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A pointwise constrained version of the Liapunov convexity theorem for single integrals

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Dedicated to Arrigo Cellina, on occasion of his 70th birthday.
António has been maybe the second—out of a long list yet to end—earning a PhD under Arrigo's supervision. He still recalls quite vividly the great pleasure aroused, back in 1985 and 1986, on his young and fresh mind, by Arrigo's lectures at the great SISSA school, e.g. those on Smale's continuous Newton method, those on Granas and Dugundji's fixpoint theory, and those on Oxtoby's Lebesgue measure versus Baire category. Arrigo still remains, to this day, António's model mathematician.

Abstract. Given any AC solution $\overline{x} : [a, b] \to \mathbb{R}^n$ to the *convex ordinary differential inclusion*

$$x'(t) \in co\{v^1(t), \dots, v^m(t)\} \quad \text{a.e. on } [a, b], \tag{*}$$

we aim at solving the associated *nonconvex inclusion*

$$x'(t) \in \{v^1(t), \dots, v^m(t)\} \quad \text{a.e.}, \ x(a) = \overline{x}(a), \ x(b) = \overline{x}(b), \tag{**}$$

under an *extra pointwise constraint* (e.g. on the first coordinate):

$$x_1(t) \le \overline{x}_1(t) \quad \forall t \in [a, b]. \tag{***}$$

While the *unconstrained inclusion* (**) had been solved already in 1940 by *Liapunov*, its *constrained* version, with (***), was solved in 1994 by *Amar* and *Cellina* in the *scalar* $n = 1$ case. In this paper we add an *extra geometrical hypothesis* which is *necessary and sufficient*, in the vector $n > 1$ case, for *existence of solution* to the *constrained inclusion* (**) and (***). We also present many examples and counterexamples to the 2×2 case.

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1. Introduction and main results

To state precisely the results of this paper, consider the space

$$\mathcal{X} := W^{1,1}([a,b],\mathbb{R}^n), \tag{1.1}$$

fix m *integrable velocities*

$$v^1(\cdot),\ldots,v^m(\cdot) \in L^1([a,b],\mathbb{R}^n), \tag{1.2}$$

fix any AC (*absolutely continuous*) *function*

$$\overline{x}(\cdot) \in \mathcal{X}^{co} := \{x(\cdot) \in \mathcal{X} :\ x'(t) \in co\{v^1(t),\ldots,v^m(t)\} \text{ a.e.}\}; \tag{1.3}$$

and consider the associated *nonconvex differential inclusion*

$$x'(t) \in \{v^1(t),\ldots,v^m(t)\} \quad \text{a.e.},\ x(a) = \overline{x}(a),\ x(b) = \overline{x}(b), \tag{1.4}$$

and its *convexified*, or *relaxed, differential inclusion*

$$x'(t) \in co\{v^1(t),\ldots,v^m(t)\} \quad \text{a.e.},\ x(a) = \overline{x}(a),\ x(b) = \overline{x}(b); \tag{1.5}$$

together with their *solution sets*

$$\overline{\mathcal{X}}^{co} := \{x(\cdot) \in \mathcal{X} :\ x(\cdot) \text{ solves } (1.5)\}, \tag{1.6}$$

$$\overline{\mathcal{X}}^{true} := \{x(\cdot) \in \mathcal{X} :\ x(\cdot) \text{ solves } (1.4)\}. \tag{1.7}$$

Fix, moreover, any "*direction*" $\omega \in \mathbb{R}^n$, e.g. $\omega = (1,0,\ldots,0)$; and consider the corresponding *constrained solution sets*:

$$\overline{\mathcal{X}}^{co}_{-} := \{x(\cdot) \in \overline{\mathcal{X}}^{co} :\ \langle x(t),\omega\rangle \le \langle \overline{x}(t),\omega\rangle\ \forall t \in [a,b]\}, \tag{1.8}$$

$$\overline{\mathcal{X}}^{true}_{-} := \overline{\mathcal{X}}^{true} \cap \overline{\mathcal{X}}^{co}_{-}, \quad \langle\cdot,\cdot\rangle = \mathbb{R}^n\text{-inner product.} \tag{1.9}$$

In order to gain some geometrical insight into our problem, taking, e.g., $|\omega| = 1$, we are given an AC factor $\overline{h}(\cdot) \in W^{1,1}([a,b])$, a *hyperplane* $\overline{H}(t)$ *orthogonal* to ω, namely

$$\overline{H}(t) := \{S \in \mathbb{R}^n :\ \langle S - \omega\overline{h}(t),\omega\rangle = 0\}, \quad t \in [a,b],$$

moving along the ω-*direction* with *relaxed velocities* $\overline{h}'(t) \in co\{\langle v^1(t),\omega\rangle,\ldots,\langle v^m(t),\omega\rangle\}$ a.e. on $[a,b]$; and an AC point $\overline{x}(t) \in \overline{H}(t)$ moving inside this *hyperplane* with *relaxed velocities* $\overline{x}'(t) \in co\{v^1(t),\ldots,v^m(t)\}$ a.e. on $[a,b]$. Our aim is to find an AC point $\widehat{x}(t)$ starting from the same initial point, $\widehat{x}(a) = \overline{x}(a) \in \overline{H}(a)$, moving with *true velocities* $\widehat{x}'(t) \in \{v^1(t),\ldots,v^m(t)\}$ a.e. on $[a,b]$, and reaching exactly the same final endpoint, $\widehat{x}(b) = \overline{x}(b) \in \overline{H}(b)$, under the *extra constraint* of (touching at will but) never crossing the *moving hyperplane* $\overline{H}(t)$.

Recall now the classical 1940 theorem of *Liapunov* [21] on *convexity* of the *range of vector measures* (which may also be seen, with a new proof, e.g.

in the more recent [4] paper). This result guarantees that

$$\forall \overline{x}(\cdot) \in \mathcal{X}^{co} \quad \exists \widehat{x}(\cdot) \in \overline{\mathcal{X}}^{true}, \quad \forall n \geq 1 \text{ in } (1.2), (1.3) \text{ and } (1.7),$$

i.e. the *unconstrained inclusion* always has a solution.

If one wishes to add the pointwise constraint in (1.8) to the inclusion (1.4), then the result of [1, lemma 3.4] tells us that

$$\forall \overline{x}(\cdot) \in \mathcal{X}^{co} \quad \exists \widehat{x}(\cdot) \in \overline{\mathcal{X}}^{true}_{-}, \quad \text{for } n = 1 \text{ in } (1.2), (1.3) \text{ and } (1.9),$$

i.e. up to now only the *scalar* $n = 1$ case of the general *constrained inclusion* had been solved. (Or, more precisely, thanks to a helpful referee info: some special cases, of this constrained inclusion with single integrals, had also been then solved, namely: in [2], the vector $n > 1$ case with constant control-polyhedron, i.e. $v^1, \ldots, v^m \in \mathbb{R}^n$ in (1.2); and, in [14], the scalar $n = 1$ case of optimal control driven by a scalar second-order linear differential inclusion.)

Our first result is a simple direct generalization to the vector $n > 1$ case of [1, lemma 3.4] which, however, does not solve $n - 1$ coordinates of the final endpoint equality in (1.4). Namely, setting

$$\overline{\mathcal{X}}_{\omega} := \{x(\cdot) \in \mathcal{X} : \ x(a) = \overline{x}(a), \quad \langle x(b), \omega \rangle = \langle \overline{x}(b), \omega \rangle\} \tag{1.10}$$

$$\overline{\mathcal{X}}^{co}_{\omega} := \{x(\cdot) \in \overline{\mathcal{X}}_{\omega} : \ x'(t) \in co\{v^1(t), \ldots, v^m(t)\} \text{ a.e.}\} \tag{1.11}$$

$$\overline{\mathcal{X}}^{true}_{\omega} := \{x(\cdot) \in \overline{\mathcal{X}}_{\omega} : \ x'(t) \in \{v^1(t), \ldots, v^m(t)\} \text{ a.e.}\}, \tag{1.12}$$

then it says:

Theorem 1.1. (Existence of true ω-extremal weak-solutions)

$$\begin{aligned} &\forall \overline{x}(\cdot) \in \mathcal{X}^{co} \ \forall \omega \in \mathbb{R}^n \ \exists x^{\min}_{\omega}(\cdot) \in \overline{\mathcal{X}}^{true}_{\omega} \ \exists x^{\max}_{\omega}(\cdot) \in \overline{\mathcal{X}}^{true}_{\omega} : \\ &\qquad \langle x^{\min}_{\omega}(\cdot), \omega \rangle \leq \langle x(\cdot), \omega \rangle \leq \langle x^{\max}_{\omega}(\cdot), \omega \rangle \quad \forall x(\cdot) \in \overline{\mathcal{X}}^{co}_{\omega}. \end{aligned} \tag{1.13}$$

The proof of Theorem 1.1 defines $x^{\min}_{\omega}(\cdot)$ and $x^{\max}_{\omega}(\cdot)$ in such a way that

$$x^{\min}_{\omega}(\cdot) \in \mathcal{X}^{ext}_{\omega} \quad \text{and} \quad x^{\max}_{\omega}(\cdot) \in \mathcal{X}^{ext}_{\omega} \ (\text{see } (3.5) \text{ and } (3.9)),$$

where the space

$$\mathcal{X}^{ext}_{\omega} := \{x(\cdot) \in \mathcal{X} : x'(t) \in \{v^{\min}_{\omega}(t), v^{\max}_{\omega}(t)\} \text{ a.e.}\} \tag{1.14}$$

depends on the also there defined

$$\omega\text{-extremal velocities} \quad v^{\min}_{\omega}(\cdot) \quad \text{and} \quad v^{\max}_{\omega}(\cdot) \ (\text{see } (3.1)). \tag{1.15}$$

Geometrically, Theorem 1.1 tells us that the final hyperplane $\overline{H}(b)$ is easily attained. Thus our real challenge, in this research, has been how to reach, more specifically, the exact final target $\overline{x}(b)$ inside $\overline{H}(b)$.

In order to solve this problem, consider a new subspace of (1.8):

$$\overline{\mathcal{X}}^{co}_{--} := \{x(\cdot) \in \overline{\mathcal{X}}^{co}_{-} : \ \langle x(t), \omega \rangle < \langle \overline{x}(t), \omega \rangle \text{ a.e. in } \mathcal{O}_x\}, \tag{1.16}$$

where $\mathcal{O}_x$ is an *open set* and satisfies

$$\mathcal{O}_x \quad \supset \quad \text{a.e.} \quad \{t \in (a, b) : \ \exists x'(t) \notin \{v^1(t), \ldots, v^m(t)\}\}, \tag{1.17}$$

e.g. $\mathcal{O}_x = (a, b)$. Then the main result of our research is:

Theorem 1.2. (Necessary and sufficient condition for ω-lower true-solutions)

$$\overline{\mathcal{X}}_{-}^{true} \neq \emptyset \Leftrightarrow \overline{\mathcal{X}}_{--}^{co} \neq \emptyset, \quad \forall \overline{x}(\cdot) \in \mathcal{X}^{co} \ \forall \omega \in \mathbb{R}^n \ \forall n > 1. \tag{1.18}$$

Remark 1.3. More precisely, the inclusion $\supset$ in (1.17) is intended in the a.e. sense, i.e. ignoring null sets. Thus, to prove the $\Rightarrow$ implication in (1.18), given any $x(\cdot)$ in $\overline{\mathcal{X}}_{-}^{true}$ one shows that such $x(\cdot)$ also belongs to $\overline{\mathcal{X}}_{--}^{co}$ by setting $\mathcal{O}_x$ in (1.16) to be the set where $\langle x(t), \omega\rangle < \langle \overline{x}(t), \omega\rangle$ (or any *open subset* of it, e.g. $\emptyset$), which trivially contains a.e. the null set in the *rhs* of (1.17). Notice also that, as an extreme weird case, one may even have $\overline{\mathcal{X}}_{--}^{co} = \overline{\mathcal{X}}_{-}^{co} = \{\overline{x}(\cdot)\} = \overline{\mathcal{X}}_{-}^{true}$.

More generally, as will be shown after (3.13), the implication $\Leftarrow$ in (1.18) remains valid if the *open set* in definition (1.17) and (1.16) is changed, to become (see (1.14) and (1.15)) any *open set*

$$\mathcal{O}_x \supset \{t \in (a,b) : \ x(t) \neq x_\omega^{\min}(t) \text{ and } x(t) \neq x_\omega^{\max}(t)\} \quad \text{or} \tag{1.19}$$

$$\mathcal{O}_x \supset \{t \in (a,b) : \ x(t) \neq \check{x}(t)\}, \quad \text{for some } \check{x}(\cdot) \in \mathcal{X}_\omega^{ext}. \tag{1.20}$$

Another possibility is to consider any *open set*

$$\mathcal{O}_x \supset \{t \in (a,b) : \ \langle x_\omega^{\min}(t), \omega\rangle < \langle x(t), \omega\rangle < \langle x_\omega^{\max}(t), \omega\rangle\}, \tag{1.21}$$

and add an *extra geometrical hypothesis*: to forbid nonsingleton faces *orthogonal* to ω, on the relative boundary of the *convex hull* of the $v^j(\cdot)$ *velocities* in (1.2); or, more precisely, for a.e. $t \in [a,b] \setminus \mathcal{O}_x$, to impose

$$\langle v^j(t), \omega\rangle = \langle v_\omega^{\min}(t), \omega\rangle = \langle v^k(t), \omega\rangle \ \Rightarrow \ v^j(t) = v^k(t), \tag{1.22}$$

$$\langle v^j(t), \omega\rangle = \langle v_\omega^{\max}(t), \omega\rangle = \langle v^k(t), \omega\rangle \ \Rightarrow \ v^j(t) = v^k(t). \tag{1.23}$$

Theorem 1.4. (ω-increasing sequence of true-solutions) *Assume, in* (1.16) *with* $\mathcal{O}_x$ *defined* (*not as in* (1.17) *but*) *as in* (1.19) *or* (1.20) *or* (1.21)*, or even by* $\mathcal{O}_x := (a,b)$*, that* $\overline{\mathcal{X}}_{--}^{co}$ *is nonempty and both* (1.22)*,* (1.23) *hold true.*

Then $\overline{x}(\cdot)$ *may be uniformly approximated by a* ω*-increasing sequence of true solutions, i.e.*

$$\begin{gathered}\exists (\widehat{x}^k(\cdot)) \subset \overline{\mathcal{X}}^{true} : \ (\widehat{x}^k(\cdot)) \to \overline{x}(\cdot) \quad \textit{and} \\ (\langle \widehat{x}^k(\cdot), \omega\rangle) \nearrow \langle \overline{x}(\cdot), \omega\rangle, \ \textit{uniformly}.\end{gathered} \tag{1.24}$$

Finally, our last result in this section delivers a stronger approximation under a stronger *extra hypothesis*: the possibility of finding a *product reordering* of the $v^j(\cdot)$ *velocities* in (1.2), namely so as to become

$$\begin{gathered} v^1(t) \leq v^2(t) \leq \cdots \leq v^m(t) \quad \text{a.e., } \textit{meaning} \\ v_i^1(t) \leq v_i^2(t) \leq \cdots \leq v_i^m(t) \quad \text{a.e. } \forall i \in \{1, \ldots, n\}. \end{gathered} \tag{1.25}$$

In this case we redefine (1.8), to become

$$\overline{\mathcal{X}}_{-}^{co} := \{x(\cdot) \in \overline{\mathcal{X}}^{co} : \ x(t) \leq \overline{x}(t) \ \forall t \in [a,b]\},$$

using the same definition of $\leq$ as in (1.26); redefine (1.14), (1.20) and (1.16) to become, for some

$$\begin{gathered}\check{x}(\cdot) \in \mathcal{X}^{ext} := \{x(\cdot) \in \mathcal{X} : \ x'(t) \in \{v^1(t), v^m(t)\} \text{ a.e.}\} \\ \text{and some } \textit{open} \quad \mathcal{O}_x \supset \{t \in (a,b) : \ x(t) \neq \check{x}(t)\},\end{gathered}$$

$$\overline{\mathcal{X}}^{co}_{--} := \{x(\cdot) \in \overline{\mathcal{X}}^{co}_{-} : \ x(t) < \overline{x}(t) \quad \text{for a.e. } t \in \mathcal{O}_x\}, \tag{1.26}$$

$x(t) < \overline{x}(t)$ now meaning $x_i(t) < \overline{x}_i(t)$ $\forall i \in \{1, \ldots, n\}$; and add an *extra hypothesis* on (1.26): for a.e. $t \in [a, b] \setminus \mathcal{O}_x$ and $\forall j \in \{1, \ldots, m\}$,

$$\text{either} \quad v^1(t) < v^j(t) \quad \text{or else} \quad v^1(t) = v^j(t), \tag{1.27}$$

$$\text{either} \quad v^j(t) < v^m(t) \quad \text{or else} \quad v^j(t) = v^m(t). \tag{1.28}$$

Theorem 1.5. (Product-increasing sequence of true-solutions) *Assume, in* (1.26)*,* $\overline{\mathcal{X}}^{co}_{--} \neq \emptyset$ *with the product ordering* (1.26) *satisfying* (1.27) *and* (1.28)*. Then* $\overline{x}(\cdot)$ *may be uniformly approximated by an increasing sequence of true solutions:*

$$\exists (\widehat{x}^k(\cdot)) \subset \overline{\mathcal{X}}^{true} : \quad (\widehat{x}^k(\cdot)) \nearrow \overline{x}(\cdot) \quad uniformly.$$

(*Notice: the hypothesis* (1.27) *and* (1.28) *is not needed for* $k = 1$.)

The multitude of potential applications of these results can easily be guessed by recalling all the research papers, using the *Liapunov convexity* theorem as main tool of the proof, which have been published on *nonconvex ordinary differential inclusions, calculus of variations* and *optimal control*, since the pioneering works [3, 12], including, e.g., dozens of papers published by the last author of the present paper, along the last quarter of century, together with, e.g., [1, 2, 8, 14, 17, 18].

Indeed, our motivation to invest such effort into this nontrivial extension of the *Liapunov* tool has been precisely the need we felt to have it available in order to unblock our road towards solving some real-life *nonconvex* problems of the *calculus of variations* and *optimal control*, involving both *single* and *multiple* integrals. Their solutions appear in our several papers (see, e.g., [9, 10]) in preparation which use (1.18) and (1.24) as a crucial tool, and yield, in particular, a sharp version of [14].

Finally, a non-exhaustive list of papers and books containing generalizations of the *Liapunov convexity* theorem [21] appears in our References below, namely [1, 2, 4–7, 13–17, 19, 20, 22–24].

2. A deeper analysis of the 2 × 2 case

While (1.18), together with either (1.17) or (1.19) or (1.20) or else (1.21)–(1.23), yields a nice and simple theoretical sufficient condition, in practical applications one is given a specific $n \times m$ matrix of velocities in (1.2) and a specific relaxed solution $\overline{x}(\cdot)$ in (1.3); and then it is not at all clear, a priori, whether the above class $\overline{\mathcal{X}}^{co}_{--}$, in (1.18) and (1.16) is, or is not, empty.

The aim of this section is to present our first contribution towards clarifying such issue, namely through necessary conditions, and sufficient conditions, to have $\overline{\mathcal{X}}^{co}_{--}$ nonempty, and to have $\overline{\mathcal{X}}^{co}_{--}$ empty. However, at least in this paper, we only do this under the most basic nontrivial framework: the special case of a 2×2 matrix of given velocities (i.e. $m = 2 = n$ in (1.2)).

More precisely, in this section we consider, for simplicity: just $m = 2$ velocities to be given in (1.2), in space-dimension $n = 2$; and with difference between these two velocities having second coordinate writable as a factor $f(t)$ times the first one. Thus we assume given

$$v^1(\cdot) \text{ and } v^2(\cdot) \in L^1([a,b],\mathbb{R}^2),\ (v^2 - v^1)(t) = v(t)(1, f(t)) \text{ a.e.,} \tag{2.1}$$

$$\begin{aligned}&\text{for some } v(\cdot) \geq 0 \text{ a.e. in } L^1(a,b) \text{ and some } f(\cdot) \text{ in}\\ &\mathcal{F} := \{f(\cdot) \in L^1(a,b) :\ (f \cdot v)(\cdot) \in L^1(a,b)\}.\end{aligned} \tag{2.2}$$

While this may, at first sight, seem like a very very special situation, in reality asking for $v(\cdot) \geq 0$ is not a loss of generality; and such very basic framework is precisely what we need for our applications, of this constrained Liapunov convexity Theorem 1.2, to the Calculus of Variations involving vector-valued functions in competition towards minimization of nonconvex single or multiple integrals, under state and gradient pointwise constrained lagrangians, in our next papers, beginning with [9,10].

To clarify the situation under this basic framework, we begin by presenting a necessary condition; hence, in particular, a counterexample illustrating the necessity of including *some* extra hypothesis, like (as in (1.18)) nonemptiness of the set $\overline{\mathcal{X}}^{co}_{--}$, in order to guarantee solvability of our pointwise constrained Liapunov problem. Again for simplicity, let us assume, in this section,

$$\omega = (1,0) \quad \text{and} \quad \mathcal{O}_x = (a,b) \quad \text{on defining } \overline{\mathcal{X}}^{co}_{--} \text{ in (1.16);} \tag{2.3}$$

and associate to each $f(\cdot) \in \mathcal{F}$ (as in (2.2)) the new class of functions

$$\mathcal{G}_f := \{g(\cdot) \in W_0^{1,1}([a,b]) :\ g(\cdot) \not\equiv 0 \text{ and } (f \cdot g')(\cdot) \in L^1(a,b)\}. \tag{2.4}$$

Theorem 2.1. (Necessary condition and counterexample) *For each given* $f(\cdot)$ *and* $\overline{x}(\cdot)$*, as in* (2.1), (2.2) *and* (1.3)*, so that*

$$\begin{aligned}&\exists \overline{\lambda}(\cdot) \in L^\infty(a,b) :\ \overline{\lambda}(t) \in [0,1] \ a.e. \quad and\\ &\overline{x}'(t) = v^1(t) + \overline{\lambda}(t)v(t) \cdot (1, f(t)) \ a.e.,\end{aligned} \tag{2.5}$$

it turns out that the class

$$\overline{\mathcal{X}}^{co}_{-} \setminus \{\overline{x}(\cdot)\} \quad (resp. \quad \overline{\mathcal{X}}^{co}_{--}) \text{ is nonempty if and only if} \tag{2.6}$$

$$\exists \overline{g}_f(\cdot) \in \mathcal{G}_f :\ \overline{g}_f(\cdot) \geq 0 \ (resp.\ \overline{g}_f(\cdot) > 0 \ a.e.) \quad and \tag{2.7}$$

$$\int_a^b f(t)\overline{g}_f{}'(t)dt = 0 \ and \ -[v \cdot (1-\overline{\lambda})](t) \leq \overline{g}_f{}'(t) \leq [v \cdot \overline{\lambda}](t) \ a.e.. \tag{2.8}$$

In particular, such necessary condition yields a simple counterexample (*independent of* $\overline{x}(\cdot)$) *involving essentially all the monotone* $f(\cdot) \in \mathcal{F}$*:*

$$f(\cdot) \ a.e. \ monotone \ and \int_a^b |f'(t)|dt > 0 \ \Rightarrow \ \overline{\mathcal{X}}^{co}_{--} = \emptyset \ (see\ (2.3)) \tag{2.9}$$

(*or, more precisely,* $\exists$ *monotone* $\check{f}(\cdot) = f(\cdot)$ *a.e. and* $\int_a^b |\check{f}'(t)|dt > 0$)*;*

$$f(\cdot) \ a.e. \ strictly \ monotone \ \Rightarrow \ \overline{\mathcal{X}}^{co}_{-} \setminus \{\overline{x}(\cdot)\} = \emptyset \ (see\ (1.8)). \tag{2.10}$$

The equality in the *lhs* of (2.8) renders natural the following definitions

$$\mathcal{F}^+ := \left\{ \begin{array}{r} f(\cdot) \in \mathcal{F} : \ \exists g_f(\cdot) \in \mathcal{G}_f \text{ with } g_f(\cdot) \geq 0 \\ \text{and } \displaystyle\int_a^b f(t) {g_f}'(t) dt = 0 \end{array} \right\} \tag{2.11}$$

$$\mathcal{F}^{++} := \left\{ \begin{array}{r} f(\cdot) \in \mathcal{F} : \ \exists g_f(\cdot) \in \mathcal{G}_f \text{ with } g_f(\cdot) > 0 \text{ a.e.} \\ \text{and } \displaystyle\int_a^b f(t) {g_f}'(t) dt = 0 \end{array} \right\}. \tag{2.12}$$

More precisely than (2.9), we prove (in [11]) that

$$f(\cdot) \text{ a.e. } \textit{monotone} \text{ and } \int_a^b |f'(t)| dt > 0 \ \Rightarrow \ f(\cdot) \notin \mathcal{F}^{++}; \tag{2.13}$$

and the next result is a kind of expectable enlarged "converse" of (2.13):

Proposition 2.2. (Density of $\mathcal{F}^{++}$ in $\mathcal{F}$)

$$\mathcal{F}^{++} \quad \textit{is a dense subset of } \mathcal{F} \tag{2.14}$$

or, more precisely, the subclass $\mathcal{F}_{pc}$ *of a.e. piecewise constant functions in* $\mathcal{F}$ (*see* (2.2)) *satisfies:*

$$\mathcal{F}_{pc} \subset \mathcal{F}^{++} \quad \textit{and} \quad \mathcal{F}_{pc} \quad \textit{is dense in } \mathcal{F}. \tag{2.15}$$

Proposition 2.3. (Partial sufficient condition via mean replacement)

$$f(\cdot) \in \mathcal{F} \textit{ and} \int_a^b f(t) \cdot t dt = \int_a^b f(t) \frac{a+b}{2} dt \quad \Rightarrow \ f(\cdot) \in \mathcal{F}^{++}. \tag{2.16}$$

(*Demand on* $f(\cdot)$*: the factor* t *should be replaceable by its mean* $\frac{a+b}{2}$*.*)

Proposition 2.4. (Partial sufficient condition for AC functions plus jumps)

$$f(\cdot) \in W^{1,1}([a,b]) \ \Rightarrow \ \int_a^b f(t) |f^0|'(t) dt = [f(a) - f(b)] \cdot \overline{|f^0|}, \tag{2.17}$$

$$\textit{in particular} \quad f(a) = f(b) \not\equiv f(\cdot) \ \Rightarrow \ f(\cdot) \in \mathcal{F}^+ \tag{2.18}$$

$$\textit{and} \quad f(a) = f(b) \neq f(t) \quad \textit{for a.e. } t \in (a,b) \ \Rightarrow \ f(\cdot) \in \mathcal{F}^{++}, \tag{2.19}$$

$$\textit{where} \quad f^0(t) := f(t) - f^{af}(t) - f(a) \ (\neq 0 \textit{ a.e. in case } (2.19)) \tag{2.20}$$

$$\textit{with} \quad f^{af}(t) := \frac{t-a}{b-a}[f(b) - f(a)] \quad \textit{and} \quad \overline{|f^0|} := \frac{1}{b-a} \int_a^b |f^0(t)| dt. \tag{2.21}$$

Moreover, the integral in (2.17) *does not change its value whenever one adds to* $f(\cdot)$ *any number of jump-discontinuities of the type*

$$f^{pc}(t) := c_i \quad \textit{for } t \in (a_i, b_i), \quad \forall i \in I, \quad \textit{with } (c_i) \textit{ bounded} \subset \mathbb{R}, \tag{2.22}$$

$$\textit{where} \quad \bigcup_{i \in I} (a_i, b_i) = \mathcal{O} := \{t \in (a,b) : \ |f^0(t)| > 0\}, \quad \textit{i.e.} \tag{2.23}$$

$$\int_a^b (f + f^{pc})(t) \cdot \ |f^0|'(t) dt = \ [f(a) - f(b)] \cdot \ \overline{|f^0|}, \tag{2.24}$$

$$\textit{hence} \quad f(a) = f(b) \neq f(t) \quad \textit{a.e.} \ \Rightarrow \ (f + f^{pc})(\cdot) \in \mathcal{F}^{++}. \tag{2.25}$$

(Propositions 2.2–2.4, together with (2.9) and (2.10), are proved in [11].)

The above (2.15), (2.16), (2.19) and (2.25) present simple hypotheses guaranteeing the *lhs* of (2.8), thus yielding, in particular, some geometrical insight on $\mathcal{F}^{++}$.

In order to generalise (2.16) to higher degree polynomial factors, instead of affine ones, we search for a polynomial $g(\cdot) \in \mathcal{G}_f$ of degree $2d$; and associate to each such $d \in \{1, 2, \ldots\}$ and to each $f(\cdot) \in \mathcal{F}$ a

$$d \times d\text{-}symmetric\ matrix \quad M_f^d \quad \text{whose entries} \quad M_f^{ij} \tag{2.26}$$

are linear combinations of pairs of iterated indefinite integrals of $f(\cdot)$; namely, for $k := i + j$ and i and $j \in \{0, 1, \ldots, d-1\}$,

$$M_f^{ij} := (-1)^k \frac{(k+1)!}{i!j!}[(k+2)f^{(-k-2)}(b) - (b-a)f^{(-k-1)}(b)], \tag{2.27}$$

using the iterated indefinite integrals of $f(\cdot)$, i.e. $f^{(0)}(t) := f(t)$ and

$$f^{(-k-1)}(t) := \int_a^t f^{(-k)}(\tau)d\tau \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.28}$$

Then, given any $\overline{x}(\cdot)$ hence a corresponding $\overline{\lambda}(\cdot)$, as in (1.3) and (2.5), we finally present our main result on the special 2×2 case:

Theorem 2.5. (Fully sufficient condition) *Under* (2.1)–(2.3) *and* (2.5), *it turns out that* (1.18) *is applicable to yield* $\overline{\mathcal{X}}_-^{true} \neq \emptyset$

$$(since \quad \exists \widetilde{x}(\cdot) \in \overline{\mathcal{X}}_{--}^{co} \quad having \quad \widetilde{x}'(\cdot) := \overline{x}'(\cdot) - g'(\cdot)(1, f(\cdot)), \tag{2.29}$$

namely $f(\cdot)$ *belongs to* $\mathcal{F}^{++}$ *as in* (2.12), *with a* $2d$ *degree polynomial* $g(\cdot) > 0$ *a.e. having* $g(a) = 0 = g(b)$ *and satisfying* (2.8))
whenever

$$(2.26) \quad has,\ for\ some\ i \in \{1, 2, \ldots\}, \quad M_f^{ii} \cdot M_f^{00} \leq 0 \tag{2.30}$$

$$(or,\ more\ generally, \quad M_f^{ii} \cdot M_f^{00} \leq (M_f^{0i})^2) \tag{2.31}$$

and, e.g., the following inequalities hold true: (*see* (2.2) *and* (2.5))

$$-ess\ inf\ [v \cdot (1 - \overline{\lambda})]([a, b]) < 0 < ess\ inf\ [v \cdot \overline{\lambda}]([a, b]). \tag{2.32}$$

Remark 2.6. See (3.52) for a still more general replacement for (2.31).

Let us clarify now conditions under which (2.31) holds true. To begin with, using the mean-value

$$m_f := \frac{1}{b-a} \int_a^b f(t)dt$$

to define the classes

$$\mathcal{F}_r^i := \{f(\cdot) \in \mathcal{F} : \ m_f = 0 \quad and \quad M_f^{ii} = r\}, \tag{2.33}$$

clearly the special case $M_f^{00} = 0$ of (2.31) yields the inclusion

$$\mathcal{F}_0^0 \subset \mathcal{F}^{++}. \tag{2.34}$$

Next we show that

$$f(\cdot) \in \mathcal{F} \Leftrightarrow \exists \widetilde{f}(\cdot) \in \mathcal{F}_0^0 \cup \mathcal{F}_1^0 \quad \exists q_f \in \mathbb{R} \setminus \{0\} : \\ f(\cdot) = m_f + q_f \cdot \widetilde{f}(\cdot); \tag{2.35}$$

and a useful simplification follows: since, obviously,

$$\int_a^b f(t)g'(t)dt = 0 \;\Leftrightarrow\; \int_a^b \widetilde{f}(t)g'(t)dt = 0, \quad \forall g(\cdot) \in \mathcal{G}_f,$$

then, for the purpose of guaranteeing existence of such $g(\cdot)$, one may, instead of the whole $\mathcal{F}$, focus attention on $\mathcal{F}_1^0$ only, due to (2.34).

To prove (2.35), just set

$$\widetilde{f}(\cdot) := \tfrac{f(\cdot)-m_f}{q_f}, \quad q_f := \begin{cases} 1, & \text{if } \; M_f^{00} = 0 \\ M_f^{00}, & \text{if } \; M_f^{00} \neq 0, \end{cases}$$

to obtain $m_{\widetilde{f}} = 0$ and $M_{\widetilde{f}}^{00} \in \{0, 1\}$ hence $\widetilde{f}(\cdot) \in \mathcal{F}_0^0 \cup \mathcal{F}_1^0$.

As shown in Sect. 4 below, one may always, by subtracting an affine function to any $f(\cdot) \in \mathcal{F}$, reduce it to a function

$$\widetilde{f}_0(\cdot) \in \mathcal{F}_0^0 \subset \mathcal{F}^{++} \quad \text{or} \quad \widetilde{f}_1(\cdot) \in \mathcal{F}_1^0 \quad \text{or} \quad \widetilde{f}_{-1}(\cdot) \in \mathcal{F}_{-1}^0. \tag{2.36}$$

Simple examples of polynomial functions not satisfying (2.30) but for which (2.31) holds true, with, e.g., $i = 1$, are also presented in Sect. 4; together with an example in which (2.31) is unapplicable and should be replaced by the more general (3.52).

Remark 2.7. On the other hand, while the inequalities (2.32) do suffice for Theorem 2.5, they may be judged too heavy. Indeed, what is really needed is

$$-[v \cdot (1 - \overline{\lambda})](t) \leq g'(t) \leq [v \cdot \overline{\lambda}](t) \quad \text{for a.e. } t \in [a, b] \tag{2.37}$$

(see (2.8)) or, more precisely, the possibility of finding a nonconstant polynomial $\varphi(\cdot)$ of degree $d - 1$ such that, setting

$$g(t) := h(t) \cdot \varphi(t) \cdot \varphi(t), \quad \text{with} \quad h(t) := (b - t)(t - a), \tag{2.38}$$

then $g'(\cdot)$ satisfies (2.37) and the nonzero vector α of Taylor coefficients of $\varphi(\cdot)$ (as in (3.50) below) solves the bilinear matricial equation

$$\alpha^T \cdot M_f^d \cdot \alpha = 0. \tag{2.39}$$

At this point it becomes natural to define, using (2.1), (2.2) and (2.12), the new class

$$\mathcal{F}_<^{++} := \{f(\cdot) \in \mathcal{F}^{++} : \; (2.32) \text{ or } (2.37) \text{ holds true}\} \tag{2.40}$$

obtaining, due to (2.6)–(2.8) and (1.18),

$$\mathcal{F}_<^{++} \neq \emptyset \;\Leftrightarrow\; \overline{\mathcal{X}}_{--}^{co} \neq \emptyset \quad (\text{with } (2.3)) \;\Rightarrow\; \overline{\mathcal{X}}_{-}^{true} \neq \emptyset. \tag{2.41}$$

3. Proofs

Proof of Theorem 1.1. Define the following measurable subsets of $[a, b]$,

$$\begin{aligned}
&E_+^0, E_+^1, \ldots, E_+^m \quad \text{and} \quad E_-^0, E_-^1, \ldots, E_-^m, \quad \text{by}: \\
&E_+^0 := E_-^0 := \emptyset; \quad \text{and,} \quad \text{for} \quad j = 1, 2, \ldots, m, \\
&\overline{E}_+^{j-1} := [a, b] \setminus E_+^0 \setminus E_+^1 \setminus \cdots \setminus E_+^{j-1}, \\
&\overline{E}_-^{j-1} := [a, b] \setminus E_-^0 \setminus E_-^1 \setminus \cdots \setminus E_-^{j-1}, \\
&E_+^j := \{t \in \overline{E}_+^{j-1} : \ \langle v^j(t), \omega\rangle \geq \langle v^k(t), \omega\rangle \ \forall k \in \{1, \ldots, m\}\}, \\
&E_-^j := \{t \in \overline{E}_-^{j-1} : \ \langle v^j(t), \omega\rangle \leq \langle v^k(t), \omega\rangle \ \forall k \in \{1, \ldots, m\}\}.
\end{aligned}$$

Using these sets E_+^j and E_-^j, and their *characteristic* functions $\mathcal{X}_{E_+^j}(\cdot)$, $\mathcal{X}_{E_-^j}(\cdot)$, define the functions

$$v_\omega^{\max}(t) := \sum_{j=1}^m \mathcal{X}_{E_+^j}(t) v^j(t), \quad v_\omega^{\min}(t) := \sum_{j=1}^m \mathcal{X}_{E_-^j}(t) v^j(t) \tag{3.1}$$

so that, by construction,

$$h(t) := \langle v_\omega^{\max}(t) - v_\omega^{\min}(t), \omega\rangle \geq 0 \quad \forall t \in [a, b]. \tag{3.2}$$

Indeed, more precisely, for any $j \in \{1, \ldots, m\}$ and any $t \in [a, b]$,

$$\begin{aligned}
\langle v_\omega^{\min}(t), \omega\rangle &= \min\{\langle v^1(t), \omega\rangle, \ldots, \langle v^m(t), \omega\rangle\} \leq \langle v^j(t), \omega\rangle \\
&\leq \max\{\langle v^1(t), \omega\rangle, \ldots, \langle v^m(t), \omega\rangle\} = \langle v_\omega^{\max}(t), \omega\rangle,
\end{aligned}$$

in particular (recall (1.13), (1.11) and (1.10))

$$\langle v_\omega^{\min}(t), \omega\rangle \leq \langle x'(t), \omega\rangle \leq \langle v_\omega^{\max}(t), \omega\rangle \quad \text{a.e.} \quad \forall x(\cdot) \in \overline{\mathcal{X}}_\omega^{co};$$

so that certainly

$$\exists \lambda(\cdot) \in \Lambda := \{\lambda(\cdot) \in L^\infty(a, b) : \ \lambda(t) \in [0, 1] \text{ a.e.}\} \tag{3.3}$$

for which one may write

$$\langle x'(t), \omega\rangle = \langle (1 - \lambda)(t) v_\omega^{\min}(t) + \lambda(t) v_\omega^{\max}(t), \omega\rangle \text{ a.e..} \tag{3.4}$$

Our first aim is to define, for $t \in [a, b]$,

$$x_\omega^{\min}(t) := \overline{x}(a) + \int_a^t (1 - \mathcal{X}_{E_-})(\tau) v_\omega^{\min}(\tau) + \mathcal{X}_{E_-}(\tau) v_\omega^{\max}(\tau) d\tau \tag{3.5}$$

where E_- is the subset of $[a, b]$ to be constructed as follows.

To begin with define, for $t \in [a, b]$, using (3.2), the function

$$H(t) := \int_t^b h(\tau) d\tau - \int_a^b h(\tau)\lambda(\tau) d\tau, \tag{3.6}$$

obtaining, by (3.6),

$$H(b) = -\int_a^b h(\tau)\lambda(\tau) d\tau \leq 0 \leq H(a) = \int_a^b h(\tau)(1 - \lambda)(\tau) d\tau$$

hence $0 \in co\{H(a), H(b)\} \subset H([a, b])$ so that, setting

$$E_- := [c_-, b], \quad \text{with} \quad c_- := \max\{t \in [a, b] : \ H(t) = 0\},$$

one gets

$$\int_{c_-}^{b} h(\tau)d\tau = \int_a^b h(\tau)\mathcal{X}_{E_-}(\tau)d\tau = \int_a^b h(\tau)\lambda(\tau)d\tau. \tag{3.7}$$

We claim that, for $t \in [a,b]$, with equality at $t = b$,

$$\int_a^t h(\tau)\mathcal{X}_{E_-}(\tau)d\tau \le \int_a^t h(\tau)\lambda(\tau)d\tau. \tag{3.8}$$

Indeed, the equality at $t = b$ appears in (3.7); while for $t \in [a, c_-]$

$$\int_a^t h(\tau)\mathcal{X}_{E_-}(\tau)d\tau = 0 \le \int_a^t h(\tau)\lambda(\tau)d\tau;$$

and, to complete the proof of (3.8), for $t \in [c_-, b]$ it is

$$\begin{aligned}\int_a^t h(\tau)\mathcal{X}_{E_-}(\tau)d\tau &= \int_{c_-}^t h(\tau)d\tau = \int_a^b h(\tau)\lambda(\tau)d\tau - \int_t^b h(\tau)d\tau \\ &\le \int_a^b h(\tau)\lambda(\tau)d\tau - \int_t^b h(\tau)\lambda(\tau)d\tau = \int_a^t h(\tau)\lambda(\tau)d\tau.\end{aligned}$$

By (3.5), (3.2), (3.8) and (3.4), $x_\omega^{\min}(a) = \overline{x}(a)$ and

$$\begin{aligned}\langle x_\omega^{\min}(t), \omega\rangle &= \langle \overline{x}(a), \omega\rangle + \int_a^t \langle v_\omega^{\min}(\tau), \omega\rangle d\tau + \int_a^t h(\tau)\mathcal{X}_{E_-}(\tau)d\tau \\ &\le \langle \overline{x}(a), \omega\rangle + \int_a^t \langle v_\omega^{\min}(\tau), \omega\rangle d\tau + \int_a^t h(\tau)\lambda(\tau)d\tau \\ &= \langle x(a), \omega\rangle + \int_a^t \langle x'(\tau), \omega\rangle d\tau = \langle x(t), \omega\rangle,\end{aligned}$$

with equality at $t = b$, thus proving (1.13) for $x_\omega^{\min}(\cdot)$.

Similarly one may construct $E_+ \subset [a,b]$ such that defining

$$x_\omega^{\max}(t) := \overline{x}(a) + \int_a^t (1 - \mathcal{X}_{E_+})(\tau)v_\omega^{\min}(\tau) + \mathcal{X}_{E_+}(\tau)v_\omega^{\max}(\tau)d\tau \tag{3.9}$$

then $x_\omega^{\max}(\cdot) \in \overline{\mathcal{X}}_\omega^{true}$ and $\langle x(t), \omega\rangle \le \langle x_\omega^{\max}(t), \omega\rangle$ $\forall t \in [a,b]$. □

Proof of Theorems 1.2, 1.4 and 1.5. (a) To simplify notation assume

$$\omega = (1, 0, \ldots, 0) \quad \text{in} \ \ (1.8); \tag{3.10}$$

hence define, as in (3.1), (3.5) and (3.9) but for this special ω,

$$v^{\min}(t) := \sum_{j=1}^m \mathcal{X}_{E_-^j}(t)v^j(t), \quad v^{\max}(t) := \sum_{j=1}^m \mathcal{X}_{E_+^j}(t)v^j(t) \tag{3.11}$$

$$x^{\min}(t) := \overline{x}(a) + \int_a^t (1 - \mathcal{X}_{E_-})(\tau)v^{\min}(\tau) + \mathcal{X}_{E_-}(\tau)v^{\max}(\tau)d\tau \tag{3.12}$$

$$x^{\max}(t) := \overline{x}(a) + \int_a^t (1 - \mathcal{X}_{E_+})(\tau)\ v^{\min}(\tau) + \mathcal{X}_{E_+}(\tau)v^{\max}(\tau)d\tau. \tag{3.13}$$

Considering the situation of (1.21) and setting

$$\overline{\mathcal{X}} := \{x(\cdot) \in \mathcal{X} :\ x(a) = \overline{x}(a) \quad \text{and} \quad x(b) = \overline{x}(b)\}, \tag{3.14}$$

then (1.6)–(1.8), (1.16), (1.21), (1.9) and (1.13) yield, respectively,

$$\begin{aligned} \overline{\mathcal{X}}^{co} &= \{x(\cdot) \in \overline{\mathcal{X}} :\ x'(t) \in co\{v^1(t), \ldots, v^m(t)\} \text{ a.e.}\} \\ \overline{\mathcal{X}}^{true} &= \{x(\cdot) \in \overline{\mathcal{X}} :\ x'(t) \in \{v^1(t), \ldots, v^m(t)\} \text{ a.e.}\} \end{aligned} \tag{3.15}$$

$$\begin{aligned} \overline{\mathcal{X}}^{co}_{-} &= \{x(\cdot) \in \overline{\mathcal{X}}^{co} :\ x_1(t) \le \overline{x}_1(t)\ \forall t \in [a,b]\} \\ \overline{\mathcal{X}}^{co}_{--} &= \{x(\cdot) \in \overline{\mathcal{X}}^{co}_{-} :\ x_1(t) < \overline{x}_1(t) \text{ for a.e. } t \in \mathcal{O}_x\} \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\mathcal{O}_x \supset \{t \in (a,b) :\ x_1^{\min}(t) < x_1(t) < x_1^{\max}(t)\} \quad \text{for } x(\cdot) \in \overline{\mathcal{X}}^{co}_{-} \\ &\overline{\mathcal{X}}^{true}_{-} = \{x(\cdot) \in \overline{\mathcal{X}}^{true} :\ x_1(t) \le \overline{x}_1(t)\ \forall t \in [a,b]\} \end{aligned} \tag{3.17}$$

$$x_1^{\min}(t) \le x_1(t) \le x_1^{\max}(t) \quad \forall t \in [a,b] \quad \forall x(\cdot) \in \overline{\mathcal{X}}^{co}; \tag{3.18}$$

while our aim (1.24) becomes

$$(\widehat{x}^k(\cdot)) \to \overline{x}(\cdot) \quad \text{and} \quad (\widehat{x}^k_1(\cdot)) \nearrow \overline{x}_1(\cdot), \quad uniformly.$$

Notice also that, using (3.3) to set

$$\Lambda_m := \left\{ (\lambda^1, \ldots, \lambda^m)(\cdot) \in \Lambda^m :\ \sum_{j=1}^{m} \lambda^j(t) = 1 \text{ a.e.} \right\},$$
$$\text{fixing any } \widetilde{x}(\cdot) \in \overline{\mathcal{X}}^{co}_{--} \quad \text{and} \quad \text{setting } \mathcal{O} := \mathcal{O}_{\widetilde{x}}, \tag{3.19}$$

then clearly

$$\exists (\widetilde{\lambda}^1, \ldots, \widetilde{\lambda}^m)(\cdot) \in \Lambda_m :\ \widetilde{x}'(t) = \sum_{j=1}^{m} \widetilde{\lambda}^j(t) v^j(t) \text{ a.e.}, \tag{3.20}$$

$$\text{hence} \quad \widetilde{x}_1'(t) = \sum_{j=1}^{m} \widetilde{\lambda}^j(t) v_1^j(t) \quad \text{for a.e. } t \in [a,b].$$

Moreover, (3.19) and (3.18) for $\widetilde{x}(\cdot)$ yield, on $[a,b] \setminus \mathcal{O}$,

$$\widetilde{x}_1(t) \in \{x_1^{\min}(t), x_1^{\max}(t)\}$$

so that, by (3.12), (3.13) and (3.11),

$$\begin{aligned} \widetilde{x}_1'(t) &\in \{x_1^{\min\prime}(t), x_1^{\max\prime}(t)\} \\ &\subset \{v_1^1(t), \ldots, v_1^m(t)\} \text{ a.e. on } [a,b] \setminus \mathcal{O}; \end{aligned}$$

which implies, by (3.20), (1.22) and (1.23),

$$\widetilde{x}'(t) \in \{v^1(t), \ldots, v^m(t)\} \quad \text{for a.e. } t \in [a,b] \setminus \mathcal{O}, \tag{3.21}$$

whenever $\mathcal{O}$ is as in (3.19).

Thus, in order to reach a $\widehat{x}(\cdot) \in \overline{\mathcal{X}}^{true}_{-}$, as intended in (1.18) to prove Theorem 1.2 using (1.21), it is enough to define

$$\widehat{x}(t) := \widetilde{x}(t) \quad \text{at each} \quad t \in [a,b] \setminus \mathcal{O} \tag{3.22}$$

to get at once, by (3.21),

$$\begin{aligned} &\widehat{x}'(t) \in \{v^1(t), \ldots, v^m(t)\} \quad \text{for a.e. } t \in [a,b] \setminus \mathcal{O}, \\ &\widehat{x}_1(t) \leq \overline{x}_1(t) \ \forall t \in [a,b] \setminus \mathcal{O}; \end{aligned} \tag{3.23}$$

so that it suffices then (recalling (3.15) and (3.17))

$$\begin{aligned} &\textit{to extend such } \widehat{x}(\cdot) \textit{ into } \mathcal{O} \quad \text{so as to satisfy :} \\ &\widehat{x}'(t) \in \{v^1(t), \ldots, v^m(t)\} \quad \text{for a.e. } t \in \mathcal{O}, \\ &\widehat{x}_1(t) \leq \overline{x}_1(t) \ \forall t \in \mathcal{O}, \end{aligned} \tag{3.24}$$

using the hypothesis in the *rhs* of (1.18), namely, by (3.19) and (3.16),

$$\widetilde{x}_1(t) < \overline{x}_1(t) \quad \text{for a.e. } t \in \mathcal{O}. \tag{3.25}$$

On the other hand, using, in (3.22), $\mathcal{O} := \mathcal{O}_{\widetilde{x}}$ defined as in (1.17) or (1.19), obviously again (3.23) holds true hence again we just need (3.24) using (3.25).

Moreover recalling that (1.14) now yields, by (3.11),

$$\mathcal{X}_1^{ext} := \{x(\cdot) \in \mathcal{X} : \ x'(t) \in \{v^{\min}(t), v^{\max}(t)\} \text{ a.e.}\},$$

if one wishes to use, instead of (1.19), the definition (1.20) for $\mathcal{O}$, namely

$$\mathcal{O} \supset \{t \in (a,b) : \ \widetilde{x}(t) \neq \check{x}(t)\}, \quad \text{for some } \check{x}(\cdot) \in \mathcal{X}_1^{ext},$$

then we clearly have

$$\widetilde{x}'(t) = \check{x}'(t) \in \{v^1(t), \ldots, v^m(t)\} \quad \text{for a.e. } t \in [a,b] \setminus \mathcal{O}$$

so that, using (3.22), again (3.23) leads to (3.24) using (3.25).

(b) This part is dedicated to prove, at last, that

$$(3.25) \quad \text{allows} \quad (3.24). \tag{3.26}$$

To simplify notation we assume here, without loss of generality,

$$\mathcal{O} = (a,b) \quad \text{in (3.25)}. \tag{3.27}$$

Indeed, once (3.24) is performed for this special case $\mathcal{O} = (a,b)$, then the general case of *any open set* $\mathcal{O}$ can be similarly solved: just apply the same recipe to each one of the *maximal intervals* (a_i, b_i) that form $\mathcal{O}$.

Let us begin the proof of (3.26) under (3.27). As in (3.20),

$$\exists (\overline{\lambda}^1, \ldots, \overline{\lambda}^m)(\cdot) \in \Lambda_m : \ \overline{x}'(t) = \sum_{j=1}^{m} \overline{\lambda}^j(t) v^j(t) \text{ a.e..}$$

Recalling (3.2) and (3.10) then clearly, for any $t \in [a,b]$,

$$f(t) := v^{\max}(t) - v^{\min}(t) \quad \text{yields} \quad g(t) := f_1(t) \geq 0; \tag{3.28}$$

and using such $g(\cdot)$ we define the *linear* functional

$$G : \Lambda \to \mathbb{R}, \quad G(\lambda(\cdot)) := \int_a^b g(t)\lambda(t)dt \quad \text{(see (3.3))}. \tag{3.29}$$

By (3.3), (3.4) and (3.28),

$$\exists\overline{\lambda}(\cdot)\in\Lambda:\ \overline{x}_1'(t)=v_1^{\min}(t)+\overline{\lambda}(t)g(t)\quad\text{for a.e. } t\in[a,b];\tag{3.30}$$

and, using (3.29), to each $x(\cdot)\in\overline{\mathcal{X}}^{co}$ there corresponds a *unique*

$$\lambda(\cdot)\in\overline{\Lambda}:=\{\lambda(\cdot)\in\Lambda:\ G(\lambda(\cdot))=G(\overline{\lambda}(\cdot))\}.\tag{3.31}$$

Indeed, similarly to (3.30), again (3.4) yields a $\lambda(\cdot)\in\Lambda$ for which

$$x_1'(t)=v_1^{\min}(t)+\lambda(t)g(t)\quad\text{for a.e. } t\in[a,b],\tag{3.32}$$

such $\lambda(\cdot)$ is in $\overline{\Lambda}$, by (3.31), (3.29) and (3.14), and we set $\lambda(\cdot):=1$ where $g(\cdot)=0$. Hence one may define the *linear* function

$$L:\overline{\mathcal{X}}^{co}\to\overline{\Lambda},\quad L(x(\cdot)):=\lambda(\cdot)\quad\text{satisfying (3.32)}.\tag{3.33}$$

Then, setting $\widetilde{\lambda}(\cdot):=L(\widetilde{x}(\cdot))$ and considering the *open set*

$$\widetilde{\mathcal{O}}:=\{t\in(a,b):\ \widetilde{x}_1(t)<\overline{x}_1(t)\}\quad\text{(see (3.16))},\tag{3.34}$$

$$\text{we have}\quad |\widetilde{\mathcal{O}}|=b-a,\quad\text{i.e.}\tag{3.35}$$

$$\int_a^t g(\tau)\widetilde{\lambda}(\tau)d\tau<\int_a^t g(\tau)\overline{\lambda}(\tau)d\tau\quad\text{a.e. in } (a,b).$$

In order to fulfil our aim (3.24) in (3.26), we proceed (recalling (1.1) and (1.9)) to construct a $\widehat{x}(\cdot)\in\overline{\mathcal{X}}_-^{true}$, namely a

$$\begin{aligned}&\widehat{x}(\cdot)\in\mathcal{X}:\ \widehat{x}(a)=\overline{x}(a)\quad\text{and}\quad\widehat{x}(b)=\overline{x}(b)\quad\text{and}\\&\widehat{x}'(t)\in\{v^1(t),\dots,v^m(t)\}\quad\text{and}\quad\widehat{x}_1(t)<\overline{x}_1(t),\ \text{a.e. in } (a,b).\end{aligned}\tag{3.36}$$

To further simplify notation (beyond (3.27)), again without loss of generality we assume

$$\widetilde{\mathcal{O}}=(a,b)\quad\text{in (3.35) and (3.36)}.\tag{3.37}$$

Indeed, once $\widehat{x}(\cdot)$ has been constructed along (a,b), assuming (3.37), to treat a general *open set* $\widetilde{\mathcal{O}}$ satisfying (3.36) one starts by defining

$$\widehat{x}(\cdot):=\widetilde{x}(\cdot)\quad\text{on the \textit{null set} } [a,b]\setminus\widetilde{\mathcal{O}};$$

and then such construction on (a,b) can easily be adapted to similarly construct $\widehat{x}(\cdot)$ on each one of the *countably many maximal intervals* (a_i,b_i) that form $\widetilde{\mathcal{O}}$.

To construct $\widehat{x}(\cdot)$, we begin by partitioning the *open interval* (a,b) where, by (3.37), the strict inequality in (3.35) holds, namely

$$\widetilde{x}_1(t)<\overline{x}_1(t)\quad\forall t\in(a,b),\tag{3.38}$$

into *countably many subintervals by points*

$$a_1:=b_1:=\frac{a+b}{2}\quad\text{and,}\quad\text{for } k=1,2,\dots,$$
$$a_{k+1}:=\frac{a+a_k}{2}\quad\text{and}\quad b_{k+1}:=\frac{b_k+b}{2},$$

thus obtaining *strictly monotone convergent* sequences

$$\begin{aligned} &(a_k) \searrow a \quad \text{and} \quad (b_k) \nearrow b, \quad \text{with} \\ &\bigcup_{k=1}^{\infty} [a_{k+1}, a_k) = \left(a, \frac{a+b}{2}\right), \quad \bigcup_{k=1}^{\infty} (b_k, b_{k+1}] = \left(\frac{a+b}{2}, b\right). \end{aligned} \tag{3.39}$$

We claim first that

$$\begin{aligned} &\exists x^1(\cdot) \in \mathcal{X} : \; x^1(\cdot) := \widetilde{x}(\cdot) \text{ on } [a,b] \setminus (b_1, b_2), \\ &x^{1\prime}(t) \in \{v^1(t), \ldots, v^m(t)\} \text{ a.e. in } (b_1, b_2), \\ &x_1^1(t) < \overline{x}_1(t) \;\forall t \in (a,b). \end{aligned} \tag{3.40}$$

Indeed, in order to construct such $x^1(\cdot)$ we start by defining $x^1(\cdot)$ as in the first line of (3.40), then partition the subinterval $[b_1, b_2]$ into N *equal-length* subsubintervals by points

$$b_1^\ell := b_1 + \ell \frac{b_2 - b_1}{N}, \quad \ell = 0, 1, \ldots, N,$$

where $N \in \mathbb{N}$ is chosen *large enough* so as to get, by (3.28) and (3.38),

$$\begin{aligned} &0 \le \int_I g(t)dt < \min \; (\overline{x}_1 - \widetilde{x}_1)([b_1, b_2]) \\ &\forall I \subset [b_1, b_2] : \; |I| \le \frac{b_2 - b_1}{N}; \end{aligned} \tag{3.41}$$

and we claim that, for $\ell = 1, 2, \ldots, N$,

$$\begin{aligned} &\exists x^{1\ell}(\cdot) \in \mathcal{X} : \; x^{1\ell}(\cdot) := \widetilde{x}(\cdot) \text{ on } [a,b] \setminus (b_1^{\ell-1}, b_1^\ell), \\ &x^{1\ell\prime}(t) \in \{v^1(t), \ldots, v^m(t)\} \text{ a.e. in } (b_1^{\ell-1}, b_1^\ell), \\ &x_1^{1\ell}(t) < \overline{x}_1(t) \;\forall t \in (a,b). \end{aligned} \tag{3.42}$$

Indeed, consider first the case $\ell = 1$. In order to construct such $x^{11}(\cdot)$, begin by defining it as in the first line of (3.42) for $\ell = 1$; then apply the *Liapunov* theorem (see, e.g., [15, 16.1.v]) to obtain measurable sets $E_{11}^1, \ldots, E_{11}^m \subset [b_1^0, b_1^1]$, with *characteristic* functions $\lambda_{11}^1(\cdot) := \mathcal{X}_{E_{11}^1}(\cdot), \ldots, \lambda_{11}^m(\cdot) := \mathcal{X}_{E_{11}^m}(\cdot)$, satisfying (see (3.20))

$$\int_{b_1^0}^{b_1^1} \sum_{j=1}^m \lambda_{11}^j(t) v^j(t) dt = \int_{b_1^0}^{b_1^1} \sum_{j=1}^m \widetilde{\lambda}^j(t) v^j(t) dt;$$

and finally set, on $[b_1^0, b_1^1]$,

$$x^{11}(t) := \widetilde{x}(b_1^0) + \int_{b_1^0}^t \sum_{j=1}^m \lambda_{11}^j(\tau) v^j(\tau) d\tau.$$

Using (3.33), let $\lambda^{11}(\cdot) := L(x^{11}(\cdot))$. Then, by (3.28) and (3.41), for $t \in [b_1^0, b_1^1]$, clearly $x_1^{11}(t) - \overline{x}_1(t)$ equals

$$\int_{b_1^0}^t g(\tau)\lambda^{11}(\tau)d\tau + \int_a^{b_1^0} g(\tau)\widetilde{\lambda}(\tau)d\tau - \int_a^t g(\tau)\overline{\lambda}(\tau)d\tau$$
$$< \min(\overline{x}_1 - \widetilde{x}_1)([b_1, b_2]) - \int_a^{b_1^0} g(\tau)(\overline{\lambda} - \widetilde{\lambda})(\tau)d\tau \leq 0,$$
$$\text{i.e.} \quad x_1^{11}(t) < \overline{x}_1(t) \quad \forall t \in [b_1^0, b_1^1].$$

We have thus proved the special case $\ell = 1$ of (3.42), as one easily checks. For $\ell = 2, \ldots, N$ the proof is similar, thus completing the proof of (3.42).

We claim now that, for $k = 1, 2, \ldots,$

$$\begin{aligned} &\exists x^k(\cdot) \in \mathcal{X} : \ x^k(\cdot) := \widetilde{x}(\cdot) \quad \text{on } [a,b] \setminus (b_k, b_{k+1}), \\ &x^{k\prime}(t) \in \{v^1(t), \ldots, v^m(t)\} \text{ a.e. in } (b_k, b_{k+1}), \\ &x_1^k(t) < \overline{x}_1(t) \ \forall t \in (a,b). \end{aligned} \tag{3.43}$$

Indeed, for $k = 1$, after having defined $x^1(\cdot)$ as in the first line of (3.40), it suffices to define

$$x^1(t) := x^{1\ell}(t) \quad \text{for } t \in [b_1^{\ell-1}, b_1^\ell], \quad \ell = 1, 2, \ldots, N,$$

and use (3.42); while for $k = 2, 3, \ldots$ the reasoning is similar. The special case $k = 1$ of (3.43) is just our initial (3.40), which is thus also proved.

We claim now that, for $k = 1, 2, \ldots,$

$$\begin{aligned} &\exists x^{-k}(\cdot) \in \mathcal{X} : \ x^{-k}(\cdot) := \widetilde{x}(\cdot) \quad \text{on } [a,b] \setminus (a_{k+1}, a_k), \\ &x^{-k\prime}(t) \in \{v^1(t), \ldots, v^m(t)\} \quad \text{a.e. in } (a_{k+1}, a_k), \\ &x_1^{-k}(t) < \overline{x}_1(t) \ \forall t \in \ (a,b). \end{aligned} \tag{3.44}$$

Indeed, the strategy is the same as to prove (3.43), after proving a statement similar to (3.42) but adapted to $(a_1^\ell, a_1^{\ell-1})$ instead of $(b_1^{\ell-1}, b_1^\ell)$.

We have, at last, all our building blocks ready; so that, in order to complete the proof of (3.36), it suffices now to glue together the functions in (3.43) and (3.44) by:

$$\widehat{x}(t) := \begin{cases} x^k(t) & \text{for } t \in [b_k, b_{k+1}], \quad k = 1, 2, \ldots \\ x^{-k}(t) & \text{for } t \in [a_{k+1}, a_k], \quad k = 1, 2, \ldots, \end{cases}$$

as one easily checks using (3.39).

Thus we have proved (3.36), hence (3.26), i.e. (1.18), including the second paragraph of Remark 1.3.

(c) Finally, in order to prove (1.24) it suffices to consider the sequence

$$\widetilde{x}^k(\cdot) := \left(1 - \frac{1}{k}\right)\overline{x}(\cdot) + \frac{1}{k}\widetilde{x}(\cdot).$$

Indeed, taking the above construction for $\widetilde{x}^1(\cdot) = \widetilde{x}(\cdot)$ and adapting it to

$$\widetilde{x}^k(\cdot) \quad \text{for } k = 2, 3, \ldots,$$

one obtains, as is easily checked using (1.22) and (1.23), a sequence

$$(\widehat{x}^k(\cdot)) \to \overline{x}(\cdot), \quad \text{with} \quad (\widehat{x}_1^k(\cdot)) \nearrow \overline{x}_1(\cdot), \ uniformly,$$

thus proving (1.24).

One may similarly prove Theorem 1.5. □

Proof of Theorem 2.1. Take any

$$\widetilde{x}(\cdot) \in \overline{\mathcal{X}}_-^{co} \setminus \{\overline{x}(\cdot)\} \text{ (see (1.8)), } \quad \text{so that} \tag{3.45}$$

$$\begin{aligned} &\exists \widetilde{\lambda}(\cdot) \in L^\infty(a,b) : \ \widetilde{\lambda}(t) \in [0,1] \quad \text{a.e.} \quad \text{and} \\ &\widetilde{x}'(t) = v^1(t) + \widetilde{\lambda}(t) v(t) \cdot (1, f(t)) \text{ a.e.,} \end{aligned} \tag{3.46}$$

subtract $\widetilde{\lambda}(t)v(t) = \widetilde{x}_1'(t) - v_1^1(t)$ from $\overline{\lambda}(t)v(t) = \overline{x}_1'(t) - v_1^1(t)$ to get

$$g(t) := \int_a^t (\overline{x}_1' - \widetilde{x}_1')(\tau) d\tau = \int_a^t (\overline{\lambda} - \widetilde{\lambda})(\tau) v(\tau) d\tau; \tag{3.47}$$

then subtract $\widetilde{x}_2'(t) = v_2^1(t) - f(t)v_1^1(t) + f(t)\widetilde{x}_1'(t)$ from $\overline{x}_2'(t) = v_2^1(t) - f(t)v_1^1(t) + f(t)\overline{x}_1'(t)$ and integrate from a to b to reach (2.7) and (2.8), since $\widetilde{\lambda}(t) \in [0,1]$; thus proving (2.6).

Finally, (2.9) and (2.10) are proved in [11]. □

Proof of Theorem 2.5. As one easily checks, using (2.28),

$$f(t)g'(t) = \frac{d}{dt} \int f(t)g'(t) = \frac{d}{dt} \int f^{(-1)\prime}(t) g'(t)$$

hence

$$f(t)g'(t) = \frac{d}{dt} \sum_{k=0}^{\infty} (-1)^k f^{(-k-1)}(t) g^{(k+1)}(t)$$

$$\int_a^b f(t)g'(t)dt = \sum_{k=0}^{\infty} (-1)^k f^{(-k-1)}(b) g^{(k+1)}(b).$$

Looking, in particular, for a polynomial $g(\cdot)$ of the form (2.38) one gets

$$\begin{aligned} g^{(k+1)}(t) &= h(t)(\varphi\ \varphi)^{(k+1)}(t) \\ &\quad + (k+1)h'(t)(\varphi\ \varphi)^{(k)}(t) + k\frac{k+1}{2} h''(t)(\varphi\ \varphi)^{(k-1)}(t) \\ g^{(k+1)}(b) &= -(b-a)(k+1)(\varphi\ \varphi)^{(k)}(b) - k(k+1)(\varphi\ \varphi)^{(k-1)}(b) \end{aligned}$$

$$\begin{aligned} \int_a^b f(t)g'(t)dt &= \sum_{k=0}^{\infty} (-1)^{k+1}(k+1)(b-a) f^{(-k-1)}(b)(\varphi\ \varphi)^{(k)}(b) \\ &\quad + \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \ell(\ell+1) f^{(-\ell-1)}(b)(\varphi\ \varphi)^{(\ell-1)}(b) \end{aligned} \tag{3.48}$$

$$\int_a^b f(t)g'(t)dt = \sum_{i=0}^{\infty} \alpha_i \sum_{j=0}^{\infty} \alpha_j M_f^{ij}, \tag{3.49}$$

using (2.27) and the general polynomial

$$\varphi(t) := \sum_{k=0}^{d-1} \frac{(t-b)^k}{k!} \alpha_k, \text{ where } \alpha_k := \varphi^{(k)}(b) \text{ and } \alpha_0 := \varphi(b), \tag{3.50}$$

so that one may insert in (3.49) the iterated derivatives

$$(\varphi\ \varphi)^{(k)}(b) = \sum_{\substack{i,j=0 \\ i+j=k}}^{d-1} \frac{k!}{i!\,j!} \alpha_i\ \alpha_j.$$

But, using (2.26), the *rhs* of (3.49) may also be written as

$$\int_a^b f(t)\ g'(t)\ dt = \alpha^T \cdot M_f^d \cdot \alpha, \tag{3.51}$$

$$\alpha^T \quad \text{the } \textit{transpose} \text{ of } \quad \alpha := (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}).$$

Clearly the matrix M_f^d has d real eigenvalues; and there exists a nontrivial solution to the bilinear matricial equation (2.39) if and only if this matrix M_f^d in

$$(2.26) \quad \text{has } \textit{eigenvalues } \beta_-,\ \beta_+ \quad \text{with} \quad \beta_- \cdot \beta_+ \leq 0, \tag{3.52}$$

as one easily checks.

We now claim that (2.39) has a solution $\alpha \neq 0$ whenever, simply, (2.30) holds. Indeed, setting

$$d := i + 1 \quad \text{and} \quad \alpha = (\alpha_0, \ldots, \alpha_{d-1}) = (\alpha_0, 0, \ldots, 0, \alpha_i),$$

while the case $M_f^{ii} \cdot M_f^{00} = 0$ is trivially solved (with $\alpha_0\ \alpha_i = 0$), in case $M_f^{ii} \cdot M_f^{00} < 0$ one sets

$$\alpha_i := 1 \quad \text{and} \quad \alpha_0 := \frac{\sqrt{(M_f^{0i})^2 - M_f^{ii} M_f^{00}} \quad - M_f^{0i}}{M_f^{00}} \tag{3.53}$$

to obtain, with $\theta := \sqrt{(M_f^{0i})^2 - M_f^{ii}\ M_f^{00}}$,

$$\begin{aligned} \alpha^T \cdot M_f^d \cdot \alpha &= \alpha_0 (M_f^{00} \alpha_0 + 2 M_f^{0i}) + M_f^{ii} \\ &= \frac{\theta - M_f^{0i}}{M_f^{00}} (\theta + M_f^{0i}) + M_f^{ii} = \frac{-M_f^{ii} \cdot M_f^{00}}{M_f^{00}} + M_f^{ii} = 0, \end{aligned}$$

i.e. (2.39), thus proving that

$$(2.30) \ \Rightarrow\ \exists \alpha \neq 0 \quad \text{satisfying} \quad (2.39) \tag{3.54}$$

which, together with (3.52), completes the proof. □

4. Examples of functions $f(\cdot) \in \mathcal{F}^{++}$ in (2.12)

To prove (2.36), we have, using (2.33),

$$f(\cdot) \in \mathcal{F} \;\Rightarrow\; \widetilde{f}_0(\cdot) := f(\cdot) - m_f - M_f^{00} \cdot \psi(\cdot) \in \mathcal{F}_0^0 \subset \mathcal{F}^{++},$$

$$\text{where} \quad \psi(t) := \frac{3!}{(b-a)^3}\left(\frac{a+b}{2} - t\right) \in \mathcal{F}_1^0;$$

$$\widetilde{f}_1(\cdot) := f(\cdot) - m_f - (M_f^{00} - 1) \cdot \psi(\cdot) \in \mathcal{F}_1^0$$

$$\text{and} \quad \widetilde{f}_{-1}(\cdot) := f(\cdot) - m_f - (M_f^{00} + 1) \cdot \psi(\cdot) \in \mathcal{F}_{-1}^0.$$

Indeed clearly $M_1^{ij} = 0$; while

$$M_\psi^{ij} = \frac{(-1)^{i+j}(b-a)^{i+j}}{(i+j+2)(i+j+3)}\frac{3!}{i!j!}$$

$$\text{since} \quad \psi^{(-k)}(t) = 3\frac{(t-a)^k}{(b-a)^3}\frac{(k-1)(b-a)+2(b-t)}{(k+1)!},$$

$$\text{in particular} \quad \left(\frac{(b-a)^i}{(i+1)!}\right)^2 \le M_\psi^{ii} \le \left(\frac{(b-a)^i}{i!}\right)^2,$$

as one easily checks by differentiation.

Notice also that the function $h(\cdot)$ in (2.38) has $M_h^{00} = 0$, since

$$h^{(-k)}(t) = \frac{(t-a)^{k+1}}{(k+1)!}\frac{k(b-a)+2(b-t)}{k+2} \quad \text{and} \quad h^{(-k)}(b) = k\frac{(b-a)^{k+2}}{(k+2)!}.$$

Finally, for $p \in \{2, 3, \dots, 8\}$, with an adequate c_p, the function

$$f_p(t) := -h(t)\frac{(p+1)(p+2)(p+3)}{(p-1)(b-a)^{p+3}}(t-a)^{p-1} - c_p$$

does not satisfy (2.30); while (2.31) holds true yielding, via (3.53), a solution $\alpha^p = (\alpha_0^p, \alpha_1^p) \neq 0$ for (2.39). Thus $f_p(\cdot) \in \mathcal{F}_1^0 \cap \mathcal{F}^{++}$, for those p. Indeed, with $c_p^k(t) := c_p \cdot (t-a)^k \cdot (k!)^{-1}$,

$$f_p^{(-k)}(t) = \frac{-(p+3)!}{(b-a)^{p+3}}\frac{(t-a)^{k+p}}{(k+p)!}\frac{(p+1)(b-t)+k(b-a)}{(p-1)(k+p+1)} - c_p^k(t)$$

$$M_{f_p}^{ij} = \frac{(-1)^{i+j}(i+j+1)!(b-a)^{i+j}(p+3)![p(i+j+1)-1]}{i!j! \qquad (i+j+p+3)! \qquad (p-1)}$$

$$M_{f_p}^{01} = -2\frac{b-a}{p+4}\frac{2p-1}{p-1} \quad \text{and} \quad M_{f_p}^{11} = \frac{6(b-a)^2(3p-1)}{(p-1)(p+4)(p+5)}$$

$$\frac{M_{f_p}^{11}}{(M_{f_p}^{01})^2} = \frac{3}{2}\frac{p-1}{(2p-1)^2}\frac{(p+4)(3p-1)}{p+5} < 1, \quad \text{for } 2 \le p \le 8. \tag{4.1}$$

For $p > 8$ the quotient in (4.1) is > 1 not only for $i = 1$ but even for any i. In particular (2.31) is never satisfied by $f_9(\cdot)$; while surely (3.52) holds true for $f_9(\cdot)$ with $d = 5$, and $f_9(\cdot) \in \mathcal{F}^{++}$ (recall (2.40), (2.41) and (2.12)).

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