := \mathcal{A}_1 u \in V'$, where \mathcal{A}_1 is defined as in the proof of Proposition 5.2. More precisely,

$$\begin{aligned} |a(x, \nabla u)| &\leq \mu_2 \|\nabla u\|_{L^\infty(\Omega)}^{p-1} + \psi_2 \leq c_u + \psi_2 \\ |b(x, u)| &\leq \psi_1 (\|u\|_{L^\infty(\Omega)}^{p-1} + 1) \leq c_u \psi_1 \\ |h(x, u)| &\leq \psi_4 (\|u\|_{L^\infty(\Omega)}^{p-1} + 1) \leq c_u \psi_4 \end{aligned}$$

and hence

$$\begin{aligned} \varphi(v) &= \int_\Omega v u + \int_\Omega \nabla v a(x, \nabla u) + \int_\Omega v b(x, u) + \int_{\partial\Omega} v h(x, u) \\ &\leq c_u \|v\|_{L^1(\Omega)} + \|\nabla v\|_{L^{\frac{N}{N-p+1}}(\Omega)} \|c_u + \psi_2\|_{L^{\frac{N}{p-1}}(\Omega)} \\ &\quad + c_u \|v\|_{L^{\frac{N}{N-p}}(\Omega)} \|\psi_1\|_{L^{\frac{N}{p}}(\Omega)} + c_u \|v\|_{L^{\frac{N-1}{N-p}}(\partial\Omega)} \|\psi_4\|_{L^{\frac{N-1}{p-1}}(\partial\Omega)} \\ &\leq c_u \|v\|_{W^{1, \frac{N}{N-p+1}}(\Omega)} \end{aligned}$$

for all $v \in V$. Thus φ extends to a bounded functional on $W^{1, \frac{N}{N-p+1}}(\Omega)$. Consequently, there exist $k \in L^{\frac{N}{p-1}}(\Omega)$ and $K \in L^{\frac{N}{p-1}}(\Omega; \mathbb{R}^N)$ such that

$$\varphi(v) = \int_\Omega v k + \int_\Omega \nabla v K$$

for all $v \in V$, see [34, Section 4.3]. Pick sequences $k_n \in C_c^\infty(\Omega)$ and $K_n \in C_c^\infty(\Omega; \mathbb{R}^N)$ such that $k_n \rightarrow k$ and $K_n \rightarrow K$ in the $L^{\frac{N}{p-1}}$ -norm. Then $f_n := k_n - \operatorname{div} K_n$ is in $C_c^\infty(\Omega) \subset C(\overline{\Omega})$. Thus by Propositions 5.2 and the additional claim in (5.3) there exists $u_n \in D(\mathcal{A}^c)$ such that $(I + \mathcal{A}^c)u_n = f_n$.

Define $\varphi_n(v) := (f_n | v)_{L^2(\Omega)}$, which can equivalently be written as $\varphi_n : = \mathcal{A}_1 u_n$. Then

$$\begin{aligned} |\varphi_n(v) - \varphi(v)| &\leq \int_\Omega |v| |k_n - k| + \int_\Omega |\nabla v| |K_n - K| \\ &\leq \|v\|_{W^{1,p}(\Omega)} \left(\|k_n - k\|_{L^{\frac{p}{p-1}}(\Omega)} + \|K_n - K\|_{L^{\frac{p}{p-1}}(\Omega)} \right). \end{aligned}$$

This shows that $\varphi_n \rightarrow \varphi$ in V' . In particular we see that (φ_n) is bounded in V' , which implies that

$$\langle \mathcal{A}_1 u_n, u_n \rangle = \varphi_n(u_n) \leq c_u \|u_n\|_V.$$

By (5.7) this implies that (u_n) is bounded in V . Thus passing to a subsequence we can assume that (u_n) converges weakly to some $\tilde{u} \in V$.

By Minty’s theorem [30, Proposition II.2.2] we have

$$\langle \mathcal{A}_1 v - \varphi_n, v - u_n \rangle \geq 0$$

for all $v \in V$. Since $\varphi_n \rightarrow \varphi$ in V' and $u_n \rightharpoonup \tilde{u}$ in V we obtain by taking limits that

$$\langle \mathcal{A}_1 v - \varphi, v - \tilde{u} \rangle \geq 0$$

for all $v \in V$. Using Minty’s theorem once again we deduce from this that $\mathcal{A}_1 \tilde{u} = \varphi = \mathcal{A}_1 u$. By the uniqueness assertion of Proposition 5.2 this implies $\tilde{u} = u$. Thus we have shown that $u_n \rightharpoonup u$ in V .

We have seen that (u_n) is bounded in V and hence in particular in $L^p(\Omega)$. Since by construction u_n is a weak solution of

$$\begin{cases} -\operatorname{div} a(x, \nabla u_n) + b(x, u_n) + u_n = k_n - \operatorname{div} K_n & \text{on } \Omega \\ a(x, \nabla u_n) \cdot \nu + h(x, u_n) = K_n \cdot \nu & \text{on } \partial\Omega, \end{cases}$$

we obtain from Theorem 4.4 that (u_n) is bounded in $C^{0,\alpha}(\Omega)$. Passing to a subsequence we thus can assume that (u_n) has a limit in $C(\overline{\Omega})$. Since $u_n \rightharpoonup u$ in V , this limit is u . Hence we have shown that for each $u \in C^\infty(\overline{\Omega})$ there exists a sequence $u_n \in D(\mathcal{A}^c)$ such that $u_n \rightarrow u$ in $C(\overline{\Omega})$. This proves that $C^\infty(\overline{\Omega})$ is contained in the closure of $D(\mathcal{A}^c)$ in $C(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is dense in $C(\overline{\Omega})$, we have shown that $D(\mathcal{A}^c)$ is dense in $C(\overline{\Omega})$. \square

In order to state the main result of this section it is convenient to introduce the notion of a non-linear contraction C_0 -semigroup. The definition is consistent with the linear case, i.e., a family of linear operators on a Banach space X is a non-linear contraction C_0 -semigroup if and only if it is a linear contraction C_0 -semigroup in the usual sense of for example [20].

Definition 5.5. Let X be a Banach space and let $\mathcal{B}: X \supset D(\mathcal{B}) \rightarrow X$ be an m -accretive operator on X , i.e., for all $\alpha > 0$ the operator $I + \alpha\mathcal{B}: D(\mathcal{B}) \rightarrow X$ is bijective with $(I + \alpha\mathcal{B})^{-1}: X \rightarrow X$ being a contraction. Then by the Crandall–Liggett theorem [30, Section IV.8] the limit $S(t)u_0 = \lim_{n \rightarrow \infty} (I + \frac{t}{n}\mathcal{B})^{-n}u_0$ exists for $u_0 \in C := \overline{D(\mathcal{B})}$ and $t \geq 0$ and the mappings $S(t)$ satisfy

- (i) $S(t): C \rightarrow C$ is contractive for every $t \geq 0$;
- (ii) $S(0) = \operatorname{id}_C$;
- (iii) $S(t + s) = S(t) \circ S(s)$ for all $t, s \geq 0$;
- (iv) $t \mapsto S(t)u_0$ is continuous for all $u_0 \in C$.

We say that $(S(t))_{t \geq 0}$ is a non-linear contraction C_0 -semigroup on C and call $-\mathcal{B}$ its generator.

The following remark about non-linear contraction C_0 -semigroups and their generators justifies that we regard the trajectories of a non-linear contraction C_0 -semigroup with generator $-\mathcal{B}$ as the unique solutions of the problem $u'(t) + \mathcal{B}u(t) = 0$. For the definition of a C^0 -solution, which is frequently also called mild solution in the literature, and proofs of the following facts we refer to [30, Section IV.3 and Section IV.8] or [7].

Remark 5.6. If \mathcal{B} is m-accretive and $(S(t))_{t \geq 0}$ is the semigroup generated by $-\mathcal{B}$, then for each $u_0 \in C := \overline{D(\mathcal{B})}$ and $T > 0$ the unique C^0 -solution of $u'(t) + \mathcal{B}u(t) = 0, u(0) = u_0$ on $[0, T]$ is given by $u(t) = S(t)u_0$.

Moreover, if X is a Hilbert space and $u_0 \in D(\mathcal{B})$, then this unique solution u is Lipschitz continuous, $u(t) \in D(\mathcal{B})$ for all $t \geq 0$ and $u'(t) + \mathcal{B}u(t) = 0$ for almost every $t > 0$. In this situation we say that u is a strong solution of $u'(t) + \mathcal{B}u(t) = 0$.

Now regard $L^2(\Omega)$ as a subspace of V' . Then the part \mathcal{A}^H of \mathcal{A} in $L^2(\Omega)$, i.e., the restriction of \mathcal{A} to

$$D(\mathcal{A}^H) := \{u \in V : \mathcal{A}u \in L^2(\Omega)\},$$

acts as an m-accretive operator on $L^2(\Omega)$ by Propositions 5.2 and 5.3. Moreover, the set $D(\mathcal{A}^H)$ is dense in $L^2(\Omega)$ by Proposition 5.4 since $D(\mathcal{A}^c) \subset D(\mathcal{A}^H)$ and $C(\overline{\Omega})$ is continuously and densely embedded into $L^2(\Omega)$. Hence $-\mathcal{A}^H$ generates a non-linear contraction C_0 -semigroup $(S(t))_{t \geq 0}$ on $L^2(\Omega)$. Remark 5.6 then leads to the following notion of an L^2 -solution.

Definition 5.7. For every $\tau > 0$ the restriction of $u(t) := S(t)u_0$ to $[0, \tau]$ is in $W^{1,\infty}(0, \tau; L^2(\Omega)) \cap L^\infty(0, \tau; W^{1,p}(\Omega))$ for all $\tau > 0$ and satisfies $u'(t) + \mathcal{A}^H u(t) = 0$ for almost every $t \geq 0$. This function u is the unique solution of (5.1) for the initial value $u_0 \in L^2(\Omega)$. We refer to u as the L^2 -solution of (5.1).

The following is our main result.

Theorem 5.8. Assume (5.2) and (5.3). Then \mathcal{A}^c is m-accretive on $C(\overline{\Omega})$, the semigroup generated by $-\mathcal{A}^c$ being the restriction of $(S(t))_{t \geq 0}$ to $C(\overline{\Omega})$. Thus for $u_0 \in C(\overline{\Omega})$ the unique L^2 -solution u of (5.1) is continuous on the closed parabolic cylinder $[0, \infty) \times \overline{\Omega}$, i.e., continuous up to the parabolic boundary.

Proof. By Propositions 5.2 and 5.3 the operator \mathcal{A}^c is m-accretive. Hence $-\mathcal{A}^c$ generates a non-linear contraction C_0 -semigroup $(S^c(t))_{t \geq 0}$ on $C(\overline{\Omega})$, see Definition 5.5 and Proposition (5.4). Since $(I + \alpha \mathcal{A}^c)^{-1}$ is the restriction of $(I + \alpha \mathcal{A}^H)^{-1}$ to $C(\overline{\Omega})$, see the additional assertion in Proposition 5.3, the operator $S^c(t)$ is the restriction of $S(t)$ to $C(\overline{\Omega})$. Thus $t \mapsto S^H(t)u_0$ is continuous as a function with values in $C(\overline{\Omega})$ provided that $u_0 \in C(\overline{\Omega})$. In this case $u(t, x) := (S(t)u_0)(x)$ is jointly continuous with respect to $t \geq 0$ and $x \in \overline{\Omega}$. \square

By the general theory about m-accretive operators, Theorem 5.8 provides a regularity result also for inhomogeneous parabolic equations. The following definition explains the situation in Hilbert spaces as laid out in [30, Section IV.4].

Definition 5.9. Let H be a Hilbert space, let $\mathcal{B}: H \supset D(\mathcal{B}) \rightarrow H$ be m-accretive, and let $T > 0$. For all $u_0 \in D(\mathcal{B})$ and $f \in W^{1,1}(0, T; H)$ there exists a unique function $u \in W^{1,1}(0, T; H) \subset C([0, T]; H)$ with $u'(t) \in D(\mathcal{B})$ almost everywhere that satisfies

$$\begin{cases} u'(t) + \mathcal{B}u(t) = f(t) \\ u(0) = u_0 \end{cases} \tag{5.8}$$

for almost every $t \in [0, T]$. This function u is Lipschitz continuous and right-differentiable, and we call u the *strong solution* of (5.8) on $[0, T]$.

Given $f \in L^1(0, T; H)$ and $u_0 \in \overline{D(\mathcal{B})}$, there exists a unique *generalized solution* $u \in C([0, T]; H)$ of (5.8) on $[0, T]$, i.e., $u(0) = u_0$ and there exist sequences (f_n) in $L^1(0, T; H)$ and (u_n) in $C([0, T]; H)$ such that $f_n \rightarrow f$ in $L^1(0, T; H)$, $u_n \rightarrow u$ in $C([0, T]; H)$ and u_n is a strong solution of $u'_n(t) + \mathcal{B}u_n(t) = f_n(t)$ on $[0, T]$ for every $n \in \mathbb{N}$. We say that $u: [0, \infty) \rightarrow H$ is a generalized solution of (5.8) if $u|_{[0, T]}$ is a generalized solution of (5.8) on $[0, T]$ for every $T > 0$.

From Definition 5.9 and Proposition 5.1 and 5.2 we obtain that for every $f \in L^1_{\text{loc}}([0, \infty); L^2(\Omega))$ and $u_0 \in L^2(\Omega)$ there exists a unique generalized solution $u \in C([0, \infty); L^2(\Omega))$ of $u'(t) + \mathcal{A}^H u(t) = f(t)$ on $[0, \infty)$ with $u(0) = u_0$. We thus regard this function u as the unique L^2 -solution of

$$\begin{cases} u_t(t, x) - \operatorname{div} a(x, \nabla u(t, x)) + b(x, u(t, x)) = f(t, x) & t > 0, x \in \Omega \\ a(x, \nabla u(t, x)) \cdot \nu + h(x, u(x)) = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases} \tag{5.9}$$

From what we have already shown, we can now easily deduce the following result about regularity of solutions of the inhomogeneous parabolic problem.

Theorem 5.10. *For $u_0 \in C(\overline{\Omega})$ and $f \in L^1_{\text{loc}}([0, \infty); C(\overline{\Omega}))$ the unique L^2 -solution u of (5.9) is in $C([0, \infty); C(\overline{\Omega}))$, i.e., u is continuous up to the parabolic boundary.*

Proof. Since \mathcal{A}^c is m -accretive by Theorem 5.8, we obtain from [30, Corollary IV.8.4] that there exists a unique C^0 -solution of $u \in C([0, \infty); C(\overline{\Omega}))$ of

$$\begin{cases} u'(t) + \mathcal{A}^c u(t) = f(t) \\ u(0) = u_0. \end{cases}$$

The definition of a C^0 -solution allows to check easily that $u \in C([0, \infty); L^2(\Omega))$ is a C^0 -solution of

$$\begin{cases} u'(t) + \mathcal{A}^H u(t) = f(t) \\ u(0) = u_0. \end{cases} \tag{5.10}$$

Moreover, the unique L^2 -solution of (5.9) is a C^0 -solution of (5.10) as well by [30, Propositions IV.8.2 and IV.8.1 (d)]. Since the C^0 -solution of (5.10) is unique by [30, Corollary IV.8.4] we have thus proved the claim. \square

Let us finally look at a class of examples that satisfy the conditions (5.2) and (5.3). Since in the diffusion Eq. (5.1) the function $a(x, \nabla u)$ is the flux, it is natural to assume that it points into the direction of steepest descent, i.e., into the direction of ∇u . If we assume in addition that the magnitude of the flux depends only on the steepness of u and possibly on the location $x \in \Omega$, i.e., if we assume that $a(x, z) = m(x, |z|)z$, then we have a simple criterion to check condition (5.3).

Lemma 5.11. *Assume that $a(x, z) = m(x, |z|)z$ for a measurable function $m: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then the first condition in (5.3) is satisfied if and only if*

$y \mapsto m(x, y) y$ is nondecreasing for every $x \in \Omega$. Similar assertions hold for b and h .

Proof. For all $x \in \Omega$ and all z_1 and z_2 in \mathbb{R}^N we have

$$\begin{aligned} & (z_1 - z_2) (a(x, z_1) - a(x, z_2)) \\ &= m(x, |z_1|) |z_1|^2 - (m(x, |z_1|) + m(x, |z_2|)) z_1 \cdot z_2 + m(x, |z_2|) |z_2|^2 \\ &\geq m(x, |z_1|) |z_1|^2 - (m(x, |z_1|) + m(x, |z_2|)) |z_1| |z_2| + m(x, |z_2|) |z_2|^2 \\ &= (|z_1| - |z_2|) (m(x, |z_1|) |z_1| - m(x, |z_2|) |z_2|) \end{aligned}$$

with equality if z_1 and z_2 point into the same direction.

If $y \mapsto m(x, y) y$ is nondecreasing, then both factors in the last expression have the same sign. Thus the product is nonnegative and the first condition in (5.3) is fulfilled.

Conversely, if the first condition in (5.3) is fulfilled, then by choosing z_2 to be a positive multiple of z_1 , we see that

$$(a - b) (m(x, a) a - m(x, b) a) \geq 0$$

for all $a, b > 0$, which implies that $y \mapsto m(x, y) y$ is nondecreasing. □

As a consequence of Lemma 5.11 we see that an important and commonly encountered class of equations of p -Laplace-type satisfy conditions (5.2) and (5.3). For simplicity we consider only examples with $p < N$, but similar assertions hold true for $p \geq N$ with slightly different integrability assumptions on the coefficients.

Example 5.12. Let $p \in (1, N)$. Set $a(x, z) := a_0(x)(s + |z|^{p-2}) z$ or $a(x, z) := a_0(x)(s^2 + |z|^2)^{\frac{p-2}{2}} z$ with a constant $s \geq 0$ and a measurable function $a_0: \Omega \rightarrow [\mu_1, \mu_2]$, where $0 < \mu_1 \leq \mu_2$. Set $b(x, u) := b_0(x)|u|^{p-2}u$ and $h(x, u) := h_0(x)|u|^{p-2}u$ with nonnegative measurable functions $b_0 \in L^{\frac{N}{p-\varepsilon}}(\Omega)$ and $h_0 \in L^{\frac{N-1}{p-1-\varepsilon}}(\partial\Omega)$ for some $\varepsilon > 0$. Then the assumption in (5.2) and (5.3) are satisfied and thus Theorem 5.8 applies. In the special case $s = 0$ and $a_0(x) = 1$ for all $x \in \Omega$ we obtain that the equation

$$\begin{cases} u_t(t, x) - \Delta_p u(t, x) + b_0(x)|u|^{p-2}u = f(t, x) & t > 0, x \in \Omega \\ |\nabla u(t, x)|^{p-2} \frac{\partial u(t, x)}{\partial \nu} + h_0(x)|u|^{p-2}u = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega \end{cases}$$

has a unique C^0 -solution $u \in C([0, \infty) \times \bar{\Omega})$.

The strategy of this section applies also to certain dynamic boundary conditions, which are often called Wentzell–Robin boundary conditions, if we carry out the arguments on a suitable product space, confer also [2] where this idea was originally introduced to the literature for linear equations. More precisely, we consider the equation

$$\begin{cases} u_t(t, x) - \operatorname{div} a(x, \nabla u(t, x)) + b(x, u(t, x)) = 0 & t > 0, x \in \Omega \\ \beta u_t(t, x) + a(x, \nabla u(t, x)) \cdot \nu + h(x, u(t, x)) = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases} \tag{5.11}$$

At least formally, we can use the first line in (5.11) to express u_t in terms of u in the second line. Then we arrive at what is classically called Wentzell–Robin boundary conditions.

We can show well-posedness of (5.11) in the space of continuous functions. Since the arguments are very similar to what we did before, we only sketch the proof of the following theorem, but see [27] for a detailed proof in the linear case.

Theorem 5.13. *Assume (5.2) and (5.3). Let $\beta: \partial\Omega \rightarrow \mathbb{R}$ be measurable and such that*

$$0 < \operatorname{ess\,inf}_{\partial\Omega} \beta \leq \operatorname{ess\,sup}_{\partial\Omega} \beta < \infty.$$

For $u_0 \in C(\overline{\Omega})$, problem (5.11) has a unique solution. This solution is continuous on the parabolic cylinder $[0, \infty) \times \overline{\Omega}$.

Proof. Define the reflexive, separable Banach space

$$\mathcal{V} := \{(u, u|_{\partial\Omega}) : u \in W^{1,p}(\Omega) \cap L^2(\Omega), u|_{\partial\Omega} \in L^2(\partial\Omega)\}$$

and let $\mathcal{A}_W: \mathcal{V} \rightarrow \mathcal{V}'$ be defined by the formal expression (5.4). The proof of Proposition 5.1 shows that \mathcal{A}_W is bounded, continuous and monotone. For $q \in [2, \infty]$ consider $L^q(\Omega) \oplus_q L^q(\partial\Omega)$, which for $q \in [2, \infty)$ is equipped with the norm given by

$$\|(u, g)\|_{L^q(\Omega) \oplus_q L^q(\partial\Omega)}^q := \|u\|_{L^q(\Omega)}^q + \|g\|_{L^q(\partial\Omega; \beta \, d\sigma)}^q,$$

whereas for $q = \infty$ we set

$$\|(u, g)\|_{L^\infty(\Omega) \oplus_\infty L^\infty(\partial\Omega)} := \|u\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega; \beta \, d\sigma)}.$$

Here σ denotes the surface measure on $\partial\Omega$, i.e., the $(N - 1)$ -dimensional Hausdorff measure. Set $\mathcal{H} := L^2(\Omega) \oplus L^2(\partial\Omega)$, and equip $\mathcal{C} := \{(u, u|_{\partial\Omega}) : u \in C(\overline{\Omega})\}$ with the norm of $L^\infty(\Omega) \oplus_\infty L^\infty(\partial\Omega)$. Then both of these spaces are subspaces of V' via

$$\langle (u, g), (v, v|_{\partial\Omega}) \rangle_{V', V} := \operatorname{scalar}[L^2(\Omega)]uv + (g | v)_{L^2(\partial\Omega; \beta \, d\sigma)}$$

for $v \in V$ and $(u, g) \in \mathcal{H}$ or $(u, g) \in \mathcal{C}$, respectively. We consider the parts $\mathcal{A}_W^{\mathcal{H}}$ and $\mathcal{A}_W^{\mathcal{C}}$ of \mathcal{A}_W in \mathcal{H} and \mathcal{C} , respectively, with domains

$$D(\mathcal{A}_W^{\mathcal{H}}) := \{u \in \mathcal{V} : \mathcal{A}_W u \in \mathcal{H}\}$$

$$D(\mathcal{A}_W^{\mathcal{C}}) := \{u \in \mathcal{V} \cap \mathcal{C} : \mathcal{A}_W u \in \mathcal{C}\}$$

Then similar arguments as in the proof of Propositions 5.2 and 5.3 show that $\mathcal{A}_W^{\mathcal{H}}$ and $\mathcal{A}_W^{\mathcal{C}}$ are m-accretive on \mathcal{H} and \mathcal{C} , respectively. Let $(\mathcal{S}(t))_{t \geq 0}$ be the semigroup generated by $-\mathcal{A}_W^{\mathcal{H}}$. If $U_0 \in D(\mathcal{A}_W^{\mathcal{H}})$ and $U(t) := \mathcal{S}(t)U_0$, then $U'(t) + \mathcal{A}_W^{\mathcal{H}} U(t) = 0$ for almost every $t \geq 0$. Writing $U(t) = (u(t), u(t)|_{\partial\Omega})$ this means that

$$\begin{aligned} & \int_{\Omega} \nabla \eta \, a(x, \nabla u(t)) + \int_{\Omega} \eta \, b(x, u(t)) + \int_{\partial\Omega} \eta \, h(x, u) \\ & = \langle \mathcal{A}_W^{\mathcal{H}} (u(t), \eta|_{\partial\Omega}) \rangle = (u'(t) | (\eta, \eta|_{\partial\Omega}))_{\mathcal{H}} = - \int_{\Omega} \eta \, u'(t) - \int_{\partial\Omega} \eta \, u'(t) \beta \end{aligned}$$

for all $\eta \in C^\infty(\overline{\Omega}) \subset \mathcal{V}$ and almost every $t \geq 0$. Hence for almost every $t \geq 0$, the function $u(t) \in W^{1,p}(\Omega)$ is a weak solution of (5.11) with t fixed. This justifies that we call the unique C^0 -solution of $u'(t) + \mathcal{A}_W^t u(t) = 0$ with $u(0) = u_0 \in \overline{D(\mathcal{A}_W^t)}$, or rather its first component, the (unique) solution of (5.11).

In the proof of Proposition 5.4 we have seen that $\{u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \mathcal{A}u \in C_c^\infty(\Omega)\}$ is dense in $C(\overline{\Omega})$, which implies that \mathcal{A}^t and \mathcal{A}^c are densely defined. Now the same arguments as for Theorem 5.8 show that for every $u_0 \in C(\overline{\Omega})$ the unique L^2 -solution of (5.11) is continuous on $[0, \infty) \times \overline{\Omega}$. \square

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