Lax type formulas with lower semicontinuous initial data and hypercontractivity results

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Abstract. In Avantaggiati and Loreti [Ric. Mat. **57**(2):171–202, 2008] we studied the Cauchy problem for a class of Hamilton–Jacobi equations with initial data verifying the Lipschitz condition. In this paper we extend those results to the case in which the initial data are lower semicontinuous [in the following lsc], and are lower bounded and semiconvex. Here we prove hypercontractivity results and new Logarithm Sobolev Inequalities (shortly, LSI).

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1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t(x, t) + \alpha x Du(x, t) + H(Du(x, t)) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where α is a positive real number and $H \in C^1(\mathbb{R}^n)$, convex, p -homogeneous, and even. In the previous paper [2] we studied the problem (1.1) taking $H(x) = \frac{1}{2}|x|^2$ and $n = 1$, in [4] we considered the problem for more general H ; in the paper [3] we have set $\sum_j \alpha_j x_j \partial_{x_j} u(x, t)$ instead of $\alpha x Du(x, t)$, and we studied the case

$$\begin{cases} u_t(x, t) + \sum_j \alpha_j x_j \partial_{x_j} u(x, t) + \frac{1}{2}|Du(x)|^2 = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n. \end{cases} \quad (1.2)$$

In the papers [2–4] we assumed that u_0 is Lipschitz continuous in all the space $\mathbb{R}^n$ and *admissible*, which means

$$\int_{\mathbb{R}^n} \exp[u_0(x)] dx < +\infty. \quad (1.3)$$

The assumption (1.3) is mainly used to prove LSI and it appears in several papers in the literature. For an earlier study of LSI in connection with Hamilton–Jacobi equations we refer to the papers [5, 11, 12, 14].

In this paper, we extend some of the results of the papers [2–4], and we explain here, shortly, the content of it.

In Sect. 2 we adapt the definition of lower boundedness [see definition (2.1)]. This definition was suggested by the estimates obtained in our previous papers.

The goodness of such definition is proved in the same section. In fact, if u_0 is lsc and so lower bounded, the function

$$(Q_t u_0)(x) = \inf_{y \in \mathbb{R}^n} \left(u_0(y) + \left(\frac{\alpha p}{1 - e^{-2\alpha p t}} \right)^{q-1} H^*(y - e^{-\alpha t} x) \right) \quad (1.4)$$

attains the minimum [see theorem (2.2)]. Furthermore, we prove that $(Q_t u_0)(x)$ is lsc and it is viscosity solution of (1.1); see the theorem 2.4 and the propositions 2.7, and 2.8.

In the next sections, we are concerned with the hypercontractivity and the Logarithmic Sobolev Inequalities and we are mainly interested to improve the results of [3].

In such sections we make on u_0 a particular type of semiconvexity hypothesis. In Sect. 3, under such hypothesis, we represent the solution of the problem (1.2) as difference of two functions, one of which solution of the Hamilton–Jacobi equation, more precisely as the difference of a C^∞ function and a function with Lipschitz continuous gradient; see Theorem 3.6.

In Sects. 4 and 5, we replace the assumption of admissibility assuming that u_0 has the behaviour

$$u_0(x) = o(H(x)), \quad \text{for } |x| \rightarrow +\infty.$$

With such framework we get new hypercontractivity weighted estimates and Logarithmic Sobolev Inequalities, which extend those of papers [3]. In Theorems 4.4, 4.6, 5.1 and 5.3 some new results are stated and proved.

2. Problem and main assumptions

The assumptions used in the paper on the function H are

$$\begin{aligned} H \text{ is } C^1(\mathbb{R}^n), H \text{ is an even, nonnegative, convex function,} \\ \text{positive – homogeneous of degree } p > 1 \end{aligned} \quad (2.1)$$

We recall that H^* denote the conjugate of H , i.e. its Legendre transform, defined by

$$H^*(\zeta) = \sup_{y \in \mathbb{R}^n} \{y\zeta - H(y)\} \quad \zeta \in \mathbb{R}^n.$$

As well known, $H^\star$ is an even, non negative, and convex function, positively homogeneous of degree q , with $\frac{1}{p} + \frac{1}{q} = 1$. We consider

$$u(x, t) = \inf_{y \in \mathbb{R}^n} \left(u_0(y) \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) + \frac{1 - e^{-\alpha p t}}{\alpha p} H^\star(y) \right), \quad (2.2)$$

or the equivalent formula,

$$u(x, t) = \inf_{y \in \mathbb{R}^n} \left(u_0(y) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^\star(y - e^{-\alpha t} x) \right). \quad (2.3)$$

We adapt the definition of lower boundedness to our problem

Definition 2.1. We say that u_0 is $(H^\star, \alpha)$ lower bounded if there exist $A \in]0, \alpha]$ and $x_0 \in \mathbb{R}^n$, such that

$$u_0(x) \geq -(Ap)^{q-1} H^\star(x - x_0). \quad (2.4)$$

The assumption on the function u_0 is

u_0 is real valued functions lsc on all the space $\mathbb{R}^n$, $(H^\star, \alpha)$ lower bounded.

2.1. Regularity properties of u

We have the result

Theorem 2.2. If (2.1), (2.4) hold, then the function u , defined by (2.3), has the following property

$$\begin{aligned} & \forall x \in \mathbb{R}^n \forall t \in]0, +\infty[, \exists y_* \in \mathbb{R}^n \text{ s.t.} \\ & u(x, t) = u_0(y_*) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^\star(y_* - e^{-\alpha t} x). \end{aligned}$$

Proof. Let x, t fixed; first of all, we observe that

$$u(x, t) \leq u_0(e^{-\alpha t} x).$$

On the other hand, from the hypothesis, since $0 < A \leq \alpha$, we get

$$\lim_{y \rightarrow \infty} \left[-(Ap)^{q-1} H^\star(y - x_0) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^\star(y - e^{-\alpha t} x) \right] = +\infty.$$

Thanks to (2.4) it is possible therefore to find R such that

$$|y| > R \Rightarrow u_0(y) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^\star(y - e^{-\alpha t} x) > u_0(e^{-\alpha t} x).$$

It follows that the function U_0 defined by:

$$U_0(y) = u_0(y) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^\star(y - e^{-\alpha t} x), \quad (2.5)$$

is lower bounded on the ball $B_R = \{y \in \mathbb{R}^n \text{ s.t. } |y| \leq R\}$, and the set E of the points $y \in \mathbb{R}^n$, on which it must be computed the infimum of $U_0(y)$, may be B_R . Since $U_0(y)$ is lsc and B_R is a compact set, the thesis follows with $y_* \in B_R$. $\square$

Remark 2.3. If the assumptions of the definition (2.1) hold with $A > \alpha$, then there exists $t \in]0, +\infty[$ such that

$$\frac{\alpha}{1 - e^{-\alpha p t}} < A, \quad (2.6)$$

and the function U_0 [defined by (2.5)] may be lower unbounded. Therefore the thesis of the theorem (2.2) may be false. It is useful to observe that (2.6) is equivalent, for $A > \alpha$, to $t > \frac{1}{\alpha p} \log \frac{A}{A-\alpha}$.

Our purpose is now the following

Theorem 2.4. *If (2.1), (2.4) hold, then the function u , defined by (2.3) is lsc on $\mathbb{R}^n \times]0, +\infty[$.*

Proof. It is enough to state that all the sub-level set of $u(x, t)$ are closed in $\mathbb{R}^n \times]0, +\infty[$. Let

$$S_r(u) = \{(x, t) \in \mathbb{R}^n \times]0, +\infty[\text{ s.t. } u(x, t) \leq r\},$$

with $r \in \mathbb{R}$, such that $S_r(u) \neq \emptyset$. We have to prove that $S_r(u)$ is closed. Let us consider a sequence (x_k, t_k) such that

$$u(x_k, t_k) \leq r \quad \text{for all } k \in \mathbb{N}; x_k \rightarrow x \text{ and } t_k \rightarrow t. \quad (2.7)$$

For each $k \in \mathbb{N}$ there exists $y_k \in \mathbb{R}^n$ such that

$$u(x_k, t_k) = u_0(y_k) + \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1} H^*(y_k - e^{-\alpha t_k} x_k).$$

As first, we prove that the sequence (y_k) is bounded.

We argue by contradiction. Assume (y_k) unbounded. Without restricting the generality, we may assume that

$$\lim_{k \rightarrow \infty} |y_k| = +\infty.$$

From (2.7) it follows

$$\begin{aligned} r &\geq u(x_k, t_k) = u_0(y_k) + \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1} H^*(y_k - e^{-\alpha t_k} x_k) \\ &\geq -(Ap)^{q-1} H^*(y_k - x_0) + \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1} H^*(y_k - e^{-\alpha t_k} x_k). \end{aligned} \quad (2.8)$$

Since $e^{-\alpha p t_k} \rightarrow e^{-\alpha p t}$, $x_k \rightarrow x$ and

$$(Ap)^{q-1} < \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1}, \quad (2.9)$$

we obtain

$$\lim_{k \rightarrow \infty} -(Ap)^{q-1} H^*(y_k - x_0) + \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1} H^*(y_k - e^{-\alpha t_k} x_k) = +\infty,$$

which is impossible by (2.7). Therefore (y_k) is bounded. Then there exists a convergent subsequence. We may assume that

$$\exists y_0 \in \mathbb{R}^n \text{ such that } \lim_{k \rightarrow \infty} y_k = y_0.$$

From (2.8) and (2.9) we get

$$\begin{aligned} r &\geq \liminf_k u(x_k, t_k) = \liminf_k \left[u_0(y_k) + \left(\frac{\alpha p}{1 - e^{-\alpha p t_k}} \right)^{q-1} H^*(y_k - e^{-\alpha t_k} x_k) \right] \\ &\geq u_0(y_0) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}} \right)^{q-1} H^*(y_0 - e^{-\alpha t} x) \geq u(x, t), \end{aligned}$$

and this complete the proof. $\square$

We have to prove now that $u(x, t)$, for $t \rightarrow 0$, takes the initial datum u_0 . Actually the meaning of such assertion is given by the following

Theorem 2.5. *If (2.1), (2.4) hold, then for every $x' \in \mathbb{R}^n$*

$$\lim_{r \rightarrow 0} \inf_{(y, s)} \{u(y, s); \quad |y - x'| < r, 0 < s < r\} = u_0(x').$$

Proof. We fix $x' \in \mathbb{R}^n$, and we introduce the function

$$v(x, t, y) = \varphi(t) H^* \left(\frac{y - e^{-\alpha t} x}{\varphi(t)} \right), \quad \text{where } \varphi(t) = \frac{1 - e^{-\alpha p t}}{\alpha p}, \quad t \in [0, +\infty[.$$

Such function v is convex with respect to both the variables x, y . On the other hand we know that for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ there exists $y_* \in \mathbb{R}^n$ such that $u(x, t) = u_0(y_*) + v(x, t, y_*) \geq u_0(y_*) + s(y_* - e^{-\alpha t} x) - \varphi(t) H(s)$, $\forall s \in \mathbb{R}^n$.

We define

$$s(r) = \frac{r}{q} (DH^*)(y_* - e^{-\alpha t} x),$$

and observe, using $s(r)$, the lower estimation of u_0 , the q -homogeneity and the convexity of H^* , that

$$\begin{aligned} u(x, t) &\geq -(Ap)^{q-1} H^*(y_* - x_0) + r H^*(y_* - e^{-\alpha t} x) - \varphi(t) H(s(r)) \\ &\geq (r - (2Ap)^{q-1}) H^*(y_* - e^{-\alpha t} x) - (2Ap)^{q-1} H^*(e^{-\alpha t} x - x_0) - \varphi(t) H(s(r)) \end{aligned}$$

We fix ϵ , and find δ so that

$$|y - x'| < 4\delta \Rightarrow u_0(y) > u_0(x') - \epsilon.$$

We introduce the set

$$\mathcal{E} = B(x', \delta) \times (0, \tau)$$

We now observe that, when $(x, t) \in \mathcal{E}$, e.g., (x, t) is in a bounded set, also y_* is in a bounded set, then there exists $R(\mathcal{E}) \in \mathbb{R}_+$ such that

$$|y_*| \leq R(\mathcal{E}), \quad \forall (x, t) \in \mathcal{E}. \quad (2.10)$$

For $(x, t) \in \mathcal{E}$ and τ small enough,

$$|e^{-\alpha t} x - x'| = |e^{-\alpha t} x - x| + |x - x'| \leq (1 - e^{-\alpha t})|x| + \delta < 2\delta,$$

so that $e^{-\alpha t} x \in B(x', 2\delta)$. It follows that $\forall (x, t) \in \mathcal{E}$

$$|y_* - x'| > 4\delta \Rightarrow |y_* - e^{-\alpha t} x| > 2\delta. \quad (2.11)$$

As consequence there exists a constant c_1 such that

$$H^*(y_* - e^{-\alpha t} x) \geq c_1 > 0, \quad \forall (x, t) \in \mathcal{E}, \quad |y_* - x'| > 4\delta$$

and (2.10) implies for some positive constant c_2 and c_3

$$\left| \frac{1}{q} (DH^*)(y_* - e^{-\alpha t} x) \right| \leq c_2,$$

$$H^*(e^{-\alpha t} x - x_0) \leq c_3$$

We fix $r \in \mathbb{R}_+$ such that

$$(r - (2Ap)^{q-1})c_1 - (2Ap)^{q-1}c_3 > u_0(x'),$$

then, $\forall (x, t) \in \mathcal{E}$, and $|y_* - x'| > 4\delta$,

$$\begin{aligned} & (r - (2Ap)^{q-1})H^*(y_* - e^{-\alpha t} x) - (2Ap)^{q-1}H^*(e^{-\alpha t} x - x_0) \\ & \geq (r - (2Ap)^{q-1})c_1 - (2Ap)^{q-1}c_3 > u_0(x'). \end{aligned}$$

Next, we choose τ so that, for r verifying $(r - (2Ap)^{q-1})c_1 - (2Ap)^{q-1}c_3 > u_0(x')$,

$$\varphi(\tau) \max_{|s| \leq rc_2} H(s) \leq \epsilon,$$

and we get

$$(r - (2Ap)^{q-1})H^*(y_* - e^{-\alpha t} x) - (2Ap)^{q-1}H^*(e^{-\alpha t} x - x_0) - \varphi(t)H(s) \geq u_0(x') - \epsilon.$$

Set $m = \min\{m_1, m_2\}$, with

$$\begin{aligned} m_1 &= \inf_{|y_* - x'| > 4\delta} (u_0(y_*) + rH^*(y_* - e^{-\alpha t} x) - \varphi(t)H(s(r))) \\ &\geq \inf_{|y_* - x'| > 4\delta} ((r - (2Ap)^{q-1})H^*(y_* - e^{-\alpha t} x) - (2Ap)^{q-1}H^*(e^{-\alpha t} x - x_0) \\ &\quad - \varphi(t)H(s(r))) \geq u_0(x') - \epsilon \end{aligned}$$

and

$$\begin{aligned} m_2 &= \inf_{|y_* - x'| \leq 4\delta} (u_0(y_*) + rH^*(y_* - e^{-\alpha t} x) - \varphi(t)H(s(r))) \\ &\geq \inf_{|y_* - x'| \leq 4\delta} (u_0(y_*) - \varphi(t)H(s(r))) \geq u_0(x') - 2\epsilon. \end{aligned}$$

Since

$$u(x, t) \geq m,$$

we deduce that

$$\lim_{r \rightarrow 0} \inf_{(y, s)} \{u(y, s); \quad |y - x'| < r, 0 < s < r\} \geq u_0(x').$$

On the other hand

$$u(x, t) \leq u_0(e^{-\alpha t} x),$$

for any $(x, t) \in \mathcal{E}$, and by the lsc of u_0 , we deduce that

$$\lim_{r \rightarrow 0} \inf_{(y, s)} \{u(y, s); \quad |y - x'| < r, 0 < s < r\} \leq u_0(x'),$$

and the result follows. $\square$

Next we observe that

$$(Q_s u_0)(x) = \min_w \left\{ u_0 \left[e^{-\alpha s} x + \left(\frac{1 - e^{-\alpha p s}}{\alpha p} \right)^{\frac{1}{p}} w \right] + H^*(w) \right\}.$$

We can conclude this section with

Theorem 2.6. *If u_0 is lsc and H is convex, even, p -homogeneous with $p > 1$, then*

$$Q_t(Q_s)(u_0) = Q_{t+s}(u_0) \quad \forall t, s \in \mathbb{R}_+.$$

Proof. We give here a short sketch of the proof (see [4]). By

$$\begin{aligned} Q_t(Q_s)(x) &= \min_z \left\{ \min_w \left\{ u_0 \left[e^{-\alpha s} \left[e^{-\alpha t} x + \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{\frac{1}{p}} z \right] \right. \right. \right. \\ &\quad \left. \left. + \left(\frac{1 - e^{-\alpha p s}}{\alpha p} \right)^{\frac{1}{p}} w \right] + H^*(w) \right\} + H^*(z) \right\} \\ &= \min_{(z, w)} \left\{ u_0 \left[e^{-\alpha(t+s)} x + e^{-\alpha s} \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{\frac{1}{p}} z + \left(\frac{1 - e^{-\alpha p s}}{\alpha p} \right)^{\frac{1}{p}} w \right] \right. \\ &\quad \left. + H^*(w) + H^*(z) \right\}. \end{aligned} \quad (2.12)$$

We make the change of variables

$$\begin{cases} e^{-\alpha s} \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{\frac{1}{p}} z + \left(\frac{1 - e^{-\alpha p s}}{\alpha p} \right)^{\frac{1}{p}} w = \left(\frac{1 - e^{-\alpha p(t+s)}}{\alpha p} \right)^{\frac{1}{p}} u \\ - \left(\frac{1 - e^{-\alpha p s}}{\alpha p} \right)^{\frac{1}{q}} z + \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{\frac{1}{q}} e^{-\alpha(p-1)s} w = \left(\frac{1 - e^{-\alpha(t+s)p}}{\alpha p} \right)^{\frac{1}{q}} v. \end{cases}$$

We put

$$A = e^{-\alpha p s} \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right); \quad B = \frac{1 - e^{-\alpha p s}}{\alpha p}.$$

We observe that

$$A + B = \frac{1 - e^{-\alpha p(t+s)}}{\alpha p}.$$

Introducing

$$\lambda = \frac{A}{A+B}, \quad 1 - \lambda = \frac{B}{A+B},$$

the previous transformation may be written

$$\begin{cases} \lambda^{\frac{1}{p}} z + (1 - \lambda)^{\frac{1}{p}} w = u \\ -(1 - \lambda)^{\frac{1}{q}} z + \lambda^{\frac{1}{q}} w = v, \end{cases}$$

whose inverse is given by

$$\begin{cases} \lambda^{\frac{1}{q}} u - (1 - \lambda)^{\frac{1}{p}} v = z \\ (1 - \lambda)^{\frac{1}{q}} u + \lambda^{\frac{1}{p}} v = w. \end{cases}$$

From (2.12), it results

$$Q_t(Q_s u_0)(x) = \min_u \left\{ u_0 \left[e^{-\alpha(t+s)} x + \left(\frac{1 - e^{-\alpha p(t+s)}}{\alpha p} \right)^{\frac{1}{p}} u \right] \right. \\ \left. + \min_v \left[H^*(\lambda^{\frac{1}{q}} u - (1 - \lambda)^{\frac{1}{p}} v) + H^*((1 - \lambda)^{\frac{1}{q}} u + \lambda^{\frac{1}{p}} v) \right] \right\} \quad (2.13)$$

By an easy computation, using the convexity of H^* , we have

$$\min_v [H^*(\lambda^{\frac{1}{q}} u - (1 - \lambda)^{\frac{1}{p}} v) + H^*((1 - \lambda)^{\frac{1}{q}} u + \lambda^{\frac{1}{p}} v)] = H^*(u).$$

Then from (2.12), (2.13) the thesis follows. $\square$

2.2. Viscosity solutions

In this subsection we show that $u(x, t)$ defined by (2.2), is a viscosity solution of the problem (1.1). For definition and main properties of viscosity solutions we refer to [7], [8] and, as reference closer to our case, to [9, 10].

Proposition 2.7. *Assume (2.1), (2.4). Then u defined by (2.2) is a viscosity subsolution of (1.1).*

Proof. Given a C^1 test function $\varphi(x, t)$, let us assume $u - \varphi$ has maximum relative point (x_0, t_0) , and without leading the generality, we may assume

$$u(x_0, t_0) = \varphi(x_0, t_0). \quad (2.14)$$

Using the semigroup property of Q_t , we have

$$Q_s(Q_{t_0-s} u_0)(x_0) = Q_{t_0} u_0(x_0)$$

In explicit form

$$u(x_0, t_0) = Q_s u(x_0, t_0 - s) \\ = \min_y \left\{ u(e^{-\alpha s} x_0 + \frac{1 - e^{-\alpha p s}}{\alpha p} y, t_0 - s) + \frac{1 - e^{-\alpha p s}}{\alpha p} H^*(y) \right\}.$$

We fix a neighborhood I_0 of (x_0, t_0) such that

$$u(x, t) - \varphi(x, t) \leq 0 \quad \forall (x, t) \in I_0.$$

From (2.14), for any fixed $y \in R^n$ we have

$$\varphi(x_0, t_0) = u(x_0, t_0) \leq u(e^{-\alpha s} x_0 + \frac{1 - e^{-\alpha p s}}{\alpha p} y, t_0 - s) + \frac{1 - e^{-\alpha p s}}{\alpha p} H^*(y). \quad (2.15)$$

We can find $\delta > 0$ such that $\forall s \in]0, \delta[$:

$$\left(e^{-\alpha s} x_0 + \frac{1 - e^{-\alpha p s}}{\alpha p} y, t_0 - s \right) \in I_0.$$

Therefore by (2.15) we get

$$\varphi(x_0, t_0) = u(x_0, t_0) \\ \leq \varphi \left(e^{-\alpha s} x_0 + \frac{1 - e^{-\alpha p s}}{\alpha p} y, t_0 - s \right) + \frac{1 - e^{-\alpha p s}}{\alpha p} H^*(y), \quad \forall s \in [0, \delta].$$

Then the function

$$[0, \delta] \ni s \rightarrow \varphi \left(e^{-\alpha s} x_0 + \frac{1 - e^{-\alpha p s}}{\alpha p} y, t_0 - s \right) + \frac{1 - e^{-\alpha p s}}{\alpha p} H^*(y) =: \psi(s),$$

has a minimum at $s = 0$. Therefore we have $\psi'(0) \geq 0$, that is

$$(-\alpha x_0 + y) \cdot D\varphi(x_0, t_0) - \varphi_t(x_0, t_0) + H^*(y) \geq 0$$

or, equivalently

$$\varphi_t(x_0, t_0) + \alpha x_0 D\varphi(x_0, t_0) + y(-D\varphi(x_0, t_0)) - H^*(y) \leq 0.$$

Since this inequality is true for all $y \in \mathbb{R}^n$ we obtain that $u(x, t)$ is viscosity subsolution of the Hamilton–Jacobi equation (1.1). $\square$

We can prove that $u(x, t)$ is a viscosity supersolution of (1.1), if the lsc datum u_0 has an approximation property (see [9] where such property is used), as it is shown in the following

Theorem 2.8. Assume (2.1), (2.4). If there exists a family of C^1 functions: $(u_\epsilon(x))_{\epsilon \in]0,1]}$ such that

$$u_0(x) = \inf_{\epsilon \in]0,1]} u_\epsilon(x) \quad \forall x \in \mathbb{R}^n,$$

then $u(x, t)$ defined by (2.2) is a viscosity supersolution of (1.1)

Proof. We observe, by (2.2), that

$$\begin{aligned} u(x, t) &= \inf_y \left(u_0 \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) + \frac{1 - e^{-\alpha p t}}{\alpha p} H^*(y) \right) \\ &= \inf_{(y, \epsilon)} \left\{ u_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) + \frac{1 - e^{-\alpha p t}}{\alpha p} H^*(y) \right\}. \end{aligned}$$

Let us consider the $C^1(\mathbb{R}^n \times \mathbb{R}_+)$ functions

$$u^\epsilon(x, t; y) = u_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) + \frac{1 - e^{-\alpha p t}}{\alpha p} H^*(y)$$

and we observe that

$$\begin{aligned} u_t^\epsilon(x, t; y) &= (-\alpha e^{-\alpha t} x - e^{-\alpha p t} y) Du_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) + e^{-\alpha p t} H^*(y) \\ Du^\epsilon(x, t; y) &= e^{-\alpha t} Du_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right); \\ H(Du^\epsilon(x, t; y)) &= e^{-\alpha p t} H \left(Du_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) \right). \end{aligned}$$

It follows

$$\begin{aligned} u_t^\epsilon(x, t; y) &= -\alpha x Du^\epsilon(x, t; y) - e^{-\alpha p t} \left[y Du_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) - H^*(y) \right] \\ &\geq -\alpha x Du^\epsilon(x, t; y) - e^{-\alpha p t} H \left(Du_\epsilon \left(e^{-\alpha t} x - \frac{1 - e^{-\alpha p t}}{\alpha p} y \right) \right) \\ &= -\alpha x Du^\epsilon(x, t; y) - H(Du^\epsilon(x, t; y)). \end{aligned}$$

Therefore we have, for every $y \in \mathbb{R}^n$ and $\epsilon \in (0, \delta)$:

$$u_t^\epsilon(x, t; y) + \alpha x Du^\epsilon(x, t; y) + H(Du^\epsilon(x, t; y)) \geq 0.$$

We get that, for any y , and $\epsilon \in (0, 1]$, $u^\epsilon(x, t; y)$ is a viscosity supersolution of (1.1). Then the result follows. $\square$

Hence u , under the assumptions of the theorem (2.8), is a viscosity solution of the problem (1.1).

3. Complementary results and assumptions

First of all we wish to show some simple properties of the semigroup $t \rightarrow Q_t$, with respect to initial datum u_0 . It is a kind of monotonicity. Given two initial data u_1 and u_2 , it is easy to prove the implication

$$u_1 \leq u_2 \Rightarrow (Q_t u_1) \leq (Q_t u_2). \quad (3.1)$$

Keeping in mind the condition of lower boundedness imposed to the initial datum u_0 , i.e.

$$u_0(x) \geq -(Ap)^{q-1} H^*(x - x_0), \quad (3.2)$$

for every $x \in \mathbb{R}^n$, and $A \leq \alpha$ and x_0 is a fixed point of $\mathbb{R}^n$. By the monotonicity property (3.1), we get

$$u(x, t) \geq Q_t \left(-(Ap)^{q-1} H^*(x - x_0) \right).$$

By an easy computation, we get

$$u(x, t) \geq - \left(\frac{Ap\alpha}{\alpha - A(1 - e^{-\alpha pt})} \right)^{q-1} H^*(e^{-\alpha t} x - x_0).$$

We point out that the right hand side is the regular solution

$$Q_t(-(Ap)^{q-1} H^*(x - \bar{x})) = u^*(x, t),$$

which in [4], verified equality and show the optimality of hypercontractivity and ultracontractivity inequalities.

As general reference to the following notions see [1, 6, 13].

We now recall the definition of semiconvex function.

Definition 3.1. $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is semiconvex, or convex up to a square, if there exists some constant $c \geq 0$ such that

$$f(x) + c|x|^2$$

is convex.

3.1. The particular case $H(x) = \frac{1}{2}|x|^2$

We are concerned, in this subsection, with the Hamilton–Jacobi equation

$$u_t(x, t) + \sum_{i=1}^n \alpha_i x_i \partial_{x_i} u(x, t) + \frac{1}{2} \sum_{i=1}^n (\partial_{x_i} u(x, t))^2 = 0 \quad (3.3)$$

where all the α_i are positive real numbers, and we put $\alpha = (\alpha_1, \dots, \alpha_n)$. Next we adapt the definition of (H^*, α) lower boundedness and we give a notion a semiconvexity connected to the class of Hamilton–Jacobi equation we are considering.

Definition 3.2. We say that u_0 is α lower bounded if

$$\exists B \in]0, \max_{i=1, \dots, n} \alpha_i] \quad \text{and} \quad x_0 \in \mathbb{R}^n,$$

such that

$$u_0(x) \geq -B|x - x_0|^2.$$

Definition 3.3. Let u_0 a real valued function, we say that u_0 is α -semiconvex, [strictly α -semiconvex], if $\exists A_j \in \mathbb{R}_+, A_j \leq \alpha_j, [< \alpha_j]$ for $j = 1, \dots, n$, such that the function

$$\varphi_A(x) = u_0(x) + \sum_{i=1}^n A_i x_i^2$$

is convex in $\mathbb{R}^n$.

The utility of the definition (3.3) is stated in the next two remarks

Remark 3.4. If u_0 is α -semiconvex with constants $(A_1, \dots, A_n)$, then for every $(A'_1, \dots, A'_n)$, such that $A_i \leq A'_i, i = 1, \dots, n$ the function

$$\varphi_{A'}(x) = u_0 + \sum_{i=1}^n A'_i x_i^2$$

is also convex in $\mathbb{R}^n$.

Moreover it follows

Remark 3.5. If u_0 is strictly α -semiconvex with constants $(A_1, \dots, A_n)$ the function

$$\varphi_\alpha(x) := u_0(x) + \sum_{i=1}^n \alpha_i x_i^2,$$

is strongly convex on $\mathbb{R}^n$ with modulus $c = 2 \min(\alpha_j - A_j)$, because

$$\varphi_\alpha(x) - \frac{c}{2}|x|^2 = \varphi_A + \sum_{i=1}^n \left(\alpha_i - A_i - \frac{c}{2} \right) x_i^2,$$

is convex on $\mathbb{R}^n$.

We are now interested to state regularity properties of u , defined by the semigroup $t \rightarrow Q_t$ on the initial datum u_0 and solution of the Hamilton Jacobi equation (3.3). Our result is suggested by [1].

Theorem 3.6. *Let u_0 strictly α -semiconvex, α lower bounded. Then the solution $u(x, t) = (Q_t u_0)(x)$ may be written*

$$u(x, t) = K(x, t) - v(x, t)$$

i.e. as difference of two functions, which are convex, the first one $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, solution of the Hamilton–Jacobi equation (3.3), independent of the initial datum u_0 ; the later one v continuous differentiable and its gradient is Lipschitzian with Lipschitz constant $L = \frac{1}{\min_i(\alpha_i - A_i)}$.

Proof. We begin the proof using the weak convexity

$$\begin{aligned} u(x, t) &= \min_{y \in \mathbb{R}^n} \left[u_0(y) + \sum_{j=1}^n \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 \right] \\ &= \min_{y \in \mathbb{R}^n} \left[\varphi_\alpha(y) - \sum_{j=1}^n \alpha_j y_j^2 + \sum_{j=1}^n \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j - e^{-\alpha_j t} x_j)^2 \right] \\ &= \min_{y \in \mathbb{R}^n} \left[\varphi_\alpha(y) - \sum_{j=1}^n \alpha_j y_j^2 + \sum_{j=1}^n \frac{\alpha_j}{1 - e^{-2\alpha_j t}} (y_j^2 + e^{-2\alpha_j t} x_j^2 - 2e^{-\alpha_j t} x_j y_j) \right] \\ &= \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j^2 + \min_{y \in \mathbb{R}^n} \left[\varphi_\alpha(y) + \sum_{j=1}^n \left[\frac{\alpha_j}{1 - e^{-2\alpha_j t}} - \alpha_j \right] y_j^2 \right. \\ &\quad \left. - \sum_{j=1}^n \frac{2\alpha_j e^{-\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j y_j \right] = \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j^2 \\ &\quad - \max_{y \in \mathbb{R}^n} \left[-\varphi_\alpha(y) - \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} y_j^2 + \sum_{j=1}^n \frac{2\alpha_j e^{-\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j y_j \right] \\ &= \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j^2 - \max_{y \in \mathbb{R}^n} \left[\sum_{j=1}^n \frac{2\alpha_j e^{-\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j y_j \right. \\ &\quad \left. - \left(\sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} y_j^2 + \varphi_\alpha(y) \right) \right] \end{aligned}$$

Define

$$K(x, t) = \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j^2 = \sum_{j=1}^n \frac{\alpha_j}{e^{2\alpha_j t} - 1} x_j^2$$

Then

$$u(x, t) = K(x, t) - [\varphi_\alpha(\cdot) + K(\cdot, t)]^*(s),$$

and

$$s_j = \frac{2\alpha_j e^{-\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j \quad j = 1 \dots n.$$

The result follows, taking

$$v(x, t) = [\varphi_\alpha(\cdot) + K(\cdot, t)]^* \left(\dots, \frac{2\alpha_j e^{-\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j, \dots \right),$$

observing that $K(x, t) \geq 0$ and φ_α is strongly convex in $\mathbb{R}^n$. Indeed we can apply theorem 4.2.1. Chapter E in [13]. $\square$

4. Hypercontractivity and LSI

4.1. Weighted hypercontractivity results

Let us recall that the nonlinear semigroup we are concerned with, is given by

$$(Q_t u_0)(x) = \min_z \left\{ u_0 \left(e^{-\alpha t} x + \left(\frac{1 - e^{-\alpha p t}}{\alpha p} \right)^{\frac{1}{p}} z \right) + H^*(z) \right\},$$

and that in paper [4] we get hypercontractivity results, assuming that the initial data u_0 were admissible, so that

$$\|\exp(u_0(x))\|_{L^r(\mathbb{R}^n, dx)} < +\infty.$$

We are dealing here with more general u_0 ; we assume that

$$u_0(x) = o(H^*(x)) \quad \text{for } |x| \rightarrow +\infty. \quad (4.1)$$

Then, for positive real number b , by (4.1), the following conditions holds

$$\int_{\mathbb{R}^n} \exp(r u_0(x) - b H^*(x)) dx < +\infty.$$

for every $r \geq 1$. With the assumption (4.1), in this paper, every Lipschitz function can be taken as initial datum u_0 .

Theorem 4.1. *Assume that u_0 is lsc, (H^*, α) lower bounded function and that u_0 satisfies (4.1). We fix $1 \leq r \leq s < +\infty$, then there exists a constant C , such that*

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \exp[s e^{\alpha t} (Q u_0)(x) - \left(\frac{s}{r}\right)^q b H^*(x)] dx \right)^{\frac{1}{s e^{\alpha t}}} \\ & \leq C \left(\int_{\mathbb{R}^n} \exp[r u_0(x) - b H^*(x)] dx \right)^{\frac{1}{r}}; \end{aligned}$$

the C constant will be given below.

Proof. We put: $a = \frac{r}{s}e^{-\alpha t}$, and build the situation in order to apply the Prekopa–Leinder inequality. For positive γ (to be fixed later) and b , we introduce the functions

$$\begin{aligned} u(x) &= \exp \left[se^{\alpha t} Q_t u_0(x) - bH^* \left(\frac{s}{r}x \right) \right], \\ v(x) &= \exp \left[-\gamma H^*(x) - bH^* \left(\frac{s}{r}x \right) \right], \\ w(x) &= \exp \left[ru_0 \left(\frac{s}{r}x \right) - bH^* \left(\frac{s}{r}x \right) \right]. \end{aligned}$$

It is easy to get that the functions u, v , and w are $L^1(\mathbb{R}^N)$, we see that

$$\begin{aligned} (u(x))^a &= \exp \left[r(Q_t u_0)(x) - abH^* \left(\frac{s}{r}x \right) \right] \\ &\leq \exp \left\{ ru_0 \left(e^{-\alpha t}x + \left(\frac{1 - e^{-\alpha pt}}{\alpha p} \right)^{\frac{1}{p}} y \right) + rH^*(y) \right\} - abH^* \left(\frac{s}{r}x \right) \} \\ &= \exp \left\{ ru_0 \left[\frac{s}{r} \left(\frac{r}{s}e^{-\alpha t}x + \frac{r}{s} \left(\frac{1 - e^{-\alpha pt}}{\alpha p} \right)^{\frac{1}{p}} y \right) \right] + rH^*(y) - abH^* \left(\frac{s}{r}x \right) \right\} \end{aligned}$$

for all $y \in \mathbb{R}^n$. We make a change of variable setting

$$y = \frac{s}{r} \left(\frac{\alpha p}{1 - e^{-\alpha pt}} \right)^{\frac{1}{p}} (1 - a)z.$$

We get

$$\begin{aligned} (u(x))^a (v(z))^{1-a} &\leq \exp \left\{ ru_0 \left(\frac{s}{r}(ax + (1 - a)z) \right) + rH^* \left(\frac{s}{r} \left(\frac{\alpha p}{1 - e^{-\alpha pt}} \right)^{\frac{1}{p}} (1 - a)z \right) \right. \\ &\quad \left. - abH^* \left(\frac{s}{r}x \right) - (1 - a)\gamma H^*(z) - (1 - a)bH^* \left(\frac{s}{r}z \right) \right\}. \end{aligned}$$

We fix γ in order to eliminate the second and the fourth terms of the last inequality. It gives

$$\gamma = r \left(\frac{s}{r} \right)^q \left[\frac{\alpha p(1 - a)}{1 - e^{-\alpha pt}} \right]^{q-1}.$$

Then we get

$$\begin{aligned} (u(x))^a (v(z))^{1-a} &\leq \exp \left\{ ru_0 \left(\frac{s}{r}(ax + (1 - a)z) \right) - b \left[aH^* \left(\frac{s}{r}x \right) + (1 - a)H^* \left(\frac{s}{r}z \right) \right] \right\} \\ &\leq \exp \left\{ ru_0 \left(\frac{s}{r}(ax + (1 - a)z) \right) - bH^* \left(\frac{s}{r}(ax + (1 - a)z) \right) \right\} \\ &= w(ax + (1 - a)z). \end{aligned}$$

Then, by Prekopa–Leinder inequality, we get

$$\left(\int_{\mathbb{R}^n} u(x) dx \right)^a \left(\int_{\mathbb{R}^n} v(x) dx \right)^{1-a} \leq \int_{\mathbb{R}^n} w(x) dx. \quad (4.2)$$

To find the constant C , we compute

$$\begin{aligned}\int_{\mathbb{R}^n} w(x)dx &= \int_{\mathbb{R}^n} \exp \left[ru_0 \left(\frac{s}{r}x \right) - bH^* \left(\frac{s}{r}x \right) \right] dx \\ &= \left(\frac{r}{s} \right)^n \int_{\mathbb{R}^n} \exp[ru_0(x) - bH^*(x)]dx.\end{aligned}\tag{4.3}$$

Furthermore, it results

$$\begin{aligned}\int_{\mathbb{R}^n} v(x)dx &= \int_{\mathbb{R}^n} \exp \left[- \left(\gamma + b \left(\frac{s}{r} \right)^q \right) H^*(x) \right] dx \\ &= \left[\gamma + b \left(\frac{s}{r} \right)^q \right]^{-\frac{n}{q}} \int_{\mathbb{R}^n} \exp[-H^*(x)]dx.\end{aligned}\tag{4.4}$$

Therefore from (4.2), (4.3), (4.4) we get

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} \exp[se^{\alpha t} Q_t u_0(x) - b \left(\frac{s}{r} \right)^q H^*(x)]dx \right)^{\frac{1}{se^{\alpha t}}} \\ &\leq C \left(\int_{\mathbb{R}^n} \exp[ru_0(x) - bH^*(x)]dx \right)^{\frac{1}{r}},\end{aligned}$$

and the constant C , by (4.3) and (4.4), is given by

$$\begin{aligned}C &= \left(\int_{\mathbb{R}^n} v(x)dx \right)^{-\frac{1-a}{r}} \left(\frac{r}{s} \right)^{\frac{n}{r}} \\ &= \left[\left(\frac{s}{r} \right)^q \left[b + r \left(\frac{\alpha p(1-a)}{1-e^{-\alpha p t}} \right)^{q-1} \right] \right]^{\frac{n(1-a)}{qr}} \\ &\quad \left(\frac{r}{s} \right)^{\frac{n}{r}} \left[\int_{\mathbb{R}^n} \exp - H^*(x)dx \right]^{-\frac{1-a}{r}} \\ &= \left(\frac{r}{s} \right)^{\frac{na}{r}} \left[\left(b + r \left(\frac{\alpha p(1-a)}{1-e^{-\alpha p t}} \right)^{q-1} \right)^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \exp(-H^*(x))dx \right)^{-1} \right]^{\frac{1-a}{r}}.\end{aligned}$$

We may conclude with the condition of hypercontractivity given by

$$b + r \left(\frac{\alpha p(1-a)}{1-e^{-\alpha p t}} \right)^{q-1} \leq \left(\int_{\mathbb{R}^n} \exp(-H^*(x))dx \right)^{\frac{q}{n}}, \quad \forall t > 0.\tag{4.5}$$

Since b is positive, from (4.5), we obtain the necessary condition

$$r \left(\frac{\alpha p(1-a)}{1-e^{-\alpha p t}} \right)^{q-1} < \left(\int_{\mathbb{R}^n} \exp(-H^*(x))dx \right)^{\frac{q}{n}},$$

and the thesis follows with $C \leq 1$. $\square$

4.2. A weighted hypercontractivity result.

The hypercontractivity results on the semigroup $t \rightarrow Q_t$ of the paper [3], require the admissibility of the initial datum u_0 of the Cauchy problem (3.3).

In this paper we state new results, assuming that u_0 verifies the hypothesis

$$u_0(x) = o(|x|^2) \quad \text{for } |x| \rightarrow +\infty. \quad (4.6)$$

Then, by (4.6), for any fixed $b_j > 0, j = 1, \dots, n$, it results

$$\int_{\mathbb{R}^n} \exp[ru_0(x) - \sum_{j=1}^n b_j x_j^2] dx < +\infty,$$

for every $r \in [1, +\infty)$. As particular case, taking

$$b_j = r \frac{a\alpha_j}{1 - e^{-2\alpha_j t}}, \quad a = \frac{r}{s} e^{-\beta t},$$

we also have

$$\int_{\mathbb{R}^n} \exp[ru_0(x) - \sum_{j=1}^n r \frac{a\alpha_j}{1 - e^{-2\alpha_j t}} x_j^2] dx < +\infty,$$

for every $r \in [1, +\infty)$.

We are going to describe one of the results of the paper.

Theorem 4.2. *We fix $\beta \geq \alpha_j, j = 1, \dots, n$, and $1 \leq r \leq s < +\infty, a = \frac{r}{s} e^{-\beta t}$. Assume that u_0 is a lsc, α lower bounded function and that u_0 satisfies (4.6). Then there exists a real constant C , such that*

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \exp [se^{\beta t} ((Q_t u_0)(x) - K(x, t))] dx \right)^{\frac{1}{se^{\beta t}}} \\ & \leq C \left(\int_{\mathbb{R}^n} \exp [r(u_0(x) - aL(x, t))] dx \right)^{\frac{1}{r}}, \end{aligned}$$

where

$$K(x, t) = \sum_{j=1}^n \frac{\alpha_j e^{-2\alpha_j t}}{1 - e^{-2\alpha_j t}} x_j^2; \quad L(x, t) = \sum_{j=1}^n \frac{\alpha_j}{1 - e^{-2\alpha_j t}} x_j^2.$$

This theorem follows from a more general result, which we give here.

Theorem 4.3. *Assume that u_0 is a lsc, α lower bounded function and that u_0 satisfies (4.6). We fix $\beta \geq \alpha_j, j = 1, \dots, n$, and $1 \leq r \leq s < +\infty, a = \frac{r}{s} e^{-\beta t}$. Then, for every fixed $\xi \in \mathbb{R}^n$, there exist $C \in \mathbb{R}_+$, depending also by ξ , such that*

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \exp \left[se^{\beta t} (Q_t u_0)(x) - \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right] dx \right)^{\frac{1}{se^{\beta t}}} \\ & \leq C \left(\int_{\mathbb{R}^n} \exp \left[ru_0(x) - \sum_{j=1}^n b_j (x_j - \xi_j)^2 \right] dx \right)^{\frac{1}{r}} \end{aligned}$$

Proof. We recall that in this particular case

$$(Q_t u_0)(x) = \min_y \left\{ u_0 \left(\dots, e^{-\alpha_j t} x_j + \sqrt{1 - e^{-2\alpha_j t}} y_j, \dots \right) + \sum_{j=1}^n \alpha_j y_j^2 \right\}. \quad (4.7)$$

Then, for every $y \in \mathbb{R}^n$, we have

$$(Q_t u_0)(x) \leq u_0 \left(\dots, e^{-\alpha_j t} x_j + \sqrt{1 - e^{-2\alpha_j t}} y_j, \dots \right) + \sum_{j=1}^n \alpha_j y_j^2. \quad (4.8)$$

In order to apply the Prekopa–Leinder inequality, we introduce the functions

$$\begin{aligned} u(x) &= \exp \left[s e^{\beta t} Q_t u_0(x) - \sum b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right], \\ v(x) &= \exp \left[- \sum \gamma_j x_j^2 - \sum b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right], \\ w(x) &= \exp \left[r u_0 \left(\dots, \frac{s}{r} e^{(\beta - \alpha_j)t} x_j, \dots \right) - \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right]. \end{aligned}$$

It is easy to get, by (4.6), that the functions u, v , and w are $L^1(\mathbb{R}^n)$.

For positive γ_j (to be fixed later) and b_j ,

$$\begin{aligned} (u(x))^a &\leq \exp \left[r u_0 \left(\dots, \frac{s}{r} e^{(\beta - \alpha_j)t} \left(\frac{r}{s} e^{-\beta t} x_j + \frac{r}{s} e^{-(\beta - \alpha_j)t} \sqrt{1 - e^{-2\alpha_j t}} y_j \right), \dots \right) \right. \\ &\quad \left. + r \sum_{j=1}^n \alpha_j y_j^2 - a \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right]. \end{aligned} \quad (4.9)$$

We make the change of variables

$$y_j = \frac{s e^{(\beta - \alpha_j)t} (1 - a)}{r \sqrt{1 - e^{-2\alpha_j t}}} z_j.$$

We now compute starting from (4.9)

$$\begin{aligned} (u(x))^a (v(z))^{1-a} &\leq \exp \left[r u_0 \left(\dots, e^{(\beta - \alpha_j)t} \frac{s}{r} [a x_j + (1 - a) z_j], \dots \right) \right. \\ &\quad \left. + r \sum_{j=1}^n \alpha_j \frac{s^2 e^{2(\beta - \alpha_j)t} (1 - a)^2}{r^2 (1 - e^{-2\alpha_j t})} z_j^2 - a \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} x_j - \xi_j \right)^2 \right. \\ &\quad \left. - (1 - a) \sum_{j=1}^n \gamma_j z_j^2 - (1 - a) \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta - \alpha_j)t} z_j - \xi_j \right)^2 \right]. \end{aligned} \quad (4.10)$$

Here we fix γ_j as follows

$$\gamma_j = \alpha_j \frac{s^2 e^{2(\beta - \alpha_j)t} (1 - a)}{r (1 - e^{-2\alpha_j t})}.$$

From (4.10) and the convexity of quadratic function, we get

$$(u(x))^a (v(z))^{1-a} \leq \exp \left[ru_0 \left(\dots, e^{(\beta-\alpha_j)t} [ax_j + (1-a)z_j], \dots \right) - \sum_{j=1}^n b_j \left(\frac{s}{r} e^{(\beta-\alpha_j)t} [ax_j + (1-a)z_j] - \xi_j \right)^2 \right],$$

i.e.

$$(u(x))^a (v(z))^{1-a} \leq w(ax + (1-a)z). \quad (4.11)$$

By Prekopa–Leinder inequality, (4.11) gives:

$$\left(\int_{\mathbb{R}^n} u(x) dx \right)^{\frac{1}{se^{\beta t}}} \leq C \left(\int_{\mathbb{R}^n} w(x) dx \right)^{\frac{1}{r}}. \quad (4.12)$$

Since it is easy to see that

$$\int_{\mathbb{R}^n} w(x) dx = \prod_{j=1}^n \frac{r}{s} e^{-(\beta-\alpha_j)t} \int_{\mathbb{R}^n} \exp \left[ru_0(x) - \sum_{j=1}^n b_j (x_j - \xi_j)^2 \right], \quad (4.13)$$

from (4.12) we obtain the result, with the constant C given by

$$C = \left(\int_{\mathbb{R}^n} v(x) dx \right)^{-\frac{1-a}{r}} \left(\prod_{j=1}^n \frac{r}{s} e^{-(\beta-\alpha_j)t} \right)^{\frac{1}{r}}.$$

We are now concerned with the case $\xi = 0$. We can obtain explicitly the constant C , useful for the hypercontractivity, just computing the integral of the function v . Indeed an easy computation gives

$$v(x) = \exp \left[- \sum_{j=1}^n m_j^2 x_j^2 \right]$$

with

$$m_j = \frac{s}{r} e^{(\beta-\alpha_j)t} \sqrt{(1-a)r \frac{\alpha_j}{1 - e^{-2\alpha_j t}} + b_j}.$$

It is easy to get

$$\int_{\mathbb{R}^n} v(x) dx = \prod_{j=1}^n \int_{\mathbb{R}} e^{-m_j^2 x_j^2} dx_j = \frac{\pi^{\frac{n}{2}}}{\prod_{j=1}^n m_j}.$$

It follows from previous results and by some computations

$$C = \prod_{j=1}^n \left\{ \left(\frac{r}{s} e^{-(\beta-\alpha_j)t} \right)^{\frac{1}{r}} \left(\frac{1}{\pi} \left[\frac{r(1-a)\alpha_j}{1 - e^{-2\alpha_j t}} + b_j \right] \right)^{\frac{1}{2} \left(\frac{1}{r} - \frac{1}{se^{\beta t}} \right)} \right\}$$

□

It result that for $b_j = 0, j = 1, \dots, n$, such constant is the same of that given in [3]. We need to investigate for conditions in order to have actual hypercontractivity, i.e. $C \leq 1$. Assume: $s = r$ and $a = e^{-\beta t}$. Since $\beta \geq \alpha_j, j = 1, \dots, n$, it seems that the more meaningful conditions are

$$\frac{r(1 - e^{-\beta t})\alpha_j}{1 - e^{-2\alpha_j t}} + b_j \leq \pi; \quad \alpha_j < \pi; \quad \beta \leq 2\alpha_j; \quad j = 1, \dots, n. \quad (4.14)$$

We observe that, if for every t we require the hypercontractivity property, from (4.14), by the positivity of all the b_j , it necessary to have

$$r\alpha_j < \pi; \quad \text{for } j = 1, \dots, n.$$

We can now get an analogous result to that one obtained in [3].

Theorem 4.4. *Under the assumptions*

$$\alpha_j < \pi; \text{ and } \max_j \{\alpha_j\} \leq \min_j \{2\alpha_j\}, \quad (4.15)$$

we fix β and b_j verifying

$$\max\{\alpha_j\} \leq \beta \leq \min\{2\alpha_j\}; \quad 0 < b_j \leq \frac{1}{2}(\pi - \alpha_j) \quad \text{for } j = 1, \dots, n. \quad (4.16)$$

Then, for every function u_0 which is lsc, α lower bounded, satisfying (4.6), and for every r such that

$$r \in \left[1, \frac{1}{2} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right) \right]$$

the following hypercontractivity property holds true

$$\left\| \exp \left[Q_t u_0 - \sum_{j=1}^n \frac{b_j}{r} e^{(\beta - 2\alpha_j)t} x_j^2 \right] \right\|_{L^{re^{\beta t}}(\mathbb{R}^n)} \leq \left\| \exp \left[u_0 - \sum_{j=1}^n \frac{b_j}{r} x_j^2 \right] \right\|_{L^r(\mathbb{R}^n)}. \quad (4.17)$$

Proof. Thanks to the assumptions the following inequality holds true

$$b_j + r\alpha_j \leq \pi \quad \text{for } j = 1, \dots, n.$$

Then by (4.14) we have $C \leq 1$; then (4.17) follows by the previous result. $\square$

4.3. LSI

We introduce the functional

$$G(t) = \left(\int_{\mathbb{R}^n} \exp[re^{\beta t}(Q_t u_0)(x) - \sum_{j=1}^n b_j e^{2(\beta - \alpha_j)t} x_j^2] dx \right)^{\frac{1}{re^{\beta t}}}.$$

We can state

Theorem 4.5. *Assume u_0 lsc, α lower bounded, and $u_0(x) = o(|x|^2)$ as $|x| \rightarrow +\infty$. Furthermore assume (4.15) (4.16). Then the function $G(t)$ is non increasing in the interval $[0, t_\star]$, with*

$$t_\star = \frac{1}{\beta} \ln \frac{1}{2r_\star} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right).$$

where

$$r_\star \in \left[1, \frac{1}{2} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right) \right]$$

Proof. We fix $0 < t_1 < t_2 < t_\star$. Recalling that (4.17) is true for every r taken in the interval $\left[1, \frac{1}{2} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right) \right]$, we fix $r_\star$ in a such interval. We observe that

$$\begin{aligned} r_\star e^{\beta t_1} &< r_\star e^{\beta t_\star} = \frac{1}{2r} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right). \\ G(t_2) &= \left(\int_{\mathbb{R}^n} \exp \left[(r_\star e^{\beta t_1}) e^{\beta(t_2-t_1)} Q_{t_2-t_1}(Q_{t_1}u_0)(x) \right. \right. \\ &\quad \left. \left. - \sum_{j=1}^n b_j e^{2(\beta-\alpha_j)t_1} e^{2(\beta-\alpha_j)(t_2-t_1)} x_j^2 \right] \right)^{\frac{1}{(r_\star e^{\beta t_1}) e^{\beta(t_2-t_\star)}}} \end{aligned}$$

We can apply (4.17) taking $r = r_\star e^{\beta t_1}$, $Q_{t_1}u_0$ as u_0 , $t = t_2 - t_1$, if it results

$$A = b_j e^{2(\beta-\alpha_j)t_1} + r_\star e^{\beta t_1} \max_j \{a_j\} \leq \pi. \quad (4.18)$$

To get (4.18) we impose

$$b_j \leq r_\star \max_j \{a_j\} \frac{\pi - \max_j \{a_j\}}{\pi + \max_j \{a_j\}}.$$

Indeed

$$\begin{aligned} A &= r_\star e^{\beta t_1} \left(\frac{b_j}{r_\star} e^{(\beta-2\alpha_j)t_1} + \max_j \{a_j\} \right) \leq r_\star e^{\beta t_1} \left(\frac{b_j}{r_\star} + \max_j \{a_j\} \right) \\ &\leq \frac{1}{2} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right) \left(\frac{\pi - \max_j \{a_j\}}{\pi + \max_j \{a_j\}} \max_j \{a_j\} + \max_j \{a_j\} \right) \\ &= \frac{1}{2} (\pi - \max\{\alpha_j\}) + \frac{1}{2} (\pi + \max\{\alpha_j\}) = \pi. \end{aligned}$$

Then we apply (4.17) to obtain

$$G(t_2) \leq G(t_1)$$

and the thesis follows. $\square$

Theorem 4.6. Assume u_0 strictly α -semiconvex, α lower bounded, and $u_0(x) = o(|x|^2)$ as $|x| \rightarrow +\infty$. Assume (4.15) (4.16). Fix r such that

$$r \in \left[1, \frac{1}{2} \left(\frac{\pi}{\max\{\alpha_j\}} + 1 \right) \right]$$

Then

$$\begin{aligned} &\beta E \left(e^{ru_0 - \sum_{j=1}^n b_j x_j^2} \right) + \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} e^{ru_0 - \sum_{j=1}^n b_j x_j^2} dx \\ &\leq r \int_{\mathbb{R}^n} e^{\frac{1}{2} ru_0 - \sum_{j=1}^n b_j x_j^2} |Du_0(x)|^2 dx + \beta \sum_{i=1}^n b_i \int_{\mathbb{R}^n} e^{ru_0 - \sum_{j=1}^n b_j x_j^2} x_i^2 dx. \end{aligned}$$

Proof. We recall the definition of the entropy functional

$$E(h) := \int_{\mathbb{R}^n} h(x) \ln h(x) dx - \int_{\mathbb{R}^n} h(x) dx \ln \int_{\mathbb{R}^n} h(x) dx$$

We introduce the function

$$g(x) = \left[re^{\beta t} (Q_t u_0)(x) - \sum b_j e^{2(\beta - \alpha_j)t} x_j^2 \right],$$

with

$$G(t) = \left(\int_{\mathbb{R}^n} \exp[re^{\beta t} (Q_t u_0)(x) - \sum_{j=1}^n b_j e^{2(\beta - \alpha_j)t} x_j^2] dx \right)^{\frac{1}{re^{\beta t}}}.$$

We consider $G(t)^{re^{\beta t}}$, we make its derivative, and, using $G'(t) \leq 0$ for every $t \in (0, t_*)$ (with the notation of the previous theorem), we get

$$\begin{aligned} & \beta E(e^g) + \sum_{j=1}^n \alpha_j \int_{\mathbb{R}^n} e^{g(x)} dx \\ & \leq re^{\beta t} \int_{\mathbb{R}^n} \exp\left(\frac{1}{2}g(x)\right) |DQ_t u_0(x)|^2 dx + \beta \int_{\mathbb{R}^n} e^{g(x)} \left(\sum_{j=1}^n b_j e^{2(\beta - \alpha_j)t} x_j^2 \right) dx. \end{aligned}$$

The result follows as t goes to 0^+ . $\square$

5. LSI from general hypercontractivity

In all this section we assume $DH \in C_{loc}^{0,1}(\mathbb{R}^n)$

Theorem 5.1. Assume u_0 lsc, (H^*, α) lower bounded and $u_0(x) = o(H^*(x))$ as $|x| \rightarrow +\infty$. Assume

$$(p\alpha)^{q-1} < \left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}} =: B,$$

and take

$$b = \frac{1}{2}(B - (p\alpha)^{q-1}).$$

Then, for any fixed

$$\begin{aligned} r \in \left[1, \frac{1}{2} \left(\frac{B}{(p\alpha)^{q-1}} + 1 \right) \right], \\ \left(\int_{\mathbb{R}^n} \exp[re^{\alpha t} (Q_t u_0(x)) - bH^*(x)] dx \right)^{\frac{1}{re^{\alpha t}}} \leq \left(\int_{\mathbb{R}^n} \exp[ru_0(x) - bH^*(x)] dx \right)^{\frac{1}{r}} \end{aligned} \quad (5.1)$$

Proof. By the assumptions

$$b + r(\alpha p)^{q-1} \leq \left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}},$$

and using that

$$\psi(t) = \frac{1 - e^{-\alpha t}}{1 - e^{-\alpha p t}},$$

is an increasing function on the positive real, we have that the hypercontractivity condition

$$b + r \left(\frac{\alpha p (1 - e^{-\alpha t})}{1 - e^{-\alpha p t}} \right)^{q-1} \leq \left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}}, \quad (5.2)$$

is satisfied, hence the result follows. $\square$

We define the function

$$G_H(t) = \left(\int_{\mathbb{R}^n} \exp(r e^{\alpha t} Q_t u_0(x) - b H^*(x)) dx \right)^{\frac{1}{r e^{\alpha t}}}, \quad (5.3)$$

with a fixed r in the set

$$\left[1, \frac{1}{2} \left(\frac{B}{(p\alpha)^{q-1}} + 1 \right) \right)$$

and we set

$$t_\star = \frac{1}{\alpha} \ln \left(\frac{1}{2r} \left(\frac{B}{(p\alpha)^{q-1}} + 1 \right) \right).$$

We recall the definition of the entropy function with respect to

$$d\Gamma = \exp(-b H^*) dx,$$

$$\mathcal{E}(h, d\Gamma) := \int_{\mathbb{R}^n} h(x) \ln h(x) d\Gamma - \int_{\mathbb{R}^n} h(x) d\Gamma \ln \int_{\mathbb{R}^n} h(x) d\Gamma$$

Theorem 5.2. *Assume u_0 lsc and (H^*, α) lower bounded and $u_0(x) = o(H^*(x))$ as $|x| \rightarrow +\infty$. Assume*

$$(p\alpha)^{q-1} < \left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}},$$

then the function $G_H(t)$, defined by (5.3), is non increasing in $(0, t_\star)$.

Proof. We fix $0 < t_1 < t_2 < t_\star$.

$$G_H(t_2) = \left(\int_{\mathbb{R}^n} \exp(r e^{\alpha t_1} e^{\alpha(t_2-t_1)} Q_{t_2-t_1} Q_{t_1} u_0(x) - b H^*(x)) dx \right)^{\frac{1}{r e^{\alpha t_1} e^{\alpha(t_2-t_1)}}}.$$

On the other hand, since $t_1 < t_\star$,

$$r e^{\alpha t_1} < r^{\alpha t_\star} = \frac{1}{2} \left(\frac{B}{(p\alpha)^{q-1}} + 1 \right)$$

We can apply (5.1) taking $r = r_\star e^{\beta t_1}$, $Q_{t_1} u_0$ as u_0 , $t = t_2 - t_1$, and the thesis follows. $\square$

Theorem 5.3. *To simplify, we assume $u_0 \in C^{1,1}(\mathbb{R}^n)$, (H^*, α) bounded and $u_0(x) = o(H^*(x))$ as $|x| \rightarrow +\infty$. Assume*

$$(p\alpha)^{q-1} < \left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}},$$

fix

$$b = \frac{1}{2} \left(\left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}} - (p\alpha)^{q-1} \right),$$

and

$$r \in \left[1, \frac{1}{2} \left(\frac{\left(\int_{\mathbb{R}^n} \exp(-H^*(x)) dx \right)^{\frac{q}{n}}}{(p\alpha)^{q-1}} + 1 \right) \right).$$

Then the following LSI holds

$$\begin{aligned} & \alpha \mathcal{E}(e^{ru_0(x)}; d\Gamma) + \alpha n \int_{\mathbb{R}^n} e^{ru_0(x)} d\Gamma \\ & \leq \frac{p^p}{r^{p-1}} \int_{\mathbb{R}^n} H \left(D e^{\frac{r}{p}} u_0(x) \right) d\Gamma + \alpha q b \int_{\mathbb{R}^n} e^{ru_0(x)} H^*(x) d\Gamma \end{aligned}$$

Proof. We consider

$$(G_H(t))^{re^{\alpha t}} = \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x) - bH^*(x)) dx.$$

Making the derivative, we get

$$\begin{aligned} & re^{\alpha t} (G_H(t))^{re^{\alpha t}-1} G'_H(t) = -\alpha (G_H(t))^{re^{\alpha t}} \ln(G_H(t))^{re^{\alpha t}} \\ & + \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x) - bH^*(x)) \alpha re^{\alpha t} Q_t u_0(x) dx \\ & + \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x) - bH^*(x)) re^{\alpha t} \partial_t Q_t u_0(x) dx. \end{aligned}$$

By $G'_H(t) \leq 0$ and by the definition of $\mathcal{E}$ it results

$$\begin{aligned} & \alpha \mathcal{E}(\exp(re^{\alpha t} Q_t u_0(x)); d\Gamma) \\ & \leq \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x)) re^{\alpha t} [\alpha x(DQ_t u_0)(x) + H(DQ_t u_0)(x)] d\Gamma \\ & = \alpha \int_{\mathbb{R}^n} \{D \exp(re^{\alpha t} Q_t u_0(x))\} x d\Gamma \\ & + \frac{re^{\alpha t}}{\left(\frac{r}{p} e^{\alpha t}\right)^p} \int_{\mathbb{R}^n} H(D \exp(\frac{r}{p} e^{\alpha t} Q_t u_0(x))) d\Gamma \\ & = -\alpha n \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x)) d\Gamma \\ & - \alpha \int_{\mathbb{R}^n} \exp(re^{\alpha t} Q_t u_0(x)) (xD(-bH^*(x))) d\Gamma \end{aligned}$$

$$+ \frac{re^{\alpha t}}{\left(\frac{r}{p}e^{\alpha t}\right)^p} \int_{\mathbb{R}^n} H(D\exp\left(\frac{r}{p}e^{\alpha t}Q_t u_0(x)\right)) d\Gamma$$

Since

$$xD(H^*(x)) = qH^*(x) \quad (5.4)$$

$$\begin{aligned} & \alpha \mathcal{E}(\exp(re^{\alpha t}Q_t u_0(x)); d\Gamma) + \alpha n \int_{\mathbb{R}^n} \exp(re^{\alpha t}Q_t u_0(x)) d\Gamma \\ & \leq \frac{p^p}{r^{p-1}e^{\alpha(p-1)t}} \int_{\mathbb{R}^n} H\left(De^{\frac{r}{p}e^{\alpha t}Q_t u_0}\right) d\Gamma + \alpha q b \int_{\mathbb{R}^n} e^{re^{\alpha t}(Q_t u_0)(x)} H^*(x) d\Gamma \end{aligned} \quad (5.5)$$

The result follows sending $t \rightarrow 0^+$. $\square$

6. Appendix

Here we show some properties of the semigroup $t \rightarrow Q_t$ helpful to get LSI inequalities in the limit as $t \rightarrow 0^+$ in (5.4).

Proposition 6.1. *Assume $DH, Du_0 \in C_{loc}^{0,1}(\mathbb{R}^n)$, then for any $R \in (0, +\infty)$ there exists $t_0 \in (0, +\infty)$ such that for any $x \in B_R^0 = \{z \in \mathbb{R}^n : |z| < R\}$, $t \in (0, t_0]$ there exists a unique $y = y(x, t) \in B_R^0$ such that*

$$Q_t u_0(x) = u_0(y) + \left(\frac{\alpha p}{1 - e^{-\alpha p t}}\right)^{q-1} H^*(y - e^{-\alpha t}x). \quad (6.1)$$

Furthermore

$$\lim_{t \rightarrow 0^+} y(x, t) = x, \quad \text{uniformly in } B_R.$$

Proof. Let L_H and L_0 the local Lipschitz constants of, respectively, DH, Du_0 , then we set

$$M = \max\{|DH(Du_0(y))| : y \in B_R\}.$$

To find y we need to solve the equation

$$Du_0(y) + DH^*\left(\left(\frac{\alpha p}{1 - e^{-\alpha p t}}\right)(y - e^{-\alpha t}x)\right) = 0,$$

equivalent to

$$y = e^{-\alpha t}x - \left(\frac{1 - e^{-\alpha p t}}{\alpha p}\right) DH(Du_0(y))$$

We fix t_1 and t_2 in $(0, +\infty)$ such that

•

$$M \left(\frac{1 - e^{-\alpha p t_1}}{\alpha p}\right) < R$$

•

$$\left(\frac{1 - e^{-\alpha p t_2}}{\alpha p}\right) L_H L_0 < 1$$

Set

$$t_0 = \min\{t_1, t_2\}.$$

The application

$$y \rightarrow e^{-\alpha t}x - \left(\frac{1 - e^{-\alpha p t}}{\alpha p}\right) DH(Du_0(y)) \quad t \in [0, t_0]$$

is a contraction. Indeed consider the iterative sequence

$$\begin{cases} y_{n+1} = e^{-\alpha t}x - \left(\frac{1 - e^{-\alpha p t}}{\alpha p}\right) DH(Du_0(y_n)), & n \in \mathbb{N} \\ y_0 = e^{-\alpha t}x, \end{cases}$$

then

$$|y_{n+1} - y_n| \leq \left(\frac{1 - e^{-\alpha p t}}{\alpha p}\right) L_H L_0 |y_n - y_{n-1}| < \left(\frac{1 - e^{-\alpha p t}}{\alpha p}\right)^{n+1} (L_H L_0)^n M$$

From this we have the existence and uniqueness of y and

$$\lim_{t \rightarrow 0^+} y_n(x, t) = x, \quad \forall n \in \mathbb{N},$$

uniformly in $B_R \times [0, t_0]$. $\square$

Theorem 6.2. Assume $DH, Du_0 \in C_{loc}^{0,1}(\mathbb{R}^n)$, $y = y(x, t)$ as in proposition (6.1), then for any $(x, t) \in B_R \times [0, t_0]$

$$D(Q_t u_0) = e^{-\alpha t} Du_0(y(x, t))$$

and, passing to the limit,

$$\lim_{t \rightarrow 0^+} D(Q_t u_0) = \lim_{t \rightarrow 0^+} Du_0(y(x, t)) = Du_0(x)$$

Proof. Set

$$w(x, t) = y(x, t) - e^{-\alpha t}x.$$

Then

$$Q_t u_0(x_1) - Q_t u_0(x) \leq u_0(e^{-\alpha t}x_1 + w(x, t)) - u_0(e^{-\alpha t}x + w(x, t)) \quad (6.2)$$

$$Q_t u_0(x_1) - Q_t u_0(x) \geq u_0(e^{-\alpha t}x_1 + w(x_1, t)) - u_0(e^{-\alpha t}x + w(x_1, t)). \quad (6.3)$$

Let $e_i = (\delta_{1i}, \dots, \delta_{ni})$ and $h > 0$ we consider

$$x_1 = x + h e_i.$$

By inequality (6.2) and (6.3) the following limit exists and it results

$$\lim_{h \rightarrow 0^+} \frac{Q_t u_0(x_1) - Q_t u_0(x)}{h} = e^{-\alpha t} \partial_{x_i} u_0(e^{-\alpha t}x + w(x, t))$$

A similar argument applies to negative h . Hence the result follows. $\square$

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