The first Darboux problem for wave equations with nonlinear dissipative term

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Abstract. The first Darboux problem for wave equations with nonlinear dissipative term is considered. The uniqueness, local and global existence and blow-up of solutions of the problem mentioned are investigated. The paper's originality is the coalescence of two standard methods: a priori estimate of solutions in the class of continuous functions is given by energetic methods; basing on this result a priori estimate in the class of continuously differentiable functions using classical method of characteristics is obtained.

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1. Introduction

In the plane of independent variables x and t consider nonlinear wave equation of the following form [1], [2, p. 57]

$$Lu := u_{tt} - u_{xx} + g(x, t, u)u_t = f(x, t), \quad (1.1)$$

where g, f are given, and u - unknown real functions.

Denote by $D_T := \{(x, t) : 0 < x < t, 0 < t < T\}$ triangular domain, bounded by characteristic segment $\gamma_{1,T} : x = t, 0 \leq t \leq T$ and by segments $\gamma_{2,T} : x = 0, 0 \leq t \leq T, \gamma_{3,T} : t = T, 0 \leq x \leq T$.

For Eq. (1.1) in domain D_T consider the first Darboux problem on determination of solution $u(x, t)$ by boundary conditions [3, p. 228]

$$u|_{\gamma_{i,T}} = 0, \quad i = 1, 2. \quad (1.2)$$

Note that on questions of existence, uniqueness and blow-up of solutions of initial, mixed, nonlocal and other problems posed for nonlinear hyperbolic type equations there are devoted a number of papers (see, e.g. [1–18]). In linear case, i. e. when $g(x, t, u) = g(x, t)$, problem (1.1), (1.2), as it is known, is well-posed and a global solvability takes place in corresponding function spaces

(see, e.g. [3, 19]). In previous literature [1, 2, 4–13], basically are studied the Cauchy initial and mixed initial-boundary value problems in functional spaces different from spaces considered in this paper. Moreover, in papers [1, 2] are considered the particular cases when $g(x, t, s) = |s|^\rho$. In papers [14–18] there are considered the similar boundary value conditions, but for hyperbolic differential equations with nonlinear source terms.

Below we show that for certain requirements for nonlinear function $g(x, t, s)$ with respect to variable s problem (1.1), (1.2) is locally solvable; the conditions for global solvability are obtained, violation of which, generally speaking, can cause a blow-up within finite interval of time.

Definition 1.1. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$, $\mathbb{R} := (-\infty, +\infty)$. Function u is called a strong generalized solution of problem (1.1), (1.2) of class C^1 in domain D_T , if $u \in C^1(\overline{D}_T)$ and there exists such the sequence of functions $u_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, that $u_n \rightarrow u$ and $Lu_n \rightarrow f$ for $n \rightarrow \infty$ in spaces $C^1(\overline{D}_T)$ and $C(\overline{D}_T)$, respectively, where $\mathring{C}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v|_{\Gamma_T} = 0\}$, $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Remark 1.1. It is clear that if $u \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$ is a classical solution of problem (1.1), (1.2), then it is a strong generalized solution of this problem of class C^1 in domain D_T . In turn, if a strong generalized solution of problem (1.1), (1.2) of class C^1 in domain D_T belongs to space $C^2(\overline{D}_T)$, then it also is a classical solution of the problem.

Definition 1.2. Let $f \in C(\overline{D}_\infty)$, $g \in C(\overline{D}_\infty \times \mathbb{R})$. We say that problem (1.1), (1.2) is globally solvable in the class C^1 , if for any finite $T > 0$ it has a strong generalized solution of the class C^1 in domain D_T .

The paper is organized as follows. In Sect. 2 it is obtained a priori estimates of the solution of problem (1.1), (1.2) in spaces $C(\overline{D}_T)$ and $C^1(\overline{D}_T)$. In Sect. 3 problem (1.1), (1.2) is equivalently reduced to the system of nonlinear Volterra type integral equations and the local solvability of this problem is proved. In Sect. 4 it is shown the global solvability of the considered problem, and in Sect. 5 the uniqueness of the solution of the problem is proved. Finally, in the last Sect. 6 the nonexistence of global solvability of this problem is shown.

2. A priori estimates of the solution of problem (1.1), (1.2) in spaces $C(\overline{D}_T)$ and $C^1(\overline{D}_T)$

Lemma 2.1. Let $f \in C(\overline{D}_T)$, $g \in C(\overline{D}_T \times \mathbb{R})$ and

$$g(x, t, s) \geq -M_T, \quad (x, t, s) \in \overline{D}_T \times \mathbb{R}, \quad M_T := \text{const} > 0. \quad (2.1)$$

Then for a strong generalized solution of problem (1.1), (1.2) of class C^1 in domain D_T it is valid the following a priori estimate

$$\|u\|_{C(\overline{D}_T)} \leq c_0 \|f\|_{C(\overline{D}_T)} \quad (2.2)$$

with positive constant $c_0 = c_0(T, M_T)$, not dependent on u and f .

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of class C^1 in domain D_T . Then due to Definition 1.1 there exists the sequence of functions $u_n \in \overset{\circ}{C}^2(\overline{D}_T, \Gamma_T)$, such that

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{C^1(\overline{D}_T)} = 0, \quad \lim_{n \rightarrow \infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0, \quad (2.3)$$

and therefore

$$\lim_{n \rightarrow \infty} \|g(x, t, u_n)u_{nt} - g(x, t, u)u_t\|_{C(\overline{D}_T)} = 0. \quad (2.4)$$

Consider function $u_n \in \overset{\circ}{C}^2(\overline{D}_T, \Gamma_T)$, as a solution of the following problem

$$Lu_n = f_n, \quad (2.5)$$

$$u_n|_{\Gamma_T} = 0. \quad (2.6)$$

Here

$$f_n := Lu_n. \quad (2.7)$$

Multiplying the both sides of equality (2.5) by u_{nt} and integrating the received in domain $D_\tau := \{(x, t) \in D_T : 0 < t < \tau\}$, $0 < \tau \leq T$ we have

$$\frac{1}{2} \int_{D_\tau} (u_{nt}^2)_t dx dt - \int_{D_\tau} u_{nxx} u_{nt} dx dt + \int_{D_\tau} g(x, t, u_n) u_{nt}^2 dx dt = \int_{D_\tau} f_n u_{nt} dx dt.$$

Assume that $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \leq T$. Then by virtue of (2.6), integrating by parts the left-hand side of the last equality, we obtain

$$\begin{aligned} \int_{D_\tau} f_n u_{nt} dx dt &= \frac{1}{2} \int_{\gamma_{1,\tau}} \nu_t^{-1} [(u_{nx} \nu_t - u_{nt} \nu_x)^2 + u_{nt}^2 (\nu_t^2 - \nu_x^2)] ds \\ &\quad + \frac{1}{2} \int_{\Omega_\tau} (u_{nx}^2 + u_{nt}^2) dx + \int_{D_\tau} g(x, t, u_n) u_{nt}^2 dx dt, \end{aligned} \quad (2.8)$$

where $\nu := (\nu_x, \nu_t)$ is unit vector of outer normal to ∂D_τ and $\gamma_{1,\tau} := \gamma_{1,T} \cap \{t \leq \tau\}$.

Taking into account the fact that operator $\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}$ is an interior differential operator on $\gamma_{1,T}$, due to (2.6) we receive

$$(u_{nx} \nu_t - u_{nt} \nu_x)|_{\gamma_{1,\tau}} = 0. \quad (2.9)$$

Further, it is clear that

$$(\nu_t^2 - \nu_x^2)|_{\gamma_{1,\tau}} = 0. \quad (2.10)$$

Therefore, by virtue of (2.8)–(2.10) we get

$$w_n(\tau) := \int_{\Omega_\tau} (u_{nx}^2 + u_{nt}^2) dx \leq 2 \int_{D_\tau} f_n u_{nt} dx dt + 2M_T \int_{D_\tau} u_{nt}^2 dx dt. \quad (2.11)$$

Taking into account inequality

$$2f_n u_{nt} \leq u_{nt}^2 + f_n^2,$$

due to (2.11) we have

$$w_n(\tau) \leq (1 + 2M_T) \int_{D_\tau} u_{nt}^2 dx dt + \int_{D_\tau} f_n^2 dx dt.$$

According to the form of function $w_n(\tau)$ it follows that

$$w_n(\tau) \leq m_T \int_0^\tau w_n(\sigma) d\sigma + \|f_n\|_{L_2(D_\tau)}^2, \quad m_T := 1 + 2M_T.$$

Whence, having the fact that the value $\|f_n\|_{L_2(D_\tau)}^2$, as a function of τ is non-decreasing, by the Gronwall's lemma [20, p. 13] we receive

$$w_n(\tau) \leq \exp(m_T \tau) \|f_n\|_{L_2(D_\tau)}^2. \quad (2.12)$$

If $(x, t) \in \overline{D}_T$, then by virtue of the condition (2.6) the following equality is valid

$$u_n(x, t) = u_n(x, t) - u_n(0, t) = \int_0^x u_{nx}(\sigma, t) d\sigma,$$

thus due to (2.12) we have

$$\begin{aligned} |u_n(x, t)|^2 &\leq \int_0^x d\sigma \int_0^x u_{nx}^2(\sigma, t) d\sigma \leq x \int_{\Omega_t} u_{nx}^2(\sigma, t) d\sigma \leq x w_n(t) \leq t w_n(t) \\ &\leq T \exp(m_T T) \|f_n\|_{C(\overline{D}_T)}^2 \text{mes } D_T = 2^{-1} T^3 \exp(m_T T) \|f_n\|_{C(\overline{D}_T)}^2. \end{aligned} \quad (2.13)$$

From this it follows that

$$\|u_n\|_{C(\overline{D}_T)} \leq T \sqrt{\frac{T}{2}} \exp\left(\frac{m_T T}{2}\right) \|f_n\|_{C(\overline{D}_T)}.$$

Passing in this inequality to limit for $n \rightarrow \infty$, and due to (2.3), (2.7), we have

$$\|u\|_{C(\overline{D}_T)} \leq T \sqrt{\frac{T}{2}} \exp\left(\frac{m_T T}{2}\right) \|f\|_{C(\overline{D}_T)}. \quad (2.14)$$

This proves estimate (2.2). $\square$

Remark 2.1. From (2.14) it follows that constant c_0 in estimate (2.2) can be taken

$$c_0 := T \sqrt{\frac{T}{2}} \exp\left(\frac{m_T T}{2}\right). \quad (2.15)$$

Below, taking into account estimate (2.2) and using the classical method of characteristics, we receive a priori estimate in space $C^1(\overline{D}_T)$ for a strong generalized solution u of problem (1.1), (1.2) of class C^1 in domain D_T .

Lemma 2.2. *Under the conditions of Lemma 2.1 for a strong generalized solution u of problem (1.1), (1.2) of class C^1 in domain D_T it is valid the following a priori estimate*

$$\|u\|_{C^1(\overline{D}_T)} \leq c_1 \quad (2.16)$$

with positive constant $c_1 = c_1(T, c_0, \|f\|_{C(\overline{D}_T)})$, not dependent on u , where $\|u\|_{C^1(\overline{D}_T)} := \max\{\|u\|_{C(\overline{D}_T)}, \|u_x\|_{C(\overline{D}_T)}, \|u_t\|_{C(\overline{D}_T)}\}$.

Proof. Let u be a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T . Then the limit equalities (2.3), (2.4) are valid, where u_n can be considered as a solution of problem (2.5), (2.6) with f_n given by (2.7).

Denote

$$l_i := \frac{\partial}{\partial t} + (-1)^i \frac{\partial}{\partial x}, \quad i = 1, 2.$$

For fixed natural number n let us introduce the following functions

$$v_{in} := l_i u_n, \quad v_{3n} := u_n, \quad i = 1, 2, \quad (2.17)$$

which by taking into account (2.2) satisfy the following boundary conditions

$$v_{1n}(0, t) = -v_{2n}(0, t), \quad v_{2n}(t, t) = 0, \quad v_{3n}(t, t) = 0, \quad 0 \leq t \leq T. \quad (2.18)$$

With respect to unknown functions v_{in} , $i = 1, 2, 3$ by virtue of (1.1) and (2.17) we have the following system of first order partial differential equations

$$\begin{cases} l_2 v_{1n} = f_n(x, t) - \frac{1}{2} g(x, t, v_{3n})(v_{1n} + v_{2n}), \\ l_1 v_{2n} = f_n(x, t) - \frac{1}{2} g(x, t, v_{3n})(v_{1n} + v_{2n}), \\ l_1 v_{3n} = v_{1n}. \end{cases} \quad (2.19)$$

Integrating the received system along the corresponding characteristic curves, due to boundary conditions (2.18), we have

$$\begin{cases} v_{1n}(x, t) - v_{1n}(0, t) = \int_{t-x}^t (f_n(P_\tau) - \frac{1}{2} g(P_\tau, v_{3n}(P_\tau))(v_{1n}(P_\tau) + v_{2n}(P_\tau))) d\tau, \\ v_{2n}(x, t) = \int_{\frac{x+t}{2}}^t (f_n(Q_\tau) - \frac{1}{2} g(Q_\tau, v_{3n}(Q_\tau))(v_{1n}(Q_\tau) + v_{2n}(Q_\tau))) d\tau, \\ v_{3n}(x, t) = \int_{\frac{x+t}{2}}^t v_{1n}(Q_\tau) d\tau, \end{cases}$$

where $P_\tau := (x - t + \tau, \tau)$, $Q_\tau := (x + t - \tau, \tau)$.

From the second equation of the received system and the first equality of (2.18), taking into account notation $Q_{\tau_0} := (t - \tau, \tau)$ this system can be rewritten as follows

$$\begin{cases} v_{1n}(x, t) = -\frac{1}{2} \int_{t-x}^t g(P_\tau, v_{3n}(P_\tau))(v_{1n}(P_\tau) + v_{2n}(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\frac{x+t}{2}}^t g(Q_{\tau_0}, v_{3n}(Q_{\tau_0}))(v_{1n}(Q_{\tau_0}) + v_{2n}(Q_{\tau_0})) d\tau + F_{1n}(x, t), \\ v_{2n}(x, t) = -\frac{1}{2} \int_{\frac{x+t}{2}}^t g(Q_\tau, v_{3n}(Q_\tau))(v_{1n}(Q_\tau) + v_{2n}(Q_\tau)) d\tau + F_{2n}(x, t), \\ v_{3n}(x, t) = \int_{\frac{x+t}{2}}^t v_{1n}(Q_\tau) d\tau. \end{cases} \quad (2.20)$$

Here

$$F_{1n}(x, t) := \int_{t-x}^t f_n(P_\tau) d\tau - \int_{\frac{x+t}{2}}^t f_n(Q_{\tau_0}) d\tau, \quad F_{2n}(x, t) := \int_{\frac{x+t}{2}}^t f_n(Q_\tau) d\tau. \quad (2.21)$$

By passing in equalities (2.20), (2.21) to limit as $n \rightarrow \infty$ in the space $C(\overline{D}_T)$ and taking into account (2.3), (2.4), (2.7) and (2.17) we have

$$\begin{cases} v_1(x, t) = -\frac{1}{2} \int_{t-x}^t g(P_\tau, v_3(P_\tau)) (v_1(P_\tau) + v_2(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\frac{t}{2}}^t g(Q_{\tau_0}, v_3(Q_{\tau_0})) (v_1(Q_{\tau_0}) + v_2(Q_{\tau_0})) d\tau + F_1(x, t), \\ v_2(x, t) = -\frac{1}{2} \int_{\frac{x+t}{2}}^t g(Q_\tau, v_3(Q_\tau)) (v_1(Q_\tau) + v_2(Q_\tau)) d\tau + F_2(x, t), \\ v_3(x, t) = \int_{\frac{x+t}{2}}^t v_1(Q_\tau) d\tau, \end{cases} \quad (2.22)$$

where $v_i := \lim_{n \rightarrow \infty} v_{in}$ (by the norm of space $C(\overline{D}_T)$), $i = 1, 2, 3$, and

$$F_1(x, t) := \int_{t-x}^t f(P_\tau) d\tau - \int_{\frac{t}{2}}^t f(Q_{\tau_0}) d\tau, \quad F_2(x, t) := \int_{\frac{x+t}{2}}^t f(Q_\tau) d\tau. \quad (2.23)$$

□

Remark 2.2. It is clear that $v_3 = u$, which is a strong generalized solution of the problem (1.1), (1.2) of class C^1 in domain D_T , besides

$$v_1 = u_t - u_x, \quad v_2 = u_t + u_x. \quad (2.24)$$

Let $G_T := \{(x, t, s) \in \mathbb{R}^3 : (x, t) \in \overline{D}_T, |s| \leq c_0 \|f\|_{C(\overline{D}_T)}\}$ and

$$K = K(T, c_0, \|f\|_{C(\overline{D}_T)}) := \frac{1}{2} \sup_{(x, t, s) \in G_T} |g(x, t, s)| < +\infty, \quad (2.25)$$

where c_0 is defined by (2.15). Then due to a priori estimate (2.2) for a strong generalized solution $u = v_3$ of problem (1.1), (1.2) of class C^1 in domain D_T we receive

$$|g(x, t, v_3(x, t))| \leq 2K, \quad (x, t) \in \overline{D}_T. \quad (2.26)$$

Let

$$\omega_i(t) := \sup_{(\xi, \tau) \in \overline{D}_t} |v_i(\xi, \tau)|, \quad i = 1, 2, 3, \quad F(t) := \sup_{(\xi, \tau) \in \overline{D}_t} |f(\xi, \tau)|. \quad (2.27)$$

From (2.22), due (2.23), (2.26) and (2.27) it follows that

$$\begin{aligned} |v_1(x, t)| &\leq 2K \int_0^t (\omega_1(\tau) + \omega_2(\tau)) d\tau + 2tF(t), \\ |v_2(x, t)| &\leq K \int_0^t (\omega_1(\tau) + \omega_2(\tau)) d\tau + tF(t), \quad |v_3(x, t)| \leq \int_0^t \omega_1(\tau) d\tau. \end{aligned}$$

Whence for $(\xi, \tau) \in \overline{D}_t$ we receive

$$\begin{aligned} |v_1(\xi, \tau)| &\leq 2K \int_0^\tau (\omega_1(\tau_1) + \omega_2(\tau_1)) d\tau_1 + 2\tau F(\tau), \\ |v_2(\xi, \tau)| &\leq K \int_0^\tau (\omega_1(\tau_1) + \omega_2(\tau_1)) d\tau_1 + \tau F(\tau), \quad |v_3(\xi, \tau)| \leq \int_0^\tau \omega_1(\tau_1) d\tau_1. \end{aligned}$$

Thus due to (2.27) and the fact that $tF(t)$ is non-decreasing function it follows that

$$\begin{aligned}\omega_1(t) &\leq 2K \int_0^t (\omega_1(\tau) + \omega_2(\tau)) d\tau + 2tF(t), \\ \omega_2(t) &\leq K \int_0^t (\omega_1(\tau) + \omega_2(\tau)) d\tau + tF(t), \quad \omega_3(t) \leq \int_0^t \omega_1(\tau) d\tau.\end{aligned}$$

Setting that $\omega(t) := \max_{1 \leq i \leq 3} \omega_i(t)$, $0 \leq t \leq T$, from the inequalities given above it follows that

$$\omega(t) \leq (4K + 1) \int_0^t \omega(\tau) d\tau + 2tF(t), \quad 0 \leq t \leq T,$$

whence by virtue of Gronwall's Lemma we have

$$\omega(t) \leq 2tF(t) \exp((4K + 1)t) \leq 2T \exp((4K + 1)T) \|f\|_{C(\overline{D}_T)}, \quad 0 \leq t \leq T.$$

Now from (2.24) it is easy to receive

$$\|u\|_{C^1(\overline{D}_T)} \leq \|\omega\|_{C[0,T]} \leq 2T \exp((4K + 1)T) \|f\|_{C(\overline{D}_T)}.$$

The Lemma 2.2 is proved, besides

$$c_1 := 2T \exp((4K + 1)T) \|f\|_{C(\overline{D}_T)}, \quad (2.28)$$

where K is defined by (2.25).

3. The equivalency of the problem (1.1), (1.2) and the system of nonlinear Volterra type integral equations (2.22) and the local solvability of the problem (1.1), (1.2)

In Sect. 2 we have reduced the problem (1.1), (1.2) to the Volterra type system of nonlinear integral equations (2.22). Before consideration the solvability of the problem (1.1), (1.2), let us prove the following lemma.

Lemma 3.1. *If a function $u \in C^1(\overline{D}_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T , then $v_1 := l_1 u$, $v_2 := l_2 u$, $v_3 := u$ is the continuous solution of the Volterra type system of nonlinear integral equations (2.22) and vice versa, if v_1, v_2, v_3 is the continuous solution of the system (2.22), then $u := v_3$ is a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T , and the equalities $v_1 = l_1 u$, $v_2 = l_2 u$ are valid.*

Proof. Indeed, let $v_1, v_2, v_3 \in C(\overline{D}_T)$ is the solution of the system of equations (2.22). It is clear that functions v_1, v_2, v_3 satisfy the following system of differential equations in partial derivatives of the first order

$$l_2 v_1 = H, \quad l_1 v_2 = H, \quad l_1 v_3 = v_1, \quad (3.1)$$

where

$$2H := 2f(x, t) - g(x, t, v_3)(v_1 + v_2). \quad (3.2)$$

Moreover, let us show that

$$l_2 v_3 = v_2. \quad (3.3)$$

Indeed, due to the equalities (3.1) for any test function $w \in C_0^\infty(D_T)$ it is valid the following equalities in the functional sense due to the distribution theory of L. Schwartz [21]:

$$\begin{aligned} \langle l_1(v_2 - l_2 v_3), w \rangle &= -\langle v_2 - l_2 v_3, l_1 w \rangle = -\langle v_2, l_1 w \rangle + \langle l_2 v_3, l_1 w \rangle \\ &= \langle l_1 v_2, w \rangle - \langle v_3, l_2 l_1 w \rangle = \langle l_1 v_2, w \rangle - \langle v_3, l_1 l_2 w \rangle \\ &= \langle l_1 v_2, w \rangle + \langle l_1 v_3, l_2 w \rangle = \langle l_1 v_2, w \rangle + \langle v_1, l_2 w \rangle \\ &= \langle l_1 v_2, w \rangle - \langle l_2 v_1, w \rangle = \langle l_1 v_2 - l_2 v_1, w \rangle = 0. \end{aligned}$$

Whence, due to the Theorem 1.4.2 from [21] we conclude that (3.3) takes place in the classical sense. Therefore, in view of (3.1) we have $v_3 \in C^1$ and

$$v_1 + v_2 = 2v_{3t}. \quad (3.4)$$

Let us extend functions $f \in C(\overline{D}_T)$ and $g \in C(\overline{D}_T \times \mathbb{R})$ continuously in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, saving the same notations. Analogously extend functions $v_i \in C(\overline{D}_T)$, $i = 1, 2$ and $v_3 \in C^1(\overline{D}_T)$ with the same class of smoothness $\mathbb{R}^2$, not changing their notations. Then function H will be extended on whole plane $\mathbb{R}^2$ [22, 23].

Let

$$\theta_\varepsilon(y) = \begin{cases} C_\varepsilon \exp \left[-\frac{\varepsilon^2}{\varepsilon^2 - |y|^2} \right], & |y| < \varepsilon, \\ 0, & |y| \geq \varepsilon, \end{cases} \quad (3.5)$$

where $y := (x, t) \in \mathbb{R}^2$, and C_ε positive constant, which is defined from the equality

$$C_\varepsilon \varepsilon^2 \int_{|\xi| < 1} \exp \left[-\frac{1}{1 - |\xi|^2} \right] d\xi = 1, \quad \xi := (\xi_1, \xi_2);$$

ε is any positive number, and $|\cdot|$ is the norm in the Euclidian space $\mathbb{R}^2$.

Consider the sequence of functions

$$\overset{n}{v}_i := v_i * \theta_{\frac{1}{n}} \in C^\infty(\mathbb{R}^2), \quad i = 1, 2, 3, \quad \overset{n}{H} := H * \theta_{\frac{1}{n}}, \quad (3.6)$$

where $*$ is the known convolution operation [21], and function θ_ε is defined from (3.5).

In view of the properties of convolution we have [21]

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\overset{n}{v}_i - v_i\|_{C(\overline{D}_T)} &= 0, \quad i = 1, 2, \quad \lim_{n \rightarrow \infty} \|\overset{n}{v}_3 - v_3\|_{C^1(\overline{D}_T)} = 0, \\ \lim_{n \rightarrow \infty} \|\overset{n}{H} - H\|_{C(\overline{D}_T)} &= 0. \end{aligned} \quad (3.7)$$

Below we prove that the sequence of function $u_n := \overset{n}{v}_3|_{\overline{D}_T} \in C^2(\overline{D}_T)$ satisfies the conditions (2.3), (2.4), (2.6) and, therefore, $u := v_3$ is a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T due to Definition 1.1.

According to the properties of convolution and the equalities (3.1), (3.3), functions $\overset{n}{v}_i$, $i = 1, 2, 3$, defined from (3.6), satisfy the following equalities

$$l_2 \overset{n}{v}_1 = \overset{n}{H}, \quad l_1 \overset{n}{v}_2 = \overset{n}{H}, \quad l_1 \overset{n}{v}_3 = \overset{n}{v}_1, \quad l_2 \overset{n}{v}_3 = \overset{n}{v}_2. \quad (3.8)$$

Further, according to the definition of operator L from (1.1) and the equalities (3.8) we have

$$L \overset{n}{v}_3 = l_2 l_1 \overset{n}{v}_3 + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t} = l_2 \overset{n}{v}_1 + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t} = \overset{n}{H} + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t}.$$

Whence due to the equalities (3.7) it follows that $L \overset{n}{v}_3 \rightarrow H + g(x, t, v_3) v_{3t}$ for $n \rightarrow \infty$ in the space $C(\overline{D}_T)$. At the same time due to (3.2), (3.4) we have $H + g(x, t, v_3) v_{3t} = f$. Therefore

$$\lim_{n \rightarrow \infty} \|L \overset{n}{v}_3 - f\|_{C(\overline{D}_T)} = 0. \quad (3.9)$$

Now let us verify the validity of the boundary conditions (1.2). According to the properties of convolution and the equalities (2.21), (2.22) we have

$$\overset{n}{v}_3(t, t) \rightarrow v_3(t, t) = 0 \quad (3.10)$$

for $n \rightarrow \infty$ in the space $C(\gamma_{1,T})$.

Analogously, due to the properties of convolution and the equalities (3.4), (2.22) and (2.23) we have

$$\overset{n}{v}_{3t}(0, \cdot) \rightarrow v_{3t}(0, \cdot) = \frac{1}{2}[v_1(0, \cdot) + v_2(0, \cdot)] = 0$$

for $n \rightarrow \infty$ in the space $C(\gamma_{2,T})$. Integrating the received equality with respect to the variable t , due to the (3.10) we obtain

$$v_3(0, \cdot) = 0. \quad (3.11)$$

The equalities $v_1 = l_1 u$, $v_2 = l_2 u$ immediately follow from above-brought considerations, which together with (3.9)–(3.11) prove Lemma 3.1, since the inverse proposition is obvious.

Now let us prove the local solvability of the system of Volterra type non-linear integral equations (2.22).

Let

$$f \in C(\overline{D}_\infty), \quad f_\infty := \sup_{(x,t) \in \overline{D}_\infty} |f(x, t)| < +\infty, \quad g \in C(\overline{D}_\infty \times \mathbb{R}) \quad (3.13)$$

and for $(x, t) \in \overline{D}_\infty$, $|s|, |s_1|, |s_2| \leq R$

$$|g(x, t, s)| \leq M(R), \quad |g(x, t, s_2) - g(x, t, s_1)| \leq c(R)|s_2 - s_1|, \quad (3.14)$$

where $M(R)$ and $c(R)$ are some non-negative continuous functions of argument $R \geq 0$. $\square$

Theorem 3.1. *Let the functions f and g satisfy conditions (3.13), (3.14). Then there exists positive number $T_* := T_*(f, g)$, such that for $T \leq T_*$ the problem (1.1), (1.2) will have at least one strong generalized solution u of the class C^1 in domain D_T*

Proof. According to Lemma 3.1 the problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of Volterra type nonlinear integral equations (2.22) in the space $C(\overline{D}_T)$. Below we will prove the unique solvability of system (2.22) by the principle of contracted mappings [see e. g., 24, p. 390].

Let $V := (v_1, v_2, v_3)$. Consider vectorial operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$, acting by the formula

$$\begin{cases} (\Phi_1 V)(x, t) = -\frac{1}{2} \int_{t-x}^t g(P_\tau, v_3(P_\tau))(v_1(P_\tau) + v_2(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\frac{t}{2}}^t g(Q_{\tau_0}, v_3(Q_{\tau_0}))(v_1(Q_{\tau_0}) + v_2(Q_{\tau_0})) d\tau + F_1(x, t), \\ (\Phi_2 V)(x, t) = -\frac{1}{2} \int_{\frac{x+t}{2}}^t g(Q_\tau, v_3(Q_\tau))(v_1(Q_\tau) + v_2(Q_\tau)) d\tau \\ \quad + F_2(x, t), \\ (\Phi_3 V)(x, t) = \int_{\frac{x+t}{2}}^t v_1(Q_\tau) d\tau. \end{cases} \quad (3.15)$$

Then the system (2.22) can be rewritten as follows

$$V = \Phi V. \quad (3.16)$$

Let $\|V\|_{X_T} := \max_{1 \leq i \leq 3} \{\|v_i\|_{C(\overline{D}_T)}\}$, $V \in X_T := C(\overline{D}_T; \mathbb{R}^3)$, where $C(\overline{D}_T; \mathbb{R}^3)$ is the set of all continuous vector-functions $V : \overline{D}_T \rightarrow \mathbb{R}^3$. Let us denote by $B_R := \{V \in X_T : \|V\|_{X_T} \leq R\}$ closed ball of radius $R > 0$ in Banach space X_T with a center in the null element.

Below we prove that: (i) Φ maps a ball B_R into itself; (ii) Φ is a contracted mapping on B_R .

Indeed, due to the first inequality of (3.14) from (3.15) for $V : \|V\|_{X_T} \leq R$ we have

$$\begin{aligned} |(\Phi_1 V)(x, t)| &\leq 2T(RM(R) + \|f\|_{C(\overline{D}_T)}), \\ |(\Phi_2 V)(x, t)| &\leq T(RM(R) + \|f\|_{C(\overline{D}_T)}), \quad |(\Phi_3 V)(x, t)| \leq TR. \end{aligned}$$

From these estimates it follows that

$$\|\Phi V\|_{X_T} \leq 2T \left(RM(R) + R + \|f\|_{C(\overline{D}_T)} \right) \leq 2T (RM(R) + R + f_\infty),$$

where f_∞ is defined in (3.13).

Assume that the value of T with fixed $R > 0$ is too small that

$$2T (RM(R) + R + f_\infty) \leq R, \quad (3.17)$$

so $\Phi V \in B_R$ and therefore condition (i) is fulfilled.

Further, due to (3.14) from (3.15) for $V^i : \|V^i\|_{X_T} \leq R$, $i = 1, 2$, we have

$$\begin{aligned} & |(\Phi_1 V^2 - \Phi_1 V^1)(x, t)| \\ & \leq \frac{1}{2} \int_{t-x}^t \left(|g(P_\tau, v_3^2(P_\tau)) - g(P_\tau, v_3^1(P_\tau))| |v_1^2(P_\tau) \right. \\ & \quad \left. + v_2^2(P_\tau)| + |g(P_\tau, v_3^1(P_\tau))| |v_1^2(P_\tau) - v_1^1(P_\tau) + v_2^2(P_\tau) - v_2^1(P_\tau)| \right) d\tau \\ & \quad + \frac{1}{2} \int_{\frac{t}{2}}^t \left(|g(Q_{\tau_0}, v_3^2(Q_{\tau_0})) - g(Q_{\tau_0}, v_3^1(Q_{\tau_0}))| |v_1^2(Q_{\tau_0}) + v_2^2(Q_{\tau_0})| \right. \\ & \quad \left. + |g(Q_{\tau_0}, v_3^1(Q_{\tau_0}))| |v_1^2(Q_{\tau_0}) - v_1^1(Q_{\tau_0}) + v_2^2(Q_{\tau_0}) - v_2^1(Q_{\tau_0})| \right) d\tau \\ & \leq 2T(Rc(R) + M(R)) \|V^2 - V^1\|_{X_T}. \end{aligned}$$

Analogously

$$\begin{aligned} & |(\Phi_2 V^2 - \Phi_2 V^1)(x, t)| \leq \frac{1}{2} \int_{\frac{x+t}{2}}^t \left(|g(Q_\tau, v_3^2(Q_\tau)) - g(Q_\tau, v_3^1(Q_\tau))| |v_1^2(Q_\tau) \right. \\ & \quad \left. + v_2^2(Q_\tau)| + |g(Q_\tau, v_3^1(Q_\tau))| |v_1^2(Q_\tau) - v_1^1(Q_\tau) + v_2^2(Q_\tau) - v_2^1(Q_\tau)| \right) d\tau \\ & \leq T(Rc(R) + M(R)) \|V^2 - V^1\|_{X_T} \end{aligned}$$

and

$$|(\Phi_3 V^2 - \Phi_3 V^1)(x, t)| \leq \int_{\frac{x+t}{2}}^t |v_1^2(Q_\tau) - v_1^1(Q_\tau)| d\tau \leq T \|V^2 - V^1\|_{X_T}.$$

Assume that for fixed $R > 0$ number T is too small that

$$\max(T, 2T(Rc(R) + R + M(R))) \leq \frac{1}{2} < 1, \quad (3.18)$$

and thus $\|\Phi V^2 - \Phi V^1\|_{X_T} \leq \frac{1}{2} \|V^2 - V^1\|_{X_T}$. Thus operator Φ is a contracted mapping on set B_R , i.e. condition (ii) is fulfilled.

From (3.17) and (3.18), in turn, follows that, if $T \leq T_*$, where

$$T_* := \min \left\{ \frac{R}{2(RM(R) + R + f_\infty)}, \frac{1}{2}, \frac{1}{4(Rc(R) + R + M(R))} \right\}, \quad (3.19)$$

then $\|\Phi V\|_{X_T} \leq R$ and $\|\Phi V^2 - \Phi V^1\|_{X_T} \leq \frac{1}{2} \|V^2 - V^1\|_{X_T}$ for $V, V^1, V^2 \in B_R$.

The application of the contracted mapping principle shows that there exists a solution V of (3.16) in $C(\overline{D_T}; \mathbb{R}^3)$ for $0 < T \leq T_*$. Theorem 3.1 is proved completely. $\square$

4. The case of global solvability of the problem (1.1), (1.2)

Theorem 4.1. *Let the conditions (2.1), (3.13) and (3.14) be valid. Then for any $T > 0$ the problem (1.1), (1.2) has a strong generalized solution of the class C^1 in the domain D_T .*

Proof. As it was noted in Lemma 3.1 the problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of nonlinear integral equations (2.22) in the space $C(\overline{D}_T)$. In view of (3.13), (3.14) the truth of the Theorem 4.1 for sufficiently small T , namely for $T \leq T_*$, where T_* is given by the equality (3.19) follows from the Theorem 3.1. Let now $T > T_*$, and $V^{T_*} := (v_1^{T_*}, v_2^{T_*}, v_3^{T_*})$ is a solution of the system of nonlinear integral equations (2.22), or, the same, of vector equation (3.16) in the domain D_{T_*} of the space $C(\overline{D}_{T_*})$ according to Theorem 3.1. For $t > \Delta t_1 := T_*$ rewrite system (2.22) as follows

$$\begin{cases} v_1(x, t) = -\frac{1}{2} \int_{\alpha_1(x, t, \Delta t_1)}^t g(P_\tau, v_3(P_\tau))(v_1(P_\tau) + v_2(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\alpha_2(x, t, \Delta t_1)}^t g(Q_{\tau_0}, v_3(Q_{\tau_0}))(v_1(Q_{\tau_0}) + v_2(Q_{\tau_0})) d\tau + F_{1, \Delta t_1}(x, t), \\ v_2(x, t) = -\frac{1}{2} \int_{\alpha_3(x, t, \Delta t_1)}^t g(Q_\tau, v_3(Q_\tau))(v_1(Q_\tau) + v_2(Q_\tau)) d\tau \\ \quad + F_{2, \Delta t_1}(x, t), \\ v_3(x, t) = \int_{\alpha_3(x, t, \Delta t_1)}^t v_1(Q_\tau) d\tau + F_{3, \Delta t_1}(x, t), \end{cases} \quad (4.1)$$

where

$$\begin{aligned} \alpha_1(x, t, \Delta t_1) &:= \max(\Delta t_1, t - x), \quad \alpha_2(x, t, \Delta t_1) := \max\left(\Delta t_1, \frac{t}{2}\right), \\ \alpha_3(x, t, \Delta t_1) &:= \max\left(\Delta t_1, \frac{x + t}{2}\right); \end{aligned}$$

$$\begin{cases} F_{1, \Delta t_1}(x, t) := -\frac{1}{2} \int_{t-x}^{\alpha_1(x, t, \Delta t_1)} g(P_\tau, v_3^{T_*}(P_\tau))(v_1^{T_*}(P_\tau) + v_2^{T_*}(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\frac{t}{2}}^{\alpha_2(x, t, \Delta t_1)} g(Q_{\tau_0}, v_3^{T_*}(Q_{\tau_0}))(v_1^{T_*}(Q_{\tau_0}) + v_2^{T_*}(Q_{\tau_0})) d\tau + F_1(x, t), \\ F_{2, \Delta t_1}(x, t) := -\frac{1}{2} \int_{\frac{x+t}{2}}^{\alpha_3(x, t, \Delta t_1)} g(Q_\tau, v_3^{T_*}(Q_\tau))(v_1^{T_*}(Q_\tau) + v_2^{T_*}(Q_\tau)) d\tau \\ \quad + F_2(x, t), \\ F_{3, \Delta t_1}(x, t) := \int_{\frac{x+t}{2}}^{\alpha_3(x, t, \Delta t_1)} v_1^{T_*}(Q_\tau) d\tau. \end{cases} \quad (4.2)$$

Since the conditions of Lemma 2.2 are fulfilled, then for any positive $\tau \leq T$ for solution of vector equation (3.16) in the domain D_τ of the space $C(\overline{D}_\tau)$ it is valid a priori estimate

$$\|V\|_{C(\overline{D}_\tau)} \leq R^T \left(\|f\|_{C(\overline{D}_\tau)} \right), \quad (4.3)$$

where $R^T = R^T(s)$ is a non-decreasing continuous function of its argument $s \geq 0$.

Let $R_* := R^T \left(\|f\|_{C(\overline{D}_T)} \right)$. As the second step Δt_2 with respect to t we take

$$\Delta t_2 := \min \left\{ \frac{1}{4M(R_1)R_1}, \frac{1}{4c(R_1)R_1} \right\}, \quad (4.4)$$

where

$$R_1 := 1 + 2TM(R_*)R_* + \|F\|_{C(\overline{D}_T)}, \quad F := (F_1, F_2, F_3). \quad (4.5)$$

Rewrite the system of equations (4.1) for $t \in [T_*, T_* + \Delta t_2]$ in the form of one vector equation

$$V = \Psi V, \quad (4.6)$$

where operator $\Psi := (\Psi_1, \Psi_2, \Psi_3)$ acts by formula

$$\left\{ \begin{array}{l} (\Psi_1 V)(x, t) = -\frac{1}{2} \int_{\alpha_1(x, t, \Delta t_1)}^t g(P_\tau, v_3(P_\tau))(v_1(P_\tau) + v_2(P_\tau)) d\tau \\ \quad + \frac{1}{2} \int_{\alpha_2(x, t, \Delta t_1)}^t g(Q_{\tau_0}, v_3(Q_{\tau_0}))(v_1(Q_{\tau_0}) + v_2(Q_{\tau_0})) d\tau + F_{1, \Delta t_1}(x, t), \\ (\Psi_2 V)(x, t) = -\frac{1}{2} \int_{\alpha_3(x, t, \Delta t_1)}^t g(Q_\tau, v_3(Q_\tau))(v_1(Q_\tau) + v_2(Q_\tau)) d\tau \\ \quad + F_{2, \Delta t_1}(x, t), \\ (\Psi_3 V)(x, t) = \int_{\alpha_3(x, t, \Delta t_1)}^t v_1(Q_\tau) d\tau + F_{3, \Delta t_1}(x, t). \end{array} \right. \quad (4.7)$$

First we show that operator Ψ maps the ball $B([T_1, T_2]; R_1) := \{V \in C(\overline{D}_{T_1, T_2}) : \|V\|_{C(\overline{D}_{T_1, T_2})} \leq R_1\}$ into itself, where $T_1 = T_*$ and $T_2 = T_* + \Delta t_2$, $\overline{D}_{T_1, T_2} := \overline{D} \cap \{T_1 \leq t \leq T_2\}$.

Indeed, due to (3.14), (4.2)–(4.5) and (4.7) we have

$$\begin{aligned} \|\Psi_1 V\|_{C(\overline{D}_{T_1, T_2})} &\leq 2M(R_1)R_1\Delta t_2 + 2M(R_*)R_*\Delta t_1 + \|F_1\|_{C(\overline{D}_T)} \\ &\leq 2^{-1} + 2TM(R_*)R_* + \|F\|_{C(\overline{D}_T)} \leq R_1. \end{aligned}$$

Analogously: $\|\Psi_i V\|_{C(\overline{D}_{T_1, T_2})} \leq R_1$, $i = 2, 3$, and finally we receive that

$$\|\Psi V\|_{C(\overline{D}_{T_1, T_2})} \leq R_1.$$

Now let us show that operator Ψ is a contracted mapping in this ball. Indeed, for $(x, t) \in \overline{D}_{T_1, T_2}$ due to (3.14), (4.4) and (4.7) we have

$$\begin{aligned}
& |(\Psi_1 V^2 - \Psi_1 V^1)(x, t)| \\
& \leq \frac{1}{2} \int_{\alpha_1(x, t, \Delta t_1)}^t \left(|g(P_\tau, v_3^2(P_\tau)) - g(P_\tau, v_3^1(P_\tau))| (v_1^2(P_\tau) + v_2^2(P_\tau)) \right. \\
& \quad \left. + |g(P_\tau, v_3^1(P_\tau))| (v_1^2(P_\tau) - v_1^1(P_\tau) + v_2^2(P_\tau) - v_2^1(P_\tau)) \right) d\tau \\
& \quad + \frac{1}{2} \int_{\alpha_1(x, t, \Delta t_1)}^t \left(|g(Q_{\tau_0}, v_3^2(Q_{\tau_0})) - g(Q_{\tau_0}, v_3^1(Q_{\tau_0}))| (v_1^2(Q_{\tau_0}) + v_2^2(Q_{\tau_0})) \right. \\
& \quad \left. + |g(Q_{\tau_0}, v_3^1(Q_{\tau_0}))| (v_1^2(Q_{\tau_0}) - v_1^1(Q_{\tau_0}) + v_2^2(Q_{\tau_0}) - v_2^1(Q_{\tau_0})) \right) d\tau \\
& \leq 2c(R_1)R_1\Delta t_2 \|v_3^2 - v_3^1\|_{C(\overline{D}_{T_1, T_2})} + 2M(R_1)\Delta t_2 \|V^2 - V^1\|_{C(\overline{D}_{T_1, T_2})} \\
& \leq \frac{1}{2} \|v_3^2 - v_3^1\|_{C(\overline{D}_{T_1, T_2})} + \frac{1}{2R_1} \|V^2 - V^1\|_{C(\overline{D}_{T_1, T_2})} \\
& \leq \left(\frac{1}{2} + \frac{1}{2R_1} \right) \|V^2 - V^1\|_{C(\overline{D}_{T_1, T_2})} = q_1 \|V^2 - V^1\|_{C(\overline{D}_{T_1, T_2})},
\end{aligned}$$

where $q_1 := \frac{1}{2} \left(1 + \frac{1}{R_1} \right) < 1$, since $R_1 > 1$ in view of (4.5).

Analogously we receive, that

$$\begin{aligned}
& |(\Psi_i V^2 - \Psi_i V^1)(x, t)| \leq q_i \|V^2 - V^1\|_{C(\overline{D}_{T_1, T_2})}, \\
& 0 < q_i := \text{const} < 1, \quad i = 2, 3.
\end{aligned}$$

Thus, due to the theorem about contracted mapping it follows the solvability of vector equation (4.6) in space $C(\overline{D}_{T_1, T_2})$. $\square$

Continuing this process step by step, and taking into account the fact that in view of global a priori estimate (4.3), and also (2.28), the length of each step Δt_i does not depend on i . Thus we receive the global solvability of the system of equations (2.22), and therefore of problem (1.1), (1.2) in domain D_T for any $T > 0$.

Remark 4.1. One may present the examples of functions $g = g(x, t, s)$, satisfying the conditions of the Theorem 4.1, or, the same, the conditions (2.1), (3.13) and (3.14). Such function is

$$g(x, t, s) = \sum_{k=1}^n \alpha_k(x, t) |s|^{\beta_k},$$

where $\alpha_k \in C(\overline{D}_\infty)$, $k = 1, \dots, n$; $\alpha_1(x, t) > 0$ for $(x, t) \in \overline{D}_\infty$ and $\beta_1 > \beta_i \geq 1$, $i = 2, \dots, n$; and also function $g(x, t, s) = \alpha(x, t)g_0(s)$, where $\alpha \in C(\overline{D}_\infty)$, $g_0 \in Lip_{loc}(\mathbb{R})$ and $\liminf_{|s| \rightarrow +\infty} g_0(s) > -\infty$, satisfy the conditions of Theorem 4.1.

5. Uniqueness of the solution of the problem (1.1), (1.2)

Lemma 5.1. *Let the conditions (3.13), (3.14) be fulfilled. Then for any $T > 0$ the problem (1.1), (1.2) cannot have more than one strong generalized solution of the class C^1 in domain D_T .*

Proof. Indeed, suppose that the problem (1.1), (1.2) has two different possible strong generalized solutions u^1 and u^2 of the class C^1 in the domain D_T . Then, according to Definition 1.1 there exists the sequence of functions $u_n^i \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n^i - u^i\|_{C^1(\overline{D}_T)} &= 0, \quad \lim_{n \rightarrow \infty} \|Lu_n^i - f\|_{C(\overline{D}_T)} = 0, \\ \lim_{n \rightarrow \infty} \|g(x, t, u_n^i)u_{nt}^i - g(x, t, u^i)u_t^i\|_{C(\overline{D}_T)} &= 0, \quad i = 1, 2. \end{aligned} \quad (5.1)$$

Let us use well-known notation $\square := \partial^2/\partial t^2 - \partial^2/\partial x^2$ and assume that $\omega_n := u_n^2 - u_n^1$. It is easy to see that function $\omega_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$ is a classical solution of the following problem

$$\square \omega_n + g_n = f_n, \quad (5.2)$$

$$\omega_n|_{\Gamma_T} = 0, \quad (5.3)$$

where

$$g_n := g(x, t, u_n^2)u_{nt}^2 - g(x, t, u_n^1)u_{nt}^1, \quad f_n := Lu_n^2 - Lu_n^1. \quad (5.4)$$

Due to the first equality from (5.1) there exists the number $A := \text{const} > 0$, not dependent on indices i and n , such that

$$\|u_n^i\|_{C^1(\overline{D}_T)} \leq A. \quad (5.5)$$

According to the second equalities from (5.1) and (5.4) it follows that

$$\lim_{n \rightarrow \infty} \|f_n\|_{C(\overline{D}_T)} = 0. \quad (5.6)$$

From (3.14), (5.5) and the first equality of (5.4) it is clear that

$$\begin{aligned} g_n^2 &= (g(x, t, u_n^2)\omega_{nt} + [g(x, t, u_n^2) - g(x, t, u_n^1)]u_{nt}^1)^2 \\ &\leq 2M^2(A)\omega_{nt}^2 + 2A^2c^2(A)\omega_n^2. \end{aligned} \quad (5.7)$$

Multiplying the both sides of (5.2) by ω_{nt} and integrating the received equality in the domain D_τ , due to boundary conditions (5.3), as it was in receiving of (2.11) from (2.5), (2.6) we shall have

$$w_n(\tau) := \int_{\Omega_\tau} (\omega_{nx}^2 + \omega_{nt}^2) dx = 2 \int_{D_\tau} (f_n - g_n) \omega_{nt} dx dt. \quad (5.8)$$

Due to estimate (5.7) and the inequality of Cauchy we shall have

$$\begin{aligned}
 2 \int_{D_\tau} (f_n - g_n) \omega_{nt} dx dt &\leq \int_{D_\tau} (f_n - g_n)^2 dx dt + \int_{D_\tau} \omega_{nt}^2 dx dt \\
 &\leq 2 \int_{D_\tau} f_n^2 dx dt + 2 \int_{D_\tau} g_n^2 dx dt + \int_{D_\tau} \omega_{nt}^2 dx dt \\
 &\leq 2 \int_{D_\tau} f_n^2 dx dt + 4A^2 c^2(A) \int_{D_\tau} \omega_n^2 dx dt + [1 + 4M^2(A)] \int_{D_\tau} \omega_{nt}^2 dx dt. \quad (5.9)
 \end{aligned}$$

Further, from equality $\omega_n(x, t) = \int_x^t \omega_{nt}(x, \tau) d\tau$, $(x, t) \in \overline{D}_T$, which follows from (5.3), by use of standard considerations we receive inequality [25, p. 63]

$$\int_{D_\tau} \omega_n^2 dx dt \leq \tau^2 \int_{D_\tau} \omega_{nt}^2 dx dt. \quad (5.10)$$

From (5.8)–(5.10) it follows that

$$\begin{aligned}
 w_n(\tau) &\leq [1 + 4M^2(A) + 4\tau^2 A^2 c^2(A)] \int_{D_\tau} \omega_{nt}^2 dx dt + 2 \int_{D_\tau} f_n^2 dx dt \\
 &\leq [1 + 4M^2(A) + 4T^2 A^2 c^2(A)] \int_0^\tau w_n(\sigma) d\sigma + 2 \int_{D_T} f_n^2 dx dt.
 \end{aligned}$$

Whence by the Gronwall's lemma we receive that

$$w_n(\tau) \leq c_2 \|f_n\|_{L_2(D_T)}^2, \quad 0 < \tau \leq T, \quad (5.11)$$

where $c_2 := 2 \exp [1 + 4M^2(A) + 4T^2 A^2 c^2(A)] T$.

Conducting the same considerations, as those used for receiving of (2.13), and taking into account obvious inequality

$$\|f_n\|_{L_2(D_T)}^2 \leq \|f_n\|_{C(\overline{D}_T)}^2 \text{mes } D_T,$$

and also due to (5.11), for $(x, t) \in \overline{D}_T$ we have

$$|\omega_n(x, t)|^2 \leq t w_n(t) \leq T c_2 \text{mes } D_T \|f_n\|_{C(\overline{D}_T)}^2 = \frac{c_2 T^3}{2} \|f_n\|_{C(\overline{D}_T)}^2.$$

From this inequality it follows immediately that

$$\|\omega_n\|_{C(\overline{D}_T)} \leq T \sqrt{\frac{c_2 T}{2}} \|f_n\|_{C(\overline{D}_T)}. \quad (5.12)$$

Recalling the definition of function ω_n , according to the first equality from (5.1) we have $\lim_{n \rightarrow \infty} \|\omega_n\|_{C^1(\overline{D}_T)} = \|u^2 - u^1\|_{C^1(\overline{D}_T)}$ and all the more

$$\lim_{n \rightarrow \infty} \|\omega_n\|_{C(\overline{D}_T)} = \|u^2 - u^1\|_{C(\overline{D}_T)}. \quad (5.13)$$

Due to this equality and (5.6), passing in (5.12) to limit for $n \rightarrow \infty$ we receive $\|u^2 - u^1\|_{C(\overline{D}_T)} = 0$, i.e. $u^1 = u^2$, which proves Lemma 5.1. $\square$

6. The case of nonexistence of a global solution of the problem (1.1), (1.2)

Below we will show that violation of condition (2.1) may cause the absence of global solvability of the problem (1.1), (1.2) in the sense of Definition 1.2. Indeed, let $g(x, t, s) = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$.

Lemma 6.1. *Let u be a strong generalized solution of the problem (1.1), (1.2) of the class C^1 in the domain D_T in the sense of Definition 1.1. Then it is valid the following integral equality*

$$\int_{D_T} u \square \varphi dx dt = \int_{D_T} |u|^\alpha u u_t \varphi dx dt + \int_{D_T} f \varphi dx dt \quad (6.1)$$

for any function φ , such that

$$\varphi \in C^2(\overline{D}_T), \quad \varphi|_{t=T} = 0, \quad \varphi_t|_{t=T} = 0, \quad \varphi|_{\gamma_{2,T}} = 0. \quad (6.2)$$

Proof. According to the definition of a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T , function $u \in C^1(\overline{D}_T)$ and there exists the sequence of functions $u_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, such that the equalities of (2.3) and (2.4) for $g(x, t, s) = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$ are valid.

Suppose that $f_n := Lu_n$. Let us multiply the both sides of equality $Lu_n = f_n$ by function φ and integrate the received equality in the domain D_T . As a result of integration by parts of the left-hand side of this equality, due to (6.2) and boundary conditions (2.6) we receive

$$\int_{D_T} u_n \square \varphi dx dt = \int_{D_T} |u_n|^\alpha u_n u_{nt} \varphi dx dt + \int_{D_T} f_n \varphi dx dt.$$

By passing to limit in this equality for $n \rightarrow \infty$, according to (2.3)–(2.5) we receive (6.1). Thus Lemma 6.1 is proved.

Let us introduce into consideration function $\varphi^0 := \varphi^0(x, t)$ such that [9, pp. 10–12]

$$\varphi^0 \in C^2(\overline{D}_\infty), \quad \varphi^0 + \varphi_t^0 \leq 0, \quad \varphi^0|_{D_{T=1}} > 0, \quad \varphi^0|_{\gamma_{2,\infty}} = 0, \quad \varphi^0|_{t \geq 1} = 0 \quad (6.3)$$

and

$$\kappa_0 := \int_{D_{T=1}} \frac{|\square \varphi^0|^{p'}}{|\varphi^0|^{p'-1}} dx dt < +\infty, \quad p' = \frac{\alpha + 2}{\alpha + 1}. \quad (6.4)$$

It is easy to verify that in the role of function φ^0 , satisfying conditions (6.3) and (6.4), one may use function

$$\varphi^0(x, t) = \begin{cases} x^n(1-t)^m, & (x, t) \in D_{T=1}, \\ 0, & t \geq 1, \end{cases}$$

for sufficiently large positive numbers n and m .

Suppose that $\varphi_T(x, t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, $T > 0$. Due to (6.3) it is easy to see that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T + T\varphi_{Tt} \leq 0, \quad \varphi_T|_{D_T} > 0, \quad \varphi_T|_{\gamma_{2,T}} = 0, \quad (6.5)$$

$$\varphi_T|_{t=T} = 0, \quad \varphi_{Tt}|_{t=T} = 0.$$

Supposing that function f is fixed, let us introduce into consideration a function of one variable T

$$\zeta(T) := \int_{D_T} f\varphi_T dxdt, \quad T > 0. \quad (6.6)$$

The following theorem on the nonexistence of a global solution of the problem (1.1), (1.2) is valid. $\square$

Theorem 6.1. *Let $g(x, t, s) = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$, $f \in C(\overline{D}_\infty)$ and $f \geq 0$ in the domain D_∞ . If*

$$\liminf_{T \rightarrow +\infty} \zeta(T) > 0, \quad (6.7)$$

then there exists positive number $T^ := T^*(f)$, such that for $T > T^*$ the problem (1.1), (1.2) cannot have a strong generalized solution u of the class C^1 in the domain D_T .*

Proof. Suppose, that in conditions of this theorem there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T . Then according to Lemma 6.1 equality (6.1) holds, where due to (6.5) in the role of function φ can be taken function $\varphi = \varphi_T$, i.e.

$$\int_{D_T} u \square \varphi_T dxdt = \int_{D_T} |u|^\alpha u u_t \varphi_T dxdt + \int_{D_T} f \varphi_T dxdt. \quad (6.8)$$

In view of (1.2) and (6.5) we have

$$\begin{aligned} \int_{D_T} |u|^\alpha u u_t \varphi_T dxdt &= \frac{1}{\alpha + 2} \int_{D_T} \varphi_T (|u|^{\alpha+2})_t dxdt = -\frac{1}{\alpha + 2} \int_{D_T} |u|^{\alpha+2} \varphi_{Tt} dxdt \\ &\geq \frac{1}{(\alpha + 2)T} \int_{D_T} |u|^{\alpha+2} \varphi_T dxdt. \end{aligned}$$

Due to this inequality and (6.6) from (6.8) it follows, that

$$\frac{1}{pT} \int_{D_T} |u|^p \varphi_T dxdt \leq \int_{D_T} u \square \varphi_T dxdt - \zeta(T), \quad p := \alpha + 2 > 1. \quad (6.9)$$

If in Young equality with parameter $\varepsilon > 0$, $ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'-1}} b^{p'}$; $a, b \geq 0$, $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$ we shall take $a = |u| \varphi_T^{\frac{1}{p}}$, $b = \frac{|\square \varphi_T|}{\varphi_T^{\frac{1}{p'}}$, $\varepsilon = \frac{1}{T}$, then since $\frac{p'}{p} = p' - 1$ we obtain

$$|u \square \varphi_T| = |u| \varphi_T^{\frac{1}{p}} \frac{|\square \varphi_T|}{\varphi_T^{\frac{1}{p'}}} \leq \frac{1}{pT} |u|^p \varphi_T + \frac{T^{p'-1}}{p'} \frac{|\square \varphi_T|^{p'}}{\varphi_T^{p'-1}}.$$

According to this inequality and (6.9) we have

$$0 \leq \frac{T^{p'-1}}{p'} \int_{D_T} \frac{|\square \varphi_T|^{p'}}{\varphi_T^{p'-1}} dx dt - \zeta(T). \quad (6.10)$$

Due to (6.3), (6.4), after changing variables $x = Tx_1$, $t = Tt_1$, it is easy to verify that

$$\int_{D_T} \frac{|\square \varphi_T|^{p'}}{\varphi_T^{p'-1}} dx dt = \frac{1}{T^{2(p'-1)}} \int_{D_{T=1}} \frac{|\square \varphi^0|^{p'}}{|\varphi^0|^{p'-1}} dx_1 dt_1 = \frac{\kappa_0}{T^{2(p'-1)}}.$$

In view to this from (6.10) we receive

$$0 \leq \frac{\kappa_0}{p'T^{p'-1}} - \zeta(T). \quad (6.11)$$

Since $p' = \frac{p}{p-1} > 1$ due to (6.4) we have

$$\lim_{T \rightarrow +\infty} \frac{\kappa_0}{p'T^{p'-1}} = 0. \quad (6.12)$$

Therefore, by virtue of (6.7) there exists positive number $T^* := T^*(f)$, such that for $T > T^*$ the right-hand side of (6.11) will be negative, whereas the left-hand side of this inequality is zero. This means that if there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C^1 in the domain D_T , then necessarily $T \leq T^*$, which proves the Theorem 6.1. $\square$

Remark 6.1. It is easy to verify that if $f \in C(\overline{D}_\infty)$, $f \geq 0$ and $f(x, t) \geq ct^{-m}$ for $t \geq 1$, where $c := \text{const} > 0$, $0 \leq m := \text{const} \leq 2$, then condition (6.7) will be fulfilled, and so for $g(x, t, s) = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$ the problem (1.1), (1.2) for sufficiently large T will not have a strong generalized solution u of the class C^1 in the domain D_T .

Indeed, let us introduce in (6.6) the transformation of independent variables x and t by formula $x = Tx_1$, $t = Tt_1$, after some estimations we have

$$\begin{aligned} \zeta(T) &= T^2 \int_{D_{T=1}} f(Tx_1, Tt_1) \varphi^0(x_1, t_1) dx_1 dt_1 \\ &\geq cT^{2-m} \int_{D_{T=1} \cap \{t_1 \geq T^{-1}\}} t_1^{-m} \varphi^0(x_1, t_1) dx_1 dt_1 \\ &\quad + T^2 \int_{D_{T=1} \cap \{t_1 < T^{-1}\}} f(Tx_1, Tt_1) \varphi^0(x_1, t_1) dx_1 dt_1 \end{aligned}$$

in supposition that $T > 1$. Further, let $T_1 > 1$ be any fixed number. Then from the last inequality for function ζ we have

$$\begin{aligned} \zeta(T) &\geq cT^{2-m} \int_{D_{T=1} \cap \{t_1 \geq T^{-1}\}} t_1^{-m} \varphi^0(x_1, t_1) dx_1 dt_1 \\ &\geq c \int_{D_{T=1} \cap \{t_1 \geq T_1^{-1}\}} t_1^{-m} \varphi^0(x_1, t_1) dx_1 dt_1, \end{aligned}$$

if $T \geq T_1 > 1$. From the latter inequality immediately follows the validity of (6.7).

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