Nonlinear Differ. Equ. Appl. 20 (2013), 977–1009

1021-9722/13/030977-33
published online September 13, 2012
DOI 10.1007/s00030-012-0193-y

Nonlinear Differential Equations and Applications NoDEA

Existence of entire solutions for a class of quasilinear elliptic equations

Giuseppina Autuori and Patrizia Pucci

Abstract. The paper deals with the existence of entire solutions for a quasilinear equation $(\mathcal{E})_\lambda$ in $\mathbb{R}^N$, depending on a real parameter λ, which involves a general elliptic operator in divergence form **A** and two main nonlinearities. The competing nonlinear terms combine each other, being the first subcritical and the latter supercritical. We prove the existence of a critical value $\lambda^* > 0$ with the property that $(\mathcal{E})_\lambda$ admits nontrivial non-negative entire solutions if and only if $\lambda \geq \lambda^*$. Furthermore, when $\lambda > \overline{\lambda} \geq \lambda^*$, the existence of a second independent nontrivial non-negative entire solution of $(\mathcal{E})_\lambda$ is proved under a further natural assumption on **A**.

Mathematics Subject Classification (2010). Primary 35J62, 35J70; Secondary 35J20.

Keywords. Quasilinear elliptic equations, Variational methods, Divergence type operators, Weighted Sobolev spaces, Ekeland's principle.

1. Introduction

In [3] Ambrosetti et al. studied the existence and multiplicity of solutions for semilinear elliptic Dirichlet problems in bounded domains, analyzing the combined effects of concave and convex nonlinearities with respect to a real parameter λ. Later, Alama and Tarantello in [2] studied a related semilinear Dirichlet problem in a bounded domain, with weighted nonlinear terms. In [2] also solvability and multiplicity were proved under various assumptions on the weights and on the parameter $\lambda \in \mathbb{R}$. The famous results of [3] were partially extended by De Figueiredo et al. to indefinite nonlinearities for the semilinear case in [10] and for the p-Laplacian operator in [11]. For recent contributions on related semilinear Dirichlet problems in bounded domains we refer to [8,18] and on equations in the entire $\mathbb{R}^N$ to [19], and to the references therein. The equation considered here is in the spirit of the previous papers, even if most of them deal with problems not directly comparable to ours. The present work is more related to the results in [2,20], although in [20] different weights were considered.

We study the one parameter elliptic equation in $\mathbb{R}^N$,

$$-\mathrm{div}\mathbf{A}(x,\nabla u)+a(x)|u|^{p-2}u=\lambda w(x)|u|^{q-2}u-h(x)|u|^{r-2}u \quad (\mathcal{E})_\lambda$$

where $\lambda\in\mathbb{R}$ and $\mathbf{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ *admits a potential* $\mathscr{A}$, *with respect to its second variable* ξ, *satisfying the following assumption*

($\mathcal{A}$) *The potential* $\mathscr{A}=\mathscr{A}(x,\xi)$ *is a continuous function in* $\mathbb{R}^N\times\mathbb{R}^N$, *with continuous derivative with respect to* ξ, $\mathbf{A}=\partial_\xi\mathscr{A}$, *and verifies:*
  - (a) $\mathscr{A}(x,0)=0$ *and* $\mathscr{A}(x,\xi)=\mathscr{A}(x,-\xi)$ *for all* $(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N$;
  - (b) $\mathscr{A}(x,\cdot)$ *is strictly convex in* $\mathbb{R}^N$ *for all* $x\in\mathbb{R}^N$;
  - (c) *There exist constants* $c,C>0$ *and an exponent* p, *with* $1<p<N$, *such that for all* $(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N$

$$c|\xi|^p\le\mathbf{A}(x,\xi)\cdot\xi \quad\text{and}\quad |\mathbf{A}(x,\xi)|\le C|\xi|^{p-1}.$$

Clearly $\mathscr{A}(x,\xi)=|\xi|^p/p$ satisfies ($\mathcal{A}$) for all $p>1$, that is the usual p-Laplacian operator $\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is covered for all $p>1$.

The nonlinear terms in $(\mathcal{E})_\lambda$ *are related to the main elliptic part by* the request that

$$\max\{2,p\}<q<\min\{r,p^*\}, \tag{1.1}$$

where $p^*=Np/(N-p)$ is the critical Sobolev exponent. *The coefficient* a *is supposed to be in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *and to satisfy for a.a.* $x\in\mathbb{R}^N$

$$\nu(x)=\max\{a(x),(1+|x|)^{-p}\},\qquad a(x)\ge c_1\,\nu(x), \tag{1.2}$$

for some constant $c_1\in(0,1]$. *The weight* w *verifies*

$$w\in L^\wp(\mathbb{R}^N)\cap L^\sigma_{\mathrm{loc}}(\mathbb{R}^N),\quad \textit{with } \wp=p^*/(p^*-q) \textit{ and } \sigma>\wp, \tag{1.3}$$

while *h is a positive weight of class* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. *Finally,* h *and* w *are related by the condition*

$$\int_{\mathbb{R}^N}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)}dx=H\in\mathbb{R}^+. \tag{1.4}$$

Assumption (1.4) already appears in [2, condition (1.4) of the existence Theorem 1.1] for positive solutions of semilinear elliptic Dirichlet problems in bounded domains and in [20] with $w=1$ for existence of solutions of quasilinear elliptic equations in $\mathbb{R}^N$. Actually, [20] is the first attempt to establish existence of nontrivial non-negative entire solutions for $(\mathcal{E})_\lambda$ in $\mathbb{R}^N$, when $A(x,\xi)=|\xi|^{p-2}\xi$ and $a=w=1$. Here, we solve the problem under conditions (1.1)–(1.4).

Sections 2–4 of the paper are devoted to the proof of the following main existence

Theorem A. *Under the above hypotheses there exists* $\lambda^*>0$ *such that equation* $(\mathcal{E})_\lambda$ *admits at least a nontrivial non-negative entire solution if and only if* $\lambda\ge\lambda^*$.

By the comments above it is clear that Theorem A extends Theorem 1.1 of [2] to quasilinear elliptic equations in $\mathbb{R}^N$. Theorem 1.2 of [2] is a complete statement when $2 < q < 2^*$ and w satisfies (1.3) for $p = 2$, rather than Theorem 1.1 of [2]. Hence, it still remains an open problem the extension of Theorem 1.2 of [2] to quasilinear elliptic equations in $\mathbb{R}^N$, that is when (1.3) is replaced by the weaker condition $w(w/h)^{(q-p)/(r-q)} \in L^{N/p}(\mathbb{R}^N)$. In any case, Theorem A extends the first part of Theorem 1.2 of [2] under condition (1.4).

In Sect. 5, under a further natural assumption on the potential $\mathscr{A}$, see $(\mathcal{A})$-(d), we prove the second main result in terms of a critical parameter $\overline{\lambda} \geq \lambda^* > 0$.

Theorem B. *For all $\lambda > \overline{\lambda}$ equation $(\mathcal{E})_\lambda$ admits at least two nontrivial nonnegative entire solutions.*

In the Appendix we present the auxiliary results largely used throughout the paper, which seem not to be so well-known. In particular, we establish in Theorem A.3 the existence of a Palais–Smale sequence via the Ekeland variational principle in a variant of the geometrical structure of the Mountain Pass theorem of Ambrosetti and Rabinowitz. For the standard result based on this technique we refer to [16]. Theorem A.3 is the key tool to construct a second independent nontrivial entire solution in the proof of Theorem B.

2. Preliminaries and non-existence for λ small

Conditions $(\mathcal{A})$-(a) and (b) imply that

$$\mathscr{A}(x,\xi) \leq \mathbf{A}(x,\xi)\cdot\xi \quad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Furthermore, $(\mathcal{A})$-(b) is weaker than the request that $\mathscr{A}$ is *p-uniformly convex*, i.e. that *there exists a constant $k > 0$ such that*

$$\mathscr{A}\left(x, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2}\mathscr{A}(x,\xi) + \frac{1}{2}\mathscr{A}(x,\eta) - k|\xi-\eta|^p \tag{b'}$$

for all $x \in \mathbb{R}^N$ and $\xi, \eta \in \mathbb{R}^N$. Condition ($b'$) is usually assumed in this context in the literature and forces $p \geq 2$, when $\mathscr{A}(x,\xi) = |\xi|^p/p$, cf. [13].

By $(\mathcal{A})$-(a) and (c)

$$\mathscr{A}(x,\xi) = \int_0^1 \frac{d}{dt}\mathscr{A}(x,t\xi)\,dt = \int_0^1 \frac{1}{t}\mathbf{A}(x,t\xi)\cdot t\xi dt \geq \frac{c}{p}|\xi|^p,$$

that is for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$

$$p\mathscr{A}(x,\xi) \geq c|\xi|^p. \tag{2.1}$$

Hence $c \leq C$ by $(\mathcal{A})$-(c).

Lemma 2.1. *Let $\xi, (\xi_n)_n \in \mathbb{R}^N$ be such that*

$$(\mathbf{A}(x,\xi_n) - \mathbf{A}(x,\xi))\cdot(\xi_n - \xi) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Then $(\xi_n)_n$ converges to ξ.

Proof. We are inspired in this proof by Lemma 2.4 of [9], see also the special case of Lemma 3 in [12]. First, we assert that $(\xi_n)_n$ is bounded. Otherwise, up to a subsequence, still denoted by $(\xi_n)_n$, we would have $|\xi_n| \to \infty$ and so

$$(\mathbf{A}(x,\xi_n) - \mathbf{A}(x,\xi)) \cdot (\xi_n - \xi) \geq c(|\xi_n|^p + |\xi|^p) - C(|\xi_n|^{p-1}|\xi| + |\xi|^{p-1}|\xi_n|) \\ \sim c|\xi_n|^p \to \infty,$$

as $n \to \infty$. This is impossible by (2.2). Therefore, $(\xi_n)_n$ is bounded and possesses a subsequence, still denoted by $(\xi_n)_n$, which converges to some $\eta \in \mathbb{R}^N$. Thus $(\mathbf{A}(x,\eta) - \mathbf{A}(x,\xi)) \cdot (\eta - \xi) = 0$ by (2.2). Moreover, the strict convexity of $\mathscr{A}(x,\cdot)$ for all $x \in \mathbb{R}^N$ implies that $\eta = \xi$. This also shows that actually the entire sequence $(\xi_n)_n$ converges to ξ. □

The space E denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} \nu(x)|u|^p dx \right)^{1/p},$$

and X the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\|u\|_E^p + \|u\|_{r,h}^p \right)^{1/p}, \quad \text{where} \|u\|_{r,h}^r = \int_{\mathbb{R}^N} h(x)|u|^r dx.$$

From now on B_R will denote the ball in $\mathbb{R}^N$ of center zero and radius $R > 0$.

From the structural assumptions (1.2)–(1.4) all the coefficients a, w, h in $(\mathcal{E})_\lambda$ are weights in $\mathbb{R}^N$. We indicate with $L^p(\mathbb{R}^N; a)$, $L^q(\mathbb{R}^N; w)$ and $L^r(\mathbb{R}^N; h)$ the corresponding weighted Lebesgue spaces. See the Appendix for the main properties.

Lemma 2.2. *The embeddings $X \hookrightarrow E \hookrightarrow D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ are continuous, with $\|\nabla u\|_p \leq \|u\|_E$ for all $u \in E$, $\|u\|_E \leq \|u\|$ for all $u \in X$ and*

$$\|u\|_{p^*} \leq C_{p^*} \|\nabla u\|_p \quad \text{for all } u \in D^{1,p}(\mathbb{R}^N). \tag{2.3}$$

Moreover, for any $R > 0$ the embeddings $E \hookrightarrow\hookrightarrow L^\varrho(B_R)$ and $X \hookrightarrow\hookrightarrow L^\varrho(B_R)$ are compact for all $\varrho \in [1, p^*)$.

Proof. The first two embeddings $X \hookrightarrow E \hookrightarrow D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ are obviously continuous and $\|\nabla u\|_p \leq \|u\|_E$ for all $u \in E$ and $\|u\|_E \leq \|u\|$ for all $u \in X$, and the third one is classical, with C_{p^*} the Talenti best constant of the embedding, cf. [24].

Let $R > 0$ be fixed. By the first part of the lemma the embedding $E \hookrightarrow W^{1,p}(B_R)$ is continuous, since $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ in (1.2), so that $0 < k_1 \leq \nu(x) \leq k_2$ for a.a. $x \in B_R$ and for some positive numbers k_1 and k_2 depending only on R. Since the embedding $W^{1,p}(B_R) \hookrightarrow\hookrightarrow L^\varrho(B_R)$ is compact for all $\varrho \in [1, p^*)$, also the embeddings $E \hookrightarrow\hookrightarrow L^\varrho(B_R)$ and $X \hookrightarrow\hookrightarrow L^\varrho(B_R)$ are compact. □

Lemma 2.3. *The embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N; w)$ is continuous, with*

$$\|u\|_{q,w} \leq \mathfrak{C}_w \|\nabla u\|_p \quad \text{for all } u \in D^{1,p}(\mathbb{R}^N), \tag{2.4}$$

and $\mathfrak{C}_w = C_{p^*} \|w\|_\wp^{1/q} > 0$.

The embeddings $E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N; w)$ *and* $X \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N; w)$ *are compact.*

Proof. By (1.3), Hölder's and Sobolev's inequalities, for all $u \in D^{1,p}(\mathbb{R}^N)$,

$$\|u\|_{q,w} \leq \left(\int_{\mathbb{R}^N} w(x)^{\wp} dx\right)^{1/\wp q} \cdot \left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{1/p^*} \leq C_{p^*} \|w\|_{\wp}^{1/q} \|\nabla u\|_p,$$

that is (2.4) holds.

In order to prove the last part of the lemma it is enough to show that $E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N; w)$. In other words, we show that if $u_n \rightharpoonup u$ in E, then $\|u_n - u\|_{q,w} \to 0$ as $n \to \infty$. By Hölder's inequality,

$$\int_{\mathbb{R}^N \setminus B_R} w(x)|u_n - u|^q dx \leq M \left(\int_{\mathbb{R}^N \setminus B_R} w(x)^{\wp} dx\right)^{1/\wp} = o(1)$$

as $R \to \infty$, being $w \in L^{\wp}(\mathbb{R}^N)$ by (1.3) and $\sup_n \|u_n - u\|_{p^*}^q = M < \infty$. For all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so large that $\sup_n \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q dx < \varepsilon/2$. Moreover, by Hölder's inequality we have as $n \to \infty$

$$\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx \leq \|w\|_{L^\sigma(B_{R_\varepsilon})} \|u_n - u\|^q_{L^{\sigma' q}(B_{R_\varepsilon})} = o(1),$$

since $E \hookrightarrow\hookrightarrow L^{\sigma' q}(B_{R_\varepsilon})$, being $\sigma' q < p^*$ by (1.3). Therefore, there exists $N_\varepsilon > 0$ such that $\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx < \varepsilon/2$ for all $n \geq N_\varepsilon$. In conclusion, for all $n \geq N_\varepsilon$

$$\|u_n - u\|_{q,w}^q = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx + \int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx < \varepsilon,$$

as required. □

Lemma 2.4. *For all* $u \in E$

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u\, dx + \|u\|_{p,a}^p \geq \kappa \|u\|_E^p, \\ &\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx + \frac{1}{p}\|u\|_{p,a}^p \geq \frac{\kappa}{p}\|u\|_E^p, \end{aligned} \tag{2.5}$$

where $\kappa = \min\{c, c_1\} > 0$. *Moreover, if* $u \in X \setminus \{0\}$ *and* $\lambda \in \mathbb{R}$ *satisfy*

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u\, dx + \|u\|_{p,a}^p + \|u\|_{r,h}^r = \lambda \|u\|_{q,w}^q, \tag{2.6}$$

then $0 < \kappa \|u\|_E^p \leq \lambda \|u\|_{q,w}^q, \lambda > 0$ *and*

$$\kappa_1 \lambda^{1/(p-q)} \leq \|u\|_{q,w} \leq \kappa_2 \lambda^{r/p(r-q)}, \tag{2.7}$$

where κ_1 *and* κ_2 *are positive constants independent of* u.

Proof. Take $u \in E$. By $(\mathcal{A})$-(c) and (1.2), it follows that

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u\, dx + \|u\|_{p,a}^p \geq c\|\nabla u\|_p^p + c_1 \|u\|_{p,\nu}^p.$$

In conclusion, $(2.5)_1$ holds, with c given in $(\mathcal{A})$-(c). Similarly, using (1.2) and (2.1), we get

$$\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx + \frac{1}{p}\|u\|_{p,a}^p \geq \frac{1}{p}\left\{c\|\nabla u\|_p^p + c_1\|u\|_{p,\nu}^p\right\}$$

for all $u \in E$, which immediately gives $(2.5)_2$.

Let $u \in X \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy (2.6). By $(2.5)_1, 0 < \kappa\|u\|_E^p \leq \lambda\|u\|_{q,w}^q$. Hence $\lambda > 0$. Therefore, by (2.4), $(2.5)_1$ and (2.6), we have

$$\begin{aligned}\|u\|_{q,w}^p \leq \mathfrak{C}_w^p\|u\|_E^p &\leq \frac{\mathfrak{C}_w^p}{\kappa}\left\{\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u dx + \|u\|_{p,a}^p\right\} \\ &\leq \frac{\lambda\mathfrak{C}_w^p}{\kappa}\|u\|_{q,w}^q. \end{aligned} \tag{2.8}$$

By Young's inequality,

$$ab \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta},$$

with $a = h(x)^{q/r}|u|^q \geq 0, b = \lambda w(x)h(x)^{-q/r} \geq 0, \alpha = r/q > 1$ and $\beta = r/(r-q) > 1$, we find

$$\lambda w(x)|u|^q \leq \frac{q}{r}h(x)|u|^r + \frac{r-q}{r}\left(\frac{\lambda w(x)}{h(x)^{q/r}}\right)^{r/(r-q)}.$$

Integration over $\mathbb{R}^N$ gives

$$\lambda\|u\|_{q,w}^q \leq \frac{q}{r}\|u\|_{r,h}^r + \frac{r-q}{r}\, H\, \lambda^{r/(r-q)}.$$

Thus, by (2.6) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla u dx + \|u\|_{p,a}^p &\leq \frac{q-r}{r}\|u\|_{r,h}^r + \frac{r-q}{r}\, H\, \lambda^{r/(r-q)} \\ &\leq \frac{r-q}{r}\, H\, \lambda^{r/(r-q)},\end{aligned}$$

being $q < r$. Hence, since $u \not\equiv 0$ by assumption, the last inequality and (2.8) yield (2.7), with

$$\kappa_1 = (\kappa/\mathfrak{C}_w^p)^{1/(q-p)} \quad \text{and} \quad \kappa_2 = [(r-q)\mathfrak{C}_w^p H/r\kappa]^{1/p}.$$

This completes the proof. □

We say that $u \in X$ is a *(weak) entire solution* of $(\mathcal{E})_\lambda$ if

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla v\, dx + \int_{\mathbb{R}^N} a(x)|u|^{p-2}uv\, dx = \lambda\int_{\mathbb{R}^N} w(x)|u|^{q-2}uv\, dx \\ - \int_{\mathbb{R}^N} h(x)|u|^{r-2}uv\, dx\end{aligned} \tag{2.9}$$

for all $v \in X$.

Hence the entire solutions of $(\mathcal{E})_\lambda$ correspond to the critical points of the energy functional $\Phi_\lambda : X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx + \frac{1}{p}\|u\|_{p,a}^p - \frac{\lambda}{q}\|u\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r.$$

If $(\mathcal{E})_\lambda$ admits a *nontrivial* entire solution $u \in X$, then $\lambda > 0$ by Lemma 2.4, and $\lambda \geq \lambda_0$ by (2.7), where

$$\lambda_0 = (\kappa_1/\kappa_2)^{p(r-q)(q-p)/q(r-p)} > 0.$$

Define

$$\lambda^* = \sup\{\lambda > 0 \,:\, (\mathcal{E})_\mu \textit{ admits only the trivial solution for all } \mu < \lambda\}.$$

Clearly $\lambda^* \geq \lambda_0 > 0$. In Sects. 3 and 4 we show that λ^* is exactly the critical value of Theorem A.

3. Preliminary results for existence

By the results of Sect. 2 from now on we consider only the case $\lambda > 0$.

Lemma 3.1. *The functional Φ_λ is coercive in X. In particular, any sequence $(u_n)_n$ in X such that $(\Phi_\lambda(u_n))_n$ is bounded admits a weakly convergent subsequence in X.*

Proof. Let us consider the following elementary inequality: for every $k_1, k_2 > 0$ and $0 < \alpha < \beta$

$$k_1|t|^\alpha - k_2|t|^\beta \leq C_{\alpha\beta} k_1 \left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \qquad \text{for all } t \in \mathbb{R}, \tag{3.1}$$

where $C_{\alpha\beta} > 0$ is a constant depending only on α and β.

Taking $k_1 = \lambda w(x)/q, k_2 = (p-1)h(x)/pr, \alpha = q$ and $\beta = r$ in (3.1), for all $x \in \mathbb{R}^N$ we have

$$\begin{aligned}\frac{\lambda}{q}\, w(x)|u(x)|^q - \frac{(p-1)h(x)}{pr}\,|u(x)|^r &\leq C_{qr}\frac{\lambda w(x)}{q}\left[\frac{\lambda w(x)/q}{(p-1)h(x)/pr}\right]^{q/(r-q)} \\ &= \mathcal{C}\,\lambda^{r/(r-q)}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)},\end{aligned}$$

where $\mathcal{C} = C_{qr}\,[pr/q(p-1)]^{q/(r-q)}/q$. Integrating the above inequality over $\mathbb{R}^N$, we get by (1.4)

$$\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{p-1}{pr}\|u\|_{r,h}^r \leq C_\lambda,$$

where $C_\lambda = \mathcal{C} H \lambda^{r/(r-q)} > 0$.

Therefore, by $(2.5)_2$, for all $u \in X$

$$\begin{aligned}\Phi_\lambda(u) = &\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx + \frac{1}{p}\|u\|_{p,a}^p - \left[\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{p-1}{pr}\|u\|_{r,h}^r\right] \\ &-\frac{p-1}{pr}\|u\|_{r,h}^r + \frac{1}{r}\|u\|_{r,h}^r\end{aligned}$$

$$\geq \frac{\kappa}{p}\|u\|_E^p + \frac{1}{pr}\|u\|_{r,h}^r - C_\lambda \geq \frac{\kappa}{p}\|u\|_E^p + \frac{1}{pr}\left(\|u\|_{r,h}^p - 1\right) - C_\lambda$$
$$\geq \frac{\min\{\kappa, r^{-1}\}}{p}\|u\|^p - C_\lambda - \frac{1}{pr}.$$

In conclusion, Φ_λ is coercive in X.

The last part of the claim follows at once by the coercivity of Φ_λ and the reflexivity of the space X, see Proposition A.11. □

Lemma 3.2. *The functional $\Phi_{\mathscr{A}} : X \to \mathbb{R}, \Phi_{\mathscr{A}}(u) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx$, is convex and of class C^1. In particular, $\Phi_{\mathscr{A}}$ is sequentially weakly lower semicontinuous in X.*

Proof. The convexity is an immediate consequence of assumption $(\mathcal{A})$-(b). Let us prove the continuity. Let $(u_n)_n, u \in X$ be such that $u_n \to u$ in X and fix a subsequence $(u_{n_k})_k$ of $(u_n)_n$. Clearly $\nabla u_{n_k} \to \nabla u$ in $[L^p(\mathbb{R}^N)]^N$ and so, by Theorem 4.9 of [6], there exists a further subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ and a function $\psi \in L^p(\mathbb{R}^N)$ such that a.e. in $\mathbb{R}^N$

$$\nabla u_{n_{k_j}} \to \nabla u \text{ as } j \to \infty \quad \text{and} \quad |\nabla u_{n_{k_j}}| \leq \psi \quad \text{for all } j \in \mathbb{N}. \tag{3.2}$$

Hence, condition $(\mathcal{A})$ implies that $\mathscr{A}(x, \nabla u_{n_{k_j}}) \to \mathscr{A}(x, \nabla u)$ a.e. in $\mathbb{R}^N$ and

$$|\mathscr{A}(x, \nabla u_{n_{k_j}})| \leq |\mathbf{A}(x, \nabla u_{n_{k_j}})| \cdot |\nabla u_{n_{k_j}}| \leq C|\nabla u_{n_{k_j}}|^p \leq C\psi^p \in L^1(\mathbb{R}^N).$$

The Dominated Convergence theorem forces that $\mathscr{A}(x, \nabla u_{n_{k_j}}) \to \mathscr{A}(x, \nabla u)$ in $L^1(\mathbb{R}^N)$ as $j \to \infty$, and the arbitrariness of $(u_{n_k})_k$ guarantees that actually $\mathscr{A}(x, \nabla u_n) \to \mathscr{A}(x, \nabla u)$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$. This gives the continuity of $\Phi_{\mathscr{A}}$, and so $\Phi_{\mathscr{A}}$ is sequentially w.l.s.c. by Corollary 3.9 of [6].

Moreover, $\Phi_{\mathscr{A}}$ is Gateaux-differentiable in X and for all $u, \varphi \in X$ it results

$$\langle \Phi'_{\mathscr{A}}(u), \varphi \rangle = \int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla \varphi\, dx.$$

Now, let $(u_n)_n, u \in X$ be such that $u_n \to u$ in X as $n \to \infty$. We claim that

$$\|\Phi'_{\mathscr{A}}(u_n) - \Phi'_{\mathscr{A}}(u)\|_{X'} = \sup_{\substack{\varphi \in X \\ \|\varphi\|=1}} \left| \int_{\mathbb{R}^N} (\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla \varphi\, dx \right| = o(1)$$

as $n \to \infty$. By $(\mathcal{A})$-(c), it follows that $\mathbf{A}(x, \nabla u)$ is in $[L^{p'}(\mathbb{R}^N)]^N$ for all $u \in X$. Applying Hölder's inequality, we obtain

$$\left| \int_{\mathbb{R}^N} (\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla \varphi\, dx \right| \leq \|\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)\|_{p'} \|\nabla \varphi\|_p.$$

Hence, for all $\varphi \in X$, with $\|\varphi\| = 1$, we have

$$\|\Phi'_{\mathscr{A}}(u_n) - \Phi'_{\mathscr{A}}(u)\|_{X'} \leq \|\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)\|_{p'}. \tag{3.3}$$

Fix now a subsequence $(u_{n_k})_k$ of $(u_n)_n$. Proceeding exactly as above we find a further subsequence $(u_{n_{k_j}})_j$ and a function $\psi \in L^p(\mathbb{R}^N)$ verifying (3.2).

Therefore, by $(\mathcal{A})$-(c) we get

$$\begin{aligned} \mid \mathbf{A}(x,\nabla u_{n_{k_j}}) - \mathbf{A}(x,\nabla u) \mid^{p'} &\le 2^{p'-1}\left(|\mathbf{A}(x,\nabla u_{n_{k_j}})|^{p'} + |\mathbf{A}(x,\nabla u)|^{p'}\right) \\ &\le 2^{p'-1}C^{p'}\left\{|\nabla u_{n_{k_j}}|^p + |\nabla u|^p\right\} \\ &\le (2C)^{p'}\psi^p \in L^1(\mathbb{R}^N). \end{aligned} \tag{3.4}$$

On the other hand, $|\mathbf{A}(x,\nabla u_{n_{k_j}}) - \mathbf{A}(x,\nabla u)| \to 0$ for a.a. $x \in \mathbb{R}^N$ as $j \to \infty$, since $\mathbf{A}$ is continuous by $(\mathcal{A})$. Thus, $\mathbf{A}(x,\nabla u_{n_{k_j}}) \to \mathbf{A}(x,\nabla u)$ in $[L^{p'}(\mathbb{R}^N)]^N$ by (3.4) and the Dominated Convergence theorem. Hence the entire sequence $\mathbf{A}(x,\nabla u_n) \to \mathbf{A}(x,\nabla u)$ in $[L^{p'}(\mathbb{R}^N)]^N$ and the claim follows from (3.3).

In conclusion, $\Phi_{\mathscr{A}}$ is of class C^1, as required. □

Lemma 3.3. *The functional $\Phi_a : X \to \mathbb{R}, \Phi_a(u) = \frac{1}{p}\|u\|_{p,a}^p$, is convex, of class C^1 and sequentially weakly lower semicontinuous. Moreover, if $(u_n)_n, u \in X$ and $u_n \rightharpoonup u$ in X, then $\Phi_a'(u_n) \overset{*}{\rightharpoonup} \Phi_a'(u)$ in X'.*

Proof. The convexity of Φ_a is obvious being $p > 1$. Moreover, since the embedding $X \hookrightarrow L^p(\mathbb{R}^N;a)$ is continuous by (1.2), with $\|u\|_{p,a} \le \|u\|$ for all $u \in X$, the functional Φ_a is continuous in X. Consequently, Φ_a is sequentially w.l.s.c. by Corollary 3.9 of [6].

Moreover, Φ_a is Gateaux-differentiable in X and for all $u, \varphi \in X$ we have

$$\langle \Phi_a'(u), \varphi\rangle = \int_{\mathbb{R}^N} a(x)|u|^{p-2}u\varphi dx.$$

Now, let $(u_n)_n, u \in X$ be such that $u_n \rightharpoonup u$ in X as $n \to \infty$. Since $X \hookrightarrow E \hookrightarrow L^p(\mathbb{R}^N;\nu) \hookrightarrow L^p(\mathbb{R}^N;a)$ by (1.2), then $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^N;a)$. Let $n \mapsto v_n = |u_n|^{p-2}u_n$ and fix a subsequence $(v_{n_k})_k$ of $(v_n)_n$. By Proposition A.10 there exists a subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ converging a.e. in $\mathbb{R}^N$ to u. Furthermore, $(u_{n_{k_j}})_j$ is bounded in $L^p(\mathbb{R}^N;a)$, so that $v_{n_{k_j}} \rightharpoonup v$ in $L^{p'}(\mathbb{R}^N;a)$ by Proposition A.8-(i). This implies that the whole sequence $v_n \rightharpoonup v$ in $L^{p'}(\mathbb{R}^N;a)$. Thus, for all $\varphi \in X$ we have

$$\int_{\mathbb{R}^N} a(x)|u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^N} a(x)|u|^{p-2}u\varphi dx$$

as $n \to \infty$, that is $\langle \Phi_a'(u_n), \varphi\rangle \to \langle \Phi_a'(u), \varphi\rangle$. This shows that $\Phi_a'(u_n) \overset{*}{\rightharpoonup} \Phi_a'(u)$ in X', as claimed.

Let us prove that $\Phi_a \in C^1(X)$. Fix $(u_n)_n, u \in X$, with $u_n \to u$ in X. Hence $u_n \to u$ in $L^p(\mathbb{R}^N;a)$, since $X \hookrightarrow L^p(\mathbb{R}^N;a)$ by (1.2). Thus $|u_n|^{p-2}u_n = v_n \to v = |u|^{p-2}u$ in $L^{p'}(\mathbb{R}^N;a)$, by Proposition A.8-(ii). Therefore, for all $\varphi \in X$, with $\|\varphi\| = 1$,

$$|\langle \Phi_a'(u_n) - \Phi_a'(u), \varphi\rangle| \le \|v_n - v\|_{p',a}\|\varphi\|_{p,a} \le \|v_n - v\|_{p',a},$$

since $\|\varphi\|_{p,a} \le \|\varphi\|_{p,\nu} \le \|\varphi\|$ for all $\varphi \in X$ by (1.2). Therefore,

$$\|\Phi_a'(u_n) - \Phi_a'(u)\|_{X'} \le \|v_n - v\|_{p',a} \to 0$$

as $n \to \infty$. In conclusion, Φ_a is of class $C^1(X)$. □

Lemma 3.4. *The functional* $\Phi_w : X \to \mathbb{R}, \Phi_w(u) = \frac{1}{q}\|u\|_{q,w}^q$*, is convex, of class* C^1 *and sequentially weakly continuous in* X*. Moreover, if* $(u_n)_n, u \in X$ *and* $u_n \rightharpoonup u$ *in* X*, then* $\Phi'_w(u_n) \to \Phi'_w(u)$ *in* X'*.*

Proof. The convexity of Φ_w is obvious, being $q > 2$. Moreover, by Lemma 2.3 it is clear that Φ_w is sequentially weakly continuous, so that in particular Φ_w is continuous. Furthermore, Φ_w is Gateaux-differentiable in X and for all $u, \varphi \in X$

$$\langle \Phi'_w(u), \varphi\rangle = \int_{\mathbb{R}^N} w(x)|u|^{q-2}u\varphi\, dx.$$

Now, let $(u_n)_n, u \in X$ be such that $u_n \rightharpoonup u$ in X and fix $\varphi \in X$, with $\|\varphi\| = 1$. By Lemma 2.3 and Proposition A.8-(ii), it follows that $v_n = |u_n|^{q-2}u_n \to v = |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N; w)$. Therefore,

$$|\langle \Phi'_w(u_n) - \Phi'_w(u), \varphi\rangle| \le \|v_n - v\|_{q',w}\|\varphi\|_{q,w} \le \mathfrak{C}_w\|v_n - v\|_{q',w}$$

by (2.4). Hence,

$$\|\Phi'_w(u_n) - \Phi'_w(u)\|_{X'} \le \mathfrak{C}_w\|v_n - v\|_{q',w},$$

that is $\Phi'_w(u_n) \to \Phi'_w(u)$ in X'. In particular, this shows that Φ_w is of class $C^1(X)$ and completes the proof of the lemma. □

Clearly the conclusions of Lemmas 3.3 and 3.4 continue to hold when the functionals are defined in the bigger space E. Indeed, all the functionals are well defined in E, being $E \hookrightarrow L^p(\mathbb{R}^N; a)$ by (1.2) and $E \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N; w)$ by Lemma 2.3.

Lemma 3.5. *The functional* $\Phi_h : X \to \mathbb{R}, \Phi_h(u) = \frac{1}{r}\|u\|_{r,h}^r$ *is convex, of class* C^1 *and sequentially weakly lower semicontinuous. Moreover, if* $(u_n)_n, u \in X$ *and* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$*, then* $\Phi'_h(u_n) \overset{*}{\rightharpoonup} \Phi'_h(u)$ *in* X'*.*

Proof. The convexity of Φ_h is obvious, being $r > 2$. Moreover, the continuity of Φ_h follows from the continuity of the embedding $X \hookrightarrow L^r(\mathbb{R}^N; h)$. Hence Φ_h is sequentially w.l.s.c. by Corollary 3.9 of [6]. On the other hand, Φ_h is Gâteaux-differentiable in X and for all $u, \varphi \in X$

$$\langle \Phi'_h(u), \varphi\rangle = \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\, dx.$$

Let $(u_n)_n, u \in X$ be such that $u_n \to u$ in X. Then, $u_n \to u$ in $L^r(\mathbb{R}^N; h)$, and so $v_n = |u_n|^{r-2}u_n \to v = |u|^{r-2}u$ in $L^{r'}(\mathbb{R}^N; h)$ by Proposition A.8-(ii). Therefore,

$$\begin{aligned}\|\Phi'_h(u_n) - \Phi'_h(u)\|_{X'} &= \sup_{\substack{\varphi\in X\\ \|\varphi\|=1}} \left|\int_{\mathbb{R}^N} h(x)\left(|u_n|^{r-2}u_n - |u|^{r-2}u\right)\varphi\, dx\right|\\ &\le \sup_{\varphi\in X\, \|\varphi\|=1} \|v_n - v\|_{r',h} \cdot \|\varphi\|_{r,h} \le \|v_n - v\|_{r',h} = o(1)\end{aligned}$$

as $n \to \infty$. This gives the C^1 regularity of Φ_h.

Suppose now that $u_n \rightharpoonup u$ in X. Let $n \mapsto v_n = |u_n|^{r-2}u_n$ and fix a subsequence $(v_{n_k})_k$ of $(v_n)_n$. Of course $u_{n_k} \rightharpoonup u$ in X and by Proposition A.10

there exists a further subsequence $(u_{n_{k_j}})_j$ such that $u_{n_{k_j}} \to u$ a.e. in $\mathbb{R}^N$. Thus $v_{n_{k_j}} \to v = |u|^{r-2}u$ a.e. in $\mathbb{R}^N$. On the other hand, $(v_{n_{k_j}})_j$ is bounded in $L^{r'}(\mathbb{R}^N; h)$ since $\|v_{n_{k_j}}\|_{r',h}^{r'} = \|u_{n_{k_j}}\|_{r,h}^{r}$ and $(u_{n_{k_j}})_j$ is bounded in $L^r(\mathbb{R}^N; h)$. Therefore $v_{n_{k_j}} \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N; h)$, by Proposition A.8-(i). In conclusion, due to the arbitrariness of $(v_{n_k})_k$, the entire sequence $v_n \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N; h)$ as $n \to \infty$. Hence, in particular for all $\varphi \in X$

$$\int_{\mathbb{R}^N} h(x)|u_n|^{r-2}u_n\varphi\, dx \to \int_{\mathbb{R}^N} h(x)|u|^{r-2}u\varphi\, dx$$

as $n \to \infty$. This gives the claim and completes the proof. □

For any $(x, u) \in \mathbb{R}^N \times \mathbb{R}$ put

$$f(x, u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u, \tag{3.5}$$

so that

$$F(x, u) = \int_0^u f(x, v)dv = \frac{\lambda}{q}\, w(x)|u|^q - h(x)\frac{|u|^r}{r}. \tag{3.6}$$

Lemma 3.6. *For any fixed $u \in X$ the functional $\mathcal{F}_u : X \to \mathbb{R}$, defined by $\mathcal{F}_u(v) = \int_{\mathbb{R}^N} f(x, u(x))v(x)dx$, is in X'. In particular, if $v_n \rightharpoonup v$ in X then $\mathcal{F}_u(v_n) \to \mathcal{F}_u(v)$.*

Proof. Take $u \in X$. Clearly $\mathcal{F}_u$ is linear. Moreover, using (2.4), we get for all $v \in X$

$$\begin{aligned}|\mathcal{F}_u(v)| &\le \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-1}|v|\, dx + \int_{\mathbb{R}^N} h(x)^{(r-1)/r}|u|^{r-1} \cdot h(x)^{1/r}|v|dx \\ &\le \lambda\|u\|_{q,w}^{q-1}\|v\|_{q,w} + \|u\|_{r,h}^{r-1}\|v\|_{r,h} \le (\lambda\mathfrak{C}_w\|u\|_{q,w}^{q-1} + \|u\|_{r,h}^{r-1})\|v\|,\end{aligned}$$

and so $\mathcal{F}_u$ is continuous in X. □

In the next result we strongly use the assumption $q > 2$. An interesting open question occurs when $1 < p < q \le 2$ and $q < \min\{r, p^*\}$.

Lemma 3.7. *The functional Φ_λ is of class C^1 and sequentially weakly lower semicontinuous in X, that is if $u_n \rightharpoonup u$ in X, then*

$$\Phi_\lambda(u) \le \liminf_{n\to\infty} \Phi_\lambda(u_n). \tag{3.7}$$

Proof. We take inspiration from Lemma 2 of [20]. Lemmas 3.2–3.5 imply that $\Phi_\lambda \in C^1(X)$. Let $(u_n)_n$ and u be such that $u_n \rightharpoonup u$ in X. The definition of Φ_λ and (3.5) give

$$\begin{aligned}\Phi_\lambda(u) - \Phi_\lambda(u_n) = &\int_{\mathbb{R}^N} [\mathscr{A}(x, \nabla u) - \mathscr{A}(x, \nabla u_n)]\, dx \\ &+ \frac{1}{p}\left(\|u\|_{p,a}^p - \|u_n\|_{p,a}^p\right) + \int_{\mathbb{R}^N} [F(x, u_n) - F(x, u)]dx. \end{aligned}\tag{3.8}$$

Since $u_n \rightharpoonup u$ in X, Lemma 3.2 implies that

$$\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u_n)\, dx \tag{3.9}$$

and Lemma 3.3 yields

$$\|u\|_{p,a}^p \le \liminf_{n\to\infty} \|u_n\|_{p,a}^p.$$

Hence, by (3.8)

$$\limsup_{n\to\infty} [\Phi_\lambda(u) - \Phi_\lambda(u_n)] \le \limsup_{n\to\infty} \int_{\mathbb{R}^N} [F(x,u_n) - F(x,u)]dx. \quad (3.10)$$

By (3.5) and (3.6), for all $s \in [0,1]$,

$$\begin{aligned} F_u(x, u + s(u_n - u)) &= f(x, u + s(u_n - u)) \\ &= f(x,u) + (u_n - u)\int_0^s f_u(x, u + t(u_n - u))dt, \end{aligned} \quad (3.11)$$

where clearly

$$f_u(x,z) = \lambda(q-1)w(x)|z|^{q-2} - h(x)(r-1)|z|^{r-2}.$$

Multiplying (3.11) by $u_n - u$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} F(x,u_n) - F(x,u) &= f(x,u)(u_n - u) \\ &\quad + (u_n - u)^2 \int_0^1 \left(\int_0^s f_u(x, u + t(u_n - u))dt \right) ds. \end{aligned} \quad (3.12)$$

Now, (3.1), with $k_1 = \lambda w(x)(q-1), k_2 = h(x)(r-1), \alpha = q-2 > 0$ and $\beta = r-2 > 0$, and (1.3) force

$$f_u(x,z) \le 2C_1 w(x)^{2/q} \left[\frac{w(x)^{r/q}}{h(x)} \right]^{(q-2)/(r-q)},$$

where C_1 is a positive constant, depending only on q, r and λ. Consequently, (3.12) yields

$$\begin{aligned} \int_{\mathbb{R}^N} [F(x,u_n) - F(x,u)]dx &\le \int_{\mathbb{R}^N} f(x,u)(u_n - u)dx \\ &\quad + C_1 \int_{\mathbb{R}^N} w(x)^{2/q}(u_n - u)^2 \left[\frac{w(x)^{r/q}}{h(x)} \right]^{(q-2)/(r-q)} dx \\ &\le \int_{\mathbb{R}^N} f(x,u)(u_n - u)dx + C_1 H^{(q-2)/q} \|u_n - u\|_{q,w}^2, \end{aligned} \quad (3.13)$$

by Hölder's inequality and (1.4). Now, Lemma 3.6 gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u)(u_n - u)\, dx = 0, \quad (3.14)$$

and Lemma 2.3 implies

$$\lim_{n\to\infty} \|u_n - u\|_{q,w} = 0. \quad (3.15)$$

Putting (3.13)–(3.15) in (3.10) we get the claim (3.7). □

Lemma 3.8. *Let $(u_n)_n$ be bounded in X and $(\lambda_n)_n$ bounded in $\mathbb{R}$. Put*

$$n \mapsto g_n(x) = -a(x)|u_n|^{p-2}u_n + \lambda_n w(x)|u_n|^{q-2}u_n - h(x)|u_n|^{r-2}u_n. \tag{3.16}$$

For all compact set $K \subset \mathbb{R}^N$ there exists $C_K > 0$ such that

$$\sup_n \int_K |g_n(x)|\, dx \le C_K.$$

Proof. Let $K \subset \mathbb{R}^N$ be a compact set. Clearly $a \in L^{N/p}(K)$ by (1.2), so that by Hölder's inequality

$$\int_K a(x)|u_n|^{p-1}dx \le |K|^{1/p^*}\left(\int_K a(x)^{N/p}dx\right)^{p/N} \sup_n \|u_n\|_{p^*}^{p-1} = C_1 \tag{3.17}$$

and $C_1 = C_1(K)$. Similarly, by Hölder's inequality and (1.3), we obtain

$$\int_K w(x)|u_n|^{q-1}dx \le |K|^{1/p^*}\|w\|_{\wp} \sup_n \|u_n\|_{p^*}^{q-1} = C_2, \tag{3.18}$$

and $C_2 = C_2(K)$. Finally, since $h \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $(\|u_n\|_{r,h})_n$ is bounded, then

$$\int_K h(x)|u_n|^{r-1}dx \le \left(\int_K h(x)dx\right)^{1/r} \sup_n \|u_n\|_{r,h}^{r-1} = C_3, \tag{3.19}$$

with $C_3 = C_3(K)$. Combining (3.17)–(3.19), and recalling that $(\lambda_n)_n$ is bounded, we get the claim. □

4. Existence if λ is large

Define

$$\overline{\lambda} = \inf_{\substack{u \in X \\ \|u\|_{q,w}=1}} \left\{ q\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx + \frac{q}{p}\|u\|_{p,a}^p + \frac{q}{r}\|u\|_{r,h}^r \right\}.$$

Note that $\overline{\lambda} > 0$. Indeed, for any $u \in X$ with $\|u\|_{q,w} = 1$, by Hölder's inequality and (1.4), we have

$$\begin{aligned} 1 = \|u\|_{q,w}^q &= \int_{\mathbb{R}^N} \frac{w(x)}{h(x)^{q/r}}\, h(x)^{q/r}|u|^q dx \le \left(\int_{\mathbb{R}^N}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)} dx\right)^{(r-q)/r} \|u\|_{r,h}^q \\ &= H^{(r-q)/r}\|u\|_{r,h}^q, \end{aligned}$$

where $H > 0$ is the number introduced in (1.4). Consequently, using also $(2.5)_2$, we get

$$\begin{aligned} q\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx + \frac{q}{p}\|u\|_{p,a}^p + \frac{q}{r}\|u\|_{r,h}^r &\ge \frac{\kappa q}{p}\|u\|_E^p + \frac{q}{r}H^{(q-r)/q} \\ &\ge \frac{\kappa q}{p\mathfrak{C}_w^p} + \frac{q}{r}H^{(q-r)/q}, \end{aligned}$$

where $\mathfrak{C}_w > 0$ is given in (2.4). In other words,

$$\overline{\lambda} \ge \frac{\kappa q}{p\mathfrak{C}_w^p} + \frac{q}{r}H^{(q-r)/q} > 0.$$

Lemma 4.1. *For all $\lambda > \overline{\lambda}$ there exists a global nontrivial non-negative minimizer $e \in X$ of Φ_λ with negative energy, that is $\Phi_\lambda(e) < 0$.*

Proof. By Lemmas 3.1, 3.7 and Corollary 3.23 of [6], for each $\lambda > 0$ there exists a global minimizer $e \in X$ of Φ_λ, that is

$$\Phi_\lambda(e) = \inf_{v \in X} \Phi_\lambda(v).$$

Clearly e is a solution of $(\mathcal{E})_\lambda$. We prove that $e \not\equiv 0$ provided that $\lambda > \overline{\lambda}$. To this aim we show that $\inf_{v \in X} \Phi_\lambda(v) < 0$.

Let $\lambda > \overline{\lambda}$. Then there exists a function $\varphi \in X$, with $\|\varphi\|_{q,w} = 1$, such that

$$\lambda \|\varphi\|_{q,w}^q = \lambda > q \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla\varphi)\, dx + \frac{q}{p}\|\varphi\|_{p,a}^p + \frac{q}{r}\|\varphi\|_{r,h}^r.$$

This can be rewritten as

$$\Phi_\lambda(\varphi) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla\varphi)\, dx + \frac{1}{p}\|\varphi\|_{p,a}^p - \frac{\lambda}{q}\|\varphi\|_{q,w}^q + \frac{1}{r}\|\varphi\|_{r,h}^r < 0$$

and consequently $\Phi_\lambda(e) = \inf_{v \in X} \Phi_\lambda(v) \le \Phi_\lambda(\varphi) < 0$.

In conclusion, for any $\lambda > \overline{\lambda}$, equation $(\mathcal{E})_\lambda$ has a nontrivial solution $e \in X$ such that $\Phi_\lambda(e) < 0$. Finally, we may assume $e \ge 0$ a.e. in $\mathbb{R}^N$, since $|e| \in X$ and $\Phi_\lambda(e) = \Phi_\lambda(|e|)$ by $(\mathcal{A})$-(a). □

Define

$$\lambda^{**} = \inf\{\lambda > 0 \,:\, (\mathcal{E})_\lambda \textit{ admits a nontrivial entire solution}\}.$$

Lemma 4.1 assures that this definition is meaningful. Clearly $\overline{\lambda} \ge \lambda^{**}$.

Theorem 4.2. *For any $\lambda > \lambda^{**}$ equation $(\mathcal{E})_\lambda$ admits a nontrivial non-negative entire solution $u_\lambda \in X$.*

Proof. We take somehow inspiration from [17, Theorem 1.1] and [23, Theorem 2.4].

Fix $\lambda > \lambda^{**}$. By definition of λ^{**} there exists $\mu \in (\lambda^{**}, \lambda)$ such that Φ_μ has a nontrivial critical point $u_\mu \in X$. We assume, without loss of generality, that $u_\mu \ge 0$ a.e. in $\mathbb{R}^N$, since $|u_\mu|$ is also a solution of $(\mathcal{E})_\mu$ by $(\mathcal{A})$–(a). Of course, u_μ is a sub-solution for $(\mathcal{E})_\lambda$. Consider the following minimization problem

$$\inf_{v \in \mathcal{M}} \Phi_\lambda(v), \quad \mathcal{M} = \{v \in X \,:\, v \ge u_\mu\}.$$

First note that $\mathcal{M}$ is closed and convex, and in turn also weakly closed. Moreover, Φ_λ is coercive in $\mathcal{M}$, being coercive in X by Lemma 3.1. Finally Φ_λ is sequentially weakly lower semicontinuous in X and so in $\mathcal{M}$. Hence, Corollary 3.23 of [6] assures that Φ_λ is bounded from below in $\mathcal{M}$ and attains its infimum in $\mathcal{M}$, i.e. there exists $u_\lambda \ge u_\mu$ such that $\Phi_\lambda(u_\lambda) = \inf_{v \in \mathcal{M}} \Phi_\lambda(v)$.

We claim that u_λ is a solution of $(\mathcal{E})_\lambda$. Indeed, take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Put

$$\varphi_\varepsilon = \max\{0, u_\mu - u_\lambda - \varepsilon\varphi\} \ge 0 \quad \text{and} \quad v_\varepsilon = u_\lambda + \varepsilon\varphi + \varphi_\varepsilon,$$

so that $v_\varepsilon \in \mathcal{M}$. Of course

$$0 \le \langle \Phi'_\lambda(u_\lambda), v_\varepsilon - u_\lambda\rangle = \varepsilon\langle \Phi'_\lambda(u_\lambda), \varphi\rangle + \langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\rangle,$$

and in turn

$$\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \ge -\frac{1}{\varepsilon}\,\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\rangle. \tag{4.1}$$

Define

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N : \, u_\lambda(x) + \varepsilon\varphi(x) \le u_\mu(x) < u_\lambda(x)\}.$$

Clearly $\Omega_\varepsilon \subset \operatorname{supp}\varphi$. Since u_μ is a subsolution of $(\mathcal{E})_\lambda$ and $\varphi_\varepsilon \ge 0$ it turns out that $\langle \Phi'_\lambda(u_\mu), \varphi_\varepsilon\rangle \le 0$. Hence, using the notation of (3.5), we have

$$\begin{aligned}\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\rangle &= \langle \Phi'_\lambda(u_\mu), \varphi_\varepsilon\rangle + \langle \Phi'_\lambda(u_\lambda) - \Phi'_\lambda(u_\mu), \varphi_\varepsilon\rangle\\ &\le \int_{\Omega_\varepsilon} (\mathbf{A}(x, \nabla u_\lambda) - \mathbf{A}(x, \nabla u_\mu)) \cdot \nabla(u_\mu - u_\lambda - \varepsilon\varphi)\, dx\\ &\quad + \int_{\Omega_\varepsilon} a(x)\,(|u_\lambda|^{p-2}u_\lambda - |u_\mu|^{p-2}u_\mu)(u_\mu - u_\lambda - \varepsilon\varphi)\, dx\\ &\quad - \int_{\Omega_\varepsilon} (f(x, u_\lambda) - f(x, u_\mu))(u_\mu - u_\lambda - \varepsilon\varphi)\, dx. \end{aligned}\tag{4.2}$$

By convexity

$$\int_{\Omega_\varepsilon} (\mathbf{A}(x, \nabla u_\lambda) - \mathbf{A}(x, \nabla u_\mu)) \cdot (\nabla u_\mu - \nabla u_\lambda)\, dx \le 0,$$

while, since $0 \le u_\mu - u_\lambda - \varepsilon\varphi = u_\mu - u_\lambda + \varepsilon|\varphi| < \varepsilon|\varphi|$ in Ω_ε, we get

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon} a(x)\,(|u_\lambda|^{p-2}u_\lambda - |u_\mu|^{p-2}u_\mu)(u_\mu - u_\lambda - \varepsilon\varphi)\, dx\right|\\ &\quad\le \int_{\Omega_\varepsilon} a(x)||u_\lambda|^{p-2}u_\lambda - |u_\mu|^{p-2}u_\mu|(u_\mu - u_\lambda - \varepsilon\varphi)\, dx\\ &\quad\le \varepsilon\int_{\Omega_\varepsilon} a(x)||u_\lambda|^{p-2}u_\lambda - |u_\mu|^{p-2}u_\mu| \cdot |\varphi|\, dx,\end{aligned}$$

and similarly

$$\left|\int_{\Omega_\varepsilon} (f(x, u_\lambda) - f(x, u_\mu))(u_\mu - u_\lambda - \varepsilon\varphi) dx\right| \le \varepsilon\int_{\Omega_\varepsilon} |f(x, u_\lambda) - f(x, u_\mu)| \cdot |\varphi| dx.$$

Therefore, (4.2) yields

$$\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \le \varepsilon\int_{\Omega_\varepsilon} \psi(x) dx,$$

where $\psi = (\mathbf{A}(x, \nabla u_\mu) - \mathbf{A}(x, \nabla u_\lambda)) \cdot \nabla\varphi + (a(x)||u_\lambda|^{p-2}u_\lambda - |u_\mu|^{p-2}u_\mu| + |f(x, u_\lambda) - f(x, u_\mu)|)|\varphi|$. We claim that $\psi \in L^1(\operatorname{supp}\varphi)$. Indeed, $\mathbf{A}(x, \nabla u_\mu)$ and $\mathbf{A}(x, \nabla u_\lambda)$ are in $[L^{p'}(\mathbb{R}^N)]^N$ by $(\mathcal{A})$-(c), while $a|u_\lambda|^{p-1}$ and $a|u_\mu|^{p-1}$ are in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, also $|f(x, u_\lambda) - f(x, u_\mu)|$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, since

$$|f(x, u_\lambda) - f(x, u_\mu)| \le \lambda w(x)\left(|u_\lambda|^{q-1} + |u_\mu|^{q-1}\right) + h(x)\left(|u_\lambda|^{r-1} + |u_\mu|^{r-1}\right).$$

Therefore the claim follows from the proof of Lemma 3.8. Thus,

$$\lim_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon}\psi(x)dx = 0,$$

since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$. In conclusion, $\langle \Phi'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \le o(\varepsilon)$ as $\varepsilon \to 0^+$, so that by (4.1) it follows that $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \ge o(1)$ as $\varepsilon \to 0^+$. Therefore, $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \ge 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, that is $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $X = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|}$, we obtain that u_λ is a solution of $(\mathcal{E})_\lambda$.

Finally, u_λ is nontrivial and non-negative, since $u_\lambda \ge u_\mu$. □

Lemma 4.3. *$\lambda^{**} = \lambda^*$.*

Proof. Theorem 4.2 shows that $\lambda^{**} \ge \lambda^*$. Suppose by contradiction that $\lambda^{**} > \lambda^*$. Problem $(\mathcal{E})_\lambda$ cannot admit a nontrivial solution $u \in X$ if $\lambda < \lambda^{**}$, since this would contradict the minimality of λ^{**}. Hence, for all $\lambda \in [\lambda^*, \lambda^{**})$ the unique solution of $(\mathcal{E})_\lambda$ is $u \equiv 0$. But this is again impossible since it would contradict the maximality of λ^*. Hence $\lambda^{**} = \lambda^*$. □

Theorem 4.4. *Equation $(\mathcal{E})_{\lambda^*}$ admits a nontrivial non-negative entire solution $u \in X$.*

Proof. Let $(\lambda_n)_n$ be a strictly decreasing sequence converging to λ^* and $u_n \in X$ be a nontrivial non-negative entire solution of $(\mathcal{E})_{\lambda_n}$. By (2.9) we get

$$\int_{\mathbb{R}^N}\mathbf{A}(x,\nabla u_n)\nabla v\,dx = \int_{\mathbb{R}^N} g_n v\,dx \quad \text{for all } v\in X, \tag{4.3}$$

where $n \mapsto g_n(x) = -a(x)|u_n|^{p-2}u_n + \lambda_n w(x)|u_n|^{q-2}u_n - h(x)|u_n|^{r-2}u_n$. By $(2.5)_1$, (2.6), (2.7) and the monotonicity of $(\lambda_n)_n$, we obtain

$$\kappa\|u_n\|_E^p + \|u_n\|_{r,h}^r \le \lambda_n\|u_n\|_{q,w}^q \le \kappa_2^q\lambda_1^{1+rq/p(r-q)}.$$

Therefore $(\|u_n\|_E)_n$ and $(\|u_n\|_{r,h})_n$ are bounded, and in turn also $(\|u_n\|)_n$ is bounded. Hence, $(g_n)_n$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ by Lemma 3.8, since also $(\lambda_n)_n$ is bounded. Moreover, by $(\mathcal{A})$-(c), Lemma 2.3, Propositions A.6, A.10 and A.11 it is possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying

$$\begin{array}{lll} u_n \rightharpoonup u \ \text{ in } X; & u_n \to u \ \text{ in } L^q(\mathbb{R}^N; w); \\ u_n \rightharpoonup u \ \text{ in } L^r(\mathbb{R}^N; h); & u_n \to u \ \text{ a.e. in } \mathbb{R}^N; \\ \nabla u_n \rightharpoonup \nabla u \ \text{ in } [L^p(\mathbb{R}^N)]^N; & \mathbf{A}(x,\nabla u_n) \rightharpoonup \Theta \ \text{ in } [L^{p'}(\mathbb{R}^N)]^N, \end{array} \tag{4.4}$$

for some $u \in X$ and $\Theta \in [L^{p'}(\mathbb{R}^N)]^N$. We claim that $\Theta = \mathbf{A}(x,\nabla u)$ and that u, which is clearly non-negative by (4.4), is the solution we are looking for.

Step 1. In the sequel we somehow follow the proofs of Theorem 2.1 of [5] and Lemma 2 of [12]. Fix $R > 0$. Let $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \varphi_R \le 1$ in $\mathbb{R}^N$ and $\varphi_R \equiv 1$ in B_R. Given $\varepsilon > 0$ define for each $t \in \mathbb{R}$

$$\eta_\varepsilon(t) = \begin{cases} t, & \text{if } |t| < \varepsilon, \\ \varepsilon\dfrac{t}{|t|}, & \text{if } |t| \ge \varepsilon. \end{cases}$$

Put $v_n = \varphi_R \eta_\varepsilon \circ (u_n - u)$, so that $v_n \in X$. Taking $v = v_n$ in (4.3), we get

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi_R(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(\eta_\varepsilon \circ (u_n - u))\, dx \\ = - \int_{\mathbb{R}^N} \eta_\varepsilon \circ (u_n - u) \mathbf{A}(x, \nabla u_n) \cdot \nabla \varphi_R \, dx \\ - \int_{\mathbb{R}^N} \varphi_R \mathbf{A}(x, \nabla u) \cdot \nabla(\eta_\varepsilon \circ (u_n - u))\, dx + \int_{\mathbb{R}^N} g_n v_n \, dx. \end{aligned} \tag{4.5}$$

Observe now that

$$\int_{\mathbb{R}^N} \eta_\varepsilon \circ (u_n - u) \mathbf{A}(x, \nabla u_n) \cdot \nabla \varphi_R \, dx \to 0 \quad \text{as } n \to \infty,$$

since $\eta_\varepsilon \circ (u_n - u) \nabla \varphi_R \to 0$ in $[L^p(\operatorname{supp} \varphi_R)]^N$ and $\mathbf{A}(x, \nabla u_n) \rightharpoonup \Theta$ in $[L^{p'}(\mathbb{R}^N)]^N$ by (4.4). Furthermore, $\nabla(\eta_\varepsilon \circ (u_n - u)) \rightharpoonup 0$ in $[L^p(\mathbb{R}^N)]^N$, since $u_n \rightharpoonup u$ in X, and consequently

$$\int_{\mathbb{R}^N} \varphi_R \mathbf{A}(x, \nabla u) \cdot \nabla(\eta_\varepsilon \circ (u_n - u))\, dx \to 0 \quad \text{as } n \to \infty,$$

being $\mathbf{A}(x, \nabla u) \in [L^{p'}(\mathbb{R}^N)]^N$.

In conclusion, the first two terms in the right hand side of (4.5) go to zero as $n \to \infty$. Now, recalling that $0 \le \varphi_R \le 1$ in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} g_n v_n \, dx \le \int_{\operatorname{supp} \varphi_R} |g_n| \cdot |\eta_\varepsilon \circ (u_n - u)|\, dx \le \varepsilon \int_{\operatorname{supp} \varphi_R} |g_n| \, dx \le \varepsilon \, C_R,$$

since $(g_n)_n$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ by Lemma 3.8. By $(\mathcal{A})$-(b) and the definitions of φ_R and η_ε,

$$\varphi_R(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(\eta_\varepsilon \circ (u_n - u)) \ge 0 \quad \text{a.e. in } \mathbb{R}^N,$$

and in turn

$$\begin{aligned} \int_{B_R} \varphi_R(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(\eta_\varepsilon \circ (u_n - u)) dx \\ \le \int_{\mathbb{R}^N} \varphi_R(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(\eta_\varepsilon \circ (u_n - u)) dx. \end{aligned}$$

Combining all these facts with (4.5), we find that

$$\limsup_{n \to \infty} \int_{B_R} \varphi_R(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(\eta_\varepsilon \circ (u_n - u))\, dx \le \varepsilon \, C_R. \tag{4.6}$$

Define the non-negative function e_n by

$$e_n(x) = (\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot \nabla(u_n - u).$$

Note that $(e_n)_n$ is bounded in $L^1(\mathbb{R}^N)$. Indeed,

$$0 \le \int_{\mathbb{R}^N} e_n(x)\, dx \le \|\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)\|_{p'} \cdot \|\nabla u_n - \nabla u\|_p \le C_0, \tag{4.7}$$

where C_0 is an appropriate constant, independent of n, deriving from the boundedness of $(\nabla u_n)_n$ in $[L^p(\mathbb{R}^N)]^N$ and of $(\mathbf{A}(x, \nabla u_n))_n$ in $[L^{p'}(\mathbb{R}^N)]^N$ by (4.4).

Fix $\theta \in (0,1)$. Split the ball B_R into

$$S_n^\varepsilon(R) = \{x \in B_R : |u_n(x) - u(x)| \leq \varepsilon\}, \quad G_n^\varepsilon(R) = B_R \setminus S_n^\varepsilon(R).$$

By Hölder's inequality,

$$\begin{aligned}\int_{B_R} e_n^\theta dx &\leq \left(\int_{S_n^\varepsilon(R)} e_n dx\right)^\theta |S_n^\varepsilon(R)|^{1-\theta} + \left(\int_{G_n^\varepsilon(R)} e_n dx\right)^\theta |G_n^\varepsilon(R)|^{1-\theta}\\ &\leq (\varepsilon C_R)^\theta |S_n^\varepsilon(R)|^{1-\theta} + C_0^\theta |G_n^\varepsilon(R)|^{1-\theta},\end{aligned}$$

by (4.6), since $\varphi_R \equiv 1$ and $\nabla(\eta_\varepsilon \circ (u_n - u)) = \nabla(u_n - u)$ in $S_n^\varepsilon(R)$, and by (4.7). Moreover, $|G_n^\varepsilon(R)|$ tends to zero as $n \to \infty$. Hence

$$0 \leq \limsup_{n\to\infty} \int_{B_R} e_n^\theta \, dx \leq (\varepsilon C_R)^\theta |B_R|^{1-\theta}.$$

Letting ε tend to 0^+ we find that $e_n^\theta \to 0$ in $L^1(B_R)$ and so, thanks to the arbitrariness of R, we deduce that

$$e_n \to 0 \quad \text{a.e. in } \mathbb{R}^N$$

up to a subsequence. Thus, by Lemma 2.1

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N,$$

and so

$$\mathbf{A}(x, \nabla u_n) \to \mathbf{A}(x, \nabla u) \quad \text{a.e. in } \mathbb{R}^N,$$

by $(\mathcal{A})$. Hence, Proposition A.7 implies that $\Theta = \mathbf{A}(x, \nabla u)$ and consequently for all $v \in X$

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u_n) \cdot \nabla v \, dx \to \int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla v \, dx \tag{4.8}$$

as $n \to \infty$, since $\mathbf{A}(x, \nabla u_n) \rightharpoonup \mathbf{A}(x, \nabla u)$ in $[L^{p'}(\mathbb{R}^N)]^N$ by (4.4).

Step 2. Since $u_n \rightharpoonup u$ in X, Lemma 3.3 yields in particular that for all $v \in X$

$$\int_{\mathbb{R}^N} a(x)|u_n|^{p-2} u_n v \, dx \to \int_{\mathbb{R}^N} a(x)|u|^{p-2} uv \, dx \tag{4.9}$$

as $n \to \infty$. Moreover, Lemmas 3.4 and 3.5 imply that for all $v \in X$

$$\begin{aligned}\int_{\mathbb{R}^N} w(x)|u_n|^{q-2} u_n v \, dx &\to \int_{\mathbb{R}^N} w(x)|u|^{q-2} uv \, dx,\\ \int_{\mathbb{R}^N} h(x)|u_n|^{r-2} u_n v \, dx &\to \int_{\mathbb{R}^N} h(x)|u|^{r-2} uv \, dx,\end{aligned} \tag{4.10}$$

as $n \to \infty$. In conclusion, passing to the limit in (4.3) as $n \to \infty$, we get by (4.8)–(4.10)

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} a(x)|u|^{p-2} uv \, dx = \lambda^* \int_{\mathbb{R}^N} w(x)|u|^{q-2} uv \, dx\\ - \int_{\mathbb{R}^N} h(x)|u|^{r-2} uv \, dx\end{aligned}$$

for all $v \in X$, that is u is an entire non-negative solution of $(\mathcal{E})_{\lambda^*}$.

Step 3. We claim that $u \not\equiv 0$. Indeed, since $u_n \rightharpoonup u$ in X by (4.4), Lemma 2.3 yields in particular that $\|u\|_{q,w} = \lim_{n\to\infty} \|u_n\|_{q,w}$. Moreover, (2.7) applied to each $u_n \neq 0$ implies that $\|u_n\|_{q,w} \geq \kappa_1 \lambda_n^{1/(p-q)}$, that is

$$\|u\|_{q,w} = \lim_{n\to\infty} \|u_n\|_{q,w} \geq \kappa_1 (\lambda^*)^{1/(p-q)} > 0,$$

since $\lambda_n \searrow \lambda^*$ and $\lambda^* > 0$. Hence u is nontrivial and non-negative by (4.4). □

Proof of Theorem A. Section 2, Lemma 4.3 and Theorems 4.2 and 4.4 show the existence of $\lambda^* > 0$, with the properties required in Theorem A. □

5. Existence of a second non-negative entire solution in a special case

In this section we prove that equation $(\mathcal{E})_\lambda$ admits at least two nontrivial solutions if λ is sufficiently large, via variational methods. We start by establishing a geometrical property for the energy functional Φ_λ, which is valid for all $\lambda > 0$ and is a variant of the geometrical structure of the Mountain Pass theorem due to Ambrosetti and Rabinowitz [4]. For a similar result, obtained with a different proof and the use of the Palais–Smale compactness condition, we refer to [7].

Lemma 5.1. *For any $e \in X \setminus \{0\}$ and $\lambda > 0$ there exist $\varrho \in (0, \|e\|_E)$ and $\alpha = \alpha(\varrho) > 0$ such that $\Phi_\lambda(u) \geq \alpha$ for all $u \in X$, with $\|u\|_E = \varrho$.*

Proof. Let u be in X. By (2.4) and $(2.5)_2$

$$\Phi_\lambda(u) \geq \frac{\kappa}{p}\|u\|_E^p - \frac{\lambda}{q}\|u\|_{q,w}^q \geq \left(\frac{\kappa}{p} - \frac{\lambda}{q}\,\mathfrak{C}_w^q \|u\|_E^{q-p}\right)\|u\|_E^p.$$

Therefore, it is enough to take $0 < \varrho < \min\left\{(\kappa q/\lambda p \mathfrak{C}_w^q)^{1/(q-p)}, \|e\|_E\right\}$, so that $\alpha = (\kappa/p - \lambda \mathfrak{C}_w^q \varrho^{q-p}/q)\,\varrho^p > 0$ satisfies the assertion. □

In Lemma 4.1 we have shown that for all $\lambda > \overline{\lambda}$ there exists a nontrivial non-negative entire solution $e \in X$ of $(\mathcal{E})_\lambda$, which is a global minimizer for Φ_λ in X, with $\Phi_\lambda(e) < 0$. In this section we are looking for a second nontrivial solution of $(\mathcal{E})_\lambda$, when $\lambda > \overline{\lambda}$.

By Lemma 5.1 and the variant of the *Ekeland principle* given in Theorem A.3, for all $\lambda > \overline{\lambda}$ there exists a sequence $(u_n)_n \in X$ such that

$$\Phi_\lambda(u_n) \to c \quad \text{and} \quad \|\Phi_\lambda'(u_n)\|_{X'} \to 0 \tag{5.1}$$

as $n \to \infty$, where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

The principal aim is to prove that $(u_n)_n$ strongly converges to some $u \in X$ and that u is a second nontrivial non-negative solution of $(\mathcal{E})_\lambda$. In order to do this, we assume the further following growth condition on $\mathscr{A}$.

$(\mathcal{A})$-(d) *There exists* $k>0$ *such that*

$$|\xi-\xi_0|^p \le k \begin{cases} (\mathbf{A}(x,\xi)-\mathbf{A}(x,\xi_0))\cdot(\xi-\xi_0), & p\ge 2,\\ \{(\mathbf{A}(x,\xi)-\mathbf{A}(x,\xi_0))\cdot(\xi-\xi_0)\}^{\frac{p}{2}}\left(|\xi|^p+|\xi_0|^p\right)^{\frac{2-p}{2}}, & 1<p<2,\end{cases}$$

for all $x,\xi,\xi_0\in\mathbb{R}^N$.

Any function $\mathbf{A}(x,\xi)=d(x)|\xi|^{p-2}\xi$, with $d\in C(\mathbb{R}^N,\mathbb{R}^+)\cap L^\infty(\mathbb{R}^N)$, satisfies $(\mathcal{A})$-(d). For the p-Laplacian operator, $\mathbf{A}(x,\xi)=|\xi|^{p-2}\xi, p>1$, property $(\mathcal{A})$-(d) was proved by Simon in [22]. In particular,
$(\mathscr{S})$ *For all* $s\in(1,\infty)$ *there exists* $\hat{k}>0$, *depending only on* s, *such that*

$$|\xi-\xi_0|^s \le \hat{k} \begin{cases} (|\xi|^{s-2}\xi-|\xi_0|^{s-2}\xi_0)\cdot(\xi-\xi_0), & s\ge 2,\\ \{(|\xi|^{s-2}\xi-|\xi_0|^{s-2}\xi_0)\cdot(\xi-\xi_0)\}^{\frac{s}{2}}\left(|\xi|^s+|\xi_0|^s\right)^{\frac{2-s}{2}}, & 1<s<2,\end{cases}$$

for all $\xi,\xi_0\in\mathbb{R}^N$.

Proof of Theorem B. Fix $\lambda>\overline{\lambda}$ and let $e\in X$ be the global minimizer of Φ_λ obtained by Lemma 4.1.

Step 1. Thanks to Lemma 5.1 and the fact that $\Phi_\lambda(e)<0$, the assumptions of Theorem A.3 are satisfied for the functional Φ_λ. Hence, there exists a sequence $(u_n)_n\subset X$ such that (5.1) holds. By Lemma 3.1, the sequence $(u_n)_n$ is bounded in X and so also $(\mathbf{A}(x,\nabla u_n))_n$ is bounded in $[L^{p'}(\mathbb{R}^N)]^N$ by $(\mathcal{A})$-(c). From now on we can follow the argument of the proof of Theorem 4.4. We report here the main differences. By Lemma 2.3, Propositions A.6, A.10 and A.11, it is again possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying (4.4), for some $u\in X$ and $\Theta\in[L^{p'}(\mathbb{R}^N)]^N$. The main point of the proof is to prove that $\Theta=\mathbf{A}(x,\nabla u)$ and that u is a nontrivial solution of $(\mathcal{E})_\lambda$, with $u\ne e$.

Clearly, now

$$\langle\Phi'_\lambda(u_n),v\rangle=\int_{\mathbb{R}^N}\mathbf{A}(x,\nabla u_n)\cdot\nabla v\,dx-\int_{\mathbb{R}^N}g_n v\,dx \tag{5.2}$$

for any $v\in X$, where the sequence $(g_n)_n$, defined in (3.16) with $\lambda_n\equiv\lambda$, is in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ by Lemma 3.8, being $(\|u_n\|)_n$ bounded.

Fix $R>0$ and $\varepsilon>0$. Take $\varphi_R\in C_0^\infty(\mathbb{R}^N)$ and η_ε as in Theorem 4.4. Put again $v_n=\varphi_R\eta_\varepsilon\circ(u_n-u)\in X$. Taking $v=v_n$ in (5.2), it results that

$$\begin{aligned}\int_{\mathbb{R}^N}&\varphi_R(\mathbf{A}(x,\nabla u_n)-\mathbf{A}(x,\nabla u))\cdot\nabla(\eta_\varepsilon\circ(u_n-u))\,dx\\ &=-\int_{\mathbb{R}^N}\eta_\varepsilon\circ(u_n-u)\mathbf{A}(x,\nabla u_n)\cdot\nabla\varphi_R\,dx\\ &\quad-\int_{\mathbb{R}^N}\varphi_R\mathbf{A}(x,\nabla u)\cdot\nabla(\eta_\varepsilon\circ(u_n-u))\,dx\\ &\quad+\langle\Phi'_\lambda(u_n),v_n\rangle+\int_{\mathbb{R}^N}g_n v_n\,dx.\end{aligned} \tag{5.3}$$

Proceeding as for Theorem 4.4, we find that the first two terms in the right hand side of (5.3) go to zero as $n\to\infty$. Moreover

$$\langle \Phi'_\lambda(u_n), v_n \rangle \to 0 \quad \text{as } n \to \infty,$$

since $\Phi'_\lambda(u_n) \to 0$ in X' and $v_n \rightharpoonup 0$ in X as $n \to \infty$. Finally, recalling that also in this case $(g_n)_n$ is bounded in $L^1_{\rm loc}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} g_n v_n \, dx \le \varepsilon C_R,$$

where $C_R > 0$ is an appropriate constant dependent only on R. From now on, proceeding as in the proof of Theorem 4.4, we find that

$$\mathbf{A}(x, \nabla u_n) \to \mathbf{A}(x, \nabla u) \quad \text{as } \ n \to \infty \ \text{ for a.a. } x \in \mathbb{R}^N.$$

Hence, Proposition A.7 shows that $\Theta = \mathbf{A}(x, \nabla u)$ and so for all $v \in X$

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u_n) \cdot \nabla v \, dx \to \int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla v \, dx \quad \text{as } \ n \to \infty, \tag{5.4}$$

since $\mathbf{A}(x, \nabla u_n) \rightharpoonup \mathbf{A}(x, \nabla u)$ in $[L^{p'}(\mathbb{R}^N)]^N$ by (4.4).

The proofs of *Steps 2* and 3 of Theorem 4.4 can be repeated also in this case, obtaining (4.9) and (4.10). Hence, passing to the limit as $n \to \infty$ in (5.2), using (5.4) and the fact that $\langle \Phi'_\lambda(u_n), v \rangle \to 0$ as $n \to \infty$ for all $v \in X$, we get

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} a(x)|u|^{p-2}uv \, dx = \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-2}uv \, dx - \int_{\mathbb{R}^N} h(x)|u|^{r-2}uv \, dx$$

for all $v \in X$, that is u is an entire solution of $(\mathcal{E})_\lambda$.
Step 2. We claim that

$$\mathcal{J}_w(n) = \int_{\mathbb{R}^N} w(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) \, dx \to 0 \tag{5.5}$$

as $n \to \infty$. Indeed, $u_n \to u$ in $L^q(\mathbb{R}^N; w)$ by Lemma 2.3, since $u_n \rightharpoonup u$ in X by (4.4). Thus, $|u_n|^{q-2}u_n \to |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N; w)$ by Proposition A.8-(ii) and in turn, applying Hölder's inequality, we get

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} w(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) \, dx \\ &\le \| |u_n|^{q-2}u_n - |u|^{q-2}u\|_{q',w} \|u_n - u\|_{q,w} \to 0, \end{aligned}$$

as $n \to \infty$. This completes the proof of (5.5).
Step 3. Here we show that

$$\|u_n - u\| \to 0 \quad \text{as } \ n \to \infty. \tag{5.6}$$

Clearly, by convexity

$$\begin{aligned} \mathcal{I}_{1,n} &= \int_{\mathbb{R}^N} (\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \ge 0 \\ \mathcal{I}_{2,n} &= \int_{\mathbb{R}^N} a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \ge 0 \\ \mathcal{I}_{3,n} &= \int_{\mathbb{R}^N} h(x)(|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u) \, dx \ge 0. \end{aligned}$$

Put $\mathcal{R}_n = \mathcal{I}_{1,n} + \mathcal{I}_{2,n} + \mathcal{I}_{3,n}$. Note that $\langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in X and $\Phi'_\lambda(u_n) \to 0$ in X' as $n \to \infty$. Hence, by (5.5)

$$\mathcal{R}_n = \langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle + \lambda \mathcal{J}_w(n) = o(1) \tag{5.7}$$

as $n \to \infty$.

By $(\mathscr{S})$, with $s = r > 2$, and by (5.7) it follows that

$$\|u_n - u\|^r_{r,h} \le \hat{k}\mathcal{I}_{3,n} = o(1), \tag{5.8}$$

as $n \to \infty$.

Case $p \ge 2$. Using (1.2), $(\mathcal{A})$-(d) and $(\mathscr{S})$, with $s = p$, we get

$$\begin{aligned}\|u_n - u\|^p_E &\le k \int_{\mathbb{R}^N} (\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \cdot (\nabla u_n - \nabla u)dx \\ &\quad + \frac{1}{c_1}\|u_n - u\|^p_{p,a} \\ &\le \max\{k, \hat{k}/c_1\}\, \{\mathcal{I}_{1,n} + \mathcal{I}_{2,n}\} = o(1),\end{aligned} \tag{5.9}$$

as $n \to \infty$, by (5.7).

Case $1 < p < 2$. By $(\mathscr{S})$, with $s = p$, and Hölder's inequality we have

$$\begin{aligned}\|\nabla u_n - \nabla u\|^p_p &\le k \int_{\mathbb{R}^N} \{(\mathbf{A}(x, \nabla u_n) - \mathbf{A}(x, \nabla u)) \\ &\quad \cdot (\nabla u_n - \nabla u)\}^{p/2} (|\nabla u_n|^p + |\nabla u|^p)^{(2-p)/2} dx \\ &\le k\mathcal{I}^{p/2}_{1,n} \left(\|\nabla u_n\|^p_p + \|\nabla u\|^p_p\right)^{(2-p)/2} \\ &\le k\mathcal{I}^{p/2}_{1,n} \left(\|\nabla u_n\|^{(2-p)p/2}_p + \|\nabla u\|^{(2-p)p/2}_p\right) \\ &\le 2kM^{(2-p)p/2}\mathcal{I}^{p/2}_{1,n},\end{aligned}$$

where $M > 0$ is such that $\|\nabla u_n\|_p, \|\nabla u\|_p \le M$ for all n, being the sequence $(\|\nabla u_n\|_p)_n$ bounded by (4.4). Similarly, again by $(\mathscr{S})$, with $s = p$, we have

$$\|u_n - u\|^p_{p,a} \le \hat{k}\mathcal{I}^{p/2}_{2,n} \left(\|u_n\|^{(2-p)p/2}_{p,a} + \|u\|^{(2-p)p/2}_{p,a}\right) \le 2\hat{k}\mathfrak{K}^{(2-p)p/2}\mathcal{I}^{p/2}_{2,n},$$

where $\mathfrak{K} > 0$ is such that $\|u_n\|_{p,a}, \|u\|_{p,a} \le \mathfrak{K}$ for all n, being $(\|u_n\|_{p,a})_n$ bounded. Hence, using also (1.2), it follows that

$$\begin{aligned}\|u_n - u\|^2_E &\le 2^{(2-p)/p} \left\{\|\nabla u_n - \nabla u\|^2_p + \left(\int_{\mathbb{R}^N} \nu(x)|u_n - u|^p dx\right)^{2/p}\right\} \\ &\le 2^{(2-p)/p} \left\{\|\nabla u_n - \nabla u\|^2_p + \left(\frac{1}{c_1}\|u_n - u\|^p_{p,a}\right)^{2/p}\right\} \\ &\le \mathcal{C}\, \{\mathcal{I}_{1,n} + \mathcal{I}_{2,n}\} = o(1)\end{aligned} \tag{5.10}$$

as $n \to \infty$ by (5.7), where $\mathcal{C} = 2^{(2-p)/p} \max\{(2k)^{2/p}M^{2-p}, (2\hat{k}/c_1)^{2/p}\mathfrak{K}^{2-p}\}$.

In conclusion, for all $p > 1$, using (5.8)–(5.10), we obtain (5.6).

Step 4. Since $u_n \to u$ in X and $\Phi_\lambda \in C^1(X)$, we have that $\Phi_\lambda(u) = c = \lim_{n\to\infty} \Phi_\lambda(u_n)$. Therefore, u is a second independent nontrivial entire solution of $(\mathcal{E})_\lambda$, with $\Phi_\lambda(u) = c > 0 > \Phi_\lambda(e)$. Clearly we can assume $u \ge 0$ a.e. in $\mathbb{R}^N$, since $|u|$ is also a solution of $(\mathcal{E})_\lambda$ by $(\mathcal{A})$-(a). This concludes the proof.

Acknowledgments

The authors have been supported by the MIUR project *Metodi Variazionali ed Equazioni Differenziali alle Derivate Parziali Non Lineari*.

Appendix A

In this section we present some auxiliary results, starting by recalling this definition.

A Banach space $(V, \|\cdot\|_V)$ is said to be *locally uniformly convex* if $\|v_n\|_V = \|v\|_V = 1$ and $\lim_{n\to\infty} \|v_n + v\|_V = 2$ imply that $\lim\limits_{n\to\infty} \|v_n - v\|_V = 0$.

Proposition A.1. *Let $(V, \|\cdot\|_V)$ be a locally uniformly convex Banach space. Let $(v_n)_n$ and v be in V such that*

$$\text{(i)}\quad v_n \rightharpoonup v \text{ in } V \text{ as } n \to \infty, \qquad \text{(ii)}\quad \limsup_{n\to\infty} \|v_n\|_V \le \|v\|_V.$$

Then $v_n \to v$ in V as $n \to \infty$.

Proof. If $v = 0$ there is nothing to prove and the conclusion is obvious. Hence suppose $v \neq 0$. By (ii) and the weak lower semicontinuity of the norm, we get

$$\lim_{n\to\infty} \|v_n\|_V = \|v\|_V. \tag{5.1}$$

Thus there exists $\overline{n} \ge 0$ such that for all $n \ge \overline{n}$ it is $\|v_n\|_V > 0$. For $n \ge \overline{n}$ define $y_n = v_n/\|v_n\|_V$ and $y = v/\|v\|_V$, so that $\|y_n\|_V = \|y\|_V = 1$. For all $f \in V'$ we have

$$\begin{aligned} f(y_n - y) &= f\left(\frac{v_n}{\|v_n\|_V} - \|v_n\|_V \frac{v_n}{\|v_n\|_V}\right) + f(v_n - v) + f\left(\|v\|_V \frac{v}{\|v\|_V} - \frac{v}{\|v\|_V}\right) \\ &= \frac{1 - \|v_n\|_V}{\|v_n\|_V} f(v_n) + f(v_n - v) + \frac{\|v\|_V - 1}{\|v\|_V} f(v). \end{aligned}$$

Now, by (i), the sequence $(\|v_n\|_V)_n$ is bounded, while $f(v_n) \to f(v)$ and $f(v_n - v) \to 0$. That is $f(y_n) \to f(y)$, or, in other words, $y_n \rightharpoonup y$ in V.

We claim that $\lim_{n\to\infty} \|y_n + y\|_V = 2$. To prove this, first note that the weak convergence of y_n and the weak lower semicontinuity of the norm imply that

$$\frac{y_n + y}{2} \rightharpoonup y \quad \text{and} \quad \|y\|_V \le \frac{1}{2} \liminf_{n\to\infty} \|y_n + y\|_V.$$

Consequently,

$$\begin{aligned} 1 = \|y\|_V &\le \frac{1}{2} \liminf_{n\to\infty} \|y_n + y\|_V \le \frac{1}{2} \limsup_{n\to\infty} \|y_n + y\|_V \\ &\le \frac{1}{2} \limsup_{n\to\infty} (\|y_n\|_V + \|y\|_V) = 1, \end{aligned}$$

that is $\lim_{n\to\infty} \|y_n + y\|_V = 2$, as claimed. The local uniform convexity of V assures that $\lim_{n\to\infty} \|y_n - y\|_V = 0$. Hence,

$$\|v_n - v\|_V \le \|v_n\|_V \|y_n - y\|_V + \|y\|_V \big|\|v_n\|_V - \|v\|_V\big| \to 0 \quad \text{as } n \to \infty,$$

by (5.1) and the boundedness of $(\|v_n\|_V)_n$. The proof is complete. □

As a consequence of Proposition A.1 we have the following important

Corollary A.2. *Let* $(V, \|\cdot\|_V)$ *be a reflexive Banach space. Let* $(v_n)_n$ *and* v *be in* V *such that as* $n \to \infty$

$$\text{(i)} \quad v_n \rightharpoonup v \text{ in } V \qquad \text{(ii)} \quad \|v_n\|_V \to \|v\|_V.$$

Then $v_n \to v$ *in* V *as* $n \to \infty$.

Proof. First observe that there exists an equivalent norm on V, say $\|\cdot\|$, which makes V a locally uniformly convex Banach space, see the *Troyanski* theorem [25]. Hence, conditions (i) and (ii) hold also if we consider on V the norm $\|\cdot\|$. Therefore $\|v_n - v\| \to 0$ as $n \to \infty$, by Proposition A.1. Finally, also $\|v_n - v\|_V \to 0$, being the two norms equivalent. □

The following theorem is stated for two general Banach spaces X and E. We apply it in Theorem B, in which X and E are the special spaces defined in Section 2. The proof of Theorem A.3 is based on the *Ekeland variational principle*, see for instance [16]. For a similar generalization of the Mountain Pass theorem, with a different proof and the use of a compactness condition, we refer to Theorem 2.5 of [7].

Theorem A.3. *Let* $(X, \|\cdot\|)$ *and* $(E, \|\cdot\|_E)$ *be two Banach spaces such that* $X \hookrightarrow E$. *Let* $\Phi : X \to \mathbb{R}$ *be a* C^1 *functional with* $\Phi(0) = 0$. *Suppose that there exist* $\varrho, \alpha > 0$ *and* $e \in X$ *such that* $\|e\|_E > \varrho, \Phi(e) < \alpha$ *and* $\Phi(u) \geq \alpha$ *for all* $u \in X$ *with* $\|u\|_E = \varrho$.

Then there exists a sequence $(u_n)_n \in X$ *such that for all* n

$$c \leq \Phi(u_n) \leq c + \frac{1}{n^2} \quad \text{and} \quad \|\Phi'(u_n)\|_{X'} \leq \frac{2}{n},$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0, \ \gamma(1) = e\}.$$

Proof. Step 1. We claim that $c \geq \alpha$. Indeed, fix $\gamma \in \Gamma$. Clearly, $\gamma \in C([0,1]; E)$, since $X \hookrightarrow E$. This implies that the function $g : [0,1] \to \mathbb{R}$ defined by $g(t) = \|\gamma(t)\|_E$ is continuous and such that $g(0) = 0 < \varrho$ and $g(1) = \|e\|_E > \varrho$. Hence $\|\gamma(t_\varrho)\|_E = \varrho$ for some $t_\varrho \in (0,1)$. Consequently, $\Phi(\gamma(t_\varrho)) \geq \alpha$ and in turn $\max_{t \in [0,1]} \Phi(\gamma(t)) \geq \alpha$. Finally, due to the arbitrariness of γ, we get the claim.

Step 2. Let us consider on Γ the metric $d_\infty(\gamma, \eta) = \max_{t \in [0,1]} \|\gamma(t) - \eta(t)\|$. Thus, (Γ, d_∞) is a complete metric space. Define on Γ the functional

$$J(\gamma) = \max_{t \in [0,1]} \Phi(\gamma(t)).$$

Of course, J is bounded from below by *Step* 1, since $J(\gamma) \geq \alpha$ for all $\gamma \in \Gamma$. Moreover, J is lower semicontinuous, being the supremum of continuous functions. Therefore, by *Ekeland's variational principle*, for all $\varepsilon > 0$ there exists $\gamma_\varepsilon \in \Gamma$ such that

(i) $J(\gamma_\varepsilon) \leq \inf_{\gamma \in \Gamma} J(\gamma) + \varepsilon^2$;
(ii) $J(\gamma) \geq J(\gamma_\varepsilon) - \varepsilon d_\infty(\gamma, \gamma_\varepsilon)$ for all $\gamma \in \Gamma$.

Step 3. We claim that for all $\varepsilon > 0$ and all $\eta_\varepsilon \in C([0,1], X)$ such that

$$\eta_\varepsilon(0) = \eta_\varepsilon(1) = 0 \quad \text{and} \quad \max_{t\in[0,1]} \|\eta_\varepsilon(t)\| \le 1 \tag{A.2}$$

there exists $t_\varepsilon \in [0,1]$ such that

$$\Phi(\gamma_\varepsilon(t_\varepsilon)) = J(\gamma_\varepsilon) \quad \text{and} \quad -\varepsilon \le \langle \Phi'(\gamma_\varepsilon(t_\varepsilon)), \eta_\varepsilon(t_\varepsilon)\rangle. \tag{A.3}$$

Indeed, fix $\varepsilon > 0$ and choose a continuous function $\eta_\varepsilon : [0,1] \to X$ satisfying (A.2). Define $\gamma_{\varepsilon,h} = \gamma_\varepsilon + h\,\eta_\varepsilon$ for all $h > 0$, so that $\gamma_{\varepsilon,h} \in \Gamma$. By the continuity of Φ and $\gamma_{\varepsilon,h}$, there exists $t_{\varepsilon,h} \in [0,1]$ such that

$$J(\gamma_{\varepsilon,h}) = \Phi(\gamma_{\varepsilon,h}(t_{\varepsilon,h})).$$

Moreover, the boundedness of $(t_{\varepsilon,h})_h$ implies the existence of a convergent subsequence, still denoted in the same way, with limit $t_\varepsilon \in [0,1]$. Therefore, thanks to the continuity of Φ and the lower semicontinuity of J, we get

$$\Phi(\gamma_\varepsilon(t_\varepsilon)) \le J(\gamma_\varepsilon) \le \liminf_{h\to 0^+} J(\gamma_{\varepsilon,h}) = \lim_{h\to 0^+} \Phi(\gamma_{\varepsilon,h}(t_{\varepsilon,h})) = \Phi(\gamma_\varepsilon(t_\varepsilon)),$$

that is $(\text{A.3})_1$ holds. Moreover, by (ii) and $(\text{A.2})_2$,

$$\begin{aligned} -\varepsilon h \le J(\gamma_{\varepsilon,h}) - J(\gamma_\varepsilon) &\le \Phi(\gamma_{\varepsilon,h}(t_{\varepsilon,h})) - \Phi(\gamma_\varepsilon(t_{\varepsilon,h})) \\ &= h\langle \Phi'(\gamma_\varepsilon(t_{\varepsilon,h})), \eta_\varepsilon(t_{\varepsilon,h})\rangle + o(h) \end{aligned}$$

as $h \to 0^+$. In turn, dividing by $h > 0$ and passing to the limit as $h \to 0^+$, we find $(\text{A.3})_2$.

Step 4. We claim that for all $\varepsilon > 0$ there exists $u_\varepsilon \in X$ such that

$$\|\Phi'(u_\varepsilon)\| \le 2\varepsilon \quad \text{and} \quad \inf_{\gamma\in\Gamma} J(\gamma) \le \Phi(u_\varepsilon) \le \inf_{\gamma\in\Gamma} J(\gamma) + \varepsilon^2. \tag{A.4}$$

Fix $\varepsilon > 0$ and let $\gamma_\varepsilon \in \Gamma$ be the curve obtained in *Step 2*, satisfying (i) and (ii). Take a with $\max\{\Phi(e), 0\} < a < \alpha$. This is possible being $\Phi(e) < \alpha$. Now, for all $t \in [0,1]$ for which

$$\Phi(\gamma_\varepsilon(t)) > a \tag{A.5}$$

we find $x_t \in X$, with $\|x_t\| = 1$, such that

$$\langle \Phi'(\gamma_\varepsilon(t)), x_t\rangle \le -\frac{3}{4}\|\Phi'(\gamma_\varepsilon(t))\|.$$

Since $\Phi \in C^1(X)$ and γ_ε is continuous, then

$$\langle \Phi'(\gamma_\varepsilon(s)), x_t\rangle \le -\frac{1}{2}\|\Phi'(\gamma_\varepsilon(s))\|$$

for all s in a neighborhood of t. By compactness of $[0,1]$ there exists a finite number of open sets $U_1, \ldots, U_k$ and vectors $x_1, \ldots, x_k \in X$, with $\|x_i\| = 1$, $i = 1, \ldots, k$, such that if $\Phi(\gamma_\varepsilon(t)) > a$, then $t \in U_i$ for some $i = 1, \ldots, k$ and

$$\langle \Phi'(\gamma_\varepsilon(t)), x_i\rangle \le -\frac{1}{2}\|\Phi'(\gamma_\varepsilon(t))\|.$$

Let V be such that $U_1, \ldots, U_k, V$ cover $[0,1]$ and let $\varphi_1, \ldots, \varphi_k, \psi$ be a partition of unity subordinated to this covering. Set

$$\eta(t) = \sum_{i=1}^{k} \varphi_i(t) x_i \quad \text{for all } t \in [0,1].$$

Note that $\eta \in \operatorname{span}\{x_1, \ldots, x_k\}$ and for all $t \in [0,1]$ satisfying (A.5)

$$\begin{aligned} \langle \Phi'(\gamma_\varepsilon(t)), \eta(t) \rangle &= \sum_{i=1}^{k} \varphi_i(t) \langle \Phi'(\gamma_\varepsilon(t)), x_i \rangle \le -\frac{1}{2} \|\Phi'(\gamma_\varepsilon(t))\| \sum_{i=1}^{k} \varphi_i(t) \\ &= -\frac{1}{2} \|\Phi'(\gamma_\varepsilon(t))\|. \end{aligned} \tag{A.6}$$

Moreover, η verifies (A.2), since $\max_{t\in[0,1]} \|\eta(t)\| \le 1, \Phi(0) = 0 < a$ and $\Phi(e) < a$. Hence, by *Step* 3, there exists $t_\varepsilon \in [0,1]$ satisfying (A.3). Furthermore, $\Phi(\gamma_\varepsilon(t_\varepsilon)) = J(\gamma_\varepsilon) \ge \alpha > a$ by $(A.3)_1$ and *Step* 1. Therefore, by (A.6) and $(A.3)_1$, with $\eta_\varepsilon = \eta$, we get

$$-\varepsilon \le \langle \Phi'(\gamma_\varepsilon(t_\varepsilon)), \eta(t_\varepsilon) \rangle \le -\frac{1}{2} \|\Phi'(\gamma_\varepsilon(t_\varepsilon))\|$$

that is

$$\|\Phi'(\gamma_\varepsilon(t_\varepsilon))\| \le 2\varepsilon.$$

Finally, using (ii) of *Step* 2 and $(A.3)_1$, we obtain

$$\inf_{\gamma\in\Gamma} J(\gamma) \le \Phi(\gamma_\varepsilon(t_\varepsilon)) \le \inf_{\gamma\in\Gamma} J(\gamma) + \varepsilon^2.$$

Hence, (A.4) holds with $u_\varepsilon = \gamma_\varepsilon(t_\varepsilon) \in X$.

This concludes the proof, by taking $\varepsilon = 1/n$ and $u_n = \gamma_n(t_n)$. □

We present now some results on the weighted Lebesgue spaces. Let ω be a weight on $\mathbb{R}^N$, that is a measurable function such that $\omega > 0$ a.e. in $\mathbb{R}^N$. If $s \in \mathbb{R}^+$, following [14, Chapter V, Section 6], we put

$$L^s(\mathbb{R}^N; \omega) = \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \omega|u|^s \in L^1(\mathbb{R}^N)\}.$$

The set $L^s(\mathbb{R}^N; \omega)$ is a linear space, thanks to the inequality (1.5) given in [14, Lemma 1.1, page 222], and

$$\|u\|_{s,\omega} = \left(\int_{\mathbb{R}^N} \omega(x)|u(x)|^s dx \right)^{1/s}$$

is a *norm-like* function on $L^s(\mathbb{R}^N; \omega)$ when $s \in (0,1)$, and a norm if $s \in [1,\infty)$.

The next result is well-known in the usual Lebesgue spaces (see, for instance, Theorem 4.9 of [6]). The proof is left to the reader, since it is standard, see also [21].

Lemma A.4. *Let $s \in [1,\infty)$. If $(u_n)_n$ and u are in $L^s(\mathbb{R}^N; \omega)$ and $u_n \to u$ in $L^s(\mathbb{R}^N; \omega)$ as $n \to \infty$, then there exist a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and a function $\psi \in L^s(\mathbb{R}^N; \omega)$ such that a.e. in $\mathbb{R}^N$*

(i) $u_{n_k} \to u$ *as* $k \to \infty$; (ii) $|u_{n_k}(x)| \le \psi(x)$ *for all* $k \in \mathbb{N}$.

Proposition A.5. *Let* $\sigma \in (0,1)$*. If* $u, v \in L^\sigma(\mathbb{R}^N;\omega)$ *then*

$$\||u| + |v|\|_{\sigma,\omega} \geq \|u\|_{\sigma,\omega} + \|v\|_{\sigma,\omega}.$$

Proof. If $u = v = 0$ in $L^\sigma(\mathbb{R}^N;\omega)$ the conclusion is trivial. Hence, let us suppose $|u| + |v| > 0$ in a subset of $\mathbb{R}^N$ of positive measure, so that $\||u| + |v|\|_{\sigma,\omega} > 0$, being $\omega > 0$ a.e. in $\mathbb{R}^N$.

Let $\sigma' < 0$ be the *conjugate exponent of* σ, given by $1/\sigma + 1/\sigma' = 1$, see [14, Section V.1.2, page 222]. Since $\omega^{1/\sigma'}(|u| + |v|)^{\sigma-1} \in L^{\sigma'}(\mathbb{R}^N)$ and $\omega^{1/\sigma}|u|, \omega^{1/\sigma}|v| \in L^\sigma(\mathbb{R}^N)$, by the *reverse Hölder inequality*, see Theorem 2.6 of [1], we obtain

$$\begin{aligned}
\||u| + |v|\|_{\sigma,\omega}^\sigma &= \int_{\mathbb{R}^N} \omega(x)^{1/\sigma'} \left(|u| + |v|\right)^{\sigma-1} \omega(x)^{1/\sigma} \left(|u| + |v|\right) dx \\
&\geq \left(\int_{\mathbb{R}^N} \omega(x) \left(|u| + |v|\right)^\sigma dx \right)^{1/\sigma'} \left(\int_{\mathbb{R}^N} \omega(x)|u|^\sigma dx \right)^{1/\sigma} \\
&\quad + \left(\int_{\mathbb{R}^N} \omega(x) \left(|u| + |v|\right)^\sigma dx \right)^{1/\sigma'} \left(\int_{\mathbb{R}^N} \omega(x)|v|^\sigma dx \right)^{1/\sigma} \\
&= \||u| + |v|\|_{\sigma,\omega}^{\sigma-1} \left(\|u\|_{\sigma,\omega} + \|v\|_{\sigma,\omega}\right).
\end{aligned}$$

In conclusion, since $\||u| + |v|\|_{\sigma,\omega} > 0$, we get the assertion. □

Proposition A.6. *Let* $s \in (1,\infty)$*. The Banach space* $(L^s(\mathbb{R}^N;\omega), \|\cdot\|_{s,\omega})$ *is uniformly convex.*

Proof. We follow the proof given in [1, Corollary 2.29]. Fix $\varepsilon \in (0,2)$ and let u, v be in $L^s(\mathbb{R}^N;\omega)$ such that $\|u\|_{s,\omega} = \|v\|_{s,\omega} = 1$ and $\|u - v\|_{s,\omega} \geq \varepsilon$.
Case $s \geq 2$. By inequality (3.5) of [1, Lemma 2.27] we have

$$\begin{aligned}
\left\| \frac{u+v}{2} \right\|_{s,\omega}^s + \left\| \frac{u-v}{2} \right\|_{s,\omega}^s &= \int_{\mathbb{R}^N} \omega(x) \left\{ \left| \frac{u+v}{2} \right|^s + \left| \frac{u-v}{2} \right|^s \right\} dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} \omega(x) \left\{ |u|^s + |v|^s \right\} dx \\
&= \frac{1}{2} \left(\|u\|_{s,\omega}^p + \|v\|_{s,\omega}^s \right) = 1.
\end{aligned}$$

Therefore

$$\left\| \frac{u+v}{2} \right\|_{s,\omega}^s \leq 1 - \left(\frac{\varepsilon}{2} \right)^s,$$

and so, taking $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^s = (1-\delta)^s$, we obtain that $\|u + v\|_{s,\omega} \leq 2(1-\delta)$.
Case $1 < s < 2$. First note that $\|u\|_{s,\omega}^{s'} = \||u|^{s'}\|_{s-1,\omega}$ for all $u \in L^s(\mathbb{R}^N;\omega)$, and so, applying Proposition A.5, with $\sigma = s - 1 \in (0,1)$, we obtain

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{s,\omega}^{s'}+\left\|\frac{u-v}{2}\right\|_{s,\omega}^{s'} &= \left\|\left|\frac{u+v}{2}\right|^{s'}\right\|_{s-1,\omega}+\left\|\left|\frac{u-v}{2}\right|^{s'}\right\|_{s-1,\omega}\\ &\le \left\|\left|\frac{u+v}{2}\right|^{s'}+\left|\frac{u-v}{2}\right|^{p'}\right\|_{s-1,\omega}\\ &\le \left(\frac{1}{2}\|u\|_{s,\omega}^{s}+\frac{1}{2}\|v\|_{s,\omega}^{s}\right)^{1/(s-1)} = 1,\end{aligned}$$

where in the last step we have used [1, Lemma 2.27, inequality (3.4)]. Now, proceeding exactly as before, and taking $\delta=\delta(\varepsilon)$ such that $1-(\varepsilon/2)^{s'}=(1-\delta)^{s'}$, we get the claim. □

Proposition A.7. *Assume that $\omega\in L^1_{\text{loc}}(\mathbb{R}^N)$. Let $s\in[1,\infty)$, and let $(u_n)_n, u\in L^s(\mathbb{R}^N;\omega)$ be such that*

$$u_n \rightharpoonup u \text{ in } L^s(\mathbb{R}^N;\omega) \quad \text{and} \quad u_n\to\tilde u \text{ a.e. in } \mathbb{R}^N$$

as $n\to\infty$. Then $u=\tilde u$ a.e. in $\mathbb{R}^N$.

Proof. Denote by $A=\{x\in\mathbb{R}^N : u(x)\ne\tilde u(x)\}$ and suppose by contradiction that A has positive measure. Take $R>0$ so large that $0<|A\cap B_R|<\infty$. By the *Severini–Egoroff* theorem there exists a measurable set $B\subset A\cap B_R$, with $0<|B|<\infty$ such that $(u_n)_n$ converges uniformly to $\tilde u$ in B and so $u_n\to\tilde u$ in $L^\infty(B)$. Consequently,

$$\|u_n-\tilde u\|^s_{L^s(B;\omega)}\le\|\omega\|_{L^1(B)}\|u_n-\tilde u\|^s_{L^\infty(B)}=o(1)$$

as $n\to\infty$, since $|B|<\infty$. Therefore, $u_n\to\tilde u$ and so $u_n\rightharpoonup\tilde u$ in $L^s(B;\omega)$. Hence, $\tilde u=u$ a.e. in B, since the weak limit is unique, being $\omega>0$ a.e. in $\mathbb{R}^N$. But this occurrence is impossible, since $B\subset A$ and $|B|>0$. This contradiction concludes the proof. □

Proposition A.8. *Assume $\omega\in L^1_{\text{loc}}(\mathbb{R}^N)$. Let $s\in(1,\infty)$ and let $(u_n)_n, u$ be in $L^s(\mathbb{R}^N;\omega)$.*

(i) *If $(u_n)_n$ is bounded in $L^s(\mathbb{R}^N;\omega)$ and $u_n\to u$ a.e. in $\mathbb{R}^N$, then*

$$u_n\rightharpoonup u \quad \text{in } L^s(\mathbb{R}^N;\omega) \quad \text{and} \quad |u_n|^{s-2}u_n\rightharpoonup|u|^{s-2}u \quad \text{in } L^{s'}(\mathbb{R}^N;\omega).$$

(ii) *If $\|u_n\|_{s,\omega}\to\|u\|_{s,\omega}$ and $u_n\rightharpoonup u$ in $L^s(\mathbb{R}^N;\omega)$, then*

$$u_n\to u \quad \text{in } L^s(\mathbb{R}^N;\omega) \quad \text{and} \quad |u_n|^{s-2}u_n\to|u|^{s-2}u \quad \text{in } L^{s'}(\mathbb{R}^N;\omega).$$

Proof. Let $(u_n)_n, u$ be in $L^s(\mathbb{R}^N;\omega)$.

Case (i). Fix a subsequence $(u_{n_k})_k$ of $(u_n)_n$. By Proposition A.6 there exists a further subsequence $(u_{n_{k_j}})_j\subset(u_{n_k})_k$ and $v\in L^s(\mathbb{R}^N;\omega)$ such that $u_{n_{k_j}}\rightharpoonup v$ in $L^s(\mathbb{R}^N;\omega)$, being $(u_{n_k})_k$ bounded in $L^s(\mathbb{R}^N;\omega)$. On the other hand, $v=u$ by Proposition A.7, since $\omega\in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\omega>0$ a.e. in $\mathbb{R}^N$ by assumption. By the arbitrariness of $(u_{n_k})_k$, we deduce that the entire sequence $u_n\rightharpoonup u$ in

$L^s(\mathbb{R}^N;\omega)$. Applying the same argument to the sequence $n \mapsto |u_n|^{s-2}u_n$, we obtain that $|u_n|^{s-2}u_n \rightharpoonup |u|^{s-2}u$ in $L^{s'}(\mathbb{R}^N;\omega)$.

Case (*ii*). Corollary A.2 and Proposition A.6 imply that $u_n \to u$ in $L^s(\mathbb{R}^N;\omega)$. Now, fix a subsequence $(v_{n_k})_k$ of $n \mapsto v_n = |u_n|^{s-2}u_n$. Hence $u_{n_k} \to u$ in $L^s(\mathbb{R}^N;\omega)$, and so there exists a further subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ such that $u_{n_{k_j}} \to u$ a.e. in $\mathbb{R}^N$ by Lemma A.4. Of course, $v_{n_{k_j}} \to v = |u|^{s-2}u$ a.e. in $\mathbb{R}^N$. On the other hand, $\|v_{n_{k_j}}\|_{s',\omega}^{s'} = \|u_{n_{k_j}}\|_{s,\omega}^{s} \to \|u\|_{s,\omega}^{s} = \|v\|_{s',\omega}^{s'}$ by assumption. Therefore, $v_{n_{k_j}} \to v$ in $L^{s'}(\mathbb{R}^N;\omega)$ by the first part of this proposition and by Corollary A.2. Due to arbitrariness of $(v_{n_k})_k$, the entire sequence $(v_n)_n$ converges to v in $L^{s'}(\mathbb{R}^N;\omega)$. $\square$

From now on, E and X denote the two Banach spaces defined in Sect. 2. In the next proposition we somehow follow the ideas contained in [15, Theorem 6].

Proposition A.9. *The Banach space* $(E, \|\cdot\|_E)$ *is uniformly convex.*

Proof. Case $p \geq 2$. Fix $\varepsilon \in (0,2)$ and let $u, v \in E$ be such that $\|u\|_E = \|v\|_E = 1$ and $\|u-v\|_E \geq \varepsilon$. Using [1, Lemma 2.27, inequality (3.5)], we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_E^p + \left\|\frac{u-v}{2}\right\|_E^p &= \int_{\mathbb{R}^N}\left\{\left|\frac{\nabla u+\nabla v}{2}\right|^p + \left|\frac{\nabla u-\nabla v}{2}\right|^p\right\}dx \\ &\quad + \int_{\mathbb{R}^N}\nu(x)\left\{\left|\frac{u+v}{2}\right|^p + \left|\frac{u-v}{2}\right|^p\right\}dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N}\left\{|\nabla u|^p + |\nabla v|^p + \nu(x)\left(|u|^p+|v|^p\right)\right\}dx \\ &= \frac{1}{2}\left(\|u\|_E^p + \|v\|_E^p\right) = 1.\end{aligned}$$

Therefore,

$$\left\|\frac{u+v}{2}\right\|_E^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p,$$

and so, taking $\delta = \delta(\varepsilon)$ such that $1-(\varepsilon/2)^p = (1-\delta)^p$, we obtain that $\|u+v\|_E \leq 2(1-\delta)$.

Case $1 < p < 2$. Fix $\varepsilon \in (0, 2^{2/p})$ and let $u, v \in E$ be such that $\|u\|_E = \|v\|_E = 1$ and $\|u-v\|_E \geq \varepsilon$. First note that $|\nabla\varphi|^{p'} \in L^{p-1}(\mathbb{R}^N)$ and $\||\nabla\varphi|^{p'}\|_{p-1} = \|\nabla\varphi\|_p^{p'}$ for all $\varphi \in E$. Therefore, using the *reverse Minkowsky inequality* given in [14, Proposition 3.2], we get

$$\begin{aligned}\left\|\frac{\nabla u+\nabla v}{2}\right\|_p^{p'} + \left\|\frac{\nabla u-\nabla v}{2}\right\|_p^{p'} &= \left\|\left|\frac{\nabla u+\nabla v}{2}\right|^{p'}\right\|_{p-1} + \left\|\left|\frac{\nabla u-\nabla v}{2}\right|^{p'}\right\|_{p-1} \\ &\leq \left\|\left|\frac{\nabla u+\nabla v}{2}\right|^{p'} + \left|\frac{\nabla u-\nabla v}{2}\right|^{p'}\right\|_{p-1}\end{aligned}$$

$$= \left[\int_{\mathbb{R}^N} \left(\left| \frac{\nabla u + \nabla v}{2} \right|^{p'} + \left| \frac{\nabla u - \nabla v}{2} \right|^{p'} \right)^{p-1} dx \right]^{\frac{1}{p-1}}$$
$$\le \left(\frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p \right)^{1/(p-1)},$$

where in the last step we have used [1, Lemma 2.27, inequality (3.4)]. Hence

$$\left\| \frac{\nabla u + \nabla v}{2} \right\|_p^{p'} + \left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} \le \left(\frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p \right)^{1/(p-1)}. \quad \text{(A.7)}$$

Similarly, $\|u\|_{p,\nu}^{p'} = \||u|^{p'}\|_{p-1,\nu}$ and $\|v\|_{p,\nu}^{p'} = \||v|^{p'}\|_{p-1,\nu}$, so that, applying Proposition A.5 with $\omega = \nu$ and $\sigma = p-1$, and proceeding as before, we obtain

$$\begin{aligned} \left\| \frac{u+v}{2} \right\|_{p,\nu}^{p'} + \left\| \frac{u-v}{2} \right\|_{p,\nu}^{p'} &\le \left\| \left| \frac{u+v}{2} \right|^{p'} + \left| \frac{u-v}{2} \right|^{p'} \right\|_{p-1,\nu} \\ &\le \left(\frac{1}{2} \|u\|_{p,\nu}^p + \frac{1}{2} \|v\|_{p,\nu}^p \right)^{1/(p-1)}. \end{aligned} \quad \text{(A.8)}$$

Therefore, combining (A.7) with (A.8), we get

$$\begin{aligned} &\left\| \frac{\nabla u + \nabla v}{2} \right\|_p^{p'} + \left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} + \left\| \frac{u+v}{2} \right\|_{p,\nu}^{p'} + \left\| \frac{u-v}{2} \right\|_{p,\nu}^{p'} \\ &\le \left(\frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p \right)^{1/(p-1)} + \left(\frac{1}{2} \|u\|_{p,\nu}^p + \frac{1}{2} \|v\|_{p,\nu}^p \right)^{1/(p-1)} \\ &\le \left(\frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p + \frac{1}{2} \|u\|_{p,\nu}^p + \frac{1}{2} \|v\|_{p,\nu}^p \right)^{1/(p-1)} \\ &= \left(\frac{1}{2} \|u\|_E^p + \frac{1}{2} \|v\|_E^p \right)^{1/(p-1)}, \end{aligned}$$

since $1/(p-1) > 1$. In other words,

$$\begin{aligned} \left\| \frac{\nabla u + \nabla v}{2} \right\|_p^{p'} + \left\| \frac{u+v}{2} \right\|_{p,\nu}^{p'} &\le \left(\frac{1}{2} \|u\|_E^p + \frac{1}{2} \|v\|_E^p \right)^{1/(p-1)} \\ &\quad - \left\{ \left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} + \left\| \frac{u-v}{2} \right\|_{p,\nu}^{p'} \right\}. \end{aligned} \quad \text{(A.9)}$$

Now, since $\|u - v\|_E \ge \varepsilon$, it follows that

$$\begin{aligned} \varepsilon^{p'} &\le \left(\|\nabla u - \nabla v\|_p^p + \|u - v\|_{p,\nu}^p \right)^{1/(p-1)} \\ &= 2^{(p+1)/(p-1)} \left(\frac{1}{2} \left\| \frac{\nabla u - \nabla v}{2} \right\|_p^p + \frac{1}{2} \left\| \frac{u-v}{2} \right\|_{p,\nu}^p \right)^{1/(p-1)} \\ &\le 2^{2/(p-1)} \left(\left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} + \left\| \frac{u-v}{2} \right\|_{p,\nu}^{p'} \right), \end{aligned}$$

being again $1/(p-1) > 1$. Hence

$$\left(\frac{\varepsilon}{2^{2/p}}\right)^{p'} \leq \left\|\frac{\nabla u - \nabla v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_{p,\nu}^{p'}.$$

Therefore, choosing $\delta = \delta(\varepsilon)$ such that $1-\varepsilon/2^{2/p} = 2^{(p-2)/(p-1)}(1-\delta)^{p'}$, from (A.9) we obtain

$$\left\|\frac{\nabla u + \nabla v}{2}\right\|_p^{p'} + \left\|\frac{u+v}{2}\right\|_{p,\nu}^{p'} \leq 2^{(p-2)/(p-1)}(1-\delta)^{p'}.$$

In conclusion, being again $1/(p-1) > 1$,

$$\left\|\frac{u+v}{2}\right\|_E^{p'} \leq 2^{(2-p)/(p-1)}\left(\left\|\frac{\nabla u + \nabla v}{2}\right\|_p^{p'} + \left\|\frac{u+v}{2}\right\|_{p,\nu}^{p'}\right) \leq (1-\delta)^{p'},$$

that is $\|u+v\|_E \leq 2(1-\delta)$, as required. □

Proposition A.10. *Let $(u_n)_n, u \in X$ be such that $u_n \rightharpoonup u$ in X. Then, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$.*

Proof. Let $(u_n)_n$ and u be as in the statement. Then, $u_n \to u$ in $L^p(B_R)$ for all $R > 0$, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Indeed, $X \hookrightarrow W^{1,p}(B_R)$, since $0 < k_1 \leq \nu(x) \leq k_2$ for a.a. $x \in B_R$ and for some positive numbers k_1 and k_2 depending only on R by (1.2), and $W^{1,p}(B_R) \hookrightarrow\hookrightarrow L^s(B_R)$ for all $s \in [1, p^*)$. In particular, in correspondence to $R = 1$ we find a subsequence $(u_{1,n})_n$ of $(u_n)_n$ such that $u_{1,n} \to u$ a.e. in B_1. Clearly $u_{1,n} \rightharpoonup u$ in X and so, in correspondence to $R = 2$, there exists a subsequence $(u_{2,n})_n$ of $(u_{1,n})_n$ such that $u_{2,n} \to u$ a.e. in B_2, and so on. The diagonal subsequence $(u_{n,n})_n$ of $(u_n)_n$, constructed by induction, converges to u a.e. in $\mathbb{R}^N$ as $n \to \infty$. □

Clearly Proposition A.10 continues to hold when X is replaced by the larger space E, see the proof of Lemma 2.2.

Proposition A.11. *The Banach space $(X, \|\cdot\|)$ is reflexive.*

Proof. We follow the proof of Proposition 8.1 of [6]. First note that the product space $Y = E \times L^r(\mathbb{R}^N; h)$, endowed with the norm $\|u\|_Y = \|u\|_E + \|u\|_{r,h}$, is reflexive, being E and $L^r(\mathbb{R}^N; h)$ both uniformly convex by Propositions A.9 and A.6. Consider now the operator $T : X \to Y$ defined by $T(u) = (u, u)$. Clearly T is well defined and linear. Moreover T is an isometry, if X is endowed with the equivalent norm $\|u\|_Y$. Therefore, $T(X)$ is a closed subspace of the reflexive space Y, and so $T(X)$ is reflexive. Consequently, also $(X, \|\cdot\|_Y)$ is reflexive, being isomorphic to a reflexive space. Finally, since reflexivity is preserved under equivalent norms, we conclude that also $(X, \|\cdot\|)$ is reflexive. □

References

[1] Adams, R.A.: Sobolev spaces. Pure and Applied Mathematics, vol. 65, p. xviii+268. Academic Press, New York (1975)

[2] Alama, S., Tarantello, G.: Elliptic problems with nonlinearities indefinite in sign. J. Funct. Anal. **141**, 159–215 (1996)

[3] Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)

[4] Ambrosetti, A., Rabinowitz, P.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)

[5] Boccardo, L., Murat, F.: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal. **19**, 581–597 (1992)

[6] Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations, Universitext, p. xiv+599. Springer, New York (2011)

[7] Candela, A.M., Palmieri, G.: Infinitely many solutions of some nonlinear variational equations. Calc. Var. Partial Differ. Equ. **34**, 495–530 (2009)

[8] Chang, X.: Multiplicity of solutions for semilinear elliptic problems with combined nonlinearities. Commun. Contemp. Math. **13**, 389–405 (2011)

[9] Colasuonno, F., Pucci, P., Varga, C.: Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. **75**, 4496–4512 (special volume dedicated to Professor Lakshmikantham) (2012)

[10] De Figueiredo, D.G., Gossez, J.-P., Ubilla, P.: Local superlinearity and sublinearity for indefinite semilinear elliptic problems. J. Funct. Anal. **199**, 452–467 (2003)

[11] De Figueiredo, D.G., Gossez, J.-P., Ubilla, P.: Local “superlinearity” and “sublinearity” for the p-Laplacian. J. Funct. Anal. **257**, 721–752 (2009)

[12] Demengel, F., Hebey, E.: On some nonlinear equations on compact Riemannian manifolds. Adv. Differ. Equ. **3**, 533–574 (1998)

[13] De Nápoli, P.L., Mariani, M.C.: Mountain pass solutions to equations of p-Laplacian type. Nonlinear Anal. **54**, 1205–1219 (2003)

[14] DiBenedetto, E.: Real Analysis, Birkhäuser Advanced Texts: Basler Lehrbücher, p. xxiv+485. Birkhäuser, Boston (2002)

[15] Dinca, G., Jebelean, P., Mawhin, J.: Variational and topological methods for Dirichlet problems with p-Laplacian. Port. Math. (N.S.) **58**, 339–378 (2001)

[16] Ekeland, I.: The mountain pass theorem and some applications, in Minimax results of Lusternik–Schnirelman type and applications, Sém. Math. Sup., vol. 107, pp. 9–34. Presses Univ. Montreal, Montreal (1989)

[17] Filippucci, R., Pucci, P., Rădulescu, V.: Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions. Comm. Partial Differ. Equ. **33**, 706–717 (2008)

[18] Kristály, A., Moroşanu, G.: New competition phenomena in Dirichlet problems. J. Math. Pures Appl. **94**, 555–570 (2010)

[19] Lu, Q.: Compactly supported solutions for a semilinear elliptic problem in $\mathbb{R}^n$ with sign-changing function and non-Lipschitz nonlinearity. Proc. R. Soc. Edinburgh Sect. A **141**, 127–154 (2011)

[20] Pucci, P., Rădulescu, V.: Combined effects in quasilinear elliptic problems with lack of compactness. Rend. Lincei Mat. Appl. **22**, 189–205 (special volume dedicated to the memory of Prof. G. Prodi) (2011)

[21] Pucci, P., Servadei, R.: Existence, nonexistence and regularity of radial ground states for p-Laplacian equations with singular weights. Ann. Inst. H. Poincaré Anal. Non Linéaire **25**, 505–537 (2008)

[22] Simon, J.: Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$ [Regularity of a solution of a nonlinear equation on $\mathbb{R}^N$]. Journées d'Analyse Non Linéaire, Lectures Notes in Math., vol. 665, pp. 205–227. Springer, Berlin (1978)

[23] Struwe, M.: Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems. A Series of Modern Surveys in Mathematics, vol. 34, p. xx+302. Springer, Berlin (2008)

[24] Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)

[25] Troyanski, S.L.: On locally uniformly convex and differentiable norms in certain non-separable Banach spaces. Studia Math. **37**, 173–180 (1971)

Giuseppina Autuori and Patrizia Pucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Via Vanvitelli 1
06123 Perugia
Italy
e-mail: pucci@dmi.unipg.it

Giuseppina Autuori
e-mail: autuori@dmi.unipg.it

Received: 2 March 2012.
Accepted: 4 August 2012.