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Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity

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Abstract. In the present paper, by applying variant mountain pass theorem and Ekeland variational principle we study the existence of multiple nontrivial solutions for a class of Kirchhoff type problems with concave nonlinearity

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\triangle u = \alpha(x)|u|^{q-2}u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

A new existence theorem and an interesting corollary of four nontrivial solutions are obtained.

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1. Introduction and preliminaries

Consider the Kirchhoff type problems with concave nonlinearity

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\triangle u = \alpha(x)|u|^{q-2}u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \le \alpha(x) \in L^\infty(\Omega)$, and $\alpha \not\equiv 0$; $1 < q < 2, \Omega$ is smooth bounded domain in $R^N(N = 1,2,3)$, $a, b > 0$, and $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with $f(x,0) = 0$ a.e. $x \in \Omega$. Here we explain why it cannot be taken $N \ge 4$. The reason is the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r > 4$ plays an important role in our argument, and it is well known that the above Sobolev imbedding does not hold when $N \ge 4$. Therefore we only consider the case of $N = 1,2,3$ in the present paper.

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It is pointed out in [8] that the problem (1.1) models several physical and biological systems, where u describes a process which depends on the average of itself (for example, population density). Moreover, this problem is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2\right) \Delta u = g(x,t), \tag{1.2}$$

which was proposed by Kirchhoff [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Some early studies of Kirchhoff equations appeared in Bernstein [3] and Pohoẑaev [18]. However, Eq. (1.2) received much attention only after Lions [13] proposed an abstract framework to the problem. Later, P. D'Ancona and S. Spagnolo [10] studied the global solvability for the degenerate Kirchhoff equation with real analytic data. Recently, many authors studied the following elliptic Kirchhoff type problems

$$\begin{cases} -(a + b\int_{\Omega} |\nabla u|^2)\triangle u = f(x,u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Some interesting results were obtained. Alves [1], Ma–Rivera [14] and He–Zou [11] studied the existence of positive solutions and infinitely many positive solutions of the problem (1.3) by variational methods, respectively; Perera and Zhang [17] obtained one nontrivial solutions of (1.3) by Yang index theory; Mao–Zhang [15], Zhang and Perera [20] got three nontrivial solutions (a positive solution, a negative solution, a sign changing solution) of (1.3) by invariant sets of descent flow; Cheng–Wu [6] obtained the existence results of positive solutions of problem (1.3), also in [7] they used a three critical point theorem due to Brezis–Nirenberg [4] and a Z_2 version of the Mountain Pass Theorem due to Rabinowitz [19] to study the existence of multiple nontrivial solutions of problem (1.3) under some weaker assumptions.

In order to establish multiple solutions for problem (1.1), we make the following assumptions:

(f_1) Subcritical growth condition

$$|f(x,t)| \le C(1 + |t|^{p-1}) \quad \text{for some } 2 < p < 2^* = \begin{cases} 6, & \text{if } N = 3, \\ +\infty, & \text{if } N = 1, 2, \end{cases}$$

where C is a positive constant.

(f_2) Supercubic condition

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t^3} = +\infty, \quad \text{uniformly for a.e. } x \in \Omega.$$

(f_3) There exists $0 \le k(x) \in L^{\infty}(\Omega)$ such that $\|k\|_{\infty} < \lambda_1$ and

$$\limsup_{|t|\to 0} \frac{f(x,t)}{at} = k(x), \quad \text{uniformly for a.e. } x \in \Omega,$$

where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

(f_4) There exists $r > 0$ such that for all $s \in [0, 1]$, one has

$$rP(x,t) \geq P(x, st), \quad \text{for a.e. } x \in \Omega, \text{ and } t \in \mathbf{R},$$

where $P(x,t) = f(x,t)t - 4F(x,t), F(x,t) = \int_0^t f(x,s)\,ds$.

By the assumptions (f_1)–(f_4), we can derive the following simple facts:

(1) (f_1) and (f_2) imply that $p > 4$.

(2) If f is sufficiently regular, (f_2) implies that P is not identically zero. Indeed, if $P = 0$, then, differentiating 4 times,

$$tf''''(x,t) = 0 \quad \text{for a.e. } x \in \Omega, \ \forall\, t \in \mathbf{R},$$

therefore $f(x,t) = a(x)t^3$ for some function $a(x)$, and this contradicts (f_2).

(3) If f is sufficiently regular, (f_3) implies that

$$\|\partial_t f(x,0)\|_\infty = \sup_{x \in \Omega} |f(x,t)| < a\lambda_1.$$

(4) $P(x,0) = 0$ by the definition of P; therefore, by (f_4) (take $s = 0$), one has

$$P(x,t) \geq 0 \quad \text{for a.e. } x \in \Omega, \ \forall\, t \in \mathbf{R}.$$

(5) We are not assuming that $tf(x,t) \geq 0$. For example,

$$f(x,t) = -t^3 + t^5 \qquad (\star)$$

satisfies the four hypotheses (f_1)–(f_4), and $f < 0$ for $t > 0$ small.

(6) If f, close to $t = 0$, is

$$f(x,t) = \sum_{n=0}^{5} a_n t^n + O(t^6) \quad \text{as } t \to 0,$$

then (f_3)–(f_4) imply that $a_0 = a_1 = a_2 = a_4 = 0, a_5 \geq 0, a_3 \in \mathbf{R}$. Hence the example $(\star)$ is "less special" than how one might think.

The assumptions (f_1)–(f_3) are sufficient to guarantee the corresponding energy functional with problem (1.1) has a mountain pass geometry, (f_4) is used to establish the boundedness of Cerami sequence, and then it possess a convergent subsequence. Moreover, there are functions f satisfying all the four hypotheses, for example, $f(x,t) = \mu|t|^{p-2}t$ with $\mu > 0$ or $f(x,t)$ in $(\star)$.

Thanks to the nonlinearity contains the concave nonlinearity $\alpha(x)|u|^{q-2}u$, and the function f is a Carathéodory function which is assumed to be supercubic near infinity. Then the right hand side nonlinearity of the problem (1.1), exhibits the combined effects of "convex" and "concave" nonlinearities. Hence the problem (1.1) is more complicated than (1.3). In the present paper, we are subject to obtain a existence theorem and an interesting corollary of four solutions for the problem (1.1) without using the PS condition.

Let $X := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \quad \|u\| = (u,u)^{\frac{1}{2}}.$$

Throughout the paper, let X' denote the dual of X and $\langle \cdot, \cdot \rangle$ be the duality pairing between X' and X, we denote by $|\cdot|_r$ the usual L^r-norm. Since Ω is a bounded domain, it is well known that $X \hookrightarrow L^r(\Omega)$ continuously for $r \in [1, 2^*]$, compactly for $r \in [1, 2^*)$. Hence, for $r \in [1, 2^*]$, there exists γ_r such that

$$|u|_r \leq \gamma_r \|u\|, \quad \forall\, u \in X. \tag{1.4}$$

Recall that a function $u \in X$ is called a weak solution of (1.1) if

$$(a + b\|u\|^2) \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \left[\alpha(x)|u|^{q-2}uv + f(x, u)v\right] dx, \quad \forall\, v \in X.$$

Seeking a weak solution of problem (1.1) is equivalent to find a critical point of C^1 functional $\Phi : X \mapsto \mathbf{R}$ defined by

$$\Phi(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \Psi(u),$$

where

$$\Psi(u) := \int_\Omega \left[\frac{1}{q}\alpha(x)|u|^q + F(x, u)\right] dx, \quad \forall\, u \in X,$$

Moreover,

$$\langle \Phi'(u), v\rangle = (a + b\|u\|^2) \int_\Omega \nabla u \nabla v \, dx - \int_\Omega \left[\alpha(x)|u|^{q-2}uv + f(x, u)v\right] dx, \quad \forall\, u, v \in X.$$

Our assumptions lead us to consider the eigenvalue problems

$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

Denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ the distinct eigenvalues of the problem (1.5). It is well known that λ_1 can be characterized as

$$\lambda_1 = \inf\{\|u\|^2 : u \in X, |u|_2 = 1\},$$

and λ_1 is achieved by the first eigenfunction $\varphi_1 > 0$.

We need the following concept, which was introduced by Cerami [5]and is a weak version of the (PS) condition.

Definition 1. Let $I \in C^1(X, \mathbf{R})$. We say that I satisfies the Cerami condition at the level $c \in \mathbf{R}$ $\quad ((C)_c$ for short) if any sequence $\{u_n\} \subset X$ along with

$$I(u_n) \to c \quad \text{and } (1 + \|u_n\|)I'(u_n) \to 0, \quad \text{as } n \to \infty$$

possesses a convergent subsequence; I satisfies the (C) condition if I satisfies $(C)_c$ for all $c \in \mathbf{R}$.

Let us mention that Cerami condition is much weaker than (PS) condition, and in our proofs there exists a $(C)_c$ sequence of Φ, thus to find a critical point it is sufficient to establish the $(C)_c$ sequence has a convergent subsequence. Therefore, we use the Cerami condition instead of the classical (PS) condition.

In the present paper, the following Proposition 1.1, variant Mountain Pass Theorem, which can been seen in [2] for the original version Mountain Pass Theorem, and Proposition 1.2 (Ekeland variational principle) are our main tools, which can be found in [9] and [16], respectively.

Proposition 1.1. *Suppose that $I \in C^1(X, \mathbf{R})$ satisfies*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\alpha < \beta, \rho > 0$ and $u_1 \in X$ with $\|u_1\| > \rho$. Let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\}$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} I(\gamma(\tau)).$$

Then $c \geq \beta > \alpha$ and there exists a sequence $\{u_n\} \subset X$ such that

$$I(u_n) \longrightarrow c, \quad (1 + \|u_n\|)\|I'(u_n)\| \longrightarrow 0, \quad \text{as } n \to \infty.$$

Moreover, if I satisfies the $(C)_c$ condition, then c is a critical value of I.

Proposition 1.2. *Let M be a complete metric space with metric d and let $I : M \mapsto (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\epsilon > 0$ be given and $u \in M$ be such that*

$$I(u) \leq \inf_M I + \epsilon.$$

Then there exists $v \in M$ such that

$$I(v) \leq I(u), \quad d(u, v) \leq 1,$$

and for each $w \in M$, one has

$$I(v) \leq I(w) + \epsilon\, d(v, w).$$

Our main results are as follows:

Theorem 1. *Let conditions $(f_1) - (f_4)$ hold. For any given $\epsilon > 0$, set*

$$A = A(\epsilon) = \inf\{c > 0 : |f(x,t)| \leq a(\|k\|_\infty + \epsilon)|t| + c|t|^{p-1}, \forall\ (x,t) \in \overline{\Omega} \times \mathbf{R}\}$$

and

$$\alpha_0 = \left(\frac{b}{4}\right)^{\frac{p-q}{p-4}} \left[\frac{\gamma_q^q}{q}\left(\frac{p(4-q)\gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{q-4}{p-q}} + \frac{A\gamma_p^p}{p}\left(\frac{p(4-q)\gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{p-4}{p-q}}\right]^{-\frac{p-q}{p-4}}.$$

If $\|\alpha\|_\infty \in (0, \alpha_0)$, then the problem (1.1) has at least four nontrivial solutions: two positive and two negative solutions.

Corollary 2. *Let a, b, λ and $\mu > 0, 1 < q < 2, 4 < p < 2^*$. For any given $\epsilon > 0$, set*

$$A' = A'(\epsilon) = \inf\{c > 0 :\ \mu|t|^{p-1} \leq a\epsilon|t| + c|t|^{p-1}, \forall\ t \in \mathbf{R}\}$$

and

$$\lambda_0 = \left(\frac{b}{4}\right)^{\frac{p-q}{p-4}} \left[\frac{\gamma_q^q}{q} \left(\frac{p(4-q)\gamma_q^q}{A'(p-4)q\gamma_p^p} \right)^{\frac{q-4}{p-q}} + \frac{A'\gamma_p^p}{p} \left(\frac{p(4-q)\gamma_q^q}{A'(p-4)q\gamma_p^p} \right)^{\frac{p-4}{p-q}} \right]^{-\frac{p-q}{p-4}}.$$

If $\lambda \in (0, \lambda_0)$, then the following Kirchhoff type problem with concave and convex nonlinearity

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\triangle u = \lambda |u|^{q-2}u + \mu |u|^{p-2}u, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{cases} \tag{1.6}$$

has at least four nontrivial solutions: two positive and two negative solutions.

2. Proof of Theorem 1

We divide the proof of Theorem 1 into two subsections, the first one for the fist positive solution, the second one for the other three solutions.

2.1. The first positive solution

In this subsection, we will devote to obtain the first positive solution of the problem (1.1) by using variant Mountain Pass Theorem.

Set

$$\overline{f}(x,t) = \begin{cases} 0, & \text{if } t \le 0; \\ f(x,t), & \text{if } t > 0, \end{cases}$$

and

$$\overline{F}(x,t) = \int_0^t \overline{f}(x,s)\,ds.$$

Define the energy functional $\overline{\Phi}: \ X \mapsto \mathbf{R}$ by

$$\overline{\Phi}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{q}\int_\Omega \alpha(x)(u^+)^q\,dx - \int_\Omega \overline{F}(x,u)\,dx,$$

where $u^\pm = \max\{\pm u, 0\}$. By virtue of the hypotheses, it is easy to see that $\overline{\Phi}$ is a C^1 functional.

Following, we divide it into four steps to obtain the first positive solution.

Step 1. There exist $\rho > 0, \ \beta > 0$ such that $\overline{\Phi}(u) \ge \beta$ for all $u \in X$ with $\|u\| = \rho$.

Indeed, for any $\epsilon > 0$, by conditions (f_1) and (f_3), then we know $A = A(\epsilon) > 0$ is well defined and

$$\overline{f}(x,t) \le a(\|k\|_\infty + \epsilon)t^+ + A(t^+)^{p-1}. \tag{2.1}$$

So, by (2.1), it follows that

$$\overline{F}(x,t) \le \frac{a}{2}(\|k\|_\infty + \epsilon)(t^+)^2 + \frac{A}{p}(t^+)^p. \tag{2.2}$$

Choosing $\epsilon > 0$ such that $(\|k\|_\infty + \epsilon) < \lambda_1$ then for all $u \in X$, the combination of (1.4) and (2.2) implies that

$$\begin{aligned}\overline{\Phi}(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{q}\int_\Omega \alpha(x)(u^+)^q\,dx - \int_\Omega \overline{F}(x,u)\,dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{q}\|\alpha\|_\infty |u|_q^q - \frac{a}{2}(\|k\|_\infty + \epsilon)\int_\Omega (u^+)^2 dx - \frac{A}{p}\int_\Omega (u^+)^p\,dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{q}\|\alpha\|_\infty \gamma_q^q \|u\|^q - \frac{a}{2}\left(\frac{\|k\|_\infty + \epsilon}{\lambda_1}\right)\|u\|^2 - \frac{A}{p}\gamma_p^p\|u\|^p \\ &\geq \left(\frac{b}{4} - \frac{1}{q}\|\alpha\|_\infty \gamma_q^q\|u\|^{q-4} - \frac{A}{p}\gamma_p^p\|u\|^{p-4}\right)\|u\|^4\end{aligned} \tag{2.3}$$

Set

$$g(t) = \frac{1}{q}\|\alpha\|_\infty \gamma_q^q t^{q-4} + \frac{A}{p}\gamma_p^p t^{p-4}, \quad t > 0.$$

Note that $1 < q < 2,\ 4 < p < 2^*$, then we have

$$\lim_{t\to+\infty} g(t) = \lim_{t\to 0^+} g(t) = +\infty.$$

Therefore g is bounded below. It achieves its minimum at the point t_0 such that $g'(t_0) = 0$, namely

$$t_0 = \left(\frac{p(4-q)\|\alpha\|_\infty \gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{1}{p-q}}.$$

Hence,

$$g(t_0) = \frac{1}{q}\|\alpha\|_\infty \gamma_q^q \left(\frac{p(4-q)\|\alpha\|_\infty \gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{q-4}{p-q}} + \frac{A}{p}\gamma_p^p \left(\frac{p(4-q)\|\alpha\|_\infty \gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{p-4}{p-q}}. \tag{2.4}$$

Since

$$\begin{aligned}\|\alpha\|_\infty < \alpha_0 = \left(\frac{b}{4}\right)^{\frac{p-q}{p-4}} \Bigg[&\frac{\gamma_q^q}{q}\left(\frac{p(4-q)\gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{q-4}{p-q}} \\ &+ \frac{A\gamma_p^p}{p}\left(\frac{p(4-q)\gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{p-4}{p-q}}\Bigg]^{-\frac{p-q}{p-4}},\end{aligned}$$

then by (2.4), it follows that

$$g(t_0) < \frac{b}{4}. \tag{2.5}$$

Take

$$\rho = \rho(\|\alpha\|_\infty) = t_0 = \left(\frac{p(4-q)\|\alpha\|_\infty \gamma_q^q}{A(p-4)q\gamma_p^p}\right)^{\frac{1}{p-q}} > 0,$$

Therefore, the combination of (2.3)and (2.5) implies that

$$\overline{\Phi}(u) \geq \left(\frac{b}{4} - g(\rho)\right)\rho^4 =: \ \beta > 0, \tag{2.6}$$

whenever $u \in X$ with $\|u\| = \rho$.

Step 2. There exists $u_1 \in X$ with $\|u_1\| > \rho$ such that $\overline{\Phi}(u_1) < 0$.

In fact,according to condition (f_2), we have

$$\lim_{t\to+\infty} \frac{\overline{f}(x,t)}{t^3} = +\infty, \quad \text{uniformly for a.e. } x \in \Omega. \tag{2.7}$$

By (2.7), for every $M > 0$, there exists $T_M > 0$ such that

$$\overline{f}(x,t) \geq M(t)^3 \quad \forall\, t \geq T_M, \quad \text{for a.e. } x \in \Omega;$$

$\overline{f}$ is continuous as a function of t, so $\overline{f} - Mt^3$ has a minimum on $[0, T_M]$; and $\overline{f} - M(t^+)^3 = 0$ for all $t < 0$, hence

$$\overline{f}(x,t) \geq M(t^+)^3 - C_M, \quad \text{for a.e. } x \in \Omega \text{ and } \forall\, t \in \mathbf{R}$$

for some $C_M > 0$. It implies that

$$\overline{F}(x,t) \geq \frac{1}{4}M(t^+)^4 - C_M t^+. \tag{2.8}$$

Choose $\phi \in X$ with $\phi(x) > 0$ a.e. $x \in \Omega$ and $|\phi|_4 = 1$, (2.8) implies that

$$\begin{aligned}\liminf_{t\to+\infty} \textstyle\int_\Omega \frac{\overline{F}(x,t\phi)}{t^4}\, dx &\geq \liminf_{t\to+\infty} \textstyle\int_\Omega \left[\frac{1}{4}M(\phi(x))^4 - \frac{C_M\phi(x)}{t^3}\right] dx \\ &\geq \liminf_{t\to+\infty} \left(\tfrac{1}{4}M - \tfrac{C_M}{t^3}|\phi|_1\right) \\ &= \tfrac{1}{4}M.\end{aligned}$$

Thanks to the arbitrariness of M, we can conclude that

$$\lim_{t\to+\infty} \frac{1}{t^4}\int_\Omega \overline{F}(x,t\phi)\, dx = +\infty. \tag{2.9}$$

Consequently, (2.9) implies that

$$\begin{aligned}\lim_{t\to+\infty} \frac{\overline{\Phi}(t\phi)}{t^4} = \lim_{t\to+\infty} \Bigg(&\frac{a\|\phi\|^2}{2t^2} + \frac{b\|\phi\|^4}{4} - \frac{t^q}{qt^4}\int_\Omega \alpha(x)|\phi|^q\, dx \\ &- \frac{1}{t^4}\int_\Omega \overline{F}(x,t\phi)\, dx\Bigg) = -\infty,\end{aligned}$$

which implies that there exists $u_1 \in X$ with $\|u_1\| > \rho$ such that $\overline{\Phi}(u_1) < 0$.

From the facts of step 1 and step 2, we have

$$\max\{\overline{\Phi}(0),\ \overline{\Phi}(u_1)\} \leq 0 < \beta \leq \inf_{\|u\|=\rho} \overline{\Phi}(u).$$

Define

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\},$$

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} \overline{\Phi}(\gamma(\tau)).$$

By Proposition 1.1 we know that $c \geq \beta > 0$ and there exists a sequence $\{u_n\} \subset X$ such that

$$\overline{\Phi}(u_n) \longrightarrow c, \qquad (1 + \|u_n\|)\|\overline{\Phi}'(u_n)\| \longrightarrow 0, \text{ as } n \to \infty. \tag{2.10}$$

Step 3. The sequence $\{u_n\}$ satisfying (2.10) is bounded in X.

First, we show $\{u_n^-\}$ is bounded in X. Indeed,

$$|\langle \overline{\Phi}'(u_n), -u_n^-\rangle| = (a + b\|u_n\|^2)\|u_n^-\|^2 \geq a\|u_n^-\|^2;$$

By Cauchy–Schwartz inequality and (2.10),

$$|\langle \overline{\Phi}'(u_n), -u_n^-\rangle| \leq \|\overline{\Phi}'(u_n)\|\|u_n^-\| \leq \|\overline{\Phi}'(u_n)\|(1+\|u_n^-\|) \to 0, \text{ as } n \to \infty. \tag{2.11}$$

Therefore $u_n^- \to 0$, as $n \to \infty$, and hence for large n, it has

$$\|u_n^-\| \leq 1. \tag{2.12}$$

Following, we will prove $\{u_n^+\}$ is bounded in X. By contradiction, we assume that $\|u_n^+\| \to +\infty$, as $n \to \infty$. Set $\omega_n = \frac{u_n^+}{\|u_n^+\|}$, then $\|\omega_n\| = 1$ and $\omega_n \geq 0$ for all $n \geq 1$. Since X is a reflexive Banach space, we can assume that there exists $\omega \in X$ such that

$$\omega_n \rightharpoonup \omega \quad \text{in } X, \quad \omega_n \to \omega \quad \text{in } L^r(\Omega), \quad \omega_n(x) \to \omega(x) \quad \text{for a.e. } x \in \Omega, \tag{2.13}$$

as $n \to \infty$, where $r \in [1, 2^*)$. Observe that $u_n = u_n^+ - u_n^-$, (2.10) and (2.11) imply that

$$|\langle \overline{\Phi}'(u_n), u_n^+\rangle| = \left|\langle \overline{\Phi}'(u_n), u_n\rangle - \langle \overline{\Phi}'(u_n), -u_n^-\rangle\right| \leq |\langle \overline{\Phi}'(u_n), u_n\rangle| + |\langle \overline{\Phi}'(u_n), -u_n^-\rangle| \to 0$$

as $n \to \infty$. Therefore for large n, it has

$$\left|A - B - \int_\Omega \overline{f}(x, u_n)u_n^+ \, dx\right| \leq 1, \tag{2.14}$$

where

$$A := a\|u_n^+\|^2 + b\|u_n^+\|^4 + b\|u_n^+\|^2\|u_n^-\|^2, \quad B := \int_\Omega \alpha(x)(u_n^+)^q \, dx, \quad A \geq 0, \quad B \geq 0.$$

Then

$$-1 \leq A - B - \int_\Omega \overline{f}(x, u_n)u_n^+ \, dx \leq 1,$$

hence

$$\int_\Omega \overline{f}(x, u_n)u_n^+ \, dx \leq A - B + 1 \leq A + 1.$$

So,for large n, by (2.12),it follows that

$$\int_\Omega \overline{f}(x, u_n)u_n^+ \, dx \leq a\|u_n^+\|^2 + b\|u_n^+\|^4 + b\|u_n^+\|^2 + 1. \tag{2.15}$$

By virtue of (1.4) and (2.15), we conclude that

$$\int_\Omega \frac{\overline{f}(x,u_n^+)}{\|u_n^+\|^3}\omega_n(x)\,dx \le \frac{1}{\|u_n^+\|^4} + \frac{a}{\|u_n^+\|^2} + \frac{b}{\|u_n^+\|^2} + b. \tag{2.16}$$

Set $\Omega_1 = \{x \in \Omega : \omega(x) > 0\}$, thus $u_n^+(x) \to +\infty$ a.e. $x \in \Omega_1$, as $n \to \infty$. Moreover, By (2.7), for any $M > 0$, there exists $C_M > 0$ such that

$$\overline{f}(x,t) \ge M(t^+)^3 - C_M, \quad \text{for a.e. } x \in \Omega \text{ and } \forall\, t \in \mathbf{R},$$

hence

$$\overline{f}(x,u_n^+) \ge M(u_n^+)^3 - C_M, \quad \text{for a.e. } x \in \Omega \text{ and } \forall\, n \in \mathbf{N}, \tag{2.17}$$

Combining (2.13) and (2.17), one has

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \frac{\overline{f}(x,u_n^+)}{\|u_n^+\|^3}\omega_n(x)\,dx &= \lim_{n\to\infty}\int_\Omega \frac{\overline{f}(x,u_n^+)}{(u_n^+(x))^3}(\omega_n(x))^4\,dx \\ &\ge \lim_{n\to\infty}\left(M\int_\Omega (\omega_n(x))^4\,dx - \frac{C_M}{\|u_n^+\|^3}\int_\Omega \omega_n(x)\,dx\right) \\ &= M\int_{\Omega_1}(\omega(x))^4\,dx.\end{aligned}$$

If the measure of Ω_1, $m(\Omega_1) > 0$, we infer that

$$\int_\Omega \frac{\overline{f}(x,u_n^+)}{\|u_n^+\|^3}\omega_n(x)\,dx \to +\infty, \quad \text{as } n \to \infty. \tag{2.18}$$

Noticing that $\|u_n^+\| \to +\infty$, so (2.16) contradicts (2.18) by passing to limit in (2.16) as $n \to \infty$. Hence, $m(\Omega_1) = 0$ and $\omega(x) = 0$, for a.e. $x \in \Omega$. Moreover, by (f_1), (2.13) and Lebesgue dominated convergence theorem, one has

$$\lim_{n\to\infty}\int_\Omega \overline{F}(x,\omega_n)\,dx = 0. \tag{2.19}$$

For any given positive integer $m \ge 1$, set

$$g_n(x) = \|u_m^+\|\omega_n(x) \quad \forall\, x \in \Omega \text{ and } n \ge 1.$$

Similarly to (2.19), we can conclude that

$$\lim_{n\to\infty}\int_\Omega \overline{F}(x,g_n)\,dx = 0. \tag{2.20}$$

Note that $\|u_n^+\| \to +\infty$, then for suitable subsequence $\{u_n^+\}$, still denoted by $\{u_n^+\}$ such that

$$0 \le s_n := \frac{\|u_m^+\|}{\|u_n^+\|} \le 1.$$

Define

$$\overline{\Phi}(\eta_n u_n^+) := \max_{\eta\in[0,1]} \overline{\Phi}(\eta u_n^+).$$

By virtue of (1.4), (2.13) and (2.20), then for large n we can conclude that

$$\begin{aligned}\overline{\Phi}(\eta_n u_n^+) &\geq \overline{\Phi}(s_n u_n^+) = \overline{\Phi}(g_n)\\ &= \frac{a}{2}\|g_n\|^2 + \frac{b}{4}\|g_n\|^4 - \frac{1}{q}\int_\Omega \alpha(x)|g_n|^q\,dx - \int_\Omega \overline{F}(x,g_n)\,dx\\ &\geq \frac{a}{2}\|g_n\|^2 + \frac{b}{4}\|g_n\|^4 - \frac{\|\alpha\|_\infty}{q}\gamma_q^q\|g_n\|^q - \int_\Omega \overline{F}(x,g_n)\,dx\\ &\geq \frac{b}{8}\|u_m^+\|^4, \end{aligned} \tag{2.21}$$

which implies that

$$\overline{\Phi}(\eta_n u_n^+) \to +\infty, \quad \text{as } n\to\infty. \tag{2.22}$$

Moreover, according to (2.12) and (f_1), we have

$$\overline{\Phi}(u_n^+) = \overline{\Phi}(u_n) - \overline{\Phi}(-u_n^-) - \frac{1}{q}\int_\Omega \alpha(x)|u_n^+|^q\,dx - \frac{b}{2}\|u_n^+\|^2\|u_n^-\|^2 < D, \quad \forall n\in\mathbf{N} \tag{2.23}$$

for some constant $D > 0$. Therefore, the combination of (2.22) and (2.23) implies that $\eta_n \in (0,1)$ and

$$\langle \overline{\Phi}'(\eta_n u_n^+), \eta_n u_n^+\rangle = 0,$$

that is,

$$\int_\Omega \overline{f}(x,\eta_n u_n^+)\eta_n u_n^+\,dx = a\eta_n^2\|u_n^+\|^2 + b\eta_n^4\|u_n^+\|^4 - \int_\Omega \alpha(x)(\eta_n u_n^+)^q\,dx. \tag{2.24}$$

Hence, for large n, the combination of (2.21) and (2.24) implies that

$$\begin{aligned}&\int_\Omega \left[\frac{1}{4}f(x,\eta_n u_n^+)\eta_n u_n^+ - F(x,\eta_n u_n^+)\right]dx\\ &= \int_\Omega \left[\frac{1}{4}\overline{f}(x,\eta_n u_n^+)\eta_n u_n^+ - \overline{F}(x,\eta_n u_n^+)\right]dx\\ &= \frac{a}{4}\eta_n^2\|u_n^+\|^2 + \frac{b}{4}\eta_n^4\|u_n^+\|^4 - \frac{1}{4}\int_\Omega \alpha(x)(\eta_n u_n^+)^q\,dx - \int_\Omega \overline{F}(x,\eta_n u_n^+)\,dx\\ &= \Phi(\eta_n u_n^+) + \left(\frac{1}{q}-\frac{1}{4}\right)\int_\Omega \alpha(x)(\eta_n u_n^+)^q\,dx - \frac{a}{4}\eta_n^2\|u_n^+\|^2\\ &\geq \frac{b}{8}\|u_m^+\|^4 - \frac{a}{4}\|u_n^+\|^2. \end{aligned}$$

Therefore, for large n, by (f_4), it follows that

$$\begin{aligned}\int_\Omega \left[\frac{1}{4} f(x,u_n^+)u_n^+ - F(x,u_n^+)\right] dx &= \int_\Omega \frac{1}{4} P(x,u_n^+)\, dx \\ &\geq \int_\Omega \frac{1}{4r} P(x,\eta_n u_n^+)\, dx \\ &= \frac{1}{r}\int_\Omega \left[\frac{1}{4} f(x,\eta_n u_n^+)\eta_n u_n^+ - F(x,\eta_n u_n^+)\right] dx \\ &\geq \frac{b}{8r}\|u_m^+\|^4 - \frac{a}{4r}\|u_n^+\|^2.\end{aligned} \tag{2.25}$$

On the other hand, for large n, by virtue of (2.10) and (2.23), there exists some constant $M' > 0$ such that

$$\begin{aligned}\int_\Omega \left[\frac{1}{4} f(x,u_n^+)u_n^+ - F(x,u_n^+)\right] dx &= \int_\Omega \left[\frac{1}{4}\overline{f}(x,u_n^+)\eta_n u_n^+ - \overline{F}(x,u_n^+)\right] dx \\ &= \overline{\Phi}(u_n^+) - \frac{1}{4}\langle \overline{\Phi}'(u_n), u_n^+\rangle + C_1\int_\Omega \alpha(x)(u_n^+)^q\, dx \\ &\quad + \frac{b}{4}\|u_n^+\|^2\|u_n^-\|^2 - \frac{a}{4}\|u_n^+\|^2 \\ &\leq M' + C_1\int_\Omega \alpha(x)(u_n^+)^q\, dx + \frac{b}{a}\|u_n^+\|^2 \\ &\leq M' + C_2\|u_n^+\|^q + \frac{b}{a}\|u_n^+\|^2\end{aligned} \tag{2.26}$$

where $C_1 = (\frac{1}{q} - \frac{1}{4})$, $C_2 = C_1\gamma_q^q\|\alpha\|_\infty$. For large n, the combination of (2.25) and (2.26) implies that

$$\frac{b}{8r}\|u_m^+\|^4 \leq M' + C_2\|u_n^+\|^q + \frac{b}{a}\|u_n^+\|^2 + \frac{a}{4r}\|u_n^+\|^2.$$

This is a contradiction due to $q < 4$ and the arbitrariness of m. Hence, $\{u_n^+\}$ is bounded in X. Consequently, $\{u_n\}$ is bounded in X.

Step 4. The sequence $\{u_n\}$ satisfying (2.10) has a convergent subsequence in X.

Indeed, from step 3, there exists a positive constant B such that $\|u_n\| \leq B$. By the reflexivity of X, we can assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in } X, \quad u_n \to u \quad \text{in } L^r(\Omega), \quad u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega,$$

as $n \to \infty$, where $r \in [1, 2^*)$. Hence, by (f_1), we know that there is $C_3 > 0$ such that

$$\begin{aligned}\int_\Omega \overline{f}(x, u_n)(u - u_n)\, dx &\leq \left(\int_\Omega |\overline{f}(x, u_n)|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |u - u_n|^p\, dx\right)^{\frac{1}{p}} \\ &\leq 2C\left[\int_\Omega (|u_n|^p + 1)\, dx\right]^{\frac{p-1}{p}} \cdot |u - u_n|_p \\ &\leq C_3 |u - u_n|_p \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \alpha(x)(u_n^+)^{q-1}(u_n - u)\, dx &\leq \|\alpha\|_\infty \left(\int_\Omega (u_n^+)^q\, dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_n - u|^q\, dx\right)^{\frac{1}{q}} \\ &= \|\alpha\|_\infty (B\gamma_q)^{q-1} |u_n - u|_q \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Moreover,

$$\begin{aligned}(a + b\|u_n\|^2) \int_\Omega \nabla u_n \nabla(u - u_n)\, dx &- \int_\Omega [\alpha(x)(u_n^+)^{q-1}(u - u_n) \\ &+ \overline{f}(x, u_n)(u - u_n)]\, dx = \langle \overline{\Phi}'(u_n), (u - u_n)\rangle \to 0,\end{aligned}$$

as $n \to \infty$. Then it has

$$\|u_n\| \longrightarrow \|u\|, \quad \text{as } n \to \infty.$$

Hence, $u_n \to u$ in X due to the uniform convexity of X.

Now, since $\overline{\Phi} \in C^1$, we have

$$\overline{\Phi}'(u) = 0, \quad \text{and} \quad \overline{\Phi}(u) = c \geq \beta > 0.$$

Hence u is a nontrivial critical point of $\overline{\Phi}$. Moreover, $u(x) > 0$ for all $x \in \Omega$. Indeed,

$$(a + b\|u\|^2) \int_\Omega \nabla u \nabla u^-\, dx - \int_\Omega \left[\alpha(x)(u^+)^{q-1}u^- + \overline{f}(x, u)u^-\right]\, dx = 0. \tag{2.27}$$

Thus, it is easy to see that (2.27) implies $\|u^-\| = 0$. So $u(x) \geq 0, x \in \overline{\Omega}$. Now it follows from the strong maximum principle that $u(x) > 0$ for all $x \in \Omega$. Hence we have

$$\Phi'(u) = 0, \quad \text{and} \quad \Phi(u) = c \geq \beta > 0,$$

that is, u is a positive solution of the problem (1.1).

2.2. The other three solutions

In this subsection we will devote to obtain the other three solutions. First, we will apply the Ekeland variational principle to obtain another positive solution of the problem (1.1).

By virtue of (2.3) and (2.6), it is easy to see that

$$\inf_{\overline{B}_\rho} \overline{\Phi} > -\infty, \quad \text{and} \quad \inf_{\partial B_\rho} \overline{\Phi} \geq \beta > 0,$$

where B_ρ is the open ball in X of radius ρ and ∂B_ρ denotes its boundary. Take $0 \le \phi \in C_0^1(\Omega)\backslash\{0\}$ with $\int_\Omega \alpha(x)|\phi|^q\,dx > 0$ and $t \in (0,\ 1)$, we have

$$\begin{aligned}\overline{\Phi}(t\phi) &= \frac{at^2}{2}\|\phi\|^2 + \frac{bt^4}{4}\|\phi\|^4 - \frac{t^q}{q}\int_\Omega \alpha(x)|\phi|^q\,dx - \int_\Omega \overline{F}(x,t\phi)\,dx \\ &\le \frac{at^2}{2}\|\phi\|^2 + \frac{bt^4}{4}\|\phi\|^4 - \frac{t^q}{q}\int_\Omega \alpha(x)|\phi|^q\,dx,\end{aligned}$$

which implies that

$$\overline{\Phi}(t\phi) < 0, \quad \text{for } t \in (0,1) \text{ small}$$

due to $1 < q < 2$. Consequently,

$$-\infty < \inf_{\overline{B}_\rho} \overline{\Phi} < 0.$$

Set $\frac{1}{n} \in (0,\ \inf_{\partial B_\rho} \overline{\Phi} - \inf_{\overline{B}_\rho} \overline{\Phi}),\ n \in \mathbf{N}$, there is $v_n \in \overline{B}_\rho$ such that

$$\overline{\Phi}(v_n) \le \inf_{\overline{B}_\rho} \overline{\Phi} + \frac{1}{n}, \tag{2.28}$$

by Proposition 1.2 (Ekeland variational principle), then

$$\overline{\Phi}(v_n) \le \overline{\Phi}(u) + \frac{1}{n}\|u - v_n\|, \quad \text{for all } u \in \overline{B}_\rho. \tag{2.29}$$

while

$$\overline{\Phi}(v_n) \le \inf_{\overline{B}_\rho} \overline{\Phi} + \epsilon < \inf_{\partial B_\rho} \overline{\Phi}.$$

So $v_n \in B_\rho$. Define $\psi_n :\ X \mapsto \mathbf{R}$ by

$$\psi_n(u) = \overline{\Phi}(u) + \frac{1}{n}\|u - v_n\|.$$

By (2.29), we have $v_n \in B_\rho$ minimizes ψ_n on $\overline{B}_\rho$. Therefore, for all $\phi \in X$ with $\|\phi\| = 1$, take $t > 0$ such that $v_n + t\phi \in \overline{B}_\rho$, then

$$\frac{\psi_n(v_n + t\phi) - \psi_n(v_n)}{t} \ge 0. \tag{2.30}$$

(2.30) implies that

$$\frac{\overline{\Phi}(v_n + t\phi) - \overline{\Phi}(v_n)}{t} + \frac{1}{n} \ge 0,$$

which implies

$$\langle \overline{\Phi}'(v_n),\ \phi\rangle \ge -\frac{1}{n}.$$

Hence,

$$\|\overline{\Phi}'(v_n)\| \le \frac{1}{n}. \tag{2.31}$$

Passing to the limit in (2.28) and (2.31), we conclude that

$$\overline{\Phi}(v_n) \to \inf_{\overline{B}_\rho} \overline{\Phi}, \quad \text{and} \quad \|\overline{\Phi}'(v_n)\| \to 0, \quad \text{as } n \to \infty. \tag{2.32}$$

Since $\|v_n\| < \rho$, then (2.32) implies

$$\overline{\Phi}(v_n) \to \inf_{\overline{B}_\rho} \overline{\Phi}, \quad \text{and} \quad (1 + \|v_n\|)\|\overline{\Phi}'(v_n)\| \to 0, \quad \text{as} \quad n \to \infty. \tag{2.33}$$

As same as step 4 in the Sect. 2.1, we can obtain $\{v_n\}$ has a convergent subsequence, still denoted by $\{v_n\}$, and $v_n \to v \in \overline{B}_\rho$, as $n \to \infty$. Consequently, by (2.33), one has

$$\overline{\Phi}(v) = \inf_{\overline{B}_\rho} \overline{\Phi}, \text{ and } \overline{\Phi}'(v) = 0.$$

Next, we claim that $0 \not\equiv v \not\equiv u$. Indeed,

$$\overline{\Phi}(v) = \inf_{\overline{B}_\rho} \overline{\Phi} < 0 = \overline{\Phi}(0) < \beta \le c = \overline{\Phi}(u).$$

Therefore, v is a nontrivial critical point of $\overline{\Phi}$. Similarly, we can obtain v is another positive solution of the problem (1.1).

Lastly, in order to seek two negative solutions, we truncate f as following

$$\underline{f}(x,t) = \begin{cases} 0, & \text{if } t > 0; \\ f(x,t), & \text{if } t \le 0, \end{cases}$$

and

$$\underline{F}(x,t) = \int_0^t \underline{f}(x,s)\, ds.$$

Define $\underline{\Phi}: \ X \mapsto \mathbf{R}$ defined by

$$\underline{\Phi}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{q}\int_\Omega \alpha(x)(u^-)^q - \int_\Omega \underline{F}(x,u)\, dx.$$

By virtue of the hypotheses, it is easy to see that $\underline{\Phi}$ is a C^1 functional. Then two negative solutions of the problem (1.1) are obtained in a similar arguments of obtaining two positive solutions. This completes the proof of Theorem 1.

3. Proof of Corollary 2

Set

$$f(x,t) = \mu|t|^{p-2}t, \qquad \alpha(x) \equiv \lambda, \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbf{R}.$$

then it is easy to verify f satisfies $(f_1) - (f_4)$. Moreover, by virtue of (f_1) and (f_3), we know that $A' = A'(\epsilon) > 0$ is well defined, then the conclusion directly follows from Theorem 1 with replacing A and α_0 by A' and λ_0, respectively.

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